Rend. Sem. Mat. Univ. Padova, Vol. 144 (2020), 159–176
DOI 10.4171/RSMUP/63

A version of purity on local abelian groups

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To László Fuchs on the occasion of his 95[th] birthday

Abstract – In [6], generalizations of the standard notion of purity on p-local abelian groups were defined using functorial methods to create injective resolutions. For example, if λ is a limit ordinal, then for a group G the completion functor $L_\lambda G$ determines the notion of L_λ-purity. Another way of constructing a type of purity, called $p_{\mathrm{w}}^{<\lambda}$-*purity,* is defined using the functor $\prod_{\alpha<\lambda}(G/p^\alpha G)$. Properties of this second type of purity are studied; for example, it is shown to be hereditary if and only if λ has countable cofinality. In addition, L_λ and $p_{\mathrm{w}}^{<\lambda}$-purity are compared in a variety of contexts, for example, in the category of Warfield groups.

Mathematics Subject Classification (2010). 20K10, 20K40.

Keywords. Local abelian group, purity, balanced projectives, Warfield groups.

1. Introduction

It was through the writings of Professor Fuchs that I learned of the beauty of abelian groups. Few other topics in mathematics are so strongly identified with a single individual, and his deep and original contributions are cited in every paper I have ever published in the field. He has been unfailingly supportive and generous to me during my career, for which I will always be grateful.

In this paper the term "group" will mean a p-local abelian group (where p is a fixed prime). As in [7], it would be possible to include non-local groups in some of these considerations, but focusing on the local case simplifies matters without unduly restricting our results. Throughout, the letter λ will denote a limit

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ordinal and the letter ξ will denote the short-exact sequence $0 \to G \to X \to A \to 0$, thought of as an element of $\mathrm{Ext}(A, G)$. Unexplained terminology can be found in [2]. In particular, we will assume some basic familiarity with *balanced projective* and *Warfield* groups, see [2, Chapter 15, Sections 7–9].

Following [6], an *injective transformation* $\mathcal{F} = (\Phi, F)$ is a natural transformation Φ from the identity functor to a functor F on the category of groups (that is, a collection of natural homomorphisms $\Phi_G : G \to FG$) such that for every group G, the map $\Phi_{FG} : FG \to F(FG)$ is a split injection. We say ξ is $\mathcal{F}$*-pure* if it represents an element of the kernel of $\mathrm{Ext}(A, G) \to \mathrm{Ext}(A, FG)$; that is, if we can complete the diagram

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, "\Phi_G"'] & X \arrow[r] \arrow[dl, "?"] & A \arrow[r] & 0 \\
& LG & & &
\end{tikzcd}$$

We denote the group of $\mathcal{F}$-pure sequences by $\mathrm{Ext}_{\mathcal{F}}(A, G) \subseteq \mathrm{Ext}(A, G)$.

We will be interested in comparing different examples of these notions of purity. In particular, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two injective transformations, then a group A will be said to have the $\mathcal{F}_1 = \mathcal{F}_2$*-purity property* if any such sequence ξ is $\mathcal{F}_1$-pure if and only if it is $\mathcal{F}_2$-pure; that is, if $\mathrm{Ext}_{\mathcal{F}_1}(A, G) = \mathrm{Ext}_{\mathcal{F}_2}(A, G)$ for all groups G.

Let RG and QG denote, respectively, the kernel and cokernel of Φ_G. If D is an injective hull for RG, then there is an $\mathcal{F}$-pure injective resolution of G of the form $0 \to G \to FG \oplus D \to QG \oplus (D/RG) \to 0$, see [6, Corollary 3].

All of the examples considered in [6] have the property that $\mathcal{F}$-purity has enough projectives, as well as enough injectives. The injective transformation $\mathcal{F}$ was said to be *dense* if QG is divisible for all $G \in \mathcal{A}$. Dense injective transformations have several useful properties. For example, they are *strongly hereditary* in the sense that an arbitrary subgroup of an $\mathcal{F}$-pure projective shares that property [6, Proposition 8]. And when $\mathcal{F}$ is dense, the sequence ξ will be $\mathcal{F}$-pure if and only if the sequence $F\xi := 0 \to FG \to FX \to FA \to 0$ is splitting exact [6, Proposition 7].

We review a few important examples from [6].

1. p^α-PURITY. Following Nunke (see [3, Chapter VI]), if α is an ordinal, then consider a sequence $0 \to \mathbb{Z}_{(p)} \to T_\alpha \to H_\alpha \to 0$ where $\mathbb{Z}_{(p)} = p^\alpha T_\alpha$ is the integers (localized at p) and H_α is the "generalized Prüfer group." This determines an injective transformation $P^\alpha : G \cong \mathrm{Hom}(\mathbb{Z}_{(p)}, G) \to \mathrm{Ext}(H_\alpha, G)$ in the usual manner. For all groups A, G we have $\mathrm{Ext}_{p^\alpha}(A, G) = p^\alpha\, \mathrm{Ext}(A, G)$.

The group A is p^α-pure projective if and only if it is isomorphic to a summand of $\mathrm{Tor}(H_\alpha, A)$*, see* [3, Theorem 82].

To discuss our other examples, we introduce a slight variation on some standard terminology. If α is an ordinal, then the group B will be said to be an *α-balanced projective* if $p^\alpha B$ free and $B/p^\alpha B$ is a totally projective p-group. Similarly, we say B is *$< \alpha$-balanced projective* if $B \cong \bigoplus_{\beta<\alpha} B_\beta$, where each B_β is β-balanced projective. It follows that a group is balanced projective if and only if it is $< \alpha$-balanced projective for some α. A balanced projective group is $< \lambda$-balanced projective if and only if it is p^λ-bounded.

2. L_λ-PURITY. Let $L_\lambda G$ be the completion of G with respect to the λ-topology, which uses $\{p^\alpha G\}_{\alpha<\lambda}$ as a neighborhood basis of 0. So $L_\lambda G$ is the inverse limit of $G/p^\alpha G$ for $\alpha < \lambda$ using the natural maps between these groups. There is an obvious transformation $G \to L_\lambda G$ whose kernel is $p^\lambda G$. If λ has countable cofinality, then L_λ is a dense transformation (i.e., $L_\lambda G/(G/p^\lambda G)$ will always be divisible), but this may fail when λ has uncountable cofinality [8, Theorem 2.2].

The group A is L_λ-pure projective if and only if there is a group C and an exact sequence $0 \to A \oplus C \to B \to Z \to 0$, where B is $< \lambda$-balanced projective and Z is p^λ-pure projective [6, Theorem 4].

3. WEAK p^α-PURITY, WHICH WE REFER TO AS p^α_{w}-PURITY. If α is an ordinal, then using the canonical epimorphism we have a transformation $G \to \mathrm{W}^\alpha G := G/p^\alpha G$. This is clearly a dense injective transformation (in fact, the cokernel of $G \to G/p^\alpha G$ is 0). Using terminology from [4], we refer to this as weak p^α-purity or p^α_{w}-purity. So ξ is p^α_{w}-pure if and only if $\xi/p^\alpha\xi := 0 \to G/p^\alpha G \to X/p^\alpha X \to A/p^\alpha A \to 0$ is splitting exact.

