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Sign-changing solutions of boundary value problems for semilinear Δ_γ-Laplace equations

Duong Trong Luyen (∗)

Abstract – In this article, we study the multiplicity of weak solutions to the boundary value problem

$$\begin{cases} -G_\alpha u = g(x,y,u) + f(x,y,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 2)$, $\alpha \in \mathbb{N}$, $g(x,y,\xi)$, $f(x,y,\xi)$ are Carathéodory functions and G_α is the Grushin operator. We use the lower bounds of eigenvalues and an abstract theory on sign-changing solutions.

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1. Introduction

Boundary value problems for semilinear elliptic equations were studied in [1, 27] (see also the references therein). Many publications [4, 5, 6, 7, 8, 10, 11, 12, 18, 26, 29, 31] are devoted to the study of the existence of sign-changing solutions of classical elliptic boundary value problems such as

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \ u = 0 \quad \text{on } \partial\Omega,$$

(∗) *Indirizzo dell'A.*: Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam, and Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.
E-mail: duongtrongluyen@tdtu.edu.vn

where $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. There have been several methods developed in studying sign-changing solutions of nonlinear elliptic equations, such as the invariant sets of descending flow method developed by Liu and Sun [5, 18, 31], and the minimax method which is established by Berestycki and Lions in the classical paper [8].

One of the classes of degenerate elliptic equations that has been studied widely in recent years is the class of equations involving an operator of the Grushin type (see [14])

$$G_\alpha := \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha \geq 0.$$

Note that $G_0 \equiv \Delta$ is the Laplacian operator, and G_α, when $\alpha > 0$, is not elliptic in domains intersecting the surface $x = 0$. Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [36, 37] (see also some recent results in [2, 17, 21, 22, 23, 33, 34, 35, 20, 13, 19, 25]).

In this paper, we consider the existence of sign-changing solutions of the Dirichlet boundary value problem

$$-G_\alpha u = g(x, y, u) + f(x, y, u) \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} := \mathbb{R}^N$, N_1, N_2, $\alpha \in \mathbb{N}$, $\Omega \cap \{(x, y) \in \mathbb{R}^N : x = 0\} \neq \emptyset$, and

$$\Delta_x := \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_y := \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2}, \quad |x|^{2\alpha} := \Big(\sum_{i=1}^{N_1} x_i^2\Big)^\alpha,$$

and the nonlinearity f is a real Carathéodory function on $\Omega \times \mathbb{R}$ and satisfies the following conditions:

(A1) there exist $p \in (2, 2_\alpha^*)$, and constants $C_1, C_2 > 0$ such that

$$|f(x, y, \xi)| \leq C_1 + C_2|\xi|^{p-1} \quad \text{almost everywhere for } (x, y, \xi) \in \Omega \times \mathbb{R},$$

where $2_\alpha^* := \frac{2N_\alpha}{N_\alpha - 2}$ and $N_\alpha := N_1 + (1 + \alpha)N_2 > 2$;

(A2) $f(x, y, \xi) = \circ(|\xi|)$, uniformly in $(x, y) \in \overline{\Omega}$, as $\xi \to 0$ and $f(x, y, \xi)\xi \geq 0$ for all $\xi \in \mathbb{R}$ and a.e. $(x, y) \in \Omega$;

(A3) there exists a constant $\mu > 2$ such that

$$0 \leq \mu F(x, y, \xi) \leq \xi f(x, y, \xi), \quad \text{for all } (x, y) \in \overline{\Omega},\ \xi \in \mathbb{R} \setminus \{0\},$$

where $F(x, y, \xi) = \int_0^\xi f(x, y, \tau)d\tau$;

(A4) $f(x, y, -\xi) = -f(x, y, \xi)$ for all $(x, y, \xi) \in \overline{\Omega} \times \mathbb{R}$;

(A5) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. There exists $\sigma < \frac{\mu}{2}$ such that

$$|g(x, y, \xi)| \le C(1 + |\xi|^\sigma), \quad \text{for all } \xi \in \mathbb{R} \text{ and a.e. } (x, y) \in \Omega.$$

Moreover, $g(x, y, \xi) = \circ(|\xi|)$, uniformly in $(x, y) \in \overline{\Omega}$, as $\xi \to 0$ and $g(x, y, \xi)\xi > 0$ for all $\xi \in \mathbb{R}\backslash\{0\}$ and a.e. $(x, y) \in \Omega$.

Our main result is given by the following theorem.

THEOREM 1.1. *Assume that f, g satisfies the conditions* (A1)–(A5) *and*

$$\frac{2p}{N_\alpha(p-2)} - 1 > \frac{\mu}{\mu - \sigma - 1}. \tag{1.3}$$

Then the problem (1.1)–(1.2) *has infinitely many sign-changing solutions.*

This article is organized as follows. In section 2, we present some definitions and preliminary results. Next, combining the lower bounds of eigenvalues and an abstract theory on sign-changing solutions, we give the proof of Theorem 1.1.

2. Preliminary results

DEFINITION 2.1. By $S_1^2(\Omega)$ we will denote the set of all functions $u \in L^2(\Omega)$ such that $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$, $|x|^\alpha \frac{\partial u}{\partial y_j} \in L^2(\Omega)$, $i = 1, 2, \ldots, N_1$, $j = 1, 2, \ldots, N_2$. We define the norm in this space as follows

$$\|u\|_{S_1^2(\Omega)} = \left\{ \int_\Omega \left(|u|^2 + |\nabla_\alpha u|^2\right) \mathrm{d}X \right\}^{\frac{1}{2}},$$

where

$$\mathrm{d}X = \mathrm{d}x_1 \ldots \mathrm{d}x_{N_1} \mathrm{d}y_1 \ldots \mathrm{d}y_{N_2},$$

$$\nabla_\alpha u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{N_1}}, |x|^\alpha \frac{\partial u}{\partial y_1}, \ldots, |x|^\alpha \frac{\partial u}{\partial y_{N_2}}\right).$$

We can also define the scalar product in $S_1^2(\Omega)$ as follows

$$(u, v)_{S_1^2(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla_\alpha u, \nabla_\alpha v)_{L^2(\Omega)}.$$

The space $S_{1,0}^2(\Omega)$ is defined as the closure of $C_0^1(\Omega)$ in the space $S_1^2(\Omega)$.

The following embedding inequality was proved in [33, 37]

$$\left(\int_{\Omega} |u|^p \mathrm{d}X\right)^{\frac{1}{p}} \leq C(p,\Omega)\|u\|_{S^2_{1,0}(\Omega)},$$

where $1 \leq p \leq 2^*_\alpha, C(p,\Omega) > 0$. The number 2^*_α is the critical Sobolev exponent of the embedding $S^2_{1,0}(\Omega) \hookrightarrow L^p(\Omega)$ and when $1 \leq p < 2^*_\alpha$, the embedding is compact.

Definition 2.2. Let $\mathbb{V}$ be a real Banach space with its dual space $\mathbb{V}^*$, $\Phi \in C^1(\mathbb{V},\mathbb{R})$. We say that Φ satisfies the Palais–Smale if for any sequence $\{u_n\}_{n=1}^{n=+\infty} \subset \mathbb{V}$ such that $\Phi(u_n)$ is bounded and

$$\|\Phi'(u_n)\|_{\mathbb{V}^*} \longrightarrow 0 \quad \text{as } n \to \infty,$$

then there exists a subsequence $\{u_{n_k}\}_{k=1}^{k=+\infty}$ that converges strongly in $\mathbb{V}$.

