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Closed ideals in the uniform topology on the ring of real-valued continuous functions on a frame

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Abstract – For a completely regular frame L, the ring $\mathcal{R}L$ of real-valued continuous functions on L is equipped with the uniform topology. The closed ideals of $\mathcal{R}L$ in this topology are studied, and a new, merely algebraic characterization of these ideals is given. This result is used to describe the real ideals of $\mathcal{R}L$, and to characterize pseudocompact frames and Lindelöf frames. It is shown that a frame L is finite if and only if every ideal of $\mathcal{R}L$ is closed. Finally, we prove that every closed ideal in $\mathcal{R}L$ is an intersection of maximal ideals.

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1. Introduction

By the term ring we mean a commutative ring with identity and by the term ideal we mean a proper ideal. All topological spaces are supposed to be completely regular and Hausdorff. All frames considered here are assumed to be completely regular.

Given a space X, the ring $C(X)$ of real-valued continuous functions on X is equipped with the uniform topology (see [16] for more details). Uniformly closed ideals in classical rings $C(X)$ have a lucid characterization in terms of z-ideals

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and zero sets which often reduce calculations with these types of ideals (see [24] for more details).

The main goal of the present paper is to generalize the characterization of uniformly closed ideals of $C(X)$, pseudocompact spaces and Lindelöf spaces as found in [3, 22, 24], to the more general setting of pointfree topology, that is, frames.

Here is an overview of the paper. In the preliminaries, we collect some properties of frames that we shall use. Ring-theoretic concepts which are not ordinary will be recalled as needed.

A discussion of ω-divisible ideals in $\mathcal{R}L$ is provided in Section 3. These ideals and uniformly closed ideals in $\mathcal{R}L$ are closely related. We commence by providing Proposition 3.1 which enables us to prove that a frame L is compact if and only if every divisible ideal in $\mathcal{R}L$ is fixed (Corollary 3.3). A characterization of F-frames based on divisible ideals in $\mathcal{R}L$ is given in Theorem 3.4. We determine the relation between ω-divisible ideals in the ring $\mathcal{R}L$ and their cozero parts in Lemma 3.7. This lemma enables us to give ring-theoretic characterizations of pseudocompact frames and Lindelöf frames (see Corollary 3.9 and Theorem 3.10, respectively).

We begin Section 4 by recalling the definition of uniform topology on the ring $\mathcal{R}L$ of real-valued continuous functions on a frame L. The main characterization of uniformly closed ideals of $\mathcal{R}L$ in terms of ω-divisible ideals is provided in Proposition 4.4. This proposition is applied to describe the real ideals of $\mathcal{R}L$, and to characterize pseudocompact frames and Lindelöf frames (see Theorem 4.6, Corollary 4.7 and Theorem 4.8). Finally, in Theorem 4.10, we show that

every ideal of $\mathcal{R}L$ is closed if and only if L is a finite frame.

M-ideals of the ring $\mathcal{R}L$ equipped with the uniform topology are investigated in the last section. In Lemma 5.1, we prove that for all $I \in \beta L$, the closure of an M-ideal M^I is equal to the closure of an O-ideal O^I. This leads to the characterization of pseudocompact frame L as those for which the closure of every prime ideal of $\mathcal{R}L$ is a (necessarily maximal) ideal (Proposition 5.3). In Proposition 5.7, it is shown that fixed maximal ideals in $\mathcal{R}L$ are closed. We end by showing that for a closed ideal Q in $\mathcal{R}L$, there is $I \in \beta L$ such that $Q = M^I$ (Proposition 5.8). This proposition says (Corollary 5.9)

every closed ideal in $\mathcal{R}L$ is an intersection of maximal ideals.

2. Preliminaries

2.1 – *Frames*

For the basic terms and notations in frames, [23] is recommended. For undefined terms and notations see [5] on pointfree function rings, and see [16] on C(X). Here, we recall a few definitions and results that will be relevant for our discussion.

A *frame* is a complete lattice for which finite meets distribute over arbitrary joins. Let L be a frame. We denote the top element and the bottom element of L by $\top_L$ and $\bot_L$ respectively, and we omit the subscripts if that does not lead to ambiguity. Throughout this context L will denote a frame. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}(X)$. A *pseudocomplement* of an element $x \in L$ is denoted by x^*. An element x of L is *complemented* whenever $x \vee x^* = \top$. An element p of L is a *point* (or a *prime*) whenever $p < \top$ and $a \wedge x \leq p$ implies that $a \leq p$ or $x \leq p$. We denoted the set of all points of L by $\mathrm{Pt}(L)$.

A frame L is *completely regular* if, for each $x \in L$, $x = \bigvee\{a \in L \mid a \prec\!\!\prec x\}$, where $a \prec\!\!\prec x$ means that there are elements (c_q) indexed by the rational numbers $\mathbb{Q} \cap [0,1]$ such that $c_0 = x,\ c_1 = a$, and $c_p \prec c_q$ for $p < q$. Note that $a \prec x$ means that $x \vee a^* = \top$.

2.2 – *The Stone–Čech compactification*

An ideal J of a frame L is *completely regular* if for each $a \in J$ there is $b \in J$ such that $a \prec\!\!\prec b$. The frame of completely regular ideals of L, denoted by βL, is a compact completely regular frame. This shows that βL is spatial, that is, $\beta L \cong \mathfrak{O}(X)$ for some topological space X. The join map $\bigvee\colon \beta L \to L$ is dense onto and referred to as the Stone–Čech compactification of L. We denoted its adjoint by r_L, and recall that (1) $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}$ for each $a \in L$, and (2) r_L preserves $\prec\!\!\prec$ (see [7] for more details).

2.3 – *The Lindelöf coreflection*

Recall that a frame L is called *Lindelöf* whenever $\bigvee T = \top$ implies $\bigvee S = \top$ for some countable $S \subseteq T$. An ideal J of $\operatorname{Coz} L$ is a σ-ideal if it is closed under countable joins. The regular Lindelöf coreflection of a frame L is the frame of σ-ideals of $\operatorname{Coz} L$ denoted by λL. Now, we recall that Hewitt realcompactification of a frame L, denoted υL, is constructed in the following procedure (see [9, 21] for more details). For any $a \in L$, if $[a] = \{x \in \operatorname{Coz} L \mid x \leq a\}$, then the map

$\ell: \lambda L \to \lambda L$ with $\ell(J) = [\bigvee J] \wedge \bigwedge\{M \in \mathrm{Pt}(\lambda L) \mid J \leq M\}$ is a nucleus and $\upsilon L = \mathrm{Fix}(\ell)$. The map $n: \beta L \to \lambda L$ given by $n(I) = \langle I\rangle_\sigma$, the σ-ideal generated by I in $\mathrm{Coz}\, L$, is an onto frame homomorphism. Therefore, there exists an onto frame homomorphism $\beta L \to \upsilon L$ given by $I \mapsto \ell(\langle I\rangle_\sigma)$.

