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A matrix description for torsion free abelian groups of finite rank

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Abstract – We describe torsion free abelian groups of finite rank applying matrices with polyadic entries. This description can be considered as a modification of the classic description by A. I. Mal′cev.

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1. Introduction

L. S. Pontryagin [23], A. I. Mal′cev [22], A. G. Kurosh [19], D. Derry [5], F. Levi [21], and R. Baer [1] began research on torsion free abelian groups of finite rank. At that time the so-called "Kurosh–Mal′cev–Derry matrix description" appeared. Actually, there existed two different results: namely, the description by Mal′cev [22] and the description by Derry [5] based on the theorem by Kurosh [19]. The Kurosh–Derry theorem was included in the well known monographs by A. G. Kurosh [20] and L. Fuchs [15]. That was the reason why the Kurosh–Derry theorem was known better than the Mal′cev's theorem. The Beaumont–Pierce description of rank-2 torsion free abelian groups belonged to the same subject matter [2].They all solved in particular the problem of terminology, that is they were finding terms for a description of those groups.

The present paper develops the paper cycle [7, 8, 12] which develops in turn the classical works mentioned above. Torsion free abelian groups of finite rank

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together with mixed quotient divisible groups are described in [7, 8, 12] in terms of finite sequences of elements of finitely presented modules over the ring of polyadic numbers.

The main result of the present paper is Theorem 4.3 which gives a matrix with polyadic entries for every torsion free abelian group of finite rank with a marked basis. The obtained matrix describes simultaneously a mixed quotient divisible group which is dual to the given group in the sense of duality [14]. Thus, we obtain a convenient method to find the dual quotient divisible group. The example in Section 4.4 shows how to do it.

All considered groups are abelian. $Z, Q, \widehat{Z}_p$ denote the rings of integers, rational numbers, and p-adic integers, respectively. The ring of polyadic numbers $\widehat{Z} = \prod_p \widehat{Z}_p$ is the direct product of the rings of p-adic integers over all prime numbers p. Z_n is the ring of residue classes modulo n, Z_{p^∞} is the group of the type p-infinity.

The characteristic $\chi = (m_p)$ is a sequence of non-negative integers or symbols infinity m_p indexed by all prime numbers. Two characteristics are equivalent if they are different by no more than for finite number of finite components. The equivalence class of a characteristic is said *type*. We use the order relation on the set of characteristics, $(m_p) \leq (n_p)$ if and only if $m_p \leq n_p$ for all prime numbers p.

For every characteristic $\chi = (m_p)$ we define the ring $Z_\chi = \prod_p K_p$ as the product over all prime numbers p of the rings K_p, where $K_p = Z_{p^{m_p}}$ if $m_p < \infty$, or $K_p = \widehat{Z}_p$ if $m_p = \infty$.

If $a_1, \ldots, a_n$ are elements of a module M over a commutative ring R, then $\langle a_1, \ldots, a_n \rangle_R$ denotes the submodule generated by these elements, $\langle a_1, \ldots, a_n \rangle$ is the subgroup generated by these elements. A module M is called *finitely presented,* if there exists an exact sequence of homomorphisms

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

for some positive integers m and n. The rings Z_χ, considered as modules over the ring of polyadic numbers, are cyclic ($n = 1$) finitely presented modules. In general case, every finitely presented $\widehat{Z}$-module M is of the form

$$M \cong Z_{\chi_1} \oplus \cdots \oplus Z_{\chi_n},$$

where $\chi_1 \leq \cdots \leq \chi_n$ is a sequence of characteristics [6]. If N is a finitely generated submodule of a finitely presented $\widehat{Z}$-module M, then two modules N and M/N are finitely presented [6].

A group A is called *quotient divisible,* if it doesn't contain non-zero torsion divisible subgroups, but it contains a free subgroup $F \subset A$ of finite rank such that the quotient group A/F is torsion divisible. A free basis of the free group F is said *basis* of the quotient divisible group A. The rank of F is called *rank* of the quotient divisible group A.

We define the category $\mathfrak{D}$ of quotient divisible groups with marked bases. An object of $\mathfrak{D}$ is a pair $(A; a_1, \dots, a_n)$, where A is a quotient divisible group and $a_1, \dots, a_n$ is its basis. A morphism $f\colon (A; a_1, \dots, a_n) \to (B; b_1, \dots, b_k)$ is a group homomorphism $f\colon A \to B$ such that its matrix M_f consists of integer numbers, where the matrix M_f is defined by the following matrix equality

$$(f(a_1), \dots, f(a_n)) = (b_1, \dots, b_k) M_f.$$

All other definitions may be found in the book [16]. Our notation also coincides with the notation in this book.