The p^α_{w}-pure projectives are the groups A which have a free subgroup $F \subseteq A$ such that A/F embeds in a p^α-bounded totally projective p-group [6, Proposition 14] and [4, Theorem 2.8].

The group A is p^α_{w}-pure projective if and only if it is isomorphic to a subgroup of an α-balanced projective group. In particular, if A is torsion, then it is p^α_{w}-pure projective if and only if it is isomorphic to a subgroup of a p^α-bounded totally projective p-group [4, Corollary 2.9].

We now introduce a related injective transformation that will have a central role in our discussions.

4. $p^{<\lambda}_{\mathrm{w}}$-PURITY. Let $\mathrm{W}^{<\lambda} G = \prod_{\alpha<\lambda} G/p^\alpha G$; the diagonal map determines a transformation $G \to \mathrm{W}^{<\lambda} G$, for which $RG = p^\lambda G$. Note that for each $\alpha < \lambda$ there is a natural projection $\mathrm{W}^{<\lambda} G \to G/p^\alpha G$ which induces a homomorphism $(\mathrm{W}^{<\lambda} G)/p^\alpha(\mathrm{W}^{<\lambda} G) \to G/p^\alpha G$. These fit together to determine a left inverse to $\mathrm{W}^{<\lambda} G \to \mathrm{W}^{<\lambda}(\mathrm{W}^{<\lambda} G)$. In other words, W^λ determines an injective transformation which we denote by $p^{<\lambda}_{\mathrm{w}}$.

Proposition 1.1. *If A, G are groups, then*

$$\operatorname{Ext}_{p_{\mathrm{w}}^{<\lambda}}(A,G)=\bigcap_{\alpha<\lambda}\operatorname{Ext}_{p_{\mathrm{w}}^{\alpha}}(A,G).$$

A group is $p_{\mathrm{w}}^{<\lambda}$-pure projective if and only if it is a summand of a group of the form $\bigoplus_{\alpha<\lambda} B_\alpha$, where each B_α is a p_{w}^{α}-pure projective; *that is, each B_α is a subgroup of an α-balanced projective. There are enough $p_{\mathrm{w}}^{<\lambda}$-pure projectives.*

Proof. The $p_{\mathrm{w}}^{<\lambda}$-pure sequences will be those in the kernel of

$$\operatorname{Ext}(A,G)\longrightarrow \operatorname{Ext}\Big(A,\prod_{\alpha<\lambda} G/p^\alpha G\Big)\cong \prod_{\alpha<\lambda}\operatorname{Ext}(A,G/p^\alpha G).$$

This clearly implies the first statement. Regarding the second, for each $\alpha<\lambda$, since $\operatorname{Ext}_{p_{\mathrm{w}}^{<\lambda}}(A,G)\subseteq \operatorname{Ext}_{p_{\mathrm{w}}^{\alpha}}(A,G)$, any p_{w}^{α}-pure projective is $p_{\mathrm{w}}^{<\lambda}$-pure projective. Suppose $B_\alpha\to A$ is a p_{w}^{α}-pure projective surjection onto A. The sum map $B_{<\lambda}:=\bigoplus_{\alpha<\lambda} B_\alpha\to A$ defines a surjection and $B_{<\lambda}$ is $p_{\mathrm{w}}^{<\lambda}$-pure projective. In addition, if $\alpha<\lambda$, then $B_{<\lambda}\to A$ can be viewed as a pushout of the surjection $B_\alpha\to A$. So $B_{<\lambda}\to A$ will be p_{w}^{α}-pure for all $\alpha<\lambda$, i.e., it is a $p_{\mathrm{w}}^{<\lambda}$-pure projective resolution of A. So $p_{\mathrm{w}}^{<\lambda}$-purity has enough projectives. And if A is $p_{\mathrm{w}}^{<\lambda}$-projective, this sequence must split, completing the proof. □

If α is an ordinal, then ξ is p_{w}^{α}-pure if and only if $\xi/p^\alpha\xi$ splits. Proposition 1.1 implies that ξ is $p_{\mathrm{w}}^{<\lambda}$-pure if and only if $\xi/p^\alpha\xi$ splits for every $\alpha<\lambda$. In particular, the notion of $p_{\mathrm{w}}^{<\lambda}$-purity has certainly appeared before, but apparently not using the functorial approach presented here.

Putting this all together, for any group G and limit ordinal λ, there are natural inclusions $G/p^\lambda G\subseteq L_\lambda G\subseteq \prod_{\alpha<\lambda} G/p^\alpha G=\mathrm{W}^{<\lambda}G$. In addition, for each $\alpha<\lambda$, the kernel of the transformation $G\to\operatorname{Ext}(H_\alpha,G)$ which determines p^α-purity is $p^\alpha G$. Therefore, there will be natural embeddings

$$\mathrm{W}^{<\lambda}G=\prod_{\alpha<\lambda} G/p^\alpha G\subseteq \prod_{\alpha<\lambda}\operatorname{Ext}(H_\alpha,G)\cong \operatorname{Ext}\Big(\bigoplus_{\alpha<\lambda} H_\alpha,G\Big)=\operatorname{Ext}(H_\lambda,G).$$

So for all A we have inclusions

$$\operatorname{Ext}_{p_{\mathrm{w}}^{\lambda}}(A,G)\subseteq \operatorname{Ext}_{L_\lambda}(A,G)\subseteq \operatorname{Ext}_{p_{\mathrm{w}}^{<\lambda}}(A,G)\subseteq p^\lambda \operatorname{Ext}(A,G).$$

In this work our primary focus will be on studying $p_{\mathrm{w}}^{<\lambda}$-purity and how it compares with L_λ-purity. In particular, we discuss those groups A that have the $L_\lambda=p_{\mathrm{w}}^{<\lambda}$-purity property. Almost invariably, the cases where λ has countable and uncountable cofinality will differ greatly. Summarizing the contents of the paper, Section 2 contains preliminary material as well as some relevant examples. Our main results, contained in Section 3, are as follows.

- Theorem 3.1. The reduced group A is balanced projective if and only if $A/p^\lambda A$ is $p_{\mathrm{w}}^{<\lambda}$-pure projective for every limit ordinal λ. This is similar to Hill's characterization of the balanced projective p-groups as those that are totally projective.
- Theorem 3.2. $p_{\mathrm{w}}^{<\lambda}$-purity is hereditary if and only if λ has countable cofinality.
- Theorem 3.3. $p_{\mathrm{w}}^{<\lambda}$-purity is strongly hereditary if and only if $\lambda = \omega$.
- Theorem 3.8. We describe the Warfield groups with the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property. If λ has countable cofinality, then all Warfield groups have this property, and if λ has uncountable cofinality, then a Warfield group has this property if and only if it is p^λ-bounded.