From Theorem A in [30], we have

Proposition 2.3. *Let* $\mathbb{V}$ *be a Hilbert space and* $\Phi \in C^1(\mathbb{V},\mathbb{R})$ *be of the form* $\Phi' = \mathrm{id} - K_\Phi$ *and satisfy the Palais–Smale condition, where* K_Φ *is a continuous operator. Assume that*

$$K_\Phi(\pm\mathcal{D}_0) \subset \pm\mathcal{D}_0$$

holds, where

$$\mathcal{D}_0 = \{u \in \mathbb{V}\colon \mathrm{dist}(u,\mathcal{P}) < \mu_0\}$$

and

$$\mathcal{P} := \{u \in \mathbb{V}, u(x) \geq 0, \textit{ for a.e. } x \in \Omega\}$$

is the positive cone of $\mathbb{V}$. *Let* N, M *be two closed subspaces of* $\mathbb{V}$ *with* $\dim N < \infty$ *and* $\dim N - \operatorname{codim} M \geq 1$. *Suppose that*

$$Q(\rho) := \{u \in M\colon \|u\|_{\mathbb{V}} = \rho\} \subset \mathcal{S} := \mathbb{V}\backslash(-\mathcal{D}_0 \cup \mathcal{D}_0).$$

Define

$$N^* = N \oplus \mathrm{span}\{u^*\}, \quad u^* = \mathbb{V}\backslash N;$$

$$N^*_+ = \{u + tu^*\colon u \in N, t \geq 0\}.$$

Assume that

(i) $\Phi(0) = 0$;

(ii) *there exists a* $R_1 > \rho$ *such that* $\Phi(u) \le 0$ *for all* $u \in N$ *with* $\|u\|_{\mathbb{V}} \ge R_1$;

(iii) *there exists a* $R_2 \ge R_1$ *such that* $\Phi(u) \le 0$ *for all* $u \in N^*$ *with* $\|u\|_{\mathbb{V}} \ge R_2$.

Let

$$\Gamma = \{\phi \in C(\mathbb{V}, \mathbb{V})\colon \phi \text{ is odd, } \phi(-\mathcal{D}_0 \cup \mathcal{D}_0) \subset (-\mathcal{D}_0 \cup \mathcal{D}_0); \\ \phi(u) = u \text{ if } \max\{\Phi(u), \Phi(-u)\} \le 0\}.$$

If

$$\gamma^* = \inf_{\phi\in\Gamma} \sup_{\phi(N_+^*)\cap\mathcal{S}} \Phi > \gamma^{**} = \inf_{\phi\in\Gamma} \sup_{\phi(N)\cap\mathcal{S}} \Phi > 0,$$

then

$$\mathcal{K}[\gamma^{**}, m_0 + 1] \cap (\mathbb{V}\backslash(-\mathcal{P} \cup \mathcal{P})) \ne \emptyset,$$

that is, there is a sign-changing critical point, where

$$m_0 := \sup_{N^*} \Phi < \infty,$$

and

$$\mathcal{K}[\gamma^{**}, m_0 + 1] := \{u \in \mathbb{V}\colon \Phi'(u) = 0, \gamma^{**} \le \Phi(u) \le m_0 + 1\}.$$

3. Proof of the main result

Define the Euler–Lagrange functional associated with the problem (1.1)–(1.2) as follows

$$\Phi(u) := \frac{1}{2}\int_\Omega |\nabla_\alpha u|^2 \mathrm{d}X - \int_\Omega F(x, y, u)\mathrm{d}X,$$

and

$$\begin{aligned}\overline{\Phi}(u) &:= \frac{1}{2}\int_\Omega |\nabla_\alpha u|^2 \mathrm{d}X - \int_\Omega F(x, y, u)\mathrm{d}X - \int_\Omega G(x, y, u)\mathrm{d}X \\ &= \Phi(u) - \int_\Omega G(x, y, u)\mathrm{d}X.\end{aligned}$$

From Proposition 2.2 in [25] and f satisfies (A1), g satisfies (A4), hence $\Phi, \overline{\Phi} \in C^1(S^2_{1,0}(\Omega), \mathbb{R})$ with

$$\langle \overline{\Phi}'(u), v\rangle = \int_\Omega \nabla_\alpha u \cdot \nabla_\alpha v \mathrm{d}X - \int_\Omega f(x, y, u)v\mathrm{d}X - \int_\Omega g(x, y, u)v\mathrm{d}X$$

for all $v \in S^2_{1,0}(\Omega)$.

Recall that a function $u \in S^2_{1,0}(\Omega)$ is a *weak solution* of the problem (1.1)–(1.2) if

$$\int_\Omega \nabla_\gamma u \cdot \nabla_\gamma v \mathrm{d}X = \int_\Omega f(x,y,u) v \mathrm{d}X + \int_\Omega g(x,y,u) v \mathrm{d}X, \quad \text{for all } v \in S^2_{1,0}(\Omega).$$

One can also check that the critical points of $\overline{\Phi}$ are weak solutions of the problem (1.1)–(1.2).

From embedding theorems for weighted Sobolev spaces, it is not difficult to show that the Grushin type has discrete spectrum in $S^2_{1,0}(\Omega)$. Let $\lambda_1, \lambda_2, \lambda_3, \dots$ be the eigenvalues of the problem

$$\begin{aligned} -G_\alpha u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$. Let $\mathbf{X}_j$ be the eigenspace associated to λ_j. We set for $k \ge 2$

$$\mathbf{Y}_k := \bigoplus_{j=1}^k \mathbf{X}_j \quad \text{and} \quad \mathbf{Z}_k = \overline{\bigoplus_{j=k}^\infty \mathbf{X}_j}.$$

Let

$$\mathcal{P} := \{u \in S^2_{1,0}(\Omega) : u(x,y) \ge 0 \text{ for a.e. } (x,y) \in \Omega\}$$

then $\mathcal{P}(-\mathcal{P})$ is the positive (negative) cone of $S^2_{1,0}(\Omega)$. We are going to consider an approximation for $S^2_{1,0}(\Omega) : \mathbf{Y}_1 \subset \mathbf{Y}_2 \subset \cdots$ and $\dim \mathbf{Y}_k < \infty$ for each $k > 2$, define

$$\Phi_k := \Phi\big|_{\mathbf{Y}_k} \quad \overline{\Phi}_k := \overline{\Phi}\big|_{\mathbf{Y}_k},$$

then $\Phi_k, \overline{\Phi}_k \in C^1(\mathbf{Y}_k, \mathbb{R})$.

Lemma 3.1. *Assume conditions* (A1)–(A5) *hold. Then* $\overline{\Phi}_k$ (*and hence* Φ_k) *satisfies the Palais–Smale condition.*

Proof. The proof of this lemma is similar to the one of Lemmas 5 in [33] (or see [25]). We omit the details. □

Lemma 3.2. *Under the assumptions of Theorem 1.1, there exist* $\rho_k > 0$ *and* $C_2 > 0$ *such that*

$$\overline{\Phi}(u) \ge C_2 \lambda_k^{\frac{2(p_0-p)}{(p-2)(p_0-2)}} := \delta_k, \quad \text{for } u \in \mathbf{Q}(\rho_k) := \{u \in \mathbf{Y}_{k-1}^\perp : \|u\|_{S^2_{1,0}(\Omega)} = \rho_k\},$$

where $p < p_0 < 2^*_\alpha$, *and* C_2 *is independent of* k. *Moreover,* $\rho_k \to \infty$ *as* $k \to \infty$.

