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Milnor's alternative for finitely generated relatively free groups that are locally graded

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Abstract – Let G be a finitely generated relatively free group that is locally graded. We show that either G contains a non-trivial free subsemigroup or G is nilpotent-by-finite.

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Milnor property. A condition, introduced in 1968 by J. Milnor [10], was later called by F. Point [12] the Milnor property, and we shall use this terminology.

Definition 1. *A group G satisfies the Milnor property, if for all elements $g, h \in G$ the subgroup $\langle h^{-i} g h^{i}, i \in \mathbb{Z}\rangle$ is finitely generated.*

Milnor proved in Lemma 3 of [10], that if a finitely generated group G has this property and A is an abelian normal subgroup in G so that G/A is cyclic then A is finitely generated. In 1976 S. Rosset [13] noticed that the assumption that A is abelian can be dropped and considered G/N cyclic. From this it follows immediately that the same conclusion holds when G/N is poly-cyclic and thus in particular when G/N is nilpotent. This is a second part of the following lemma. For the first part see [13, Lemmas 2 and 3] and [9, Lemma 3 and Corollary 4].

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Lemma 1. *Let G be a finitely generated group satisfying the Milnor property.*

(i) *G' is finitely generated.*

(ii) *If G/N is nilpotent, then N is finitely generated.*

In 1995 Y. K. Kim and A. H. Rhemtulla [9] introduced the term *restrained* for the groups satisfying the Milnor property. They proved that the finitely generated groups satisfying positive laws are restrained. The same was also proved by other authors (e.g. [8], [12], [2]). Recall that a positive law is a law $u(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n)$, where u, v are distinct words in the free group $\langle x_1, x_2, \ldots \rangle$, written without negative powers of $x_1, x_2, \ldots, x_n$.

If a group satisfies a positive law, it satisfies the Milnor property. The Milnor property is necessary but not sufficient for a subexponential growth since the infinite Burnside groups satisfy positive laws (hence the Milnor property) and have exponential growth [1].

All groups below are assumed finitely generated.

Definition 2. *A class of groups satisfies the Milnor's alternative if every group in this class either has a polynomial growth or contains a free subsemgroup and then has exponential growth.*

It follows by the results of J. Milnor [10] and J. Wolf [16] that the class of virtually soluble groups satisfies the Milnor's alternative. In the article [3] of Milnor, Power et al. (1968, Problem 5603), the problem concerning the Milnor's alternative was posed for the class of all groups. J. Tits (1972) gave the positive answer for the class of linear groups [15]. Ching Chou (1980) gave the positive answer for the class of elementary amenable groups [5].

The negative answer for the class of all groups follows from existence of groups of intermediate growth found by R. Grigorchuk [6] in 1983. The other examples of classes satisfying the Milnor's alternative are not known.

We recall that a group G is called locally graded if every nontrivial, finitely generated subgroup of G has a proper subgroup of finite index. The class of locally graded groups is closed under taking subgroups, extensions and groups which are locally-or-residually 'locally graded'. The class of locally graded groups was introduced in 1970 by S. N. Černikov [4] to avoid groups such as infinite Burnside groups or Ol'shanskii–Tarski monsters.

The class of the locally graded groups does not satisfy the Milnor's alternative since it contains the residually finite (hence locally graded) groups of intermediate growth [6]. The class of the relatively free groups also does not satisfy the Milnor's alternative because the infinite Burnside groups $B(m, n)$, for $n \geq 2$, $n \geq 665$ have exponential growth [1] and do not contain a free subsemigroup.

We show that the intersection of these classes, that is the class of locally graded relatively free groups satisfies the Milnor's alternative.

Theorem 1. *Every finitely generated, locally graded, relatively free group G either has a polynomial growth or contains a free subsemigroup.*

Proof. Let G be a locally graded relatively free group. We have to prove that if G has no free subsemigroup then G has a polynomial growth.

First we show that every relatively free group G without free subsemigroup satisfies a positive law. Indeed, if G is cyclic, it is clear. If G is not cyclic, there are elements a and b in a set of free generators in G. Since by assumption the semigroup generated by a and b is not free, there is a positive relation $u(a, b) = v(a, b)$. Then in view of the properties of the relatively free groups (see [11, 13.21]) the group G satisfies the positive law $u(x, y) \equiv v(x, y)$.

By R we denote the finite residual in the group G, that is the intersection of all subgroups of finite index in G. So G satisfies a non-trivial positive law $u(x, y) \equiv v(x, y)$ and the group G/R is residually finite. Now by the well known lemma of J. Semple and A. Shalev [14], saying that *a residually finite group satisfying a nontrivial positive law is virtually nilpotent,* we conclude that G/R is nilpotent-by-finite. It suffices to show that $R = 1$. We argue by contradiction and assume that $R \neq 1$. We have $R \leq H \leq G$ where G/H is finite and H/R nilpotent. Then, being of finite index, H is finitely generated and by Lemma 1(ii) it follows that R is finitely generated. Since G is locally graded, R has a proper normal subgroup K of finite index that we can assume to be fully invariant in R. Then K is normal in G and H/K is finite-by-nilpotent, and hence nilpotent-by-finite (by P. Hall). Whence G/K is nilpotent-by-finite and thus residually finite. But this gives the contradiction with $R \leq K$.

So $R = 1$ and G is nilpotent-by-finite. Now by result of Gromov [7], G has a polynomial growth, which proves the alternative. □

There are well known locally graded relatively free groups of polynomial and of exponential growth, however from the above theorem we have:

Corollary 1. *There does not exist a locally graded relatively free group of intermediate growth.*

The general problem of existence of relatively free group of intermediate growth remains open.

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