2. χ-adic completion

Every positive integer n is of the form $n = p_1^{k_1} p_2^{k_2} \cdot \dots \cdot p_s^{k_s}$, where $p_1, p_2, \dots, p_s$ are first s prime numbers $2, 3, 5, \dots$ and $0 \leq k_i \in Z, i = 1, \dots, s$. We define *characteristic* of n as $\operatorname{char}(n) = (k_1, \dots, k_s, 0, 0, \dots)$. In particular, $\operatorname{char}(1) = (0, 0, \dots)$. It is clear that characteristics of positive integers belong to the zero type. In fact, we obtain a one-to-one correspondence between positive integers and characteristics of zero type. It is also clear that a positive integer n is a divisor of a positive integer m if and only if $\operatorname{char}(n) \leq \operatorname{char}(m)$. Let χ be a characteristic. If $\operatorname{char}(n) \leq \chi$ for a positive integer n, we write shortly $n \leq \chi$.

We introduce now the χ-adic topology on an abelian group in the similar way as the Z-adic topology, see [8, 15, 16]. The χ-adic topology on a group A is defined by the basis of neighborhoods for zero $\{nA \mid n \leq \chi\}$. It means that a subset of the group A is open in the χ-adic topology on the group A if and only if it is empty or it is a union of subsets of the form

$$a + nA, a \in A, \quad n \leq \chi.$$

We introduce the χ-adic completion $\widehat{A}_\chi$ for a group A as the inverse limit of the inverse spectrum

$$\pi_n^m\colon A/mA \longrightarrow A/nA \tag{1}$$

for all pairs (m, n) of positive integers such that $\operatorname{char}(n) \leq \operatorname{char}(m) \leq \chi$. Here $\pi_n^m(a + mA) = a + nA$.

An element $a = (a_n)_{n \leq \chi}$ of the group $\prod_{n \leq \chi} A/nA$ is called a *net* of the spectrum (1) if $\pi_n^m(a_m) = a_n$ for every pair (m, n) of positive integers such that

$\mathrm{char}(n) \leq \mathrm{char}(m) \leq \chi$. It is clear that all nets form a subgroup of the group $\prod_{n\leq\chi} A/nA$. This subgroup is called *inverse limit* of the spectrum (1) and it is the χ-adic completion $\widehat{A}_\chi$ of the group A.

Let $a \in A$ be an element of the group A. It is easy to see that the element $(a + nA)_{n\leq\chi} \in \prod_{n\leq\chi} A/nA$ is a net of the spectrum (1). Thus, we obtain the homomorphism $\mu\colon A \to \widehat{A}_\chi$, where $\mu(a) = (a + nA)_{n\leq\chi}$, which is also called *$\chi$-adic completion.*

We employ the spectrum (1) for the ring of integers Z:

$$\pi_n^m\colon Z/mZ \longrightarrow Z/nZ, \quad \mathrm{char}(n) \leq \mathrm{char}(m) \leq \chi. \tag{2}$$

Every group Z/mZ is also a ring in the natural way. Every group homomorphism $\pi_n^m\colon Z/mZ \to Z/nZ$ is a ring homomorphism as well. Thus, the χ-adic completion of the ring of integers Z is a commutative ring. This ring is the ring Z_χ mentioned above. The elements of the ring Z_χ are called *χ-adic numbers.* A set $\{m_r \mid r \leq \chi\}$ of integers defines a χ-adic number if and only if it satisfies the condition

$$m_s \equiv m_r \pmod r \tag{3}$$

for every two indices r and s such that $\mathrm{char}(r) \leq \mathrm{char}(s) \leq \chi$.

Let $\alpha = \{m_r \mid r \leq \chi\}$ be a χ-adic number and $b \in \widehat{A}_\chi$, that is $b = \{b_r \mid r \leq \chi\}$ and it is a net of the spectrum (1). It is obvious that the set $\{m_r b_r \mid r \leq \chi\}$ is also a net of the spectrum (1). And we can define $\alpha b = \{m_r b_r \mid r \leq \chi\} \in \widehat{A}_\chi$. Thus, every χ-adic completion $\widehat{A}_\chi$ is a Z_χ-module.

Let A be a torsion group such that for every element $a \in A$ its order r satisfies $r \leq \chi$. Then the group A is a Z_χ-module. Really, for a χ-adic number $\alpha = \{m_r \mid r \leq \chi\}$ we define

$$\alpha a = m_r a. \tag{4}$$

In particular, every torsion group is a module over the ring of polyadic numbers $\widehat{Z} = \prod_p \widehat{Z}_p$.