Proof. By (A1)–(A5), for any $\epsilon > 0$ small enough, there exists a $C_\epsilon > 0$ such that

$$F(x,y,\xi) + G(x,y,\xi) \leq \epsilon|\xi|^2 + C_\epsilon|\xi|^p, \quad \text{for all } \xi \in \mathbb{R} \text{ and a.e. } (x,y) \in \Omega.$$

Applying Sobolev's embedding $S^2_{1,0}(\Omega) \hookrightarrow L^{2^*_\alpha}(\Omega)$, and using the interpolation inequality, for any $u \in S^2_{1,0}(\Omega)$, we obtain

$$\begin{aligned} \bar{\Phi}(u) &\geq \frac{1}{2}\int_\Omega |\nabla_\alpha u|^2 \mathrm{d}X - \int_\Omega (\epsilon|u|^2 + C_\epsilon|u|^p)\mathrm{d}X \\ &\geq \frac{1}{4}\|u\|^2_{S^2_{1,0}(\Omega)} - C_1\|u\|^r_{L^2(\Omega)}\|u\|^{p-r}_{L^{p_0}(\Omega)} \\ &\geq \frac{1}{4}\|u\|^2_{S^2_{1,0}(\Omega)} - C_2\|u\|^r_{L^2(\Omega)}\|u\|^{p-r}_{S^2_{1,0}(\Omega)}, \end{aligned} \tag{3.1}$$

where $\frac{r}{2} + \frac{p-r}{p_0} = 1$, $p_0 \in (p, 2^*_\alpha)$.
Moreover by $u \in \mathbf{Y}^\perp_{k-1}$, hence

$$\|u\|_{L^2(\Omega)} \leq \lambda_k^{\frac{-1}{2}} \|u\|_{S^2_{1,0}(\Omega)}. \tag{3.2}$$

Combining (3.1) and (3.2), for any $u \in \mathbf{Y}^\perp_{k-1}$ such that

$$\|u\|_{S^2_{1,0}(\Omega)} = \frac{\lambda_k^{\frac{r}{2(p-2)}}}{(2C_2 p)^{\frac{1}{p-2}}} := \rho_k,$$

we have

$$\bar{\Phi}(u) \geq C_2 \lambda_k^{\frac{2(p_0-p)}{(p-2)(p_0-2)}}.$$ □

For any $m > k+2$, let $\mathcal{P}_m := \mathcal{P} \cap \mathbf{Y}_m$ be the positive cone in $\mathbf{Y}_m$ and

$$Q(\rho_k, m) := \{u \in \mathbf{Y}^\perp_{k-1} \cap \mathbf{Y}_m \colon \|u\|_{S^2_{1,0}(\Omega)} = \rho_k\}.$$

Since $Q(\rho_k, m)$is compact in $\mathbf{Y}_m$ and includes only sign-changing elements, it is easy to check that

$$\operatorname{dist}(Q(\rho_k, m), \pm\mathcal{P}_m) := d_m > 0.$$

For any $\mu_m \in \left(0, \frac{d_m}{4}\right)$. define

$$\mathcal{D}_0(m, \mu_m) := \{u \in \mathbf{Y}_m \colon \mathrm{dist}(u, \mathcal{P}_m) < \mu_m\},$$

then $\mathcal{D}_0(m, \mu_m)$ is open and convex in $\mathbf{Y}_m$, $\pm\mathcal{P}_m \subset \pm\mathcal{D}_0(m, \mu_m)$ and

$$Q(\rho, m) \subset \mathcal{S}_m := \{\mathbf{Y}_m \backslash \mathcal{D}_m\}, \tag{3.3}$$

where

$$\mathcal{D}_m := -\mathcal{D}_0(m, \mu_m) \cup \mathcal{D}_0(m, \mu_m). \tag{3.4}$$

Evidently, the gradient of $\overline{\Phi}_m$ can be expressed as

$$\overline{\Phi}' = \mathrm{id} - \mathrm{Proj}_m K_{\overline{\Phi}},$$

where $K_{\overline{\Phi}} \colon S^2_{1,0}(\Omega) \to S^2_{1,0}(\Omega)$ is given by

$$K_{\overline{\Phi}} u = -G_\alpha^{-1}(f(\cdot, u(\cdot)) + g(\cdot, u(\cdot))) \quad \text{for all } u \in S^2_{1,0}(\Omega).$$

Proj_m is the projection on $\mathbf{Y}_m$ from $S^2_{1,0}(\Omega)$ and

$$\langle K_{\overline{\Phi}} u, v\rangle := \int\limits_{\mathbb{R}^N} (f(x, y, u) + g(x, y, u)) v \mathrm{d}X, \quad \text{for all } u, v \in S^2_{1,0}(\Omega).$$

LEMMA 3.3. *Assume conditions* (A1)–(A3) *and* (A5) *hold. Then there exists a* $\mu_m \in (0, d_m/4)$ *such that* $\mathrm{Proj}_m K_{\overline{\Phi}}(\pm\mathcal{D}_0(m, \mu_m)) \subset \pm\mathcal{D}_0(m, \mu_m)$ *and* $\mathrm{Proj}_m K_{\Phi}(\pm\mathcal{D}_0(m, \mu_m)) \subset \pm\mathcal{D}_0(m, \mu_m)$.

PROOF. Write $u^+ = \max\{u, 0\}, u^- = \min\{u, 0\}$. For any $u \in \mathbf{Y}_m, t \in [2, 2^*_\alpha)$, the exists a $C_t > 0$ such that

$$\begin{aligned} \|u^\pm\|_{L^t(\Omega)} &= \min_{\omega \in \mp\mathcal{P}_m} \|u - \omega\|_{L^t(\Omega)} \\ &\le C_t \min_{\omega \in \mp\mathcal{P}_m} \|u - \omega\|_{S^2_{1,0}(\Omega)} \\ &= C_t \,\mathrm{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m). \end{aligned} \tag{3.5}$$

By assumptions (A1), (A2), and (A5), for any $\epsilon > 0$ small enough, there exists a $C_\epsilon > 0$ such that

$$f(x, y, \xi)\xi + g(x, y, \xi)\xi \le \epsilon|\xi|^2 + C_\epsilon|\xi|^p, \tag{3.6}$$

for all $\xi \in \mathbb{R}$ and a.e. $(x, y) \in \Omega$. Combining (3.5), (3.6), and $f(x, y, \xi)\xi \geq 0$, $g(x, y, \xi)\xi \geq 0$ for all $\xi \in \mathbb{R}$ and a.e. $(x, y) \in \Omega$, we have for $\epsilon > 0$ small enough

$$\begin{aligned}
&\operatorname{dist}_{S^2_{1,0}(\Omega)}(v, \mp\mathcal{P}_m)\|v^\pm\|_{S^2_{1,0}(\Omega)} \\
&\quad\leq \|v^\pm\|^2_{S^2_{1,0}(\Omega)} = \langle v, v^\pm\rangle \\
&\quad= \int\limits_{\mathbb{R}^N} (|f(x, y, u^\pm)| + |g(x, y, u^\pm)|)|v^\pm| \mathrm{d}X \\
&\quad\leq \int\limits_{\mathbb{R}^N} (\epsilon|u^\pm| + C_\epsilon|u^\pm|^{p-1})|v^\pm| \mathrm{d}X \\
&\quad\leq \Big[\frac{2}{5} \operatorname{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m) + C \operatorname{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m)^{p-1}\Big]\|v^\pm\|_{S^2_{1,0}(\Omega)},
\end{aligned}$$

that is,

$$\begin{aligned}
&\operatorname{dist}_{S^2_{1,0}(\Omega)}(\mathrm{Proj}_m K_{\overline{\Phi}}, \mp\mathcal{P}_m) \\
&\quad\leq \frac{2}{5} \operatorname{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m) + C_3 \operatorname{dist}_{S^2_{1,0}(\Omega)}(u, \mp\mathcal{P}_m)^{p-1}.
\end{aligned}$$

Therefore, there exists a $\mu_m < \frac{d_m}{4}$ such that $\operatorname{dist}_{S^2_{1,0}(\Omega)}(\mathrm{Proj}_m K_{\overline{\Phi}}, \mp\mathcal{P}_m) \leq \mu_m$ for every $u \in \mp\mathcal{D}_0(m, \mu_m)$. The conclusion follows. □

