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A note on almost maximal chain rings

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Abstract – This paper discusses maximal and almost maximal rings in a non-commutative setting. Annihilators ideals in chain rings and their relationship to the concept of self-injectivity are investigated. In particular, a two-sided chain ring is right self-injective if and only if it is right co-Hopfian and a left maximal ring. Finally, localizations of chain rings are discussed.

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Maximal and almost maximal valuation rings play an important role in commutative ring theory ([8] and [9]). It is the goal of this paper to investigate to which extent it is possible to consider such rings in a non-commutative setting. Our discussion is partly motivated by the results in [3] and [4] both of which focused on finitely generated modules over duo chain rings. However, we want to emphasize that our discussion of right (almost) maximal rings does not require the rings to be duo ring in contrasts to the previously mentioned papers. We in particular focus on [8, Section II.6] which summarizes the fundamental properties of almost maximal and maximal valuations rings.

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A ring R is *right maximal* if, for any family $\{I_\alpha \mid \alpha \in \Lambda\}$ of right ideals of R, every system of pairwise solvable congruences of the form $x \equiv x_\alpha \ (I_\alpha)$ with $\alpha \in \Lambda$ and $x_\alpha \in R$ has a simultaneous solution in R. The ring R is *right almost maximal* if such congruences have a simultaneous solution whenever $\bigcap_\Lambda I_\alpha \neq 0$. Examples of maximal rings are the ring of p-adic integers $\mathbb{Z}_{(p)}$ and the rings $\mathbb{Z}/p^n\mathbb{Z}$ for $0 < n < \omega$. However, $\mathbb{Z}$ is an almost maximal ring that is not a maximal ring [6]. Examples of non-commutative right almost maximal and right maximal rings can be found at the end of this paper (Example 12).

Chain rings are the natural extension of valuation rings to a non-commutative setting. A ring R is a *right chain ring* if its right ideals are linearly ordered by inclusion. Numerous examples of chain rings can be found in [5]. In right chain ring, the pairwise solvability of a system of congruences can easily be verified:

Lemma 1. *Let R be a right chain ring with a family $\{I_\alpha \mid \alpha \in \Lambda\}$ of right ideals of R. A system of congruences of the form $x \equiv x_\alpha \ (I_\alpha)$ with $\alpha \in \Lambda$ and $x_\alpha \in R$ is pairwise solvable if and only if $x_\beta - x_\alpha \in I_\alpha$ whenever $I_\beta \subseteq I_\alpha$.*

Proof. Suppose that $z \in R$ is a solution for the congruences $x \equiv x_\beta \ (I_\beta)$ and $x \equiv x_\alpha \ (I_\alpha)$ where $I_\beta \subseteq I_\alpha$. Then

$$x_\beta - x_\alpha = (x_\beta - z) + (z - x_\alpha) \in I_\beta + I_\alpha \subseteq I_\alpha.$$

Conversely, consider two congruences $x \equiv x_\beta \ (I_\beta)$ and $x \equiv x_\alpha \ (I_\alpha)$. Since R is a right chain ring, we may assume that $I_\beta \subseteq I_\alpha$. Clearly, x_β is a common solution for both congruences. □

The next result extends [9, Proposition 1] to the non-commutative setting. For a ring R and a subset S of R, the *right annihilator of S* is

$$\operatorname{ann}_r(S) = \{a \in R | Sa = 0\}.$$

In a similar way, we define the *left annihilator of S* denoted by $\operatorname{ann}_\ell(S)$ and the annihilator of a subset X of a module which is denoted by $ann(X)$. An element $a \in R$ is a *right zero-divisor* if $\operatorname{ann}_r(a) \neq 0$.