Every Z_χ-module is a module over the ring of polyadic numbers $\widehat{Z}$ as well, because $Z_\chi \cong \widehat{Z}/I_\chi$, where $I_\chi = \{\gamma \in \widehat{Z} \mid \mathrm{char}(\gamma) \geq \chi\}$ is an ideal of the ring $\widehat{Z}$.

Note that the p-adic topology and the Z-adic topology are particular cases of χ-adic topologies for the characteristics $\chi = (0, \dots, 0, \infty, 0, \dots)$ and $\chi = (\infty, \infty, \dots)$, respectively.

3. Locally cyclic torsion groups

A group is called *locally cyclic* if every its finitely generated subgroup is cyclic. Every locally cyclic group is either torsion free, then it is of rank 1, or torsion, then it is isomorphic to a subgroup of the group Q/Z, that is of the form $\bigoplus_p Z_{p^{m_p}}$. Thus, the torsion locally cyclic groups up to isomorphism are in one-to-one correspondence with characteristics (m_p).

Let C_χ be a locally cyclic torsion group of characteristic χ. For every positive integer r, satisfying $r \leq \chi$, the group C_χ contains a unique subgroup of order r. This subgroup is cyclic. Choosing a generator c_r for this subgroup and for all $r \leq \chi$, we obtain a set of generators $\{c_r \mid r \leq \chi\}$ for the group C_χ. The set of generators $\{c_r \mid r \leq \chi\}$ for the group C_χ is said *normal,* if $c_r = \frac{s}{r} c_s$ for every two positive integers r and s satisfying the condition $\operatorname{char}(r) \leq \operatorname{char}(s) \leq \chi$.

The homomorphism $w\colon C_\chi \to Q/Z$, where $w(c_r) = \frac{1}{r} + Z \in Q/Z$, is well defined if the set of generators $\{c_r \mid r \leq \chi\}$ is normal.

Proposition 3.1. *Let C_χ be a torsion locally cyclic group of characteristic χ. Then the group $\operatorname{Hom}_{\hat{Z}}(C_\chi, Q/Z)$ is a Z_χ-module. If $w\colon C_\chi \to Q/Z$ is the homomorphism corresponding to a normal set of generators for the group C_χ, then every homomorphism $\varphi\colon C_\chi \to Q/Z$ is uniquely presented of the form $\varphi = \alpha w$, where $\alpha \in Z_\chi$.*

Proof. First of all, $\operatorname{Hom}_{\hat{Z}}(C_\chi, Q/Z) = \operatorname{Hom}(C_\chi, Q/Z)$, because the groups C_χ and Q/Z are torsion. Since the group C_χ is a Z_χ-module (see (4)), the group $\operatorname{Hom}_{\hat{Z}}(C_\chi, Q/Z)$ is a Z_χ-module as well. Namely, $(\alpha\varphi)(x) = \varphi(\alpha(x))$ for $\varphi \in \operatorname{Hom}_{\hat{Z}}(C_\chi, Q/Z), \alpha \in Z_\chi$ and $x \in C_\chi$.

Let $\varphi\colon C_\chi \to Q/Z$ be a homomorphism and let $\{c_r \mid r \leq \chi\}$ be a normal set of generators corresponding to the homomrphism $w\colon C_\chi \to Q/Z$. Then $\varphi(c_r) = \frac{m_r}{r} + Z \in Q/Z$ for an integer m_r. Since the set of generators $\{c_r \mid r \leq \chi\}$ is normal, the set of integers $\{m_r \mid r \leq \chi\}$ satisfies the condition (3). It means that it defines a χ-adic number $\alpha \in Z_\chi$. We obtain $\varphi(c_r) = \frac{m_r}{r} + Z = m_r(\frac{1}{r} + Z) = m_r(w(c_r)) = \alpha(w(c_r))$ by (4). Since the homomorphisms φ and αw coincides on the set of generators, we obtain the equality $\varphi = \alpha w$.

If $\alpha w = 0$, then m_r is divisible by r for all $r \leq \chi$, hence $\alpha = 0$. It proves the uniqueness. □

4. Torsion-free finite-rank groups

We consider here the category $\mathcal{F}$ of torsion-free finite-rank groups with marked bases, see [7, 8, 24]. Objects of the category $\mathcal{F}$ are pairs of the form $(A; a_1, \dots, a_n)$, where A is a torsion free group of finite rank and $a_1, \dots, a_n$ is its basis. The basis here means a maximal linearly independent set of elements. In particular, the integer $n \geq 0$ is the rank of the group A.

Morphisms of the category $\mathcal{F}$ are usual group homomorphisms such that the matrix with respect to marked bases consists of integer elements.

Let A be a torsion free group with a marked basis $a_1, \dots, a_n$, that is an object of the category $\mathcal{F}$. We denote $F = \langle a_1, \dots, a_n \rangle$ is the free subgroup of the group A generated by the given set of elements.