Lemma 3.4. *Assume conditions* (A1)–(A3) *and* (A5) *hold. Then there exists a locally Lipschitz continuous map*

$$B_0\colon \widetilde{S^2_{1,0}(\Omega)} \longrightarrow S^2_{1,0}(\Omega)$$

such that

$$B_0((\pm\mathcal{D}_0(m, \mu_m)) \cap \widetilde{S^2_{1,0}(\Omega)}) \subset \pm\mathcal{D}_0(m, \mu_m)$$

and $V_m(u) := i(u)u - B_0(u)$ *is a pseudo-gradient vector field of* Φ_m*, where*

$$\widetilde{S^2_{1,0}(\Omega)} := S^2_{1,0}(\Omega)\backslash\mathcal{K}, \quad \mathcal{K} := \{u \in S^2_{1,0}(\Omega)\colon \Phi'(u) = 0\}.$$

Moreover, since Φ_m *and* i *are even functionals,* B_0 *(and hence* V_m*) can be choose to be odd.*

Proof. By Lemma 3.3, the proof of Lemma 3.4 is the same as the proof of Lemma 2.1 in [30], so we omit it. □

LEMMA 3.5. *Suppose that f satisfies* (A1) *and g satisfies* (A5). *Then*

$$\lim_{\substack{u\in\mathbf{Y}_{k+1},\\ \|u\|_{S^2_{1,0}(\Omega)}\to\infty}} \overline{\Phi}(u) = -\infty, \qquad \lim_{\substack{u\in\mathbf{Y}_{k+1},\\ \|u\|_{S^2_{1,0}(\Omega)}\to\infty}} \Phi(u) = -\infty.$$

PROOF. By the definition of $\mathbf{Y}_{k+1}$, (A1) and (A5), it is easy to verify that

$$\lim_{\substack{u\in\mathbf{Y}_{k+1},\\ \|u\|_{S^2_{1,0}(\Omega)}\to\infty}} \frac{1}{\|u\|^2_{S^2_{1,0}(\Omega)}}\int_\Omega F(x,y,u)\mathrm{d}X = \infty,$$

$$\lim_{\substack{u\in\mathbf{Y}_{k+1},\\ \|u\|_{S^2_{1,0}(\Omega)}\to\infty}} \frac{1}{\|u\|^2_{S^2_{1,0}(\Omega)}}\int_\Omega G(x,y,u)\mathrm{d}X \leq \infty.$$

Then the conclusions of this lemma follow immediately. □

LEMMA 3.6. *Suppose that f satisfies* (A1)–(A4) *and g satisfies* (A5). *Then for each fixed $m > 0$, there exists a $C_4 > 0$ such that*

$$\|u\|_{L^{\sigma+1}(\Omega)} \leq C_4 d^{\frac{1}{\mu}},$$

for all $u \in \pm U_\delta \cap \{u \in \mathbf{Y}_m\colon \overline{\Phi}_m(u) \leq d\}$, where C_4 is independent of $m, d > 0$ and

$$U_\delta := \left\{u \in \mathbf{Y}_m\colon \|\overline{\Phi}'_m(u) - \Phi'_m(u)\|_{(S^2_{1,0}(\Omega))^*} > \frac{\|\overline{\Phi}'_m(u)\|_{(S^2_{1,0}(\Omega))^*}}{\delta}\right\}.$$

PROOF. We consider two cases.

CASE 1. $u \in U_\delta \cap \{u \in \mathbf{Y}_m\colon \overline{\Phi}_m(u) \leq d\}$. We have that

$$\frac{1}{2}\|u\|^2_{S^2_{1,0}(\Omega)} - \int_\Omega F(x,y,u)\mathrm{d}X - \int_\Omega G(x,y,u)\mathrm{d}X \leq d, \tag{3.7}$$

$$\|\overline{\Phi}'_m(u)\|_{(S^2_{1,0}(\Omega))^*} < \delta\|\overline{\Phi}'_m(u) - \Phi'_m(u)\|_{(S^2_{1,0}(\Omega))^*}, \tag{3.8}$$

and

$$\begin{aligned}|\langle\overline{\Phi}'_m(u), u\rangle| &= \left|\|u\|^2_{S^2_{1,0}(\Omega)} - \int_\Omega f(x,y,u)u\mathrm{d}X - \int_\Omega g(x,y,u)u\mathrm{d}X\right| \\ &\leq \|\overline{\Phi}'_m(u)\|_{(S^2_{1,0}(\Omega))^*}\|u\|^2_{S^2_{1,0}(\Omega)} \\ &\leq \delta\|\overline{\Phi}'_m(u) - \Phi'_m(u)\|_{(S^2_{1,0}(\Omega))^*}\|u\|^2_{S^2_{1,0}(\Omega)}.\end{aligned} \tag{3.9}$$

From (A5) hence

$$\delta\|\overline{\Phi}'_m(u)-\Phi'_m(u)\|_{(S^2_{1,0}(\Omega))^*}\|u\|^2_{S^2_{1,0}(\Omega)}\le C_5(\|u\|^2_{L^2(\Omega)}+\|u\|^\sigma_{L^\mu(\Omega)}),$$

we get by (3.9) that

$$\begin{aligned}-\|u\|^2_{S^2_{1,0}(\Omega)}\le{}&-\int_\Omega f(x,y,u)u\mathrm{d}X-\int_\Omega g(x,y,u)u\mathrm{d}X\\&+C_5(\|u\|^2_{L^2(\Omega)}+\|u\|^\sigma_{L^\mu(\Omega)})\|u\|_{S^2_{1,0}(\Omega)}.\end{aligned}\tag{3.10}$$

Choose $\mu_0\in(2,\mu)$. By (3.7) and (3.10), we know that

$$\begin{aligned}\Big(\frac{\mu_0}{2}-1\Big)\|u\|^2_{S^2_{1,0}(\Omega)}\le{}&\int_\Omega[\mu_0F(x,y,u)-f(x,y,u)u]\mathrm{d}X\\&+\int_\Omega[\mu_0G(x,y,u)-g(x,y,u)u]\mathrm{d}X\\&+C_5(\|u\|^2_{L^2(\Omega)}+\|u\|^\sigma_{L^\mu(\Omega)})\|u\|_{S^2_{1,0}(\Omega)}+\mu_0d.\end{aligned}$$

This and (A3), (A5) hence give

$$\begin{aligned}&\Big(\frac{\mu_0}{2}-1\Big)\|u\|^2_{S^2_{1,0}(\Omega)}+C_6\|u\|^\mu_{L^\mu(\Omega)}\\&\quad\le\Big(\frac{\mu_0}{2}-1\Big)\|u\|^2_{S^2_{1,0}(\Omega)}+\int_\Omega[f(x,y,u)u-\mu_0F(x,y,u)]\mathrm{d}X+C_7\\&\quad\le\int_\Omega[\mu_0G(x,y,u)-g(x,y,u)u]\mathrm{d}X\\&\qquad+C_5(\|u\|^2_{L^2(\Omega)}+\|u\|^\sigma_{L^\mu(\Omega)})\|u\|_{S^2_{1,0}(\Omega)}+\mu_0d+C_7\\&\quad\le C_8\|u\|^2_{L^2(\Omega)}+C_8\|u\|^{\sigma+1}_{L^{\sigma+1}(\Omega)}\\&\qquad+C_5(\|u\|^2_{L^2(\Omega)}+\|u\|^\sigma_{L^\mu(\Omega)})\|u\|_{S^2_{1,0}(\Omega)}+\mu_0d+C_7.\end{aligned}$$