2. Preliminary ideas and examples

We begin with the most straightforward case. If $\lambda = \omega$, then p^ω-purity is just the classical notion of purity. In addition, $L_\omega G$ will be the p-adic completion of G. Since this is the generic way to construct pure injective hulls of groups, it follows that for every A and G we have

$$\mathrm{Ext}_{L_\omega}(A, G) = \mathrm{Ext}_{p_{\mathrm{w}}^{<\omega}}(A, G) = p^\omega \,\mathrm{Ext}(A, G).$$

In other words, every group A has the $L_\omega = p^\omega$-purity property. We will see that when $\lambda > \omega$ these groups of extensions are often quite different.

Therefore, the first time these types of purity can differ is at $\lambda = \omega 2$, which we consider now, as least in the case of torsion groups.

Proposition 2.1. *Suppose T is a p-group.*

a. *If λ has countable cofinality, then T is L_λ-pure projective if and only if it is p_{w}^{λ}-pure projective (i.e., a subgroup of a p^λ-bounded totally projective p-group);*

b. *T is $p_{\mathrm{w}}^{<\omega 2}$-pure projective if and only if it is $p^{\omega 2}$-pure projective.*

Proof. (a) Since $p_{\mathrm{w}}^{\lambda} \subseteq L_\lambda \subseteq p^\lambda$, if T is L_λ-pure projective, then it must be p_{w}^{λ}-pure projective. Conversely, a p^λ-pure projective is clearly L_λ-pure projective. So, since L_λ-purity is strongly hereditary, any subgroup of a p^λ-bounded totally projective p-group (i.e., a torsion p_{w}^{λ}-pure projective) must be L_λ-pure projective.

(b) Clearly, any $p^{\omega 2}$-pure projective is $p_{\mathrm{w}}^{<\omega 2}$-pure projective, so assume T is $p_{\mathrm{w}}^{<\omega 2}$-pure projective. It follows that T is a summand of a group of the form $B = \bigoplus_{k<\omega} B_k$, where each B_k is a $p_{\mathrm{w}}^{\omega+k}$-pure projective. But since $p^{\omega+k}$-purity is strongly hereditary, the torsion subgroup of a $p_{\mathrm{w}}^{\omega+k}$-pure projective is $p^{\omega+k}$-pure projective. Therefore, the torsion subgroup of B is $p^{\omega 2}$-pure projective. And since T is a summand of B, it is $p^{\omega 2}$-pure projective as well. □

Note that if A is $p_{\mathrm{w}}^{\omega 2}$-pure projective, then it is $p_{\mathrm{w}}^{<\lambda}$-pure projective for all $\lambda > \omega 2$. But there is a torsion $p_{\mathrm{w}}^{\omega 2}$-pure projective that is not p^{α}-pure projective for any ordinal α, see [9, Theorem 3.9]. Therefore, the analogue of Proposition 2.1(b) does not hold for any $\lambda > \omega 2$; that is, it is the best such result possible.

We next present a dual to the above.

Proposition 2.2. *Suppose G is a reduced cotorsion group.*

a. *If λ has countable cofinality, then G is L_λ-pure injective if and only if it is p_{w}^{λ}-pure injective (i.e., p^{λ}-bounded);*
b. *G is $p_{\mathrm{w}}^{<\omega 2}$-pure injective if and only if it is $p^{\omega 2}$-pure injective.*

Proof. (a) Again, since $p_{\mathrm{w}}^{\lambda} \subseteq L_\lambda \subseteq p^{\lambda}$, if G is L_λ-pure injective, then it must be p_{w}^{λ}-pure injective. And conversely, any p^{λ}-bounded cotorsion group must be complete in the λ-topology [8, Theorem 3.4], that is, L_λ-pure injective.

(b) Clearly, any $p^{\omega 2}$-pure injective is $p_{\mathrm{w}}^{<\omega 2}$-pure injective. Conversely, suppose G is $p_{\mathrm{w}}^{<\omega 2}$-pure injective. It follows that G is a summand of

$$P := \prod_{k<\omega} G/p^{\omega+k}G.$$

But, for each $k < \omega$, since $G/p^{\omega+k}G$ is a $p^{\omega+k}$-bounded cotorsion group, it is $p^{\omega+k}$-pure injective. Therefore, P is $p^{\omega 2}$-pure injective, so that G is, as well. □

Any group of length $\omega 2$ will be $p_{\mathrm{w}}^{<\lambda}$-pure injective for any $\lambda > \omega 2$. On the other hand, in [5], a cotorsion group of length $\omega 2$ was constructed that failed to be p^{α}-pure injective for any α less than the first measurable cardinal. In other words, Proposition 2.2(b) also fails for $\lambda > \omega 2$, at least up to the first measurable cardinal.

Our primary goal is to compare L_λ-purity and $p_{\mathrm{w}}^{<\lambda}$-purity. Recall that if λ has countable cofinality, then L_λ is a dense injective transformation, so that L_λ-purity is strongly hereditary. The following makes this more explicit.

Proposition 2.3. *If λ has countable cofinality, then a group is L_λ-pure projective if and only if it is a subgroup of a $p_{\mathrm{w}}^{<\lambda}$-pure projective.*

Proof. Suppose first that B is $p_{\mathrm{w}}^{<\lambda}$-pure projective; so B is also L_λ-pure projective. If A is a subgroup of B, then since L_λ is strongly hereditary, A must also be L_λ-pure projective.

Conversely, suppose A is L_λ-pure projective. We know that A is a summand of a group that can be embedded in a $< \lambda$-balanced projective group B. But such a B will, in fact, be $p_{\mathrm{w}}^{<\lambda}$-pure projective, completing the argument. $\square$

We now consider ordinals of uncountable cofinality. Recall that if λ has uncountable cofinality and B is a group with a decomposition $B \cong \bigoplus_{\alpha<\lambda} B_\alpha$, where each B_α is p^α-bounded, then B is complete in the λ-topology. So Proposition 1.1 implies the next statement.

Lemma 2.4. *If λ has uncountable cofinality, then a $p_{\mathrm{w}}^{<\lambda}$-pure projective group will be complete in the λ-topology.*

Clearly, if λ has countable cofinality, then since L_λ-purity is strongly hereditary, it has global projective dimension 1. On the other hand, we have the next result.

Proposition 2.5. *If λ has uncountable cofinality, then the global projective dimension of L_λ-purity is infinite.*

Proof. Let G_0 be any non-zero p-group. Next, having constructed a p-group G_n of L_λ-pure injective dimension at least n, we then show how to construct G_{n+1}.