2.4 – *The ring* $\mathcal{R}L$

Regarding the frame of the reals $\mathcal{L}(\mathbb{R})$ and the f-ring $\mathcal{R}L$ of continuous real functions on L, we use the notation of [5]. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}L$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$. An element $\alpha \in \mathcal{R}L$ is called *bounded* if $\alpha(p, q) = \top$ for some $p, q \in \mathbb{Q}$, and L is called *pseudocompact* if $\mathcal{R}L = \mathcal{R}^*L$, where $\mathcal{R}^*L$ denotes the subring of $\mathcal{R}L$ consisting of its bounded elements. Note that $\mathcal{R}^*L \cong \mathcal{R}(\beta L)$. This shows that $\mathcal{R}^*L \cong C(X)$ for some topological space X, since $C(X) \cong \mathcal{R}(\mathfrak{O}(X)) \cong \mathcal{R}(\beta L)$.

We freely use the properties of the cozero map $\mathrm{coz}: \mathcal{R}L \to L$, given by

$$\mathrm{coz}\,\alpha = \bigvee\{\alpha((p, 0) \vee (0, q)) \mid p < 0 \text{ or } q > 0\},$$

and those of $\mathrm{Coz}\, L = \{\mathrm{coz}\,\alpha \mid \alpha \in \mathcal{R}L\}$, the *cozero part* of L. Note that $\mathrm{Coz}\, L$ is a regular sub-σ-frame of L, and a frame is completely regular if and only if it is generated by its cozero part. We refer to [4, 5, 8] for general properties of cozero elements and cozero parts of frames. Here, we emphasize the following:

(1) for any $\alpha \geq \mathbf{0}$ in $\mathcal{R}L$, $\mathrm{coz}\,\alpha = \bigvee\{\mathrm{coz}((\alpha - \mathbf{r})^+) \mid 0 < r \in \mathbb{Q}\}$, and

(2) whenever $a \prec\!\!\prec x$ in L, there exists $b \in \mathrm{Coz}\, L$ such that $a \prec\!\!\prec b \prec\!\!\prec x$.

For ideals Q of $\mathcal{R}L$ and J of $\mathrm{Coz}\, L$, we write $\mathrm{coz}[Q] = \{\mathrm{coz}\,\varphi \mid \varphi \in Q\}$ and $\mathrm{coz}^{\leftarrow}[J] = \{\varphi \in \mathcal{R}L \mid \mathrm{coz}\,\varphi \in J\}$, respectively. The following statements hold (see [12]):

(1) if Q is an ideal of $\mathcal{R}L$, then $\mathrm{coz}[Q]$ is an ideal of $\mathrm{Coz}\, L$;

(2) if J is an ideal of L or $\mathrm{Coz}\, L$, then $\mathrm{coz}^{\leftarrow}[J]$ is an ideal of $\mathcal{R}L$;

(3) if M is a maximal ideal of $\mathcal{R}L$, then $\mathrm{coz}[M]$ is a maximal ideal of $\mathrm{Coz}\, L$;

(4) if J is a maximal ideal of L or $\mathrm{Coz}\, L$, then $\mathrm{coz}^{\leftarrow}[J]$ is a maximal ideal of $\mathcal{R}L$.

Recall from[12, 15] that an ideal Q of $\mathcal{R}L$ is called

z-ideal if, for any $\alpha, \beta \in \mathcal{R}L$, $\mathrm{coz}\,\alpha = \mathrm{coz}\,\beta$ and $\alpha \in Q$ imply $\beta \in Q$, or, equivalently, $\mathrm{coz}^{\leftarrow}[\mathrm{coz}[Q]] = Q$;

fixed whenever $\bigvee \mathrm{coz}[Q] < \top$, and we say that it is *free* otherwise.

3. ω-divisible ideals in $\mathcal{R}L$

An ideal I in a ring A is *divisible* if for two members x_1 and x_2 in I there exist $a \in I$ and $y_1, y_2 \in A$ such that $x_1 = ay_1$ and $x_2 = ay_2$ (see [19]). We note that if I is a divisible ideal of a ring A, then $\{xA\}_{x\in I}$ is a directed family of ideals of A, which implies that I is the direct limit of $\{xA\}_{x\in I}$ (see [17]). Therefore, a divisible ideal is exactly a directed limit of principal ideals.

For each $f \in C(X)$, the zero-set $Z(f)$ is the set of zeros of f, that is, $Z(f) = \{x \in X \mid f(x) = 0\}$. An ideal I in $C(X)$ is called *fixed* if $\bigcap Z[I] = \bigcap_{f\in I} Z(f) \neq \emptyset$. As remarked in [3, p. 325], Theorem 4.11 in [16] implies that X is compact if and only if every divisible ideal in $C(X)$ is fixed. Now, we are going to extend these results to frames. We begin with the following proposition. For the proof of this proposition, we shall use the following fact. Suppose $f_1, f_2 \in C(X)$. Let $g = (|f_1| + |f_2|)^{1/2}$. Now, for $i = 1, 2$, we define

$$(\dagger) \qquad h_i(x) = \begin{cases} \dfrac{f_i(x)}{g(x)} & \text{if } x \notin Z(g), \\ 0 & \text{if } x \in Z(g). \end{cases}$$

then it is clear that $h_i(x) \in C(X)$ and $f_i = gh_i(x)$ (see[19]).

We say an ideal Q of $\mathcal{R}L$ satisfies the property $(*)$ if $\alpha \in Q$ implies $|\alpha|^{1/2} \in Q$. Note that the property $(*)$ implies that if $\alpha \in Q$, then $|\alpha| \in Q$.

Proposition 3.1. *If Q is an ideal of $\mathcal{R}L$ which satisfies the property* $(*)$, *then Q is a divisible ideal.*

Proof. Suppose $\alpha_1, \alpha_2 \in Q$. Let $\beta_1 = \frac{\alpha_1}{1+|\alpha_1|}$, $\beta_2 = \frac{\alpha_2}{1+|\alpha_2|}$ and $\gamma = (|\beta_1| + |\beta_2|)^{1/2}$. Then β_1, β_2 and γ belong to Q. Since they are in $\mathcal{R}^*L$, and $\mathcal{R}^*L$ is isomorphic to a $C(X)$ via an f-ring isomorphism, we infer from $(\dagger)$ that $\beta_1 = \frac{\alpha_1}{1+|\alpha_1|}$ and $\beta_2 = \frac{\alpha_2}{1+|\alpha_2|}$ are multiples of γ. This implies α_1 and α_2 are a multiple of γ. Therefore Q is a divisible ideal. □

A prime ideal P of $\mathcal{R}L$ satisfies the property $(*)$. To see this, take $\alpha \in P$. Then $|\alpha||\alpha| = \alpha^2 \in P$ and hence $|\alpha|^{1/2}|\alpha|^{1/2} = |\alpha| \in P$, implying that $|\alpha|^{1/2} \in P$. Therefore every prime ideal of $\mathcal{R}L$ is a divisible ideal. In particular, every maximal ideal of $\mathcal{R}L$ is a divisible ideal. Since the set of ideals satisfying $(*)$ is clearly closed under arbitrary intersections, we have the following corollary.