By $\mu>2,\mu>\sigma+1$, applying Young's inequalities and Cauchy's inequalities, we have

$$\begin{aligned}&C_8\|u\|^2_{S^2_{1,0}(\Omega)}+C_9\|u\|^\mu_{L^\mu(\Omega)}\\&\quad\le C_5(\|u\|^2_{L^2(\Omega)}+\|u\|^\sigma_{L^\mu(\Omega)})\|u\|_{S^2_{1,0}(\Omega)}+\mu_0d+C_7\\&\quad\le C_\epsilon\|u\|^2_{L^2(\Omega)}+\epsilon\|u\|^2_{S^2_{1,0}(\Omega)}+C_{\epsilon'}\|u\|^{2\sigma}_{L^2(\Omega)}+\epsilon'\|u\|^2_{S^2_{1,0}(\Omega)}+\mu_0d+C_7,\end{aligned}$$

for all $\epsilon, \epsilon' > 0$ small enough. By the fact that $2\sigma < \mu$, we can obtain

$$\|u\|_{L^{\sigma+1}(\Omega)} \le C_{10}\|u\|_{L^{\mu}(\Omega)} \le C_4 d^{\frac{1}{\mu}}.$$

CASE 2. $u \in -U_\delta \cap \{u \in \mathbf{Y}_m : \overline{\Phi}_m(u) \le d\}$, that is,

$$\|\overline{\Phi}'_m(-u)\|_{(S^2_{1,0}(\Omega))^*} < \delta\|\overline{\Phi}'_m(-u) - \Phi'_m(-u)\|_{(S^2_{1,0}(\Omega))^*} \quad \text{and} \quad \overline{\Phi}_m(u) \le d.$$

Then

$$\begin{aligned}\overline{\Phi}_m(-u) &= \overline{\Phi}_m(u) + \int_\Omega [G(x,y,u) - G(x,y,-u)]\mathrm{d}X \\ &\le d + C_{11}\|u\|^2_{L^2(\Omega)} + C_{11}\|u\|^{\sigma+1}_{L^{\sigma+1}(\Omega)},\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\|u\|^2_{S^2_{1,0}(\Omega)} - \int_\Omega F(x,y,-u)\mathrm{d}X - \int_\Omega G(x,y,-u)\mathrm{d}X \\ &\quad \le d + C_{11}\|u\|^2_{L^2(\Omega)} + C_{11}\|u\|^{\sigma+1}_{L^{\sigma+1}(\Omega)},\end{aligned} \tag{3.11}$$

$$\begin{aligned}&|\langle\overline{\Phi}'_m(-u), -u\rangle| \\ &\quad = \left|\|u\|^2_{S^2_{1,0}(\Omega)} + \int_\Omega f(x,y,-u)u\mathrm{d}X + \int_\Omega g(x,y,-u)u\mathrm{d}X\right| \\ &\quad \le \|\overline{\Phi}'_m(-u)\|_{(S^2_{1,0}(\Omega))^*}\|u\|^2_{S^2_{1,0}(\Omega)} \\ &\quad \le \delta\|\overline{\Phi}'_m(-u) - \Phi'_m(-u)\|_{(S^2_{1,0}(\Omega))^*}\|u\|^2_{S^2_{1,0}(\Omega)}.\end{aligned} \tag{3.12}$$

Note that

$$\delta\|\overline{\Phi}'_m(-u) - \Phi'_m(-u)\|_{(S^2_{1,0}(\Omega))^*}\|u\|^2_{S^2_{1,0}(\Omega)} \le C_{12}(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)}).$$

Then we get

$$\begin{aligned}-\|u\|^2_{S^2_{1,0}(\Omega)} \le &\int_\Omega f(x,y,-u)u\mathrm{d}X + \int_\Omega g(x,y,-u)u\mathrm{d}X \\ &+ C_{12}(\|u\|^2_{L^2(\Omega)} + \|u\|^\sigma_{L^\mu(\Omega)})\|u\|_{S^2_{1,0}(\Omega)}.\end{aligned} \tag{3.13}$$

Therefore, by (3.11)–(3.13), (A3), and (A5), we obtain

$$
\begin{aligned}
&\left(\frac{\mu_0}{2}-1\right)\|u\|^2_{S^2_{1,0}(\Omega)} + C_6\|u\|^{\mu}_{L^{\mu}(\Omega)} \\
&\quad\le \left(\frac{\mu_0}{2}-1\right)\|u\|^2_{S^2_{1,0}(\Omega)} + \int_\Omega [-f(x,y,-u)u - \mu_0 F(x,y,u)]\mathrm{d}X + C_7 \\
&\quad\le \int_\Omega [\mu_0 G(x,y,-u) + g(x,y,-u)u]\mathrm{d}X \\
&\qquad + C_{12}(\|u\|^2_{L^2(\Omega)} + \|u\|^{\sigma}_{L^{\mu}(\Omega)})\|u\|_{S^2_{1,0}(\Omega)} + \mu_0 d + C_7 \\
&\quad\le C_8\|u\|^2_{L^2(\Omega)} + C_8\|u\|^{\sigma+1}_{L^{\sigma+1}(\Omega)} \\
&\qquad + C_{12}(\|u\|^2_{L^2(\Omega)} + \|u\|^{\sigma}_{L^{\mu}(\Omega)})\|u\|_{S^2_{1,0}(\Omega)} + \mu_0 d + C_7. \qquad \square
\end{aligned}
$$

This gives the desired result.

Lemma 3.7. *Assume conditions* (A1)-(A5) *hold and assume that* $u_m \in \mathbf{Y}_m$ *is sign-changing and satisfies*

$$\overline{\Phi}'_m(u_m) = 0, \qquad \sup_{m\ge 1} |\overline{\Phi}_m(u_m)| < \infty.$$

Then $\{u_m\}_{m=1}^{m=\infty}$ *has a convergent subsequence whose limit is a sign-changing critical point of* $\overline{\Phi}$.

Proof. From Lemma 3.1, we have $\{u_m\}_{m=1}^{m=\infty}$ has a convergent subsequence in $S^2_{1,0}(\Omega)$. We just prove that the limit of the subsequence is also sign-changing. Let $u_m^{\pm} := \max\{\pm u_m, 0\}$. Then

$$\|u_m^{\pm}\|^2_{S^2_{1,0}(\Omega)} = \int_\Omega [f(x,y,u_m^{\pm})u_m^{\pm} + g(x,y,u_m^{\pm})u_m^{\pm}]\mathrm{d}X.$$

By (A1), (A2), and (A5), we have for any $\epsilon > 0$, there exists a C_ϵ such that

$$f(x,y,\xi)\xi + g(x,y,\xi)\xi \le \epsilon|\xi|^2 + C_\epsilon|\xi|^p, \quad \text{for all } \xi \in \mathbb{R} \text{ and a.e. } (x,y) \in \Omega.$$

It follows that

$$\|u_m^{\pm}\|^2_{S^2_{1,0}(\Omega)} \le \epsilon\|u^{\pm}\|^2_{S^2_{1,0}(\Omega)} + C_\epsilon\|u^{\pm}\|^p_{L^p(\Omega)}.$$

Hence, $\|u_m^{\pm}\|^2_{S^2_{1,0}(\Omega)} \ge C_{13}$, where C_{13} is a constant independent of m. This implies that the limit of the subsequence is also sign-changing. $\square$

Proof of Theorem 1.1. Assume that there exists a $C_{14} > 0$ such that $\overline{\Phi}$ has no sign-changing critical point with critical value greater than C_{14}. Choose $k_0 > 0$ such that $\delta_k > C_{14}$ for all $k > k_0$, where δ_k comes from Lemma 3.2. Let $m > k + 2 > k_0 + 2$. Then $\mathbf{Y}_k \subset \mathbf{Y}_m$. Let

$$N := \mathbf{Y}_k, \quad M(m) = \mathbf{Y}_{k-1}^{\perp} \cap \mathbf{Y}_m, \quad Q(\rho_k, m) := \{u \in M(m) : \|u\|_{S^2_{1,0}(\Omega)} = \rho_k\}.$$