Lemma 2. *A right chain ring with right zero-divisors contains non-zero nilpotent elements.*

Proof. Since R has right zero-divisors, there exists $0 \neq a \in R$ with $\operatorname{ann}_r(a) \neq 0$. Because R is a right chain ring, $\operatorname{ann}_r(a) \cap aR \neq 0$. Hence, we can find a non-zero $s \in \operatorname{ann}_r(a)$ of the form $s = ar$ for some $r \in R$. If $sa = 0$, then $s^2 = (sa)r = 0$. On the other hand, if $sa \neq 0$, then $(sa)^2 = s(as)a = 0$. In either case, R contains a non-zero element b with $b^2 = 0$. □

Theorem 3. *A right chain ring R with right zero-divisors is right almost maximal if and only if it is right maximal.*

Proof. Clearly, if R is right maximal, then R is right almost maximal.

To show the converse, assume that R is almost right maximal. By Lemma 2, R contains a non-zero element b with $b^2 = 0$. Let $\{I_\alpha | \alpha \in \Lambda\}$ be a family of right ideals of R, and consider a system $x \equiv x_\alpha(I_\alpha)$ with $\alpha \in \Lambda$ of pairwise solvable congruences. To show that this system has a simultaneous solution, we may assume that $\bigcap_{\alpha\in\Lambda} I_\alpha = 0$.

Let $0 \neq b \in R$ such that $b^2 = 0$. We can select α_0 such that $b \notin I_{\alpha_0}$ because $\bigcap_{\alpha\in\Lambda} I_\alpha = 0$. Since R is a right chain ring, we have $I_{\alpha_0} \subseteq bR$. Consider $\Lambda' = \{\alpha \in \Lambda \mid I_\alpha \subseteq I_{\alpha_0}\}$. Since the congruences are pairwise solvable, the equivalences $x \equiv x_{\alpha_0}(I_{\alpha_0})$ and $x \equiv x_\alpha(I_\alpha)$ have a common solution y_α for any $\alpha \in \Lambda$. In particular, for $\alpha \in \Lambda'$, we have

$$x_\alpha - x_{\alpha_0} = (x_\alpha - y_\alpha) + (y_\alpha - x_{\alpha_0}) \in I_{\alpha_0} \subseteq bR.$$

Consequently, $x_\alpha - x_{\alpha_0} = br_\alpha$ for some $r_\alpha \in R$.

Consider the right ideal $K_\alpha = \{r \in R \mid br \in I_\alpha\}$ of R, and observe that $b^2 = 0 \in I_\alpha$ implies $b \in K_\alpha$ for all $\alpha \in \Lambda$. Therefore, $0 \neq b \in \bigcap_{\alpha\in\Lambda} K_\alpha \subseteq \bigcap_{\alpha\in\Lambda'} K_\alpha$. We now consider the system of congruences $x \equiv r_\alpha(K_\alpha)$ with $\alpha \in \Lambda'$.

Let $\alpha, \gamma \in \Lambda'$. Since R is a right chain ring, we may assume $I_\gamma \subseteq I_\alpha$. By Lemma 1, $b(r_\gamma - r_\alpha) = x_\gamma - x_\alpha \in I_\alpha$, and $r_\gamma - r_\alpha \in K_\alpha$. Thus, the congruences $x \equiv r_\alpha(K_\alpha)$ with $\alpha \in \Lambda'$ are pairwise solvable. Since $0 \neq b \in K_\alpha$ for all $\alpha \in \Lambda$, this last system admits a simultaneous solution r_0 since R is right almost maximal. But $r_0 - r_\alpha \in K_\alpha$ yields $(x_{\alpha_0} + br_0) - x_\alpha = b(r_0 - r_\alpha) \in I_\alpha$. Therefore $x = x_{\alpha_0} + br_0$ is a simultaneous solution of the original system. □

We want to remind the reader that R is *right self-injective* if it is injective as a right R-module, or equivalently if, for all right ideals I of R, every R-homomorphism $f: I \to R$ is left multiplication by a suitable element of R. A right R-module M is *co-Hopfian* if every injective endomorphism of M is an automorphism. The ring is *right co-Hopfian* if R_R is co-Hopfian. Examples of co-Hopfian modules can be found in [11], [12], and [16]. We continue our discussion with some basic properties of co-Hopfian rings.