Corollary 3.2. *If Q is an intersection of prime ideals of $\mathcal{R}L$, then Q is a divisible ideal.*

An immediate consequence of the above corollary and [13, Lemma 4.7], is the next corollary.

Corollary 3.3. *A frame L is compact if and only if every divisible ideal in $\mathcal{R}L$ is fixed.*

The next theorem demonstrates that in general, $\mathcal{R}L$ contains ideals which are not divisible ideals. Recall from [11, Proposition 3.2] that a frame L is an F-frame if and only if $\mathcal{R}L$ is a Bézout ring, meaning that every finitely generated ideal of $\mathcal{R}L$ is principal.

It is straightforward that a finitely generated divisible ideal is principal. It is also clear that every ideal of a ring is divisible if and only if every finitely generated ideal of the ring is divisible. It follows that every ideal of a ring R is divisible if and only if the ring is a Bézout ring. Now, the following theorem follows readily from [11, Proposition 3.2].

Theorem 3.4. *Every ideal of $\mathcal{R}L$ is a divisible ideal if and only if L is an F-frame.*

The main tool to be used in this paper is the concept of ω-divisible ideals, which was defined by Azarpanah in [3] under the name of strongly divisible ideals, and used there to characterize Lindelöf spaces as follows:

X is Lindelöf if and only if every ω-divisible ideal of $C(X)$ is fixed.

Definition 3.5. An ideal I of a ring A is *ω-divisible* if every at most countably generated ideal $J \subset I$ is contained in a principal ideal.

In this section, we will show that L is Lindelöf if and only if every ω-divisible ideal of $\mathcal{R}L$ is fixed. To prove the main results of this section, we need the following lemma. In order to state the following lemma, we need some background. Suppose that (α_n) is a sequence of positive elements of $\mathcal{R}L$. The set

$$\{(\alpha_1 \wedge \mathbf{2}^{-1}) + (\alpha_2 \wedge \mathbf{2}^{-2}) + \cdots + (\alpha_n \wedge \mathbf{2}^{-n}) \mid n \in \mathbb{N}\}$$

has a supremum in the poset $\mathcal{R}L$ (see [6]). This supremum is denoted by

$$\sum_{n=1}^{\infty} (\alpha_n \wedge \mathbf{2}^{-n}).$$

Proposition 3.6. [18] *Let $\alpha, \beta \in \mathcal{R}L$. If $|\alpha| \leq |\beta|^p$ for some $1 < p \in \mathbb{Q}$, then α is a multiple of β.*

LEMMA 3.7. *For an ideal Q of the ring $\mathcal{R}L$, the following statements hold.*

(1) *If Q is a z-ideal of $\mathcal{R}L$ such that* $\mathrm{coz}[Q]$ *is closed under countable join, then Q is ω-divisible.*

(2) *If Q is ω-divisible, then* $\mathrm{coz}[Q]$ *is closed under countable join.*

PROOF. (1) Let $(\alpha_n) \subseteq Q$. Then we have

$$\beta = \sum_{n=1}^{\infty} \mathbf{2}^{-n} \alpha_n^{\frac{2}{3}} (\mathbf{1} + \alpha_n^{\frac{2}{3}})^{-1} \in \mathcal{R}L$$

and it is clear that $\mathrm{coz}\,\beta = \bigvee_{n=1}^{\infty} \mathrm{coz}\,\alpha_n$. Since $\mathrm{coz}[Q]$ is closed under countable join, $\mathrm{coz}\,\beta \in \mathrm{coz}[Q]$, implying that β belongs to the z-ideal Q. But for each n, $\mathbf{2}^{-n} \frac{\alpha_n^{\frac{2}{3}}}{\mathbf{1}+\alpha_n^{\frac{2}{3}}} \leq \beta$, and so $|\alpha_n| \leq |\beta \mathbf{2}^n (\mathbf{1} + \alpha_n^{\frac{2}{3}})|^{\frac{3}{2}}$. Now, by Proposition 3.6, each α_n is a multiple of $\beta \mathbf{2}^n (\mathbf{1} + \alpha_n^{\frac{2}{3}})$, which implies that α_n is a multiple of β. Therefore Q is ω-divisible.

(2) Let $(\alpha_n) \subset Q$. Then $\{\mathrm{coz}\,\alpha_n \mid n \in \mathbb{N}\}$ is a countable subset of $\mathrm{coz}[Q]$. Since Q is ω-divisible, take $\beta \in Q$ such that β divides α_n, for each n. This shows that for each n, $\mathrm{coz}\,\alpha_n \leq \mathrm{coz}\,\beta$, implying that $\bigvee_{n=1}^{\infty} \mathrm{coz}\,\alpha_n \leq \mathrm{coz}\,\beta$. But $\mathrm{coz}[Q]$ is an ideal containing $\mathrm{coz}\,\beta$, therefore $\bigvee_{n=1}^{\infty} \mathrm{coz}\,\alpha_n$ belongs to $\mathrm{coz}[Q]$. □

In what follows, we intend to present a characterization of pseudocompact frames in terms of ω-divisible ideals. Before this characterization is proposed, we need some background. We know that, for any maximal ideal M, the field $\frac{\mathcal{R}L}{M}$ always contains a canonical copy of the field $\mathbb{R}$ of real numbers: the set of images of the constant functions under the canonical homomorphism. Now, M is a *real ideal* when the canonical copy of the field $\mathbb{R}$ is the entire field $\frac{\mathcal{R}L}{M}$ (see [16, Chapter 5] for more details). We also need the next proposition which is proved in [14].

PROPOSITION 3.8. *A frame L is pseudocompact if and only if every maximal ideal of $\mathcal{R}L$ is real.*

COROLLARY 3.9. *For a frame L, the following statements hold.*

(1) *A maximal ideal of $\mathcal{R}L$ is real if and only if it is ω-divisible.*

(2) *L is pseudocompact if and only if every maximal ideal of $\mathcal{R}L$ is ω-divisible.*

(3) *L is pseudocompact if and only if every ideal is contained in an ω-divisible z-ideal.*

Proof. (1) Let Q be a maximal ideal in $\mathcal{R}L$. By [14, Proposition 3.6], Q is real if and only if $\mathrm{coz}[Q]$ is closed under countable join if and only if Q is ω-divisible, by Lemma 3.7.