Then by (3.3), we obtain

$$Q(\rho_k, m) \subset \mathcal{S}_m.$$

Define

$$N^* = N \oplus \operatorname{span}\{u^*\}, \quad u^* \in \mathbf{Y}_{k+1}, u^* \notin \mathbf{Y}_k;$$
$$N^*_+ = \{u + tu^* : u \in N, t \geq 0\}.$$

Then $N^* \cap \mathbf{Y}_{k+1} \neq \{0\}$, and both N^* and N^*_+ are independent of m. Clearly, by Lemma 3.5, we have

(i) $\overline{\Phi}_m(0) = 0$;

(ii) there exists a $R_1 > \rho_k$ such that $\overline{\Phi}_m(u) \leq 0$ for all $u \in N$ with $\|u\|_{S^2_{1,0}(\Omega)} \geq R_1$;

(iii) there exists a $R_2 \geq R_1 > 0$ such that $\overline{\Phi}_m(u) \leq 0$ for all $u \in N^*$ with $\|u\|_{S^2_{1,0}(\Omega)} \geq R_2$;

Let

$$\Gamma_m := \{\phi \in C(\mathbf{Y}_m, \mathbf{Y}_m) : \phi \text{ is odd}, \phi(\mathcal{D}_m) \subset (\mathcal{D}_m);$$
$$\phi(u) = u \text{ if } \max\{\overline{\Phi}_m(u), \overline{\Phi}_m(-u)\} \leq 0\}.$$

Define

$$\gamma_k^*(m) := \inf_{\phi \in \Gamma_m} \sup_{\phi(N^*_+) \cap \mathcal{S}_m} \overline{\Phi}_m, \quad \gamma_k^{**}(m) := \inf_{\phi \in \Gamma_m} \sup_{\phi(N) \cap \mathcal{S}_m} \overline{\Phi}_m > 0,$$

For any $\phi \in \Gamma_m$, by Lemma 1.44 in [28] (or see [3, 32]), we have

$$\phi(N \cap B_{R_1}) \cap Q(\rho_k, m) \neq \emptyset,$$

and by Lemma 3.2 we can obtain

$$\sup_{\phi(N \cap B_{R_1}) \cap \mathcal{S}_m} \overline{\Phi}_m \leq \inf_{Q(\rho_k, m)} \overline{\Phi}_m \leq C_2 \lambda_k^{\frac{2(p_0 - p)}{(p-2)(p_0 - 2)}} := \delta_k.$$

Therefore, we get

$$\gamma_k^{**}(m) \geq C_2 \lambda_k^{\frac{2(p_0-p)}{(p-2)(p_0-2)}} := \delta_k \longrightarrow \infty, \quad \text{as } k \to \infty. \tag{3.14}$$

We consider two cases.

CASE 1. For $k \geq k_0$, if there exists a sequence $m_i \to \infty$ as $i \to \infty$ such that

$$\gamma_k^*(m_i) > \gamma_k^{**}(m_i) \quad \text{for all } i > 1,$$

then by Proposition 2.3, there exists a sign-changing critical point u_{m_i} such that

$$\overline{\Phi}'_{m_i}(u_{m_i}) = 0 \quad \text{and} \quad C_0 < \delta_k \leq \gamma_k^{**}(m_i) \leq \overline{\Phi}(u_{m_i}) \leq \sup_{N^*} \overline{\Phi} + 1$$

Here $\sup_{N^*} \overline{\Phi}$ is a constant depending on k and independent of m_i. By Lemma 3.7, $\{u_{m_i}\}_{i=1}^{i=+\infty}$ has a convergent subsequence whose limit u is a sign-changing critical point of $\overline{\Phi}$, and $\overline{\Phi}(u) \geq \delta_k > C_0$. This contradicts the assumption.

CASE 2. For all $k \geq k_0$, there exists a m_k such that

$$\gamma_k^*(m) = \gamma_k^{**}(m) \quad \text{for all } m > m_k. \tag{3.15}$$

Let $K_{\text{com}}(m)$ denote the set of common critical points of Φ_m and $\overline{\Phi}_m$. By (A5), $K_{\text{com}}(m) = \{0\}$. Define

$$V_\delta := \{u \in \mathbf{Y}_m : \|u\|_{S^2_{1,0}(\Omega)} \leq \delta\},$$

and let U_δ be as in Lemma 3.6, which contains all non-common critical points of Φ_m and $\overline{\Phi}_m$. By Lemma 3.5, there exists a $R_1 > \rho_k$ such that $\overline{\Phi}_m(u) \leq 0$ for all $u \in N$ with $\|u\|_{S^2_{1,0}(\Omega)} \geq R_1$. Here R_1 is independent of m. Combining the definition of $\gamma_k^*(m)$ and (3.14), we find a $\phi_0 \in \Gamma_m$ such that

$$\sup_{\phi_0(N_+^*) \cap \mathcal{S}_m} \overline{\Phi} = \sup_{\phi(N_+^* \cap B_{R_1}) \cap \mathcal{S}_m} \overline{\Phi} \leq \gamma_k^*(m) + \frac{1}{2}. \tag{3.16}$$

Let

$$U_\delta^*(m) := V_\delta \cup U_\delta \cup (-U_\delta).$$

Then U_δ is a symmetric set and contains all critical points of Φ_m and $\overline{\Phi}_m$. Define two non-negative continuous functions:

$$\zeta_1(u) = \begin{cases} 0, & \text{if } u \in U_{10}^*(m), \\ 1, & \text{if } u \notin U_{20}^*(m), \end{cases} \quad (\text{is even}), \quad \zeta_2(u) = \begin{cases} 0, & \text{if } u \leq 0, \\ 1, & \text{if } u \geq 1, \end{cases}$$

and a vector field

$$V_m^* := -\zeta_2(\max\{\overline{\Phi}_m(u), \overline{\Phi}_m(-u)\})\zeta_1(u)V_m(u),$$

where the pseudo gradient vector field $V_m$comes from Lemma 3.4 obtained for Φ_m. Since V_m can be choose to be odd, then V_m^* is odd.

Let $\Theta(t, u)$ denote the unique (odd in u) solution of the Cauchy initial value problem:

$$\frac{\mathrm{d}\Theta(t,u)}{\mathrm{d}t} = V_m^*(\Theta(t,u)), \quad \Theta(0,u) = u \in \mathbf{Y}_m.$$

Then $\Theta(t, u)$ is also a pseudo-gradient flow for $\overline{\Phi}_m$ and

$$\frac{\mathrm{d}\overline{\Phi}_m(\Theta(t,u))}{\mathrm{d}t} \le 0. \tag{3.17}$$

For any $u \notin U_\delta^*(m)$, we have $u \notin \pm U_\delta$ and by Lemma 3.6, we obtain

$$\|\Phi_m'(u)\|_{(S_{1,0}^2(\Omega))^*} \le \frac{\delta+1}{\delta}\|\overline{\Phi}_m'(u)\|_{(S_{1,0}^2(\Omega))^*},$$
$$\|\overline{\Phi}_m'(u)\|_{(S_{1,0}^2(\Omega))^*} \le \frac{\delta}{\delta-1}\|\Phi_m'(u)\|_{(S_{1,0}^2(\Omega))^*}.$$