4.1 – *Principal identity*

Let $c_r \in A/F$ be an element of order $r, 0 < r \in Z$. Then $c_r = \bar{z} = z + F, z \in A$, and $rc_r = 0$, that is $rz \in F$. Therefore, the following equality with integer coefficients takes place:

$$rz = m_1^{(r)} a_1 + \cdots + m_n^{(r)} a_n. \tag{5}$$

The equality (5) implies the following equality of two sets in the divisible hull $Qa_1 \oplus \cdots \oplus Qa_n$ of the group A:

$$\bar{z} = z + F = \Big(\frac{m_1^{(r)}}{r} + Z\Big) a_1 + \cdots + \Big(\frac{m_n^{(r)}}{r} + Z\Big) a_n. \tag{6}$$

Let's denote $a_1^\circ(\bar{z}) = \frac{m_1^{(r)}}{r} + Z, \dots, a_n^\circ(\bar{z}) = \frac{m_n^{(r)}}{r} + Z$. We obtain n functions $a_1^\circ, \dots, a_n^\circ \in \operatorname{Hom}(A/F, Q/Z)$ and the equality (6) becomes of the form

$$\bar{z} = a_1^\circ(\bar{z}) a_1 + \cdots + a_n^\circ(\bar{z}) a_n. \tag{7}$$

The equality (7) is called *principal identity* [7] of the group A with respect to the basis $a_1, \dots, a_n$, that is the principal identity of the given object of the category $\mathcal{F}$. The homomorphisms $a_1^\circ, \dots, a_n^\circ \colon A/F \to Q/Z$ are called *coefficients of the principal identity.*

4.2 – *Groups with one τ-adic relation*

We consider here groups A with a locally cyclic quotient A/F. Since the group A/F is torsion locally cyclic, it is determined by a characteristic χ of the type $\tau = [\chi]$. This class of groups has been introduced in [9] with the name "groups with one τ-adic relation." It contains groups satisfying the condition that every proper pure subgroup is free [10], in particular it contains also groups with free subgroups of infinite index [11].

Let $\{c_r \mid r \le \chi\}$ be a normal set of generators for the locally cyclic group A/F. We consider two elements of this set $c_r = z + F$ and $c_s = v + F$ such that $\mathrm{char}(r) \le \mathrm{char}(s) \le \chi$. It means that $s = rk$ for $k \in Z$. The equality (5) gives us $rz = m_1^{(r)} a_1 + \cdots + m_n^{(r)} a_n$ and $sv = m_1^{(s)} a_1 + \cdots + m_n^{(s)} a_n$. Since $kc_s = c_r$, it follows that $kv - z \in F$, that is $kv - z = k_1 a_1 + \cdots + k_n a_n$ for some integer coefficients. Multiplying the last equality by r, we obtain

$$sv - rz = rk_1 a_1 + \cdots + rk_n a_n = (m_1^{(s)} - m_1^{(r)}) a_1 + \cdots + (m_n^{(s)} - m_n^{(r)}) a_n$$

Therefore, $m_1^{(s)} - m_1^{(r)} = rk_1, \ldots, m_n^{(s)} - m_n^{(r)} = rk_n$.

We can see that the set of integers $\{m_i^{(r)} \mid r \le \chi\}$ satisfies the condition (3) for every $i = 1, \ldots, n$ and it determines a χ-adic number $\alpha_i \in Z_\chi$.

Thus, the chosen normal set of generators $\{c_r \mid r \le \chi\}$ for the locally cyclic group A/F gives us the set of χ-adic numbers $\alpha_1, \ldots, \alpha_n$ and the homomorphism $w\colon A/F \to Q/Z$, where $w(c_r) = \frac{1}{r} + Z, r \le \chi$.

Theorem 4.1. *Let A be a torsion free group with a marked basis $a_1, \ldots, a_n$, $F = \langle a_1, \ldots, a_n \rangle$. Let the group A/F be locally cyclic of characteristic χ with a chosen normal set of generators, which corresponds to the set of χ-adic numbers $\alpha_1, \ldots, \alpha_n$ and to the homomorphism $w\colon A/F \to Q/Z$. Then,*

1. *the following equality takes place in the Z_χ-module $\widehat{A}_\chi$:*

$$\alpha_1 \mu(a_1) + \cdots + \alpha_n \mu(a_n) = 0,$$

   where $\mu\colon A \to \widehat{A}_\chi$ is the χ-adic completion;
2. *the coefficients of the principal identity $\bar{z} = a_1^\circ(\bar{z}) a_1 + \cdots + a_n^\circ(\bar{z}) a_n$ are of the form $a_1^\circ = \alpha_1 w, \ldots, a_n^\circ = \alpha_n w$.*