(2) The combination of (1) with Proposition 3.8 implies (2).

(3) Since every maximal ideal of $\mathcal{R}L$ is a z-ideal, it is obvious by (2). □

We will use the following characterizations of Lindelöf frames to prove the next theorem. A frame L is Lindelöf if and only if $\bigvee : \lambda L \to L$ is codense, that is, if and only if $\bigvee J < \top$ for all proper σ-ideals J of $\mathrm{Coz}\, L$ (see [9]).

Theorem 3.10. *A frame L is Lindelöf if and only if every ω-divisible ideal in $\mathcal{R}L$ is fixed.*

Proof. Necessity. Suppose that Q is an ω-divisible ideal in $\mathcal{R}L$. Then $\mathrm{coz}[Q]$ is closed under countable join, by Lemma 3.7. Now suppose, by way of contradiction, that $\bigvee \mathrm{coz}[Q] = \top$. Since the frame L is Lindelöf, there is a countable $J \subseteq Q$ with $\bigvee \mathrm{coz}[J] = \bigvee \mathrm{coz}[Q] = \top$. Since the ideal Q is ω-divisible, the ideal generated by J is principal, say generated by α. It follows that $\mathrm{coz}\,\alpha = \bigvee \mathrm{coz}[J] = \top$. This implies that α is a unit element of $\mathcal{R}L$, a contradiction. Therefore $\bigvee \mathrm{coz}[Q] < \top$, this means that Q is fixed.

Sufficiency. Let J be a proper σ-ideal of $\mathrm{Coz}\, L$. Then, by Lemma 3.7, $\mathrm{coz}^{\leftarrow}[J]$ is an ω-divisible ideal and hence $\bigvee \mathrm{coz}[\mathrm{coz}^{\leftarrow}[J]] < \top$. Therefore $\bigvee J = \bigvee \mathrm{coz}[\mathrm{coz}^{\leftarrow}[J]] < \top$. □

4. Uniformly closed ideals in $\mathcal{R}L$

Let $\psi \in \mathcal{R}L$. For each $r \in \mathbb{Q}^+$, let

$$B(\psi, r) = \{\alpha \in \mathcal{R}L \mid |\alpha - \psi| \leq \mathbf{r}\} \quad \text{and} \quad B_\psi = \{B(\psi, r) \mid r \in \mathbb{Q}^+\}.$$

Then there is a unique topology on $\mathcal{R}L$ for which for any $\psi \in \mathcal{R}L$, the family $\{B(\psi, r) \mid r \in \mathbb{Q}^+\}$ forms a base for the neighborhood system of ψ. This topology is called the *uniform topology on* $\mathcal{R}L$ (briefly *u-topology on* $\mathcal{R}L$). A typical basic neighborhood in the uniform topology on the subring $\mathcal{R}^*L$ of $\mathcal{R}L$ denoted by $B^*(\psi, r)$, $\psi \in \mathcal{R}^*L$ (for more details see [2, 5]). We are going to determine the relation between ω-divisible ideals and closed ideals in the uniform topology on $\mathcal{R}L$. For this, we first provide the necessary tools in the following.

Proposition 4.1 ([10]). *Let $\alpha, \beta \in \mathcal{R}L$. If $\mathrm{coz}\,\alpha \lll \mathrm{coz}\,\beta$, then α is a multiple of β.*

If T is a subset of $\mathcal{R}L$, then $\overline{T}$ will denote its (uniform) closure in $\mathcal{R}L$. We note that the equality $||x|-|y|| \leq |x-y|$ holds true in a totally ordered ring, and, consequently, in every f-ring.

LEMMA 4.2. *Let Q be a z-ideal of $\mathcal{R}L$. If $\alpha \in \bar{Q}$, then $|\alpha| \in \bar{Q}$.*

PROOF. For every positive integer n, take $\beta_n \in Q$ such that $|\alpha - \beta_n| \leq \frac{\mathbf{1}}{\mathbf{n}}$. It follows that $||\alpha| - |\beta_n|| \leq |\alpha - \beta_n| \leq \frac{\mathbf{1}}{\mathbf{n}}$ for every $n \in \mathbb{N}$. Since Q is a z-ideal, $|\beta_n| \in Q$ for every $n \in \mathbb{N}$. Therefore $|\alpha| \in \bar{Q}$. □

LEMMA 4.3. *Let $\alpha \in \mathcal{R}L$ and $r \in \mathbb{Q}^+$. If $\beta = ((\alpha - \mathbf{r}) \vee \mathbf{0}) + ((\alpha + \mathbf{r}) \wedge \mathbf{0})$, then* $\operatorname{coz}\beta \lll \operatorname{coz}\alpha$.

PROOF. We see that

$$\operatorname{coz}((\alpha - \mathbf{r}) \vee \mathbf{0}) = \operatorname{coz}((\alpha - \mathbf{r})^+) = (\alpha - \mathbf{r})(0, -) = \alpha(r, -)$$

and

$$\begin{aligned}\operatorname{coz}((\alpha + \mathbf{r}) \wedge \mathbf{0}) &= \operatorname{coz}((-\alpha - \mathbf{r}) \vee \mathbf{0}) \\ &= \operatorname{coz}((-\alpha - \mathbf{r})^+) \\ &= (-\alpha - \mathbf{r})(0, -) \\ &= -\alpha(r, -) \\ &= \alpha(-, -r).\end{aligned}$$

On the other hand, $\operatorname{coz}\beta \leq \operatorname{coz}((\alpha - \mathbf{r}) \vee \mathbf{0}) \vee \operatorname{coz}((\alpha + \mathbf{r}) \wedge \mathbf{0})$. In consequence, $\operatorname{coz}\beta \leq \alpha((-, -r) \vee (r, -))$. Since, by [5, Corollary 1], $(-, -r) \vee (r, -) \lll (-, 0) \vee (0-)$, we have $\alpha((-, -r) \vee (r, -)) \lll \alpha((-, 0) \vee (0-)) = \operatorname{coz}\alpha$ because α preserves $\lll$. Therefore $\operatorname{coz}\beta \lll \operatorname{coz}\alpha$. □

PROPOSITION 4.4. *Let Q be an ideal of $\mathcal{R}L$. Then Q is closed in the uniform topology on $\mathcal{R}L$ if and only if Q is an ω-divisible z-ideal.*

PROOF. Necessity. Let Q be a closed ideal of $\mathcal{R}L$. We first show that Q is a z-ideal. Consider $\alpha \in \mathcal{R}L$ and $\beta \in Q$ with $\operatorname{coz}\alpha = \operatorname{coz}\beta$. For every positive integer n, let

$$\gamma_n = \left[\left(\alpha - \frac{\mathbf{1}}{\mathbf{n}}\right) \vee \mathbf{0}\right] + \left[\left(\alpha + \frac{\mathbf{1}}{\mathbf{n}}\right) \wedge \mathbf{0}\right].$$