Proposition 4. *Let R be a ring.*

a. *R_R is co-Hopfian if only if every $a \in R$ with* $\mathrm{ann}_r(a) = 0$ *is a unit.*

b. *A right self-injective ring with finite right Goldie-dimension is right co-Hopfian. This may fail if R is a right self-injective ring which does not have finite right Goldie-dimension.*

Proof. (a) Assume that every $a \in R$ with $\mathrm{ann}_r(a) = 0$ is a unit. If α is an injective endomorphism of R_R, then $\alpha(r) = \alpha(1_R)r$. Since $\mathrm{ann}_r(\alpha(1_R)) = \mathrm{Ker}\, f = 0$, we obtain that $f(1_R)$ is a unit. Thus, f is an automorphism of R. Conversely, if R_R is a co-Hopfian module and $a \in R$ satisfies $\mathrm{ann}_r(a) = 0$, then left multiplication by a is an injective endomorphism α of R_R. Because R_R is co-Hopfian, α is an isomorphism, and a is a unit.

(b) If $\alpha\colon R \to R$ is a monomorphism, then $R = \alpha(R) \oplus B$. Since R_R and $\alpha(R)_R$ have the same finite right Goldie dimension, $B = 0$. If R is the endomorphism ring of an infinite dimensional K-vector-space V, then R is right self-injective by ([14]). If $\phi\colon V \to V$ is a monomorphism which is not onto, then the induced sequence

$$0 \longrightarrow \mathrm{Hom}_K(V, V) \longrightarrow \mathrm{Hom}_K(V, V) \longrightarrow \mathrm{Hom}_K(V, V/\phi(V)) \longrightarrow 0$$

shows that R is not right co-Hopfian since $\mathrm{Hom}_K(V, V/\phi(V)) \neq 0$. □

Corollary 5. *A right self-injective right chain ring is right co-Hopfian.*

Proposition 6 ([15], Ikeda–Nakayama's Theorem). *The following properties of a ring R are equivalent*:

a. *every homomorphism $f\colon I \to R$, where I is finitely generated right ideal, has the form $f(x) = rx$ for some $r \in R$*;
b. *R satisfies*
    i. $\mathrm{ann}_\ell(A_1 \cap A_2) = \mathrm{ann}_\ell(A_1) + \mathrm{ann}_\ell(A_2)$ *for all finitely generated right ideals A_1 and A_2,*
    ii. $\mathrm{ann}_\ell(\mathrm{ann}_r(a)) = Ra$ *for all $a \in R$.*

Clearly, condition (b.i) is satisfied whenever R is a right chain ring.

Corollary 7. *The following conditions are equivalent for a right and left chain ring R*:

a. *R is right co-Hopfian*;
b. $\mathrm{ann}_\ell(\mathrm{ann}_r(a)) = \mathrm{ann}_\ell(\mathrm{ann}_r(Ra)) = Ra$ *for every $a \in R$*;
c. *every homomorphism $\alpha\colon aR \to R$ is left multiplication by an element of R.*

Proof. (a) $\Longrightarrow$ (b). Clearly, $\mathrm{ann}_r(a) = \mathrm{ann}_r(Ra)$; and the first equality holds. For $a \in R$, we have $Ra \subseteq \mathrm{ann}_\ell(\mathrm{ann}_r(Ra))$. If $b \in \mathrm{ann}_\ell(\mathrm{ann}_r(Ra))$, then $Rb \subseteq Ra$ or $Ra \subseteq Rb$ since R a left chain ring. If the first inclusion is true, then nothing needs to be shown. If the second inclusion holds, then $\mathrm{ann}_r(Rb) \subseteq \mathrm{ann}_r(Ra)$. Since $Rb \subseteq \mathrm{ann}_\ell(\mathrm{ann}_r(Ra))$, we have $\mathrm{ann}_r(Rb) \supseteq \mathrm{ann}_r(\mathrm{ann}_\ell(\mathrm{ann}_r(Ra))) = \mathrm{ann}_r(Ra)$ and $\mathrm{ann}_r(Rb) = \mathrm{ann}_r(Ra)$, and $ar = 0$ if and only if $br = 0$ for all $r \in R$.