Let H be a p-group such that $p^\lambda H \cong G_n$ and $H/p^\lambda H$ is totally projective. Let Y be a p^λ-bounded totally projective p-group such that $0 \to G_{n+1} \to Y \to H \to 0$ is a p^λ-pure projective resolution of H. So G_{n+1} is isotype in Y, and by Lemma 2.4, Y will be complete in the λ-topology. So if Z is the closure of G_{n+1} in the λ-topology, then $Z/G_{n+1} \cong p^\lambda H \cong G_n$, and we can identify Z with $L_\lambda G_{n+1}$. So $0 \to G_{n+1} \to Z \to G_n \to 0$ is our standard L_λ-pure injective resolution of G_{n+1}. It easily follows that if the L_λ-pure injective dimension of G_n is at least n, then the L_λ-pure injective dimension of G_{n+1} is at least $n+1$, completing the argument. $\square$

We now turn to a consideration of the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property for $\lambda > \omega$. Clearly, if A is $p_{\mathrm{w}}^{<\lambda}$-pure projective, then both these groups of extensions will be 0, so that A has the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property. For the same reason, if A is p^λ-pure projective, then A will have the $L_\lambda = p^\lambda$-purity property. This applies, for example, when A is a p^λ-bounded totally projective group. We now consider a simple case.

LEMMA 2.6. *If A is divisible, then it has the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property.*

PROOF. One containment being already established, suppose ξ is $p_{\mathrm{w}}^{<\lambda}$-pure, i.e., $\xi/p^\alpha\xi$ splits for all $\alpha < \lambda$. Since $A/p^\alpha A = 0$, we always have $G/p^\alpha G \cong X/p^\alpha X$. In particular, $L_\lambda G$ will be naturally isomorphic to $L_\lambda X$. Therefore, $G \to L_\lambda G$ extends to $X \to L_\lambda X \cong L_\lambda G$, i.e., ξ is L_λ-pure. □

The next result follows directly by considering the standard long-exact sequence for L_λ-purity, so we omit its (straightforward) proof.

LEMMA 2.7. *Suppose $0 \to B \to A \to C \to 0$ is L_λ-pure. If B and C have the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property (or the $L_\lambda = p^\lambda$-purity property), then so does A.*

A p-group A is said to be a *C_λ-group* if $A/p^\alpha A$ is totally projective for all $\alpha < \lambda$. It is a classical result of Nunke that a C_λ-group has the $p_{\mathrm{w}}^{<\lambda} = p^\lambda$-purity property, see [3, Theorem 93]. So by Lemma 2.6, any divisible torsion group will have the $L_\lambda = p^\lambda$-purity property.

PROPOSITION 2.8. *Suppose A is a C_λ-group. If λ has countable cofinality, then A has the $L_\lambda = p^\lambda$-purity property. However, if λ has uncountable cofinality, then A may fail to have the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property.*

PROOF. Suppose first that λ has countable cofinality. It is a classical result that A has a p^λ-pure subgroup B that is totally projective such that $C := A/B$ is divisible (i.e., B is a λ-basic subgroup of A, see [10]). Since C is a divisible torsion group, it has the $L_\lambda = p^\lambda$-purity property. So $0 \to B \to A \to C \to 0$ is also L_λ-pure. Since B and C have the $L_\lambda = p^\lambda$-purity property, by Lemma 2.7, so does A

Assume now that λ has uncountable cofinality and that A is some p^λ-bounded C_λ-group that is not totally projective. Find a p^λ-bounded totally projective p-group X so that ξ is a balanced projective resolution of A. In particular, ξ is $p_{\mathrm{w}}^{<\lambda}$-pure.

Note that G is isotype in X, which by Lemma 2.4 is complete in the λ-topology. And since $p^\lambda A = 0$, G is closed in X. Therefore, G is also complete in the λ-topology; in particular, G is L_λ-pure injective.

So, if ξ were L_λ-pure, it would have to split. This would say that A is isomorphic to a summand of X, so that it, too, is totally projective, contrary to assumption. □

We want to extend Proposition 2.8. The following simple observation will help us study these types of purity with respect to Ulm subgroups and factors.

LEMMA 2.9. *Suppose α is an ordinal and $A/p^\alpha A$ is p_{w}^α-pure projective. Then ξ splits if and only if both $p^\alpha \xi$ and $\xi/p^\alpha \xi$ are splitting exact.*

PROOF. One direction being trivial, suppose $p^\alpha \xi$ and $\xi/p^\alpha \xi$ split. Let Y be a subgroup of $p^\alpha X$ such that $p^\alpha X = p^\alpha G \oplus Y$ (so $Y \cong p^\alpha A$). It follows that $\xi' := 0 \to G \to X/Y \to A/p^\alpha A \to 0$ is exact and $\xi' \mapsto \xi$ under $\mathrm{Ext}(A/p^\alpha A, G) \to \mathrm{Ext}(A, G)$.

There is a natural isomorphism $\xi'/p^\alpha \xi' \cong \xi/p^\alpha \xi$. So, since the latter splits, so does $\xi'/p^\alpha \xi'$. This implies that $\xi' \in \mathrm{Ext}_{p_{\mathrm{w}}^\alpha}(A/p^\alpha A, G)$. But since $A/p^\alpha A$ is p_{w}^α-pure projective, we have $\mathrm{Ext}_{p_{\mathrm{w}}^\alpha}(A/p^\alpha A, G) = 0$; in particular, $\xi' = 0$. This implies that $\xi = 0$, as required. $\square$

PROPOSITION 2.10. *Suppose $\lambda = \alpha + \mu$, where $\mu \neq 0$ is also a limit ordinal. If A, G are groups and $A/p^\alpha A$ is p_{w}^α-pure projective, then $\xi \mapsto p^\alpha \xi$ gives an isomorphism* $\mathrm{Ext}_{p_{\mathrm{w}}^{<\lambda}}(A, G) \cong \mathrm{Ext}_{p_{\mathrm{w}}^{<\mu}}(p^\alpha A, p^\alpha G)$.

PROOF. If $\xi \in \mathrm{Ext}(A, G)$ is $p_{\mathrm{w}}^{<\lambda}$-pure, then we know that $p^\alpha \xi$ is exact. Now, if $G \to \mathrm{W}^{<\lambda} G$ extends to X, then $p^\alpha G \to p^\alpha(\mathrm{W}^{<\lambda} G) \cong \mathrm{W}^{<\mu}(p^\alpha G)$ extends to $p^\alpha X$. In particular, $p^\alpha \xi$ is $p_{\mathrm{w}}^{<\mu}$-pure.

Next, if $\nu \in \mathrm{Ext}_{p_{\mathrm{w}}^{<\mu}}(p^\alpha A, p^\alpha G)$, then we can use general properties of Ext to find a $\delta \in \mathrm{Ext}(A, p^\alpha G)$ that restricts to ν. We then define ξ by extending δ using a pushout on $p^\alpha G \to G$. It easily follows that $\nu = p^\alpha \xi$ and ξ is in $\mathrm{Ext}_{p_{\mathrm{w}}^\alpha}(A, G)$, i.e., $\xi/p^\alpha \xi$ splits.