Then, by Lemma 4.3, $\operatorname{coz}\gamma_n \lll \operatorname{coz}\alpha = \operatorname{coz}\beta$, for each n. Now, Proposition 4.1 implies that each γ_n is a multiple of β, thus each γ_n belongs to the ideal Q. But

$|\alpha - \gamma_n| \le \frac{\mathbf{2}}{\mathbf{n}} \to \mathbf{0}$ as $n \to \infty$, therefore γ_n converges to α in the uniform topology. This shows that $\alpha \in Q$, proving that Q is a z-ideal. It remains to show that Q is ω-divisible. Let (α_n) be a countable subset of Q. By Lemma 3.7, to prove ω-divisibility it suffices to show that $\bigvee_{n=1}^{\infty} \operatorname{coz} \alpha_n \in \operatorname{coz}[Q]$. Now, putting $\varphi_n = \sum_{i=1}^{n}(|\alpha_i| \wedge \mathbf{2}^{-i})$ for each $n \in \mathbb{N}$, it is clear that the sequence φ_n converges to $\varphi = \sum_{n=1}^{\infty}(|\alpha_n| \wedge \mathbf{2}^{-n})$ in the uniform topology. But for each n, $\operatorname{coz} \varphi_n = \bigvee_{i=1}^{n} \operatorname{coz} \alpha_i \in \operatorname{coz}[Q]$. This show that each $\varphi_n \in Q$ because Q is a z-ideal. Consequently, $\varphi \in Q$ since Q is closed. Therefore

$$\bigvee_{n=1}^{\infty} \operatorname{coz} \alpha_n = \operatorname{coz} \varphi \in \operatorname{coz}[Q].$$

Sufficiency. Consider $\mathbf{0} \le \alpha \in \bar{Q}$. Then for every positive integer n, take $\beta_n \in Q$ such that $|\alpha - \beta_n| \le \frac{\mathbf{1}}{\mathbf{n}}$. Now, we have $\alpha \le \beta_n + \frac{\mathbf{1}}{\mathbf{n}}$, hence $\left(\alpha - \frac{\mathbf{1}}{\mathbf{n}}\right)^+ \le |\beta_n|$ and therefore $\operatorname{coz}\left(\left(\alpha - \frac{\mathbf{1}}{\mathbf{n}}\right)^+\right) \le \operatorname{coz}(|\beta_n|) = \operatorname{coz} \beta_n$. In consequence, $\operatorname{coz} \alpha = \bigvee_{n=1}^{\infty} \operatorname{coz}\left(\left(\alpha - \frac{\mathbf{1}}{\mathbf{n}}\right)^+\right) \le \bigvee_{n=1}^{\infty} \operatorname{coz} \beta_n$. Next, Lemma 3.7 implies that $\bigvee_{n=1}^{\infty} \operatorname{coz} \beta_n \in \operatorname{coz}[Q]$, implying that $\operatorname{coz} \alpha \in \operatorname{coz}[Q]$. But Q is a z-ideal, so $\alpha \in Q$. Finally, for arbitrary $\alpha \in \bar{Q}$, by Lemma 4.2, $|\alpha| \in \bar{Q}$, showing that $|\alpha| \in Q$, by what we have just observed. In consequence, $\alpha \in Q$ since Q is a z-ideal. Therefore $\bar{Q} = Q$, which means that Q is closed. □

Corollary 3.8 in [10] says that the following algebraic condition is necessary and sufficient for an ideal Q in $\mathcal{R}L$ to be a z-ideal:

> given $\alpha \in \mathcal{R}L$, if there exists $\beta \in Q$ such that α belongs to every maximal ideal containing β, then $\alpha \in Q$.

Therefore, as asserted, Proposition 4.4 is algebraic in nature.

An immediate consequence of Lemma 3.7 and the foregoing proposition, is the following corollary.

Corollary 4.5. *An ideal Q of $\mathcal{R}L$ is closed if and only if it is a z-ideal such that* $\operatorname{coz}[Q]$ *is closed under countable join.*

The next result characterizes real ideals of $\mathcal{R}L$ in terms of closed and ω-divisible ideals.

Theorem 4.6. *The following are equivalent for an ideal Q of $\mathcal{R}L$:*

(1) *Q is a real ideal;*

(2) *Q is a closed maximal ideal of $\mathcal{R}L$;*

(3) *Q is a maximal closed ideal of $\mathcal{R}L$;*

(4) *Q is a maximal ω-divisible ideal of $\mathcal{R}L$;*

(5) *Q is an ω-divisible maximal ideal of $\mathcal{R}L$.*

Proof. The equivalences of (1), (2), and (5) are clear from Corollary 3.9 and Proposition 4.4.

(2) $\Longrightarrow$ (3). It is obvious.

(3) $\Longrightarrow$ (4). Suppose M is an ω-divisible ideal and let $Q \subseteq M$. Then, by Lemma 3.7, $\mathrm{coz}^{\leftarrow}[\mathrm{coz}[M]]$ is an ω-divisible z-ideal, and hence it is closed by Proposition 4.4. Since $Q \subseteq M \subseteq \mathrm{coz}^{\leftarrow}[\mathrm{coz}[M]]$, by (3), $Q = \mathrm{coz}^{\leftarrow}[\mathrm{coz}[M]]$, showing that $Q = M$. Therefore Q is a maximal ω-divisible ideal of $\mathcal{R}L$.

(4) $\Longrightarrow$ (3). Suppose M is a closed ideal and let $Q \subseteq M$. Then, by Proposition 4.4, M is ω-divisible. Therefore $Q = M$, by (4).

(3) $\Longrightarrow$ (5). Suppose, by way of contradiction, that Q is not maximal and choose a maximal ideal M of $\mathcal{R}L$ such that $Q \subset M$. Pick $\alpha \in M \setminus Q$. We claim that

$$J = \{a \in \mathrm{Coz}\, L \mid a \leq \mathrm{coz}\,\alpha \vee \mathrm{coz}\,\beta,\ \text{ for some } \beta \in Q\}$$

is an ideal of $\mathrm{Coz}\, L$ such that is closed under countable join. To see this, obviously, J is an ideal of $\mathrm{Coz}\, L$. Since Q is closed, Proposition 4.4 implies that it is ω-divisible, and hence, by Lemma 3.7, $\mathrm{coz}[Q]$ is closed under countable join. This shows that J is closed under countable join. Now, by Lemma 3.7, $\mathrm{coz}^{\leftarrow}[J]$ is an ω-divisible z-ideal and thus, by Proposition 4.4, $\mathrm{coz}^{\leftarrow}[J]$ is a closed ideal of $\mathcal{R}L$. Clearly, $Q \subseteq \mathrm{coz}^{\leftarrow}[J]$. But $\alpha \in \mathrm{coz}^{\leftarrow}[J]$ and $\alpha \notin Q$ imply that $Q \subset \mathrm{coz}^{\leftarrow}[J]$. This contradicts (3). □

The following extends Theorem 3.10 in [2] with ω-divisible ideals.