Proof. According to the definition of the χ-adic numbers

$$\alpha_1 = \{m_1^{(r)} \mid r \le \chi\}, \quad \ldots, \quad \alpha_n = \{m_n^{(r)} \mid r \le \chi\},$$

the following equality takes place in the group A for every $r \le \chi$:

$$rz = m_1^{(r)} a_1 + \cdots + m_n^{(r)} a_n, z \in A.$$

It shows that the element $m_1^{(r)} a_1 + \cdots + m_n^{(r)} a_n$ is divisible by r in the group A. Therefore,

$$m_1^{(r)}(a_1 + rA) + \cdots + m_n^{(r)}(a_n + rA) = 0$$

in the group A/rA. Since the last equality takes place for all $r \le \chi$, we obtain the equality $\alpha_1 \mu(a_1) + \cdots + \alpha_n \mu(a_n) = 0$.

Let $c_r = \bar{z} = z + F, z \in A$, be an element of the chosen normal set of generators. According to the equalities (7) and (6), we have

$$a_i^\circ(c_r) = a_i^\circ(\bar{z}) = \frac{m_i^{(r)}}{r} + Z = m_i^{(r)}\Big(\frac{1}{r} + Z\Big) = m_i^{(r)} w(c_r).$$

Note that $m_i^{(r)} w(c_r) = \alpha_i w(c_r)$ by (4). It follows that the homomorphisms a_i° and $\alpha_i w\colon A/F \to Q/Z$ coincide on the set of generators $\{c_r \mid r \leq \chi\}$. Hence they coincide for all elements of the group A/F, that is $a_i^\circ = \alpha_i w$ for all $i = 1, \dots, n$. □

4.3 – *General case*

Let A be a torsion free group with a marked basis $a_1, \dots, a_n$, $F = \langle a_1, \dots, a_n\rangle$. The quotient group A/F can be decomposed into a finite direct sum of locally cyclic groups by many different ways. For example,

$$A/F = \bigoplus_p (A/F)_p \cong \bigoplus_p \Big(\bigoplus_{i=1}^n Z(p^{m_{ip}})\Big) = \bigoplus_{i=1}^n \Big(\bigoplus_p Z(p^{m_{ip}})\Big) = \bigoplus_{i=1}^n C_{\chi_i},$$

where $\chi_1 = (m_{1p}), \dots, \chi_n = (m_{np})$ are characteristics. Supposing inequalities $m_{1p} \leq m_{2p} \leq \cdots \leq m_{np}$ hold for each prime number p, we obtain that $\chi_1 \leq \cdots \leq \chi_n$. Some first characteristics can be equal to zero.

We consider here the general case, where

$$A/F = C_1 \oplus \cdots \oplus C_m \tag{8}$$

and the groups $C_1, \dots C_m$ are locally cyclic of characteristics $\chi_1, \dots, \chi_m$ respectively.

Proposition 4.2. *If $A/F = C_1 \oplus \cdots \oplus C_m$, where the groups $C_1, \dots C_m$ are locally cyclic of characteristics $\chi_1, \dots, \chi_m$ respectively, then* $\operatorname{Hom}_{\hat{Z}}(A/F, Q/Z) \cong Z_{\chi_1} \oplus \cdots \oplus Z_{\chi_m}$. *In particular, the $\hat{Z}$-module* $\operatorname{Hom}_{\hat{Z}}(A/F, Q/Z)$ *is finitely presented.*

Proof. Applying the functor $\operatorname{Hom}_{\hat{Z}}(-, Q/Z)$, we obtain

$$\operatorname{Hom}_{\hat{Z}}(A/F, Q/Z) \cong \operatorname{Hom}_{\hat{Z}}(C_1, Q/Z) \oplus \cdots \oplus \operatorname{Hom}_{\hat{Z}}(C_m, Q/Z).$$

Since $\operatorname{Hom}_{\hat{Z}}(C_k, Q/Z) \cong Z_{\chi_k}, k = 1, \dots, m$, by Proposition 3.1, we obtain the result. □

We consider the natural short exact sequence

$$0 \longrightarrow F \longrightarrow A \xrightarrow{\sigma} C_1 \oplus \cdots \oplus C_m \longrightarrow 0,$$

where $C_1, \ldots, C_m$ are torsion locally cyclic of the characteristics $\chi_1, \ldots, \chi_m$ respectively.