Now, suppose $\gamma < \lambda$; we want to show $\xi/p^\gamma \xi$ splits. First, if $\gamma \leq \alpha$, this follows from the fact that $\xi/p^\alpha \xi$ splits. And if $\alpha < \gamma = \alpha + \delta$, then $\xi/p^\alpha \xi$ splits, and since $p^\alpha \xi$ is p_{w}^μ-pure, $p^\alpha \xi/p^{\alpha+\delta} \xi$ splits. So by Lemma 2.9, $\xi/p^\gamma \xi$ splits, as required.

It follows that $\mathrm{Ext}_{p_{\mathrm{w}}^{<\lambda}}(A, G) \to \mathrm{Ext}_{p_{\mathrm{w}}^{<\mu}}(p^\alpha A, p^\alpha G)$ is a surjective homomorphism. But a final appeal to Lemma 2.9 also implies that it is injective. $\square$

Suppose $\lambda = \alpha + \mu$ and A is a group. It is well known that if $A/p^\alpha A$ is p_{w}^α-pure projective and $p^\alpha A$ is p_{w}^μ-pure projective, then A is p_{w}^λ-pure projective. The following easy consequence of Proposition 2.10 gives a similar result.

COROLLARY 2.11. *Suppose $\lambda = \alpha + \mu$, $\mu \neq 0$ and A is a group. If $A/p^\alpha A$ is p_{w}^α-pure projective, then A is $p_{\mathrm{w}}^{<\lambda}$-pure projective if an only if $p^\alpha A$ is $p_{\mathrm{w}}^{<\mu}$-pure projective.*

If λ has countable cofinality and $\lambda = \alpha + \mu$ as above, then there are natural isomorphisms $p^\alpha L_\lambda G \cong L_\mu p^\alpha G$ and $L_\lambda G/p^\alpha L_\lambda G \cong G/p^\alpha G$. To see this, we may clearly assume that $p^\lambda G = 0$. There is an exact sequence

$$0 \longrightarrow L_\mu p^\alpha G \longrightarrow L_\lambda G \longrightarrow G/p^\alpha G \longrightarrow 0.$$

Since $L_\mu p^\alpha G/p^\alpha G$ is divisible and $G/p^\alpha G$ is p^α-bounded, the first isomorphism easily follows. And applying this isomorphism to the above exact sequence gives the second.

Using the ideas of the last paragraph, by a minor modification of the proof of Proposition 2.10 we have the next result.

Proposition 2.12. *Suppose $\lambda = \alpha + \mu$ has countable cofinality, where $\mu \neq 0$ also has countable cofinality. If A, G are groups and $A/p^\alpha A$ is p^α_{w}-pure projective, then $\xi \mapsto p^\alpha \xi$ gives an isomorphism* $\mathrm{Ext}_{L_\lambda}(A, G) \cong \mathrm{Ext}_{L_\mu}(p^\alpha A, p^\alpha G)$.

Proof. The sequence ξ will be in $\mathrm{Ext}_{L_\lambda}(A, G)$ if and only if $L_\lambda \xi$ splits. And using the fact that $A/p^\alpha A$ is p^α_{w}-pure projective and $\xi/p^\alpha \xi$ will be splitting, this can easily be seen to be equivalent to the requirement that $L_\mu p^\alpha \xi$ is splitting, i.e., $p^\alpha \xi$ is L_μ-pure. □

Corollary 2.13. *Suppose $\lambda = \alpha + \mu$ has countable cofinality, $\mu \neq 0$ and A is a group. If $A/p^\alpha A$ is p^α_{w}-pure projective, then A is L_λ-pure projective if and only if $p^\alpha A$ is L_μ-pure projective.*

When λ has countable cofinality, by combining Propositions 2.8, 2.10, and 2.12 we can produce a class of examples of groups with the $L_\lambda = p^{<\lambda}_{\mathrm{w}}$-purity property.

Corollary 2.14. *Suppose $\lambda = \alpha + \mu$ has countable cofinality, $\mu \neq 0$ and $A/p^\alpha A$ is p^α_{w}-pure projective.*

a. *A has the $L_\lambda = p^{<\lambda}_{\mathrm{w}}$-purity property if and only if $p^\alpha A$ has the $L_\mu = p^{<\mu}_{\mathrm{w}}$-purity property.*
b. *If either* (1) *$\mu = \omega$, or* (2) *$p^\alpha A$ is a C_μ-group, then for all groups G we have*

$$\mathrm{Ext}_{L_\lambda}(A, G) = p^\mu \,\mathrm{Ext}(p^\alpha A, p^\alpha G) = \mathrm{Ext}_{p^{<\lambda}_{\mathrm{w}}}(A, G).$$

On the other hand, if λ has uncountable cofinality, the following result puts a pretty severe restriction on the groups that can have the $L_\lambda = p^{<\lambda}_{\mathrm{w}}$-purity property.

Proposition 2.15. *Suppose λ has uncountable cofinality. If the group A has the $L_\lambda = p^{<\lambda}_{\mathrm{w}}$-purity property, then $p^\lambda A$ is divisible.*

Proof. Let ξ be a $p_{\mathrm{w}}^{<\lambda}$-pure projective resolution of A. In particular, G will be isotype in X, which by Lemma 2.4 will be complete in the λ-topology. This means that if $Y \subseteq X$ satisfies $Y/G \cong p^\lambda A$, then we can identify Y with the completion, $L_\lambda G$, and $0 \to G \to Y \to p^\lambda A \to 0$ is an L_λ-pure injective resolution of G.

Now ξ is $p_{\mathrm{w}}^{<\lambda}$-pure, so if A had the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property, we could conclude that ξ is also L_λ-pure. So the identity on G extends to a homomorphism $\phi: X \to Y$.

Interpret ϕ as a function $X \to X$. Since the identity map, 1_X, and ϕ agree on G, their difference, $1_X - \phi$, determines a homomorphism $\gamma: A \cong X/G \to X$. Observe that $\gamma(p^\lambda A) \subseteq p^\lambda X = 0$. This implies that 1_X and ϕ must agree not only on G, but on all of Y. In other words, ϕ will be a splitting of $Y \subseteq X$, i.e., $X = Y \oplus Z$ for some subgroup $Z \subseteq X$.

So, modding out by G, we have

$$A \cong X/G \cong (Y \oplus Z)/G \cong (Y/G) \oplus Z \cong p^\lambda A \oplus Z.$$

But $p^\lambda A$ will only be a summand of A if it is divisible. □

3. General results on $p_{\mathrm{w}}^{<\lambda}$-purity

The following application of $p_{\mathrm{w}}^{<\lambda}$-pure projectivity is clearly reminiscent of Hill's result that the balanced projective p-groups are precisely those that are totally projective [2, Theorem 6.5].

Theorem 3.1. *The reduced group A is balanced projective if and only if $A/p^\lambda A$ is $p_{\mathrm{w}}^{<\lambda}$-pure projective for all limit ordinals λ.*

Proof. It follows easily that if A is balanced projective, then it satisfies the given condition for all λ.