Corollary 4.7. *The following are equivalent for any frame L:*

(1) *L is pseudocompact;*

(2) *the closure of any ideal in $\mathcal{R}L$ is an ideal;*

(3) *every ideal in $\mathcal{R}L$ is contained in an ω-divisible ideal;*

(4) *every ideal in $\mathcal{R}L$ is contained in a closed ideal.*

Proof. The equivalences of (1), (2), and (4) are proved in [2, Theorem 3.10].

(2) $\Longrightarrow$ (3). By Proposition 4.4, it is obvious.

(3) $\Longrightarrow$ (1). It follows from (3) that every maximal ideal of $\mathcal{R}L$ is ω-divisible. This implies that the frame L is pseudocompact due to Corollary 3.9 (2). □

The next result characterizes Lindelöf frames in terms of closed ideals of $\mathcal{R}L$.

Theorem 4.8. *A frame L is Lindelöf if and only if every closed ideal in $\mathcal{R}L$ is fixed.*

Proof. Necessity. By Proposition 4.4 and Theorem 3.10, it is obvious.

Sufficiency. By Theorem 3.10, it is enough to show that every ω-divisible ideal in $\mathcal{R}L$ is fixed. Let Q be an ω-divisible ideal of $\mathcal{R}L$. Then $Q \subseteq M = \mathrm{coz}^{\leftarrow}[\mathrm{coz}[Q]]$. Since M is an ω-divisible z-ideal, Proposition 4.4 shows that it is closed. Thus the present hypothesis implies that $\bigvee M < \top$, implying that $\bigvee Q < \top$, since $\bigvee Q \leq \bigvee M$. Therefore Q is fixed. □

The next corollary is obvious by combination of Theorem 3.10 with previous theorem.

Corollary 4.9. *Every closed ideal in $\mathcal{R}L$ is fixed if and only if every ω-divisible ideal in $\mathcal{R}L$ is fixed.*

In what follows, as an application of this section, we study frames L for which every countably generated ideal of $\mathcal{R}L$ is principal. We show that all such frames are finite. We will use the following results to prove the next theorem.

(1) $\mathcal{R}L$ is a Noetherian ring if and only if L is finite (see [1, Theorem 3.11]).

(2) A frame L is a P-frame in case $a \vee a^* = \top$ for each $a \in \mathrm{Coz}\, L$. Every ideal of $\mathcal{R}L$ is a z-ideal if and only if L is a P-frame if and only if for every $\alpha, \beta \in \mathcal{R}L$, the ideal generated by $\alpha^2+\beta^2$ contains both α and β. In particular, if L is a P-frame, every finitely generated ideal of $\mathcal{R}L$ is principal (see [10, Proposition 3.9]).

Theorem 4.10. *For a frame L, the following are equivalent*:

(1) *L is finite*;

(2) *every countably generated ideal of $\mathcal{R}L$ is principal*;

(3) *every ideal of $\mathcal{R}L$ is ω-divisible*;

(4) *every ideal of $\mathcal{R}L$ is closed.*

Proof. The equivalence of (2) and (3) follows from Proposition 3.2.1 in [24].

(1) $\Longrightarrow$ (2). Suppose L is finite. Then, by complete regularity, L is a P-frame. This shows that every countably generated ideal of $\mathcal{R}L$ is principal.

(2) $\Longrightarrow$ (1). The present hypothesis implies that the $\mathcal{R}L$ is a Noetherian ring. Thus, L is finite.

(1) $\Longrightarrow$ (4). Suppose L is finite. Then L is a P-frame. That is to say every ideal of $\mathcal{R}L$ is a z-ideal. On the other hand, every ideal of $\mathcal{R}L$ is ω-divisible since (1) implies (3). Therefore every ideal of $\mathcal{R}L$ is ω-divisible z-ideal. Now, by Proposition 4.4, we can conclude that (1) implies (4).

(4) $\Longrightarrow$ (3). By Proposition 4.4, it is obvious. □

5. M-ideals in $\mathcal{R}L$ with uniform topology

We recall from [12] that the M-ideal M^I and the O-ideal O^I of $\mathcal{R}L$, where $I \in \beta L$, are defined by

$$M^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\} \quad \text{and} \quad O^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \lll I\}.$$

Obviously, $O^I \subseteq M^I$. For any $a \in L$ and $I \in \beta L$, $r_L(a) \lll I$ if and only if $a \in I$. Hence, $O^I = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \in I\}$. It is proved in [10, 12] that

(1) for all $I \in \beta L$, M^I and O^I are z-ideal;

(2) an ideal Q of $\mathcal{R}L$ is maximal if and only if there is $I \in \operatorname{Pt}(\beta L)$ such that $Q = M^I$;

(3) for any prime ideal P of $\mathcal{R}L$, there exist a unique point $I \in \operatorname{Pt}(\beta L)$ such that $O^I \subseteq P \subseteq M^I$;

(4) M^I is the unique maximal ideal containing O^I, for any $I \in \operatorname{Pt}(\beta L)$;

(5) for any $I \in \operatorname{Pt}(\beta L)$ and $\alpha \in \mathcal{R}L$, $\alpha \in O^I$ if and only if $\beta\alpha = \mathbf{0}$ for some $\beta \notin M^I$.

We are now going to investigate which prime ideals of $\mathcal{R}L$ are closed in the uniform topology. We begin with the following lemma.

Lemma 5.1. *For all $I \in \beta L$, $\overline{M^I} = \overline{O^I}$.*

Proof. Since $O^I \subseteq M^I$, it is enough to show that $M^I \subseteq \overline{O^I}$. Let $\alpha \in M^I$ and $r \in \mathbb{Q}^+$. Take $\beta_r = \big((\alpha - \frac{\mathbf{r}}{\mathbf{2}}) \vee \mathbf{0}\big) + \big((\alpha + \frac{\mathbf{r}}{\mathbf{2}}) \wedge \mathbf{0}\big)$. Then, by Lemma 4.3, we have $\operatorname{coz}\beta_r \lll \operatorname{coz}\alpha$. It follows that $r_L(\operatorname{coz}\beta_r) \lll r_L(\operatorname{coz}\alpha) \subseteq I$, implying that $\operatorname{coz}\beta_r \in \underline{\operatorname{coz}}[O^I]$. Hence, $\beta_r \in O^I$ since O^I is a z-ideal. But $|\alpha - \beta_r| \leq \mathbf{r}$, therefore $\alpha \in \overline{O^I}$. □

Using the foregoing lemma and part (3) of the above, we can conclude that no non-maximal prime ideal of $\mathcal{R}L$ is closed. However, as noted in Theorem 4.6, the closed maximal ideals of $\mathcal{R}L$ are exactly the real ideals. That is to say, for any $I \in \mathrm{Pt}(\beta L)$, M^I is closed if and only if $\ell(\langle I\rangle_\sigma) \neq \top_{\upsilon L}$ since, by [14, Corollary 3.7], we can conclude that for any point I of βL, M^I is real if and only if $\ell(\langle I\rangle_\sigma) \neq \top_{\upsilon L}$, or equivalently, $\langle I\rangle_\sigma \neq \top_{\lambda L}$. Therefore, the following proposition now follows immediately.