Since $Ra \subseteq Rb$, we have $a = sb$ for some $s \in R$. If $bt \in bR \cap \mathrm{ann}_r(s)$ for some $t \in R$, then $0 = s(bt) = (sb)t = at$. By what was shown in the last paragraph, $bt = 0$. Therefore, $bR \cap \mathrm{ann}_r(s) = 0$. Since $bR \neq 0$ and R is a right chain ring, $\mathrm{ann}_r(s) = 0$. Because R is right co-Hopfian, s is a unit of R by Proposition 4(a). Therefore, $b = s^{-1}a \in Ra$ as was to be shown.

(b) $\iff$ (c) is a direct consequence of the Ikeda–Nakayama's Theorem.

(b) $\implies$ (a). If $a \in R$ satisfy $\mathrm{ann}_r(a) = 0$, then $R = \mathrm{ann}_\ell(\mathrm{ann}_r(a)) = Ra$, so that $R = Ra$. Thus, $a \notin J(R)$. Since R is a right and left chain ring, a is a unit; and R is right co-Hopfian by Proposition 4(a). $\square$

We continue with a technical result which is a generalization of a result for commutative rings in [13].

Proposition 8. *Let R be a right and left chain ring with Jacobson radical $J = J(R)$ such that R_R is co-Hopfian.*

a. *If I is a left ideal of R which is not a left annihilator, then there is a principal left ideal Rx containing I, such that $Rx = \mathrm{ann}_\ell(\mathrm{ann}_r(I))$ and $I = Jx$.*
b. *If $A \subsetneq B$ are left ideals of R, then $\mathrm{ann}_\ell(\mathrm{ann}_r(A)) \subseteq B$.*

Proof. (a) Since I is left ideal which is not a left annihilator, we deduce that $I \subsetneq \mathrm{ann}_\ell(\mathrm{ann}_r(I))$. If $x \in \mathrm{ann}_\ell(\mathrm{ann}_r(I)) \setminus I$, then $I \subsetneq Rx$ since R is a left chain ring. In particular, $\mathrm{ann}_r(Rx) \subseteq \mathrm{ann}_r(I)$. Moreover, $I \subsetneq Rx \subseteq \mathrm{ann}_\ell(\mathrm{ann}_r(I))$ yields

$$\mathrm{ann}_r(I) = \mathrm{ann}_r(\mathrm{ann}_\ell(\mathrm{ann}_r(I))) \subseteq \mathrm{ann}_r(Rx).$$

Therefore, $\mathrm{ann}_r(Rx) = \mathrm{ann}_r(I)$, and we obtain $\mathrm{ann}_\ell(\mathrm{ann}_r(Rx)) = \mathrm{ann}_\ell(\mathrm{ann}_r(I))$. By Corollary 7b, $Rx = \mathrm{ann}_\ell(\mathrm{ann}_r(I))$.

We now show $Jx = I$. By Nakayama's Lemma, $Jx \neq Rx$a. If $Jx \subsetneq K \subsetneq Rx$ for some left ideal K of R, then $K = Lx$ where $L = \{r \in R \mid rx \in K\}$. Clearly, $J \subsetneq L \subsetneq R$, which contradicts the maximality of J. Thus, there are no left ideals properly contained between Jx and Rx. On the other hand, since R is a left chain ring, $Jx \subseteq I$ or $I \subseteq Jx$. The first inclusion implies $I = Jx$ since I is strictly contained in Rx. If the second inclusion were strict, then $I \subset Rb \subseteq Jx \subset Rx$ for some $b \in Jx \setminus I$ since R is a left chain ring. Now, $\mathrm{ann}_r(Rb) \subseteq \mathrm{ann}_r(I)$, which yields $Rx = \mathrm{ann}_\ell(\mathrm{ann}_r(I)) \subseteq \mathrm{ann}_\ell(\mathrm{ann}_r(Rb)) = Rb$ by what was shown in the first paragraph and Corollary 7. Thus, $Rx = Rb$, and $Jx = Rx$ which clearly contradicts Nakayama's Lemma.