All preimages $A_1 = \sigma^{-1}(C_1), \ldots, A_m = \sigma^{-1}(C_m)$ have the same basis $a_1, \ldots, a_n$. Obviously, $F = A_1 \cap \cdots \cap A_m$ and $A = A_1 + \cdots + A_m$. Each group A_k is a group with one τ-adic relation with respect to the characteristic χ_k and we can apply Theorem 4.1. According to this theorem, the fixation of a normal set of generators for the group C_k gives us a sequence of χ_k-adic numbers $\alpha_{k1}, \ldots, \alpha_{kn} \in Z_{\chi_k}$ and also a homomorphism $w_k \colon C_k \to Q/Z$ for every $k = 1, \ldots, m$.

Moreover, the following equality takes place in the χ_k-adic completion of the group A_k

$$\alpha_{k1}a_1 + \cdots + \alpha_{kn}a_n = 0.$$

We identify here $\mu(a) = a$, where $\mu \colon A_k \to (\widehat{A_k})_{\chi_k}$ is the χ_k-adic completion of the group A_k, for the simplification of our notation.

The coefficients of the principal identity $\bar{z} = a_1^\circ(\bar{z})a_1 + \cdots + a_n^\circ(\bar{z})a_n$ for the group A_k are of the form $a_1^\circ = \alpha_{k1}w_k, \ldots, a_n^\circ = \alpha_{kn}w_k$.

At last, we can define a matrix for every torsion free group A of rank n:

$$(9) \qquad \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{pmatrix}.$$

The k-th row

$$(\alpha_{k1}, \ldots, \alpha_{kn})$$

of the matrix consists of χ_k-adic numbers, $k = 1, \ldots, m$. The definition of the matrix depends on the choice of three things: a marked basis, a direct decomposition (8) of the group A/F and a normal set of generators of the group A/F.

We consider also the identity embeddings $i_k \colon C_k \to A/F$ and the projections $\pi_k \colon A/F \to C_k$ with respect to the direct decomposition (8), $k = 1, \ldots, m$ Supposing the choice of a normal generating set for each group C_k, we define the homomorphisms $y_1, \ldots, y_m \in \mathrm{Hom}_{\widehat{Z}}((A/F, Q/Z)$ in the following way:

$$(10) \qquad y_1 = w_1\pi_1, \quad \ldots, \quad y_m = w_m\pi_m.$$

THEOREM 4.3. *Let* (9) *correspond to a torsion free group A with a marked basis $a_1, \ldots, a_n$. Let the homomorphisms $y_1, \ldots, y_m \in \mathrm{Hom}_{\widehat{Z}}((A/F, Q/Z)$ be defined by* (10). *Then,*

1. *the following equalities take place in the* Z_{χ_k}*-modules* $\widehat{A}_{\chi_k}$

$$\alpha_{k1}a_1 + \cdots + \alpha_{kn}a_n = 0$$

for all $k = 1, \ldots, m$;

2. *the coefficients of the principal identity* $\bar{z} = a_1^\circ(\bar{z})a_1 + \cdots + a_n^\circ(\bar{z})a_n$ *for the group* A *are of the form* $a_s^\circ = \alpha_{1s}y_1 + \cdots + \alpha_{ms}y_m$ *for every* $s = 1, \ldots, n$.

Proof. By Theorem 4.1 the equality $\alpha_{k1}a_1 + \cdots + \alpha_{kn}a_n = 0$ takes place in the Z_{χ_k}-module $(\widehat{A}_k)_{\chi_k}$. It is equivalent to the fact that for every positive integer $r \leq \chi_k$ and for every integer coefficients such that

$$m_1 \equiv \alpha_{k1} \pmod r, \quad \ldots, \quad m_n \equiv \alpha_{kn} \pmod r,$$

the element $m_1a_1+\cdots+m_na_n$ is divisible by r in the group A_k. The group $A_k = \sigma^{-1}(C_k)$ is a subgroup of the group A. It follows that the element $m_1a_1+\cdots+m_na_n$ must be certainly divisible by r in the group A as well. This is the reason why the equality $\alpha_{k1}a_1 + \cdots + \alpha_{kn}a_n = 0$ holds in the Z_{χ_k}-modules $\widehat{A}_{\chi_k}$ as well.