Proposition 5.2. *The following statements hold.*

(1) *If $I \in \mathrm{Pt}(\beta L)$, then $\overline{O^I} = M^I$ if and only if $\ell(\langle I\rangle_\sigma) \neq \top_{\upsilon L}$.*

(2) *Let P be a prime ideal of $\mathcal{R}L$, and M^I the unique maximal ideal containing P, for some $I \in \mathrm{Pt}(\beta L)$. Then $\overline{P}$ is a (necessarily maximal) ideal if and only if $\ell(\langle I\rangle_\sigma) \neq \top_{\upsilon L}$.*

It is known that a frame L is pseudocompact if and only if $\beta L \cong \upsilon L$ (see [20, Corollary 2.4.10]). Thus we have the following extension of Corollary 4.7.

Proposition 5.3. *A frame L is pseudocompact if and only if the closure of any prime ideal of $\mathcal{R}L$ is a (necessarily maximal) ideal.*

Remark 5.4. For any $I \in \mathrm{Pt}(\beta L)$, if $\ell(\langle I\rangle_\sigma) \neq \top_{\upsilon L}$ and $O^I \neq M^I$, then, by Proposition 5.2, O^I is a proper dense subset of M^I. Now, Proposition 4.4 implies that O^I is not ω-divisible. Therefore even if the closure of an ideal is an ideal, it may fail to be ω-divisible.

An ideal Q in $\mathcal{R}L$ is contained in a unique maximal ideal M^I for a point I of βL if and only if $O^I \subseteq Q$. To see this, first, suppose $I \in \mathrm{Pt}(L)$ such that $O^I \subseteq Q$. We must show that M^I is the unique maximal ideal containing Q. This is clear since M^I is the unique maximal ideal containing O^I. Conversely, suppose M^I is the unique maximal ideal containing Q for a point I of βL. We must show that $O^I \subseteq Q$. Let $\alpha \in O^I$. Then $\beta\alpha = \mathbf{0}$ for some $\beta \notin M^I$. Since M^I is the only maximal ideal containing Q, we have $\mathcal{R}L = \langle Q, \beta\rangle$, the ideal generated by $Q \cup \{\beta\}$. In consequence, there exist $\gamma \in Q$ and $\delta \in \mathcal{R}L$ such that $\mathbf{1} = \gamma + \beta\delta$, implying that $\alpha = \alpha\gamma \in Q$. Therefore $O^I \subseteq Q$.

Now, as an immediate consequence from Theorem 4.6 and Lemma 5.1, we have the following theorem.

Theorem 5.5. *If an ideal Q in $\mathcal{R}L$ is contained in a unique maximal ideal, then Q is closed if and only if Q is a real ideal.*

This immediately leads to the next corollary.

COROLLARY 5.6. *If P is a prime ideal in $\mathcal{R}L$, then $\overline{P} = P$ if and only if P is a real ideal.*

For any $a \in L$ with $a < \top$, we denote $M^{r_L(a)}$ as M_a and observe that

$$M_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq a\}$$

is an ideal in $\mathcal{R}L$ [15]. The following is proved in [15].

(1) The fixed maximal ideals of $\mathcal{R}L$ are exactly the ideals M_p for $p \in \mathrm{Pt}(L)$.

(2) For any $I \in \beta L$, $\bigvee \operatorname{coz}[M^I] = \bigvee I$. Therefore the ideal M^I is fixed if and only if $\bigvee I < \top$.

PROPOSITION 5.7. *If M is a fixed maximal ideal in $\mathcal{R}L$, then it is closed.*

PROOF. Suppose M is a fixed maximal ideal in $\mathcal{R}L$. Then there is $p \in \mathrm{Pt}(L)$ such that $M = M_p = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq p\}$. We must show that $\overline{M_p} = M_p$. Since $M_p \subseteq \overline{M_p}$, it is enough to show that $\overline{M_p} \subseteq M_p$. Suppose $\beta \in \overline{M_p}$. Since M_p is a z-ideal, Lemma 4.2 implies that $|\beta| \in \overline{M_p}$. Now, let $n \in \mathbb{N}$. Then there exists an element α_n in M_p such that

$$||\beta| - \alpha_n| = |\alpha_n - |\beta|| \leq \frac{\mathbf{1}}{\mathbf{n}},$$

and so $|\beta| - \frac{\mathbf{1}}{\mathbf{n}} \leq \alpha_n$. This shows that $\left(|\beta| - \frac{\mathbf{1}}{\mathbf{n}}\right)^+ \leq |\alpha_n|$, implying that

$$\operatorname{coz}\left(|\beta| - \frac{\mathbf{1}}{\mathbf{n}}\right)^+ \leq \operatorname{coz}(|\alpha_n|) = \operatorname{coz}\alpha_n \leq p.$$

In consequence, $\left(|\beta| - \frac{\mathbf{1}}{\mathbf{n}}\right)^+ \in M_p$. Therefore

$$\operatorname{coz}\beta = \operatorname{coz}|\beta| = \bigvee_{n\in\mathbb{N}} \operatorname{coz}\left(|\beta| - \frac{\mathbf{1}}{\mathbf{n}}\right)^+ \leq p.$$

This means that $\beta \in M_p$. □

An ideal Q of $\mathcal{R}L$ is called *free* if $\bigvee \operatorname{coz}[Q] = \top$. Combining foregoing proposition and Theorem 4.6, we can conclude that if Q is a free maximal ideal of $\mathcal{R}L$, then Q is real if and only if it is closed if and only if it is ω-divisible.

Before we prove the following proposition, let us explain our choice of terminology.