(b) Note that $A \subseteq \mathrm{ann}_\ell(\mathrm{ann}_r(A))$. If equality holds, then the result is obvious. If the inclusion is strict, then A cannot be an annihilator ideal. By (a), $\mathrm{ann}_\ell(\mathrm{ann}_r(A)) = Rx$ for some $x \in R \setminus A$. For any $b \in B \setminus A$, we have $Rx \subseteq Rb$ or $Rb \subsetneq Rx$. If the first inclusion holds, then $\mathrm{ann}_\ell(\mathrm{ann}_r(A)) = Rx \subseteq Rb \subseteq B$. So, assume $Rb \subset Rx$. Since R is a right and left chain ring, $A \subsetneq Rb \subseteq Rx$. Hence, $\mathrm{ann}_r(Rx) \subseteq \mathrm{ann}_r(Rb) \subseteq \mathrm{ann}_r(A)$, which implies

$$\mathrm{ann}_\ell(\mathrm{ann}_r(A)) \subseteq \mathrm{ann}_\ell(\mathrm{ann}_r(Rb)) = Rb \subseteq B$$

by Corollary 7. □

Theorem 9. *The following conditions are equivalent for a right and left chain ring R:*

a. *R is right self-injective;*

b. i. *R_R is co-Hopfian, and*

   ii. *R is a left maximal ring.*

Proof. (a) $\Longrightarrow$ (b). Because of Corollary 5, it remains to show (ii). For this, consider a family $\{I_\alpha \mid \alpha \in \Lambda\}$ of left ideals of R. Suppose that $x \equiv x_\alpha(I_\alpha)$ with $\alpha \in \Lambda$ is a system of pairwise solvable congruences with $x_\alpha \in R$ for all $\alpha \in \Lambda$. We want to show that this system admits a simultaneous solution. Since R is a left chain ring, $x_\alpha - x_\gamma \in I_\alpha$ whenever $I_\gamma \subseteq I_\alpha$ by Lemma 1. If there is a non-zero left ideal I_{α_0} such that $I_{\alpha_0} \subseteq I_\alpha$ for all $\alpha \in \Lambda$, then $x_{\alpha_0} - x_\alpha \in I_\alpha$ for all $\alpha \in \Lambda$ implies that the system has a simultaneous solution.

Assume that there is no such left ideal. Let $J_\alpha = \mathrm{ann}_r(I_\alpha)$, and observe that $I_\gamma \subseteq I_\alpha$ implies $J_\alpha \subseteq J_\gamma$. Consider the maps $f_{x_\alpha}\colon J_\alpha \to R$ which are defined as left multiplication by x_α for $\alpha \in \Lambda$. Since R is a right chain ring, the J_α's form a chain, and $\cup_{\alpha\in\Lambda} J_\alpha$ is a right ideal of R. Suppose that $J_\alpha \subseteq J_\gamma$ for some $\alpha, \gamma \in \Lambda$, and let $z \in J_\alpha$. If $I_\gamma \subseteq I_\alpha$, then $x_\alpha - x_\gamma \in I_\alpha$ by Lemma 1, and we have $(x_\alpha - x_\gamma)z = 0$ since $J_\alpha = \mathrm{ann}_r(I_\alpha)$. On the other hand, if $I_\alpha \subseteq I_\gamma$, then $J_\gamma \subseteq J_\alpha$, and $J_\alpha = J_\gamma$. By Lemma 1, $x_\alpha - x_\gamma \in I_\gamma$. Since $J_\alpha = J_\gamma$, we have $(x_\alpha - x_\gamma)z = 0$ in this case too. Thus, $f_{x_\gamma}(z) = x_\gamma z = x_\alpha z = f_{x_\alpha}(z)$. Therefore, we can define a map $f\colon \cup_{\alpha\in\Lambda} J_\alpha \to R$ by $f(y) = x_\alpha y$ if $y \in J_\alpha$. Since R is right self-injective, f can be extended to all of R. This implies that f is a multiplication from the left by some element $t \in R$.