Further, for all $u \notin U_\delta^*(m)$, we get

$$\begin{aligned}
&\langle\overline{\Phi}_m'(u), V_m(u)\rangle \\
&\quad= \langle\Phi_m'(u), V_m(u)\rangle - \langle\Phi_m'(u) - \overline{\Phi}_m'(u), V_m(u)\rangle \\
&\quad\ge \frac{1}{2}\|\Phi_m'(u)\|^2_{(S_{1,0}^2(\Omega))^*} - 2\|\Phi_m'(u)\|^2_{(S_{1,0}^2(\Omega))^*}\|\Phi_m'(u) - V_m(u)\|^2_{(S_{1,0}^2(\Omega))^*} \\
&\quad\ge \frac{(\delta-1)^2 - 4(\delta+1)}{2\delta^2}\|\overline{\Phi}_m'(u)\|^2_{(S_{1,0}^2(\Omega))^*},
\end{aligned}$$

and

$$\|V_m(u)\|_{(S_{1,0}^2(\Omega))^*} \le 2\|\Phi_m'(u)\|^2_{(S_{1,0}^2(\Omega))^*} \le \frac{2(\delta+1)}{\delta}\|\overline{\Phi}_m'(u)\|^2_{(S_{1,0}^2(\Omega))^*}.$$

Moreover, since $u \notin U_{20}^*(m)$ implies $\zeta_1(u) = 1$, we see that $\overline{\Phi}_m(u) > 1$ implies

$$\zeta_2(\max\{\overline{\Phi}_m(u), \overline{\Phi}_m(-u)\}) = 1.$$

We obtain

(3.18)

$$\begin{aligned}\frac{d\overline{\Phi}_m(\Theta(t,u))}{dt}\Big|_{t=0} &= \Big\langle \overline{\Phi}'_m(\Theta(t,u)), \frac{d\Theta}{dt}\Big\rangle\Big|_{t=0}\\ &= \langle \overline{\Phi}'_m(\Theta(t,u)), V^*_m(\Theta(t,u))\rangle\big|_{t=0}\\ &= \langle \overline{\Phi}'_m(u), -\zeta_2(\max\{\overline{\Phi}_m(\Theta(t,u)), \overline{\Phi}_m(-\Theta(t,u))\})\\ &\qquad\qquad \zeta_1(\Theta(t,u))V_m(\Theta(t,u)))\rangle\big|_{t=0}\\ &= \langle \overline{\Phi}'_m(u), -\zeta_2(\max\{\overline{\Phi}_m(\Theta(t,u)), \overline{\Phi}_m(-\Theta(t,u))\})\\ &\qquad\qquad \zeta_1(u)V_m(u))\rangle\\ &= -\langle \overline{\Phi}'_m(u), V_m(u)\rangle \le -\frac{277}{800}\|\overline{\Phi}'_m(u)\|^2_{(S^2_{1,0}(\Omega))^*}\end{aligned}$$

for all $u \notin U^*_{20}(m)$ safisfying $\overline{\Phi}_m(u) > 1$.

We claim that $\Theta(t,\phi_0(\cdot)) \in \Gamma_m$ for any $t \ge 0$. In fact, $\Theta(t,\phi_0(u))$ is odd in u since $\phi_0(u)$ and V^*_m are odd. Recall that $\phi_0 \in \Gamma_m$. Then, $\phi_0(u) = u$ for u with $\max\{\overline{\Phi}(u), \overline{\Phi}(-u)\} \le 0$. Hence, $\Theta(t,\phi_0(u)) = \Theta(t,u)$ and $V^*_m(u) = 0$ for u with $\max\{\overline{\Phi}(u), \overline{\Phi}(-u)\} \le 0$. It follows that $\Theta(t,u) = u$ and then $\Theta(t,\phi_0(u)) = u$ for u with $\max\{\overline{\Phi}(u), \overline{\Phi}(-u)\} \le 0$. Using Theorem 1 in [9] and Lemma 3.4 similar to the one of Theorem 2.1 in [30], we obtain

$$\Theta(t,\phi_0(\mathcal{D}_m)) \subset \Theta(t,\mathcal{D}_m) \subset \mathcal{D}_m, \quad \text{for all } t \ge 0.$$

Therefore, $\Theta(t,\phi_0(u)) \in \Gamma_m$ for any $t \ge 0$. For any $t \ge 0$, we can deduce the following estimates which lead to a contradiction. In fact, by the fact that $\Theta(t,\phi_0(u)) \in \Gamma_m$ is odd,

$$\Theta(t,\phi_0(N^*_+)) \cap \mathcal{S}_m \subset \Big\{u \in \mathbf{Y}_m \colon \overline{\Phi}_m(u) \le \gamma^{**}_k(m) + \frac{1}{2}\Big\},$$

V_{20} is bounded, (3.15)–(3.18), and Lemma 3.6, we have

$$\begin{aligned}\gamma^*_k(m) + \frac{1}{2} &= \gamma^{**}_k(m) + \frac{1}{2}\\ &\ge \sup_{\phi_0(N^*_+)\cap\mathcal{S}_m} \overline{\Phi}\\ &\ge \sup_{\Theta(t,\phi_0(N^*_+))\cap\mathcal{S}_m} \overline{\Phi}\\ &\ge \sup_{\Theta(t,\phi_0(N^*))\cap\mathcal{S}_m} \overline{\Phi} - \Big(\sup_{\Theta(t,\phi_0(N^*))\cap\mathcal{S}_m} \overline{\Phi} - \sup_{\Theta(t,\phi_0(N^*_+))\cap\mathcal{S}_m} \overline{\Phi}\Big)\end{aligned}$$

$$
\begin{aligned}
&\geq \gamma_{k+1}^{**}(m) - \sup_{u\in\Theta(t,\phi_0(N_+^*))\cap\mathcal{S}_m} (\overline{\Phi}(-u) - \overline{\Phi}(u)) \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{\substack{u\in\Theta(t,\phi_0(N_+^*))\cap\mathcal{S}_m \\ \cap\{u\in\mathbf{Y}_m:\, \overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}}} (\overline{\Phi}(-u) - \overline{\Phi}(u)) \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{\substack{u\in\Theta(t,\phi_0(N_+^*))\cap\mathcal{S}_m \\ \cap\{u\in\mathbf{Y}_m:\, \overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}}} |\overline{\Phi}(-u) - \overline{\Phi}(u)| \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{u\in U_{20}^*\cap\{u\in\mathbf{Y}_m:\, \overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}} |\overline{\Phi}(-u) - \overline{\Phi}(u)| \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{\substack{u\in(-U_{20}\cup U_{20}) \\ \cap\{u\in\mathbf{Y}_m:\, \overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}}} |\overline{\Phi}(-u) - \overline{\Phi}(u)| - C_{15} \\
&\geq \gamma_{k+1}^{**}(m) - \sup_{\substack{u\in(-U_{20}\cup U_{20}) \\ \cap\{u\in\mathbf{Y}_m:\, \overline{\Phi}_m(u)\leq\gamma_k^{**}(m)+\frac{1}{2}\}}} C_8\|u\|_{L^{\sigma+1}(\Omega)}^{\sigma+1} - C_{15} \\
&\geq \gamma_{k+1}^{**}(m) - C_8(\gamma_k^{**}(m))^{\frac{1+\sigma}{\mu}} - C_{15}.
\end{aligned}
$$

Therefore, we get the inequality

$$
\gamma_{k+1}^{**}(m) \leq \gamma_k^{**}(m)(1 + C_8(\gamma_k^{**}(m))^{\frac{1+\sigma-\mu}{\mu}}), \tag{3.19}
$$

for all $k \geq k_0$.