Let Q be an ideal in $\mathcal{R}L$ and $I = \bigvee_{\alpha\in Q} r_L(\operatorname{coz}\alpha)$. Then $L \neq I \in \beta L$. To see this, suppose that $I = L$. Since βL is compact, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in Q$ such that

$$L = \top_{\beta L} = \bigvee_{i=1}^{n} r_L(\operatorname{coz}\alpha_i) \subseteq r_L\Big(\operatorname{coz}\Big(\sum_{i=1}^{n}\alpha_i^2\Big)\Big).$$

This implies that

$$L = \top_{\beta L} = r_L\Big(\operatorname{coz}\Big(\sum_{i=1}^{n}\alpha_i^2\Big)\Big),$$

implying

$$\top_L = \operatorname{coz}\Big(\sum_{i=1}^{n}\alpha_i^2\Big).$$

This means that $\sum_{i=1}^{n}\alpha_i^2$ is a unit element of $\mathcal{R}L$ which belong to Q. Therefore $Q = \mathcal{R}L$ which is a contradiction.

Proposition 5.8. *If Q is a closed ideal in $\mathcal{R}L$, then $Q = M^I$, where $I = \bigvee_{\alpha\in Q} r_L(\operatorname{coz}\alpha)$.*

Proof. Obviously, $Q \subseteq M^I$. We must show that $M^I \subseteq Q$. Suppose $\beta \in M^I$ and $n \in \mathbb{N}$. Since, by Lemma 5.1, $\overline{O^I} = \overline{M^I} \supseteq M^I$, there is an element γ_n in O^I with $|\gamma_n - \beta| \leq \frac{\mathbf{1}}{\mathbf{n}}$. Since $\gamma_n \in O^I$,

$$\operatorname{coz}\gamma_n \in I = \bigvee_{\alpha\in Q} r_L(\operatorname{coz}\alpha) = \Big\langle \bigcup_{\alpha\in Q} r_L(\operatorname{coz}\alpha)\Big\rangle.$$

In consequence, there exist $a_i \in r(\operatorname{coz}\alpha_i)$, $i = 1, 2, \ldots, m$, such that $\operatorname{coz}\gamma_n \leq \bigvee_{i=1}^{m} a_i$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in Q$. Since, for every $1 \leq i \leq m$

$$a_i \in r_L(\operatorname{coz}\alpha_i) = \{x \in L \mid x \lll \operatorname{coz}\alpha_i\},$$

we have $a_i \lll \operatorname{coz}\alpha_i$, and hence

$$\operatorname{coz}\gamma_n \leq \bigvee_{i=1}^{m} a_i \lll \bigvee_{i=1}^{m} \operatorname{coz}\alpha_i = \operatorname{coz}\Big(\sum_{i=1}^{m}\alpha_i^2\Big)$$

which implies that $\operatorname{coz}\gamma_n \lll \operatorname{coz}\big(\sum_{i=1}^{m}\alpha_i^2\big)$. Now, Proposition 4.1 shows that γ_n is a multiple of $\sum_{i=1}^{m}\alpha_i^2 \in Q$ and so $\gamma_n \in Q$. We can therefore conclude that $\beta \in \overline{Q} = Q$. Consequently, $M^I \subseteq Q$ and the proof is completed. □

Recall form [18] that an ideal of $\mathcal{R}L$ is an intersection of maximal ideals if and only if it is of the form M^I, for some $I \in \beta L$. The combination of this fact with foregoing proposition yields the following corollary.

Corollary 5.9. *Every closed ideal in $\mathcal{R}L$ is an intersection of maximal ideals.*

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References

[1] S. K. Acharyya – G. Bhunia – P. P. Ghosh, *Finite frames, P-frames and basically disconnected frames*, Algebra universalis 72 (2014), pp. 209–224.

[2] S. K. Acharyya – G. Bhunia – P. P. Ghosh, *Pseudocompact frames L versus different topologies on $\mathcal{R}L$*, Quaest. Math. 38 (2015), pp. 423–430.

[3] F. Azarpanah, *Algebraic properties of some compact spaces*, Real Anal. Exchange 25 (1999), pp. 317–328.

[4] R. N. Ball – J. Walters-Wayland, *C- and C^*-quotients in pointfree topology*, Dissertationes Math. (Rozprawy Mat.) 412 (2002), pp. 1–61.

[5] B. Banaschewski, *The real numbers in pointfree topology*, Textos de Mathemática (Séries B), No. 12, Departamento de Mathemática da Universidade de Coimbra, Coimbra, 1997.

[6] B. Banaschewski, *A new aspect of the cozero lattice in pointfree topology*, Topology Appl. 156 (2009), pp. 2028–2038.

[7] B. Banaschewski – C. J. Mulvey, *Stone–Čech compactification of locales* II, J. Pure Appl. Algebra 33 (1984), pp. 107–122.

[8] B. Banaschewski – C. Gilmour, *Pseudocompactness and the cozero part of a frame*, Comment. Math. Univ. Carolin. 37 (1996), pp. 577–588.

[9] B. Banaschewski – C. Gilmour, *Realcompactness and the cozero part of a frame*, Appl. Categ. Structures 9 (2001), pp. 395–417.

[10] T. Dube, *Concerning P-frames, essential P-frames, and strongly zero-dimensional frames*, Algebra Universalis 61 (2009), pp. 115–138.

[11] T. Dube, *Some algebraic characterizations of F-frames*, Algebra Universalis 62 (2009), pp. 273–288.

[12] T. Dube, *Some ring-theoretic properties of almost P-frames*, Algebra Universalis 60 (2009), pp. 145–162.

[13] T. Dube, *On the ideal of functions with compact support in pointfree function rings*, Acta Math. Hungar. 129 (2010), pp. 205–226.

[14] T. Dube, *Real ideals in pointfree rings of continuous functions*, Bull. Aust. Math. Soc. 83 (2011), pp. 338–352.

[15] T. Dube, *Extending and contracting maximal ideals in the function rings of pointfree topology*, Bull. Math. Soc. Sci. Math. Roumanie 55 (2012), pp. 365–374.

[16] L. Gillman – M. Jerison, *Rings of continuous functions*, Springer-Verlag, Berlin, 1976.

[17] P. A. Grillet, *Abstract algebra*, Springer-Verlag, New York, 2007.

[18] O. Ighedo, *Concerning ideals of pointfree function rings*, Ph.D. Thesis, University of South Africa, Pretoria, 2013.

[19] J. G. Horne, *On O_ω-ideals in $C(X)$*, Proc. Amer. Math. Soc. 9 (1958), pp. 511–518.

[20] N. Marcus, *Realcompactification of frames*, Sci. Thesis, University of South Africa, Pretoria, 1993.

[21] N. Marcus, *Realcompactification of frames*, Comment. Math. Univ. Carolin. 36 (1995), pp. 347–356.

[22] P. Nanzetta – D. Plank, *Closed ideals in $C(X)$*, Proc. Amer. Math. Soc. 35 (1972), pp. 601–606.

[23] J. Picado – A. Pultr, *Frames and locales*: *Topology without points*, Frontiers in Mathematics, Springer, Basel, 2012.

[24] R. Stokke, *Closed ideals in $C(X)$ and ϕ- algebras*, Topology Proc. 22 (1997), pp. 501–528.

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