To show that t is a solution for the system of congruences, let $a_\alpha \in J_\alpha$. Then

$$(t - x_\alpha)a_\alpha = ta_\alpha - x_\alpha a_\alpha = f(a_\alpha) - f_{x_\alpha}(a_\alpha) = 0.$$

Therefore,

$$t - x_\alpha \in \operatorname{ann}_\ell(J_\alpha) = \operatorname{ann}_\ell(\operatorname{ann}_r(I_\alpha)).$$

We know $I_\alpha \subseteq \operatorname{ann}_\ell(\operatorname{ann}_r(I_\alpha))$. If equality holds, then t is a solution of the congruence $x \equiv x_\alpha(I_\alpha)$. Suppose $I_\alpha \subsetneq \operatorname{ann}_\ell(\operatorname{ann}_r(I_\alpha))$. Recall that we are assuming that there is no a smallest left ideal in the family $\{I_\alpha\}_\alpha \in \Lambda$. Thus, $I_\gamma \subset I_\alpha$ for some $I_\gamma \in \Lambda$. By Proposition 8b, $\operatorname{ann}_\ell(J_\gamma) = \operatorname{ann}_\ell(\operatorname{ann}_r(I_\gamma)) \subseteq I_\alpha$. Then $t - x_\gamma \in I_\alpha$ since $t - x_\gamma \in \operatorname{ann}_\ell(J_\gamma)$ (as we showed for α). By the pairwise solvability, $x_\gamma - x_\alpha \in I_\alpha$ using Lemma 1. Thus,

$$t - x_\alpha = (t - x_\gamma) + (x_\gamma - x_\alpha) \in I_\alpha$$

for all $\alpha \in \Lambda$. Hence, x is again a solution of the congruence $x \equiv x_\alpha(I_\alpha)$. Therefore, t is a simultaneous solution of the original system of congruences.

(b) $\Longrightarrow$ (a). For a right ideal I of R, consider a R-homomorphism $f : I \to R$. Since $f(x)(\operatorname{ann}_r(x)) = f(x \ \operatorname{ann}_r(x)) = 0$ for all $x \in I$, we obtain $f(x) \in \operatorname{ann}_\ell(\operatorname{ann}_r(x))$. In view of (i) and Corollary 7, we have $\operatorname{ann}_\ell(\operatorname{ann}_r(x)) = Rx$ which yields $f(x) = r_x x$ for some $r_x \in R$. We want to show that f is left multiplication with a fixed element r of R. For this, we consider the system of congruences $x \equiv r_a(\operatorname{ann}_\ell(a))$ for $a \in R$. To show that this system is pairwise solvable, let $a, b \in R$. Since R is a right chain ring, we may assume $aR \subseteq bR$ so that $a = bs$ for some $s \in R$. Then,

$$(r_b - r_a)a = r_b(bs) - r_a(a) = f(b)s - f(a) = f(bs) - f(a) = 0.$$

Therefore $r_b - r_a \in \operatorname{ann}_\ell(a)$, and the system is pairwise solvable. By (ii), the system has a simultaneous solution $r \in R$. Then $r - r_x \in \operatorname{ann}_\ell(x)$ for all $x \in I$ implies $f(x) = r_x x = rx$ for all $x \in I$. Therefore, the morphism defined by $\hat{f}(x) = rx$ is the required extension of f; and R is right self-injective. $\square$

As a consequence, we obtain a variant of the characterization of commutative self-injective rings given by Klatt and Levy ([13], Theorem 2.3):

Corollary 10. *A valuation ring R is self-injective if only if R_R is co-Hopfian and R is a maximal ring.*