The coefficient a_s° of the principal identity $\bar{z} = a_1^\circ(\bar{z})a_1+\cdots+a_n^\circ(\bar{z})a_n$ for the group A is a homomorphism $a_s^\circ\colon A/F \to Q/Z, s = 1, \ldots, n$. The restriction of the function a_s° on the subgroup $C_k \subset A/F$ coincides with the same coefficient of the principal identity for the group $A_k = \sigma^{-1}(C_k)$. By Theorem 4.1 we obtain the equality of two functions $a_s^\circ i_k\pi_k = \alpha_{ks}w_k\pi_k\colon A/F \to Q/Z, k = 1, \ldots, m$. Summarization all these equalities gives us the following equality:

$$a_s^\circ(i_1\pi_1 + \cdots + i_m\pi_m) = \alpha_{1s}w_1\pi_1 + \cdots + \alpha_{ms}w_m\pi_m.$$

Since $i_1\pi_1 + \cdots + i_m\pi_m = id_{A/F}$ and $y_k = w_k\pi_k, k = 1, \ldots, m$, we obtain the final result $a_s^\circ = \alpha_{1s}y_1 + \cdots + \alpha_{ms}y_m$. □

Theorem 4.3 can be represented in the following more visual form:

$$\left(\begin{array}{ccccccc}
 & a_1 & a_2 & \cdots & a_n & & \\
y_1 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & = & 0 \\
y_2 & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} & = & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
y_m & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} & = & 0 \\
 & = & = & \cdots & = & & \\
 & a_1^\circ & a_2^\circ & \cdots & a_m^\circ & &
\end{array}\right)$$

Remark 4.4. The pure hull $\langle a_1^\circ, \dots, a_n^\circ \rangle_* \subset \mathrm{Hom}(A/F, Q/Z)$ consists of all elements $a \in \mathrm{Hom}(A/F, Q/Z)$ such that $ka \in \langle a_1^\circ, \dots, a_n^\circ \rangle$ for some $0 < k \in Z$. In particular, the torsion parts of the groups $\langle a_1^\circ, \dots, a_n^\circ \rangle_*$ and $\mathrm{Hom}(A/F, Q/Z)$ coincide. If the set of the elements $a_1^\circ, \dots, a_n^\circ$ is linearly independent over Z, then the group $A^\circ = \langle a_1^\circ, \dots, a_n^\circ \rangle_*$ is quotient divisible and $a_1^\circ, \dots, a_n^\circ$ is the basis of this quotient divisible group, see details in [12]. Moreover, the group A° is dual to the group A in the sense of duality [7, 24] and in the sense of duality [14] as well. The basis $a_1^\circ, \dots, a_n^\circ$ is dual to the basis $a_1, \dots, a_n$.

While the matrix (9) depends on the choice of the normal generating set of the group A/F and it gives us a way how to find the dual quotient divisible group A° by Theorem 4.3, it is clear that the dual group $A^\circ = \langle a_1^\circ, \dots, a_n^\circ \rangle_* \subset \mathrm{Hom}(A/F, Q/Z)$ doesn't depend on the choice of the normal generating set of the group A/F.

4.4 – *Example*

We denote subrings of rational numbers $Q^{(p)} = \left\{ \frac{r}{p^s} \mid r, s \in Z \right\} \subset Q$ and $Q_p = \left\{ \frac{r}{s} \mid (s, p) = 1 \right\} \subset Q$ for a prime number p. Let V be a vector space over the field of rational numbers Q with a basis a_1, a_2. We consider a subgroup A of the additive group V, which is generated by the subgroup $a_1 Q^{(2)} + a_2 Q^{(5)}$ and by the element $\frac{a_1 + a_2}{3}$. That is $A = \left\langle a_1 Q^{(2)} + a_2 Q^{(5)}, \frac{a_1 + a_2}{3} \right\rangle$. This example belongs to M. Bognar, see [16, Example 4.3, p. 432]. The group A is torsion free of rank-2, it is indecomposable into a direct sum, but it is almost completely decomposable.

The torsion group A/F is locally cyclic of characteristic $(\infty, 1, \infty, 0, \dots)$, $A/F \cong Z_{2^\infty} \oplus Z_3 \oplus Z_{5^\infty}$. Nevertheless, it is also convenient to consider the group A/F as a direct sum of three locally cyclic subgroups $A/F = C_1 \oplus C_2 \oplus C_3$ of characteristics $\chi_1 = (\infty, 0, 0, \dots)$, $\chi_2 = (0, 0, \infty, 0, \dots)$, and $\chi_3 = (0, 1, 0, \dots)$ respectively. We fix now normal generating sets for the groups C_1, C_2, C_3 in the natural way $\left\{ \frac{a_1}{2^r} + F \mid 0 < r \in Z \right\}, \left\{ \frac{a_2}{5^r} + F \mid 0 < r \in Z \right\}$, and $\left\{ \frac{a_1 + a_2}{3} + F \right\}$. It gives us three homomorphisms $y_1, y_2, y_3 \in \mathrm{Hom}_{\hat{Z}}(A/F, Q/Z)$ and the matrix

$$\begin{pmatrix} & a_1 & a_2 & & \\ y_1 & 1 & 0 & = & 0 \\ y_2 & 0 & 1 & = & 0 \\ y_3 & 1 & 1 & = & 0 \\ & = & = & & \\ & a_1^\circ & a_2^\circ & & \end{pmatrix}$$

The $\hat{Z}$-module $\mathrm{Hom}_{\hat{Z}}(A/F, Q/Z)$ is of the form

$$\mathrm{Hom}_{\hat{Z}}(A/F, Q/Z) = y_1\hat{Z}_2 \oplus y_2\hat{Z}_5 \oplus y_3 Z_3 \cong \hat{Z}_2 \oplus \hat{Z}_5 \oplus Z_3 = Z_\chi,$$

where $\chi = (\infty, 1, \infty, 0, \dots)$.