For the converse, consider the following condition, where γ is 0 or a limit ordinal. If $p^\gamma A = 0$ and $A/p^\lambda A$ is $p_{\mathrm{w}}^{<\lambda}$-pure projective for all limit ordinals $\lambda \leq \gamma$, then A is balanced projective. We verify this statement for all γ by induction, which clearly will give the result.

For the base case, if $\gamma = 0$ and $A = p^0 A = 0$, then A is trivially balanced projective.

Suppose next that $\gamma = \nu + \omega$, where $\nu < \gamma$ is 0 or a limit ordinal and A satisfies our condition. It easily follows that $A/p^\nu A$ still satisfies our condition for all limit ordinals $\lambda \leq \nu$. By induction, then, $A/p^\nu A$ is balanced projective, and hence p_{w}^{ν}-pure projective. Since $A \cong A/p^\gamma A$ is $p_{\mathrm{w}}^{<\gamma}$-pure projective, by Corollary 2.11, $p^\nu A$ is $p_{\mathrm{w}}^{<\omega}$-pure projective. That is, $p^\nu A$ is a direct sum of cyclics and so, balanced projective. Therefore, since $A/p^\nu A$ and $p^\nu A$ are balanced projectives, so is A, as required.

Finally, suppose γ is a limit of limit ordinals and A satisfies our condition. If ξ is any balanced sequence, then we want to show ξ splits. For every limit ordinal $\lambda < \gamma$, the sequence $\xi/p^\lambda\xi$ will again be balanced. And by induction, $A/p^\lambda A$ will be balanced projective. Therefore, each $\xi/p^\lambda\xi$ must split; in other words, ξ is $p_{\mathrm{w}}^{<\gamma}$-pure. But since $A \cong A/p^\gamma A$ is $p_{\mathrm{w}}^{<\gamma}$-pure projective, ξ itself must split, as required. □

THEOREM 3.2. *$p_{\mathrm{w}}^{<\lambda}$-purity is hereditary if and only if λ has countable cofinality.*

PROOF. Suppose first that λ has uncountable cofinality. Let $0 \to M \to H \to \mathbb{Z}_{p^\infty} \to 0$ be p^λ-pure, where H is a p^λ-bounded totally projective p-group; so this sequence is also $p_{\mathrm{w}}^{<\lambda}$-pure. Clearly, H is p^λ-pure projective, and hence $p_{\mathrm{w}}^{<\lambda}$-pure projective. In addition, M is not complete in the λ-topology; in fact, $L_\lambda M \cong H$. So by Lemma 2.4, M is not $p_{\mathrm{w}}^{<\lambda}$-pure projective, so that $p_{\mathrm{w}}^{<\lambda}$-purity is not hereditary.

Conversely, suppose λ has countable cofinality; let λ be the ascending limit of the ordinals α_n. Clearly, for a group G we can define $p_{\mathrm{w}}^{<\lambda}$-purity using the homomorphism $G \to \prod_{n<\omega} G/p^{\alpha_n}G$, so we identify the later group with $\mathrm{W}^{<\lambda}\, G$.

Clearly, if $\mathrm{Ext}^2_{p_{\mathrm{w}}^{<\lambda}}(A, G) = 0$ for all groups A and G, then $p_{\mathrm{w}}^{<\lambda}$-purity is hereditary. This will follow if for all G we can show that in the sequence

$$(*) \qquad 0 \longrightarrow G/p^\lambda G \longrightarrow \mathrm{W}^{<\lambda}\, G \longrightarrow QG \longrightarrow 0,$$

which we used to construct a $p_{\mathrm{w}}^{<\lambda}$-pure injective resolution of G, that QG is always $p_{\mathrm{w}}^{<\lambda}$-pure injective.

To that end, we can identify $L_\lambda G$ with the set of all vectors $\langle a_n\rangle \in \mathrm{W}^{<\lambda}\, G$ such that $\phi_n(a_{n+1}) = a_n$ for all $n < \omega$, where $\phi_n : G/p^{\alpha_{n+1}}G \to G/p^{\alpha_n}G$ is the canonical epimorphism. We can fit all the ϕ_ns into a single function $\phi : \mathrm{W}^{<\lambda}\, G \to \mathrm{W}^{<\lambda}\, G$, i.e., $\phi(\langle a_n\rangle) = \langle\phi_n(a_{n+1})\rangle$. Let $\nu : \mathrm{W}^{<\lambda}\, G \to \mathrm{W}^{<\lambda}\, G$ be defined by $\nu = \phi - 1_{\mathrm{W}^{<\lambda}\, G}$. The kernel of ν is precisely $L_\lambda G$.

We next claim that ν is surjective, so that $\mathrm{W}^{<\lambda}\, G/L_\lambda G \cong \mathrm{W}^{<\lambda}\, G$. Let $\langle a_n\rangle$ be an arbitrary element of $\mathrm{W}^{<\lambda}\, G$. Let $b_0 = 0$. Next, find $b_1 \in G/p^{\alpha_1}G$ such that $\phi_0(b_1) = a_0 + b_0 = a_0$. In general, having chosen $b_0, b_1, \ldots, b_n$, choose $b_{n+1} \in G/p^{\alpha_{n+1}}G$ so that $\phi_n(b_{n+1}) = a_n + b_n$. Therefore,

$$\nu(\langle b_n\rangle) = \langle\phi_n(b_{n+1})\rangle - \langle b_n\rangle = \langle\phi_n(b_{n+1}) - b_n\rangle = \langle a_n\rangle,$$

which shows that ν is surjective, as claimed.

We therefore have a short exact sequence

$$0 \longrightarrow L_\lambda G \longrightarrow \mathrm{W}^{<\lambda} G \longrightarrow \mathrm{W}^{<\lambda} G \longrightarrow 0.$$

Compare this with sequence ($*$). We can think of $G/p^\lambda G$ as a subgroup of $L_\lambda G$. And since λ has countable cofinality, $D := L_\lambda G/(G/p^\lambda G)$ is divisible. Therefore,

$$QG \cong \mathrm{W}^{<\lambda} G/(G/p^\lambda G) \cong (\mathrm{W}^{<\lambda} G/L_\lambda G) \oplus D \cong \mathrm{W}^{<\lambda} G \oplus D.$$

Since the last group is clearly $p_{\mathrm{w}}^{<\lambda}$-pure injective, our proof is complete. $\square$

THEOREM 3.3. *$p_{\mathrm{w}}^{<\lambda}$-purity is strongly hereditary if and only if $\lambda = \omega$.*

PROOF. We know that $p_{\mathrm{w}}^{<\omega}$-purity agrees with ordinary purity, which is strongly hereditary, so assume that $\lambda > \omega$.

Let D be a divisible torsion group of cardinality $2^{\aleph_0}$. Construct a p^λ-pure exact sequence

$$0 \longrightarrow K \longrightarrow H \xrightarrow{\pi} D \longrightarrow 0$$

where H is a p^λ-bounded totally projective p-group; we may assume K is a subgroup of H and $D = H/K$. Let $B = \bigoplus_{j<\omega} \mathbb{Z}_{p^j}$ with torsion-completion $\overline{B}$. For our purposes, the important thing about $\overline{B}$ is that its socle, $\overline{B}[p]$, is compact in the induced p-adic topology. There is an embedding $\overline{B} \to D$, which we take to be an inclusion.