An element c of a ring R is *right regular* if it is not a right zero-divisor. It is *regular* if it is both right and left regular. The symbols $C_r(R)$ and $C_\ell(R)$ denote the right and left regular elements of R respectively. A set X of regular elements of R is a *right Ore-set* if, for $r \in R$ and $c \in X$, we can find $s \in R$ and $d \in X$ such that $rd = cs$. In this case, R has a *classical right ring of quotient $Q^r(X)$* with respect to X. If X is also a left Ore-set, then the classical left ring of quotient with respect to X and the classical right ring of quotients coincide; and it is denoted as $Q(X)$. Moreover, if R is a right and left chain ring without zero-divisors, then the non-zero elements of R form a right and left Ore-set whose skew-field of quotients is denoted by $Q(R)$. Any localization of R at a right and left Ore-set can be viewed as a subring of $Q(R)$ containing R.

Theorem 11. *Let R be a right and left chain ring without zero-divisors, X a right and left Ore-subset of R, and $S = Q(X)$ the classical right and left ring of quotients with respect to X. Then the following hold*:

a. *S is a right and left chain ring such that $J(S) \subseteq R$*;

b. *if R is right* (*almost*) *maximal, then the same is true for S.*

Proof. (a) We first show that $Q(X)$ is a right and left chain ring. If $a = rc^{-1}$ and $b = sc^{-1}$ with $r, s \in R$ and $c \in X$ are elements of S, then either $r = ts$ or $s = tr$ for some $t \in R$ since R is a left chain ring. Without loss of generality, assume that the first inclusion holds. Then $a = rc^{-1} = tsc^{-1} = tb$. Hence, S is a left chain ring. Since X is also a left Ore-set, S is a right chain ring as well.

Consider $0 \neq q = rc^{-1} \in S$ with $r \in R$ and $c \in X$. Since R is a left chain ring, we can find $t \in R$ such that $r = tc$ or $c = tr$. In the first case,

$$q = rc^{-1} = tcc^{-1} = t \in R.$$

In the second case, we compute in $Q(R)$ to obtain $c^{-1} = r^{-1}t^{-1}$. Hence,

$$q = rc^{-1} = rr^{-1}t^{-1} = t^{-1}.$$

Thus every element of $Q(X)$ either belongs to R or is of the form s^{-1} for some $s \in R$. We use this to show $J(S) \subseteq R$. If $q \in J(S)\setminus R$, then we can find $r, s \in R$ and $c \in X$ such that $q = rc^{-1} = s^{-1}$ by what has just been shown. Since $q \in J(S)$, the same holds for $r = qc$. Then $c = sr \in J(S)$ which is not possible since c is a unit of S. In particular, R contains all the proper right ideals of S.

We consider a family $\{a_\alpha + J_\alpha \mid \alpha \in \Lambda\}$ with the finite intersection property where each J_α is a right ideal of S. We can eliminate all the J_α's from consideration which are not proper since each $q \in S$ belongs to $a_\alpha + S$. Thus, $J_\alpha \subseteq J(S) \subseteq R$ by (a). We fix $\gamma \in \Lambda$, and select $c \in X$ such that $ca_\gamma \in R$. Since the family $\{a_\alpha + J_\alpha \mid \alpha \in \Lambda\}$ has the finite intersection property, we can find $x_\gamma \in J_\gamma$ and $x_\alpha \in J_\alpha$ such that $a_\gamma + x_\gamma = a_\alpha + x_\alpha$ from which we get $ca_\alpha = ca_\gamma + cx_\gamma - cx_\alpha \in R$ since $J_\gamma, J_\alpha \subseteq R$. Clearly, the family $\{ca_\alpha + cJ_\alpha \mid \alpha \in \Lambda\}$ has the finite intersection property. Hence, we can find