The coefficients of the principal identity are of the form $a_1^\circ = y_1 + y_3$, $a_2^\circ = y_2 + y_3$. At last we can find the dual quotient divisible group

$$A^\circ = \langle y_1 + y_3, y_2 + y_3\rangle_* \subset \mathrm{Hom}_{\hat{Z}}(A/F, Q/Z).$$

Note that y_3 belongs to the group A° as a torsion element, hence $y_1, y_2 \in A^\circ$ as well. Therefore $A^\circ = \langle y_1, y_2, y_3\rangle_*$ and it is easy to see that

$$A^\circ = y_1 Q_2 \oplus y_2 Q_5 \oplus \langle y_3\rangle \cong Q_2 \oplus Q_5 \oplus Z_3.$$

Thus, the dual quotient divisible group A° is mixed of rank-2 and it is decomposable into a direct sum of two quotient divisible groups of rank-1, namely $A^\circ \cong Q_2 \oplus (Q_5 \oplus Z_3)$ or $A^\circ \cong Q_5 \oplus (Q_2 \oplus Z_3)$. The elements a_1°, a_2° represent the dual basis such that the object $(A^\circ, a_1^\circ, a_2^\circ)$ of the category $\mathfrak{D}$ of quotient divisible groups with marked bases is dual to the object (A, a_1, a_2) of the category $\mathfrak{F}$.

Applying the isomorphism $\mathrm{Hom}_{\hat{Z}}(A/F, Q/Z) \to Z_\chi = \hat{Z}_2 \oplus Z_3 \oplus \hat{Z}_5$, we can consider the elements a_1°, a_2° as χ-adic numbers $\gamma_1 = (1, 1, 0), \gamma_2 = (0, 1, 1) \in Z_\chi = \hat{Z}_2 \oplus Z_3 \oplus \hat{Z}_5$ respectively. Then the group A is defined by one χ-adic relation $\gamma_1 a_1 + \gamma_2 a_2 = 0$.

The quotient divisible groups, which are dual to almost completely decomposable torsion free groups, are researched also in [13]. In particular, the quotient divisible groups, which are dual to the famous Corners groups [3], are found there.

5. Conclusion

A. I. Mal′cev considers torsion free groups of finite rank in [22]. He defines a matrix for a group A with a marked basis $a_1, \dots, a_n$ and for every prime number p. Elements of this matrix belong to a "generalization" of numbers. In fact, the elements of a row of his matrix belong to either $\hat{Z}_p$ or Z_{p^m} for some $0 \le m \in Z$. Considering the same row over all prime numbers, we obtain, at first, a characteristic χ_k and, second, the row $(\alpha_{k1}, \dots, \alpha_{kn})$, which consists of χ_k-adic numbers. Moreover, the row $(\alpha_{k1}, \dots, \alpha_{kn})$ has the property that $\alpha_{k1}a_1 + \dots + \alpha_{kn}a_n = 0$ in the χ_k-adic completion of the group A. Therefore we obtain our matrix (9) from matrices by Mal′cev. The difference between our approach and his one is the following. We consider the general case here. Mal′cev chooses the normal generating sets in a special way such that his matrices are of a special form, that is also of great interest. Thus, our present paper generalizes and develops the Mal′cev's ideas.

The duality between the categories of quotient divisible groups and torsion-free finite-rank groups with quasi-homomorphisms has been introduced in [14] as well as the mixed quotient divisible groups themselves. This duality has been modified in [7, 24]. Our example shows in particular how the Mal′cev's matrix helps us to find the dual quotient divisible group. We hope that the duality together with the Mal′cev's matrices is a promising tool for the simultaneous research of torsion-free finite-rank groups and mixed quotient divisible groups.

In conclusion, we mention some results on this direction. O. I. Davydova [4] has described quotient divisible groups of rank-1. E. V. Gordeeva and A. A. Fomin [17] have transposed some classic results by R. Baer [1] about homogeneous completely decomposable torsion free groups to the mixed quotient divisible groups. E. I. Kompantseva and A. A. Fomin [18] have described quotient divisible groups, which are dual to the torsion free groups A with the finite quotient A/F.

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