Let $A = \pi^{-1}(\overline{B}) \subseteq H$, and for $\alpha < \lambda$, let $A(\alpha) = p^\alpha H \cap A$. So $A(\alpha) \cap K = A \cap p^\alpha H \cap K = p^\alpha K$. We next claim that $A(\alpha) + K = A$. The containment $\subseteq$ is obvious, so assume $a \in A$. We know that $p^\alpha H + K = H$, so $a = h + k$ where $h \in p^\alpha H$ and $k \in K$. Now, $\pi(h) = \pi(a) \in \overline{B}$, so that, in fact, $h \in A(\alpha)$. So $a = h + k \in A(\alpha) + K$.

It follows that $A(\alpha)/p^\alpha K \cong [A(\alpha) + K]/K = A/K = \overline{B}$. In other words,

$$0 \longrightarrow K/p^\alpha K \longrightarrow A/p^\alpha K \longrightarrow \overline{B} \longrightarrow 0$$

is splitting exact and $A(\alpha)/p^\alpha K$ is a complementary summand to $K/p^\alpha K$. In particular, for $\alpha \geq \omega$, we can conclude $p^\alpha A = p^\alpha K$.

Since H is p^λ-pure projective, it is $p_{\mathrm{w}}^{<\lambda}$-pure projective. We will therefore be done if we can verify that A is not $p_{\mathrm{w}}^{<\lambda}$-pure projective. Aiming for a contradiction, suppose $A \oplus M \cong \bigoplus_{\beta<\lambda} Q_\beta$, where Q_β is p_{w}^{β}-pure projective for each $\beta < \lambda$. Since any subgroup of a p_{w}^{β}-pure projective is also p_{w}^{β}-pure projective, there is no loss of generality in assuming that each Q_β is isomorphic to a subgroup of a p^β-bounded totally projective p-group.

We have an isomorphism

$$(K/p^\omega K) \oplus \overline{B} \oplus (M/p^\omega M) \cong (A \oplus M)/(p^\omega A \oplus p^\omega M) \cong \bigoplus_{\beta<\lambda} (Q_\beta/p^\omega Q_\beta).$$

By a standard argument (see, for example, [1, Lemma 71.1]), there is an $\alpha < \lambda$ such that the image of $\overline{B}[p]$ is contained in $\bigoplus_{\beta<\alpha}(Q_\beta/p^\omega Q_\beta)$. Therefore, the composition

$$(\dagger) \qquad \begin{aligned} \overline{B} &\longrightarrow (K/p^\omega K) \oplus \overline{B} \oplus (M/p^\omega M) \\ &\cong \bigoplus_{\beta<\lambda}(Q_\beta/p^\omega Q_\beta) \longrightarrow \bigoplus_{\beta<\alpha}(Q_\beta/p^\omega Q_\beta). \end{aligned}$$

is an injection. In particular, since $\overline{B}$ is unbounded, it follows that $\alpha \geq \omega$.

Next observe that the injective homomorphism in ($\dagger$) can also be expressed as

$$\begin{aligned} \overline{B} &\longrightarrow (K/p^\alpha K) \oplus \overline{B} \oplus (M/p^\alpha M) \\ &\cong (A \oplus M)/(p^\alpha A \oplus p^\alpha M) \\ &\cong \bigoplus_{\beta<\lambda}(Q_\beta/p^\alpha Q_\beta) \\ &\longrightarrow \bigoplus_{\beta<\alpha}(Q_\beta/p^\alpha Q_\beta) \\ &\cong \bigoplus_{\beta<\alpha} Q_\beta \\ &\longrightarrow \bigoplus_{\beta<\alpha} Q_\beta/p^\omega Q_\beta. \end{aligned}$$

This means that there is an injective homomorphism $\overline{B} \to \bigoplus_{\beta<\alpha} Q_\beta$. But this is impossible, since an unbounded torsion-complete group is never isomorphic to a subgroup of a (reduced) totally projective p-group. □

So an arbitrary subgroup of a $p_{\mathrm{w}}^{<\lambda}$-pure projective need not retain that property. In particular, we have the following consequence.

Corollary 3.4. *We have* $\mathrm{Ext}_{p_{\mathrm{w}}^{<\lambda}}(A, G) = \mathrm{Ext}_{L_\lambda}(A, G)$ *for all groups A and G if and only if $\lambda = \omega$.*

Proof. If $\lambda = \omega$, we know they agree; so suppose $\lambda > \omega$. If λ has countable cofinality, then L_λ-purity is strongly hereditary, whereas $p_{\mathrm{w}}^{<\lambda}$-purity is not. If λ has uncountable cofinality, then the result follows from Proposition 2.8. □

For the remainder of the paper we are going to focus on Warfield groups. These can be described as the summands of the simply presented groups. Though not strictly speaking necessary, we will assume all Warfield groups are reduced.

We denote the height of an element $x \in A$ by $|x|$; or perhaps $|x|_A$ for clarity. The height sequence of x, that is $(|p^n x|)_{n<\omega}$, has a *gap* at $|p^n x|$ if $|p^n x| + 1 < |p^{n+1} x|$. If x is an element of infinite order, we will let

$$|p^\omega x| = \sup\{|p^k x| \colon k < \omega\}.$$

We say a group A satisfies the *λ-gap condition* if $p^\lambda A = 0$ and whenever $x \in A$ is an element of infinite order whose height sequence has an infinite number of gaps, then $|p^\omega x| < \lambda$.

Proposition 3.5. *The class of groups satisfying the λ-gap condition includes the $< \lambda$-balanced projective groups and is closed with respect to direct sums and subgroups.*

Proof. If $\alpha < \lambda$ and A is α-balanced projective, then $p^\alpha A$ will be free. If $x \in A$ is an element of infinite order, then since $A/p^\alpha A$ is torsion, there will be a $k < \omega$ such that $p^k x \in p^\alpha A$. Clearly, the height sequence of x has no gaps after $|p^k x|$, so A satisfies the λ-gap condition.

The fact that direct sums of groups satisfying the λ-gap condition continue to do so is routine; in particular, a $< \lambda$-balanced projective group will satisfy the λ-gap condition.

So suppose A satisfies the λ-gap condition and $C \subseteq A$ is a subgroup. Since $p^\lambda A = 0$, it immediately follows that $p^\lambda C = 0$, as well.

Suppose $x \in C$ has infinite order and $|p^\omega x|_C = \lambda$; clearly $|p^\omega x|_A = \lambda$, as well. Since A is assumed to satisfy the λ-gap condition, we know that the height sequence of x computed in A has a finite number of gaps. In particular, $\lambda = \nu + \omega$ for some $\nu < \lambda$.