$a \in R$ such that $a \in ca_\alpha + cJ_\alpha$ for all $\alpha \in I$, say $a = ca_\alpha + cx_\alpha$. Thus, $c^{-1}a = a_\alpha + x_\alpha \in a_\alpha + J_\alpha$ for all $\alpha \in \Lambda$ since c is a unit of S. □

We continue our discussion with some examples of non-commutative right almost maximal and right maximal chain rings:

Example 12. Let p be a prime.

a. There exists a countable non-commutative ring R with a reduced torsion-free additive group whose non-zero one-sided ideals are of the form $p^n R$ for some $n < \omega$ [1]. The ring R is almost right and left maximal, but not maximal.
b. If $\widehat{R}$ be the p-adic completion of the ring R in (a), then $\widehat{R}$ is a right and left maximal ring.
c. Let R be a right Noetherian, right chain domain whose lattice of right ideals is inversely order isomorphic to an ordinal σ of uncountable cardinality. Select a non-zero ideal I of R corresponding to ω in the inverse order. Then R/I clearly is almost maximal [5].

Proof. (a) A family $\{I_\alpha \mid \alpha \in \Lambda\}$ of (one-sided) ideals of R such that $\bigcap_\Lambda I_\alpha \neq 0$ has only finitely many distinct members. Thus, R is almost maximal. If R were maximal, then it would be compact in the p-adic topology because the non-zero ideals of R are of the form $p^n R$ for some $n < \omega$. In particular, R^+ is complete in the p-adic topology. Since R^+ contains a pure subgroup isomorphic to $\mathbb{Z}_p$, it also contains a copy of its p-adic completion, which is isomorphic to J_p, the p-adic integers. This is not possible since R is countable.

(b) Let $\widehat{I}$ be a non-zero ideal of $\widehat{R}$. Then $\widehat{I} \cap R = p^n R$ for some $n < \omega$. Thus, $p^n \widehat{R} \subseteq \widehat{I}$. On the other hand, if $x \in \widehat{I}$, then there are $y \in \widehat{R}$ and $r \in R$ such that $x - p^n y = r$ since $\widehat{R}/R$ is divisible as an Abelian group. Moreover, $r \in \widehat{I} \cap R = p^n R$ so that we can find $s \in R$ such that $r = p^n s$. Hence, $\widehat{I} = p^n \widehat{R}$. Thus the only relevant linear topology to be considered for maximality is the p-adic topology. Since $\widehat{R}$ is almost maximal as in (a), completeness implies maximality as in the commutative case [8, Theorem II.6.1]. For details on completions see [2] and [10].

(c) can be shown by arguing as in (a). □

The rings in the last example have the additional property that they are right duo, i.e. each of their right ideals is two-sided. In a right and left duo ring, every multiplicatively closed set is right and left Ore. In particular, the notion of R-topology in [8, Chapter VIII.1] can be extended to such rings in order to investigate maximal and almost maximal modules. We conclude with an application of Theorems 9 and 11:

Corollary 13. *Let R be a right and left duo chain ring with $C_r(R) = C_\ell(R)$. If R is left* (*right*) *maximal, then $S = Q(C_r(R))$ is right* (*left*) *self-injective.*

Proof. Since R is duo, $C_r(R)$ is a right and left Ore-set. Suppose $q = rc^{-1} \in S \setminus R$ for some $r \in R$ and $c \in C_r(R)$. Since R is a left chain ring, we can find $t \in R$ such that $r = tc$ or $c = tr$. In the first case, $q = rc^{-1}c = r \in R$, while in the second case, r is a right regular element of R for $rx = 0$ implies $cx = 0$. Thus, r is a regular element, and hence a unit of S. In particular, $c^{-1} = r^{-1}t^{-1}$ and $q = rr^{-1}t^{-1} = t^{-1}$. Thus, we can argue as in the proof of Theorem 11 to obtain that S is right (left) maximal. Finally, S clearly is co-Hopfian. □

We need not actually distinguish between almost maximality and maximality in the last result because of Theorem 3 if R has zero-divisors. If R does not have zero-divisors, then S is a division algebra.

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