We need to show that x also has a finite number of gaps in its height sequence when computed in C. Replacing x by $p^k x$ for some $k < \omega$, we may assume $|x|_A \geq |x|_C \geq \nu$ and x has no gaps in its height sequence computed in A. Assume $|x|_A = |x|_C + m$ where $m < \omega$. Therefore, for all $n < \omega$ we have

$$|p^n x|_C \leq |p^n x|_A = |x|_A + n = |x|_C + m + n.$$

This, however, implies that computed in C, the height sequence of x has at most m gaps, as required. $\square$

Corollary 3.6. *Any L_λ-pure projective, and in particular, any $p_{\mathrm{w}}^{<\lambda}$-pure projective, satisfies the λ-gap condition.*

We are going to consider the converse of the last result. In other words, are there situations in which a group that satisfies the λ-gap condition will necessarily be $p_{\mathrm{w}}^{<\lambda}$-pure projective? Certainly, it does not hold for an arbitrary group and limit ordinal. Even if $\lambda = \omega$, then any reduced torsion-free group satisfies the ω-gap condition, but such a group will be $p_{\mathrm{w}}^{<\omega}$-pure projective (or L_ω-pure projective) if and only if it is pure projective if and only if it is free.

In a positive direction, we have the next result.

Proposition 3.7. *If A is a Warfield group, then the following are equivalent*:

1. *A is $p_{\mathrm{w}}^{<\lambda}$-pure projective*;
2. *A is L_λ-pure projective*;
3. *A satisfies the λ-gap condition.*

Proof. We certainly know (1) $\implies$ (2), and by Corollary 3.6, (2) $\implies$ (3), so assume A is Warfield and satisfies the λ-gap condition. Any p^λ-bounded totally projective p-group is $p_{\mathrm{w}}^{<\lambda}$-pure projective, so adding such a summand to A, we may assume there is a decomposition $A \cong \bigoplus_{i\in I} B_i$, where for each $i \in I$, B_i is a simply presented Warfield group (with the λ-gap condition) of (torsion-free) rank 1. Clearly, if we can prove each B_i is $p_{\mathrm{w}}^{<\lambda}$-projective, then so is A. So, without loss of generality, assume $A = B_i$ has rank 1.

Suppose $\langle x\rangle$ is a nice infinite cyclic subgroup of B such that $B/\langle x\rangle$ is totally projective (and p^λ-bounded). Suppose first that the height sequence of x has only a finite number of gaps. Replacing x by $p^k x$ for some $k < \omega$, we may assume x has no such gaps. If $\alpha := |x| < \lambda$, then it easily follows that B will be the direct sum of an α-balanced projective and a totally projective p-group. In particular, it will be $p_{\mathrm{w}}^{<\lambda}$-pure projective.

Assume next that x has an infinite number of gaps in its height sequence. Let $\alpha = |p^\omega x|$. Since B satisfies the λ-gap condition, we can conclude that $\alpha < \lambda$. It is also easily seen that B is isomorphic to $B' \oplus T$, where T is a totally projective p-group and B' is p^α-bounded. Without loss of generality, we may clearly assume $T = 0$ and $B = B'$.

Let C be an α-balanced projective such that $p^\alpha C = \langle y\rangle$ is infinite cyclic. It follows from the theory of simply presented groups that the assignment $x \mapsto y$ extends to a homomorphism $\phi\colon B \to C$.

We can define a homomorphism $X \to C' := C \oplus (X/\langle x\rangle)$ using the formula $z \mapsto (\phi(z), z + \langle x\rangle)$. Since the kernel of the second coordinate map is $\langle x\rangle$ and ϕ is injective on $\langle x\rangle$, it follows that this is an embedding of X into C'. But C' is clearly an α-balanced projective. Therefore, B is a p_{w}^{α}-pure projective, and hence a $p_{\mathrm{w}}^{<\lambda}$-pure projective, as required. □

We conclude with our promised characterization of the Warfield groups with the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property.

THEOREM 3.8. *Suppose A is a Warfield group.*

a. *If λ has countable cofinality, then A has the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property.*

b. *If λ has uncountable cofinality, then the following are equivalent*:

   1. *A has the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property*;
   2. *$p^\lambda A = 0$*;
   3. *A is $p_{\mathrm{w}}^{<\lambda}$-pure projective, so $\mathrm{Ext}_{L_\lambda}(A, G) = \mathrm{Ext}_{p_{\mathrm{w}}^{<\lambda}}(A, G) = 0$ for all G.*

PROOF. (a) Since λ has countable cofinality, it follows from Proposition 2.8 that any totally projective p-group has the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property. Since any $p_{\mathrm{w}}^{<\lambda}$-pure projective group satisfies the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property, by the same decomposition result used in the proof of Proposition 3.7, it suffices to prove the following: if B is a simply presented group of (torsion-free) rank 1 that fails to satisfy the λ-gap condition, then B has the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property. So we may assume that B has a nice infinite cyclic subgroup generated by x such that one of two things happens: (1) $|p^\omega x| = \lambda$ and x has an infinite number of gaps in its height sequence, or (2) $|p^\omega x| > \lambda$.

In either case, if we let T be the torsion subgroup of A, then we claim that $T/p^\alpha T \to A/p^\alpha A$ is an isomorphism for all $\alpha < \lambda$.

Let us first show how verifying the claim leads to a proof of (a). First off, it implies that $L_\lambda T \to L_\lambda A$ is an isomorphism. So $T \to L_\lambda T$ extends to A via $A \to L_\lambda A \cong L_\lambda T$; that is, T is L_λ-pure in A.

Next, for every $\alpha < \lambda$, we have $T/p^\alpha T \cong A/p^\alpha A$ is a torsion Warfield module, i.e., it is a totally projective p-group. In particular, T is a C_λ-group, so by Proposition 2.8, it has the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property.

Finally, since A/T is clearly divisible, by Lemma 2.6 it has the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property. So by Lemma 2.7, A also has the $L_\lambda = p_{\mathrm{w}}^{<\lambda}$-purity property.

Returning to verifying the claim, let $\alpha < \lambda$. Since T is isotype in A, $T/p^\alpha T \to A/p^\alpha A$ is injective. So assume $y \in A \setminus T$; so y has infinite order and $0 \neq p^j y \in \langle x \rangle$ for some $j < \omega$. It is elementary to see that either condition (1) or (2) implies that there is a $k \geq j$ such that $|p^k y| \geq \alpha + k$. So, if $z \in A$ satisfies $|z| \geq \alpha$ and $p^k y = p^k z$, then $y = z + (y - z) \in p^\alpha A + T$. Therefore, $T/p^\alpha T \to A/p^\alpha A$ is also surjective. This proves the claim, and completes the proof of (a).

(b) Suppose λ has uncountable cofinality. By Proposition 2.15, we know that (1) $\Longrightarrow$ (2). It follows from Proposition 3.7 and the fact that λ has uncountable cofinality that (2) $\Longrightarrow$ (3). Finally, trivially (3) $\Longrightarrow$ (1). $\square$

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Manoscritto pervenuto in redazione il 5 dicembre 2019.