Then from the condition (1.3), we can take some $p_0 \in (2, 2_\alpha^*)$ and $q \in \big(\frac{2N_\alpha}{N_\alpha+2}, 2\big)$ such that

$$
\frac{1+\sigma-\mu}{\mu}\,\frac{2(p_0-p)}{(p-2)(p_0-2)}\Big(\frac{2}{N_\alpha} - \frac{2-q}{q}\Big) < -1. \tag{3.20}
$$

From Theorem 1.3 in [15] (or see [16]), we obtain

$$
\lambda_k \geq C_{16}k^{\frac{2}{N_\alpha}}. \tag{3.21}
$$

From (3.19)–(3.21), using iteration, we get that

$$
\begin{aligned}
\gamma_{k_0+\ell}^{**}(m) &\leq \gamma_{k_0}^{**}(m)\prod_{k=k_0}^{k_0+\ell-1}(1 + C_8(\gamma_k^{**}(m))^{\frac{1+\sigma-\mu}{\mu}}) \\
&\leq \gamma_{k_0}^{**}(m)\exp\Big(\sum_{k=k_0}^{k_0+\ell-1}\ln(1 + C_8(\gamma_k^{**}(m))^{\frac{1+\sigma-\mu}{\mu}})\Big)
\end{aligned}
$$

$$\le \gamma_{k_0}^{**}(m) \exp C_8 \Big(\sum_{k=k_0}^{k_0+\ell-1} (\gamma_k^{**}(m))^{\frac{1+\sigma-\mu}{\mu}} \Big)$$

$$\le \gamma_{k_0}^{**}(m) \exp C_8 \Big(\sum_{k=k_0}^{k_0+\ell-1} k^{\frac{1+\sigma-\mu}{\mu} \frac{2(p_0-p)}{(p-2)(p_0-2)} (\frac{2}{N_\alpha} - \frac{2-q}{q})} \Big)$$

$$< \infty,$$

for all $\ell \in \mathbb{N}$, which yields the desired contradiction. Thus, $\bar{\Phi}$ possesses an unbounded sign-changing sequence of critical values. □

Corollary 3.8. *Assume that f satisfies the conditions* (A1)–(A3) *and there is a $\sigma < \frac{\mu}{2}$ such that*

$$|f(x,y,u) - f(x,y,-u)| \le C(1 + |u|^\sigma) \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } (x,y) \in \Omega.$$

Then the problem

$$-G_\alpha u = f(x,y,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{3.22}$$

has an infinite sequence of sign-changing solutions provided that (1.3) *holds.*

Corollary 3.9. *Assume that f satisfies the conditions* (A1)–(A3)*, there exists a $R > 0$ such that*

$$f(x,y,-u) = -f(x,y,u) \quad \text{for a.e. } (x,y) \in \Omega, |u| \ge R,$$

and

$$\frac{2p}{N_\alpha(p-2)} - 1 > \frac{\mu}{\mu - 1}.$$

Then the problem (3.22) *has an infinite sequence of sign-changing solutions in $S^2_{1,0}(\Omega)$.*

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References

[1] A. Ambrosetti – P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), pp. 349–381.

[2] C. T. Anh – B. K. My, *Existence of solutions to Δ_λ-Laplace equations without the Ambrosetti–Rabinowitz condition*, Complex Var. Elliptic Equ. 61 (1) (2016), pp. 137–150.

[3] A. Bahri – H. Berestycki, *A perturbation method in critical point theory and applications,* Trans. Amer. Math. Soc. 267 (1981), no. 1, pp. 1–32.

[4] S. Barile – G. M. Figueiredo, *Existence of least energy positive, negative and nodal solutions for a class of $p\&q$-problems with potentials vanishing at infinity*, J. Math. Anal. Appl. 427 (2015), pp. 1205–1233.

[5] T. Bartsch – Z. Liu – T. Weth *Sign changing solutions of superlinear Schrödinger equations*, Comm. Partial Differential Equations 29 (2004), no. 1-2, pp. 25–42.

[6] T. Bartsch – T. Weth, *Three nodal solutions of singularly perturbed elliptic equations on domains without topology*, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), no. 3, pp. 259–281.

[7] T. Bartsch – T. Weth – M. Willem, *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math. 96 (2005), pp. 1–18.

[8] H. Berestycki – P. L. Lions, *Nonlinear scalar field equations,* II, *Existence of infinitely many solutions*, Arch. Rational Mech. Anal. 82 (1983), no. 4, pp. 313–345.

[9] H. Brezis, *On a characterization of flow-invariant sets,* Comm. Pure Appl. Math 23 (1970), pp. 261–263.

[10] A. Castro – J. Cossio – J. Neuberger, *A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math., 27 (1997), no. 4, pp. 1041–1053.

[11] S. T. Chen – Y. B. Li – X. H. Tang, *Sign-changing solutions for asymptotically linear Schrodinger equation in bounded domains*, Electron. J. Differential Equations 317 (2016), pp. 1–9.

[12] S. Dipierro – M. Medina – E. Valdinoci, *Fractional elliptic problems with critical growth in the whole of $\mathbb{R}^n$,* Appunti, Scuola Normale Superiore di Pisa (Nuova Serie) 15, Edizioni della Normale, Pisa, 2017.

[13] B. Franchi – E. Lanconelli, *An embedding theorem for Sobolev spaces related to non-smooth vector fields and Harnack inequality*, Comm. Partial Differential Equations 9 (1984), no. 13, pp. 1237–1264.

[14] V. V. Grushin, *A certain class of hypoelliptic operators,* Mat. Sb. (N.S.) **83** (1970), no. 125, pp. 456–473, in Russian.

[15] C. Hua – L. Peng, *Lower bounds of Dirichlet eigenvalues for some degenerate elliptic operators,* Calc. Var. Partial Differential Equations 54 (2015), no. 3, pp. 2831–2852.

[16] C. Hua – Z. Yifu, *Lower bounds of eigenvalues for a class of bi-subelliptic operators*. J. Differential Equations 262 (2017), no. 12, pp. 5860–5879.

[17] A. E. Kogoj – E. Lanconelli, *On semilinear Δ_λ-Laplace equation,* Nonlinear Analysis 75 (2012), no. 12, pp. 4637–4649.

[18] Z. L. Liu – J. X. Sun, *Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations*, J. Differential Equations 172 (2001), no. 2, pp. 257–299.

[19] D. Lupo – K. Payne, *Critical exponents for semilinear equations of mixed elliptic-hyperbolic and degenerate types,* Comm. Pure Appl. Math. 56 (2003), no. 3, pp. 403–424.

[20] D. T. Luyen, *Two nontrivial solutions of boundary value problems for semilinear Δ_γ differential equations,* Math. Notes 101 (2017), no. 5, pp. 815–823.

[21] D. T. Luyen – N. M. Tri, *Existence of solutions to boundary value problems for semilinear Δ_γ differential equations,* Math. Notes 97 (2015), no. 1, pp. 73–84.

[22] D. T. Luyen – N. M. Tri, *Large-time behavior of solutions to damped hyperbolic equation involving strongly degenerate elliptic differential operators,* Sibirsk. Mat. Zh. 57 (2016), no. 4, pp. 809–829, in Russian; translation in Sib. Math. J. 57 (2016), no. 4, pp. 632–649.

[23] D. T. Luyen – N. M. Tri, *Global attractor of the Cauchy problem for a semilinear degenerate damped hyperbolic equation involving the Grushin operator,* Ann. Pol. Math. 117 (2) (2016), pp. 141–162.

[24] D. T. Luyen – N. M. Tri, *Existence of infinitely many solutions for semilinear degenerate Schrödinger equations,* J. Math. Anal. Appl. 461 (2018), no. 2, pp. 1271–1286.

[25] D. T. Luyen – N. M. Tri, *On the existence of multiple solutions to boundary value problems for semilinear elliptic degenerate operators*. Complex Var. Elliptic Equ. 64 (2019), no. 6, pp. 1050–1066.

[26] E. S. Noussair – J. Wei, *On the effect of domain geometry on the existence of nodal solutions in singular perturbations problems*, Indiana Univ. Math. J., 46 (1997), pp. 1255–1271.

[27] S. I. Pokhozaev, *Eigenfunctions for the equation $\Delta u + \lambda f(u) = 0$*, Russian Dokl. Akad. Nauk SSSR 165 (1965), pp. 33–36.

[28] P. H. Rabinowitz, *Multiple critical points of perturbed symmetric functionals*, Trans. Amer. Math. Soc. 272 (1982), no. 2, pp. 753–769.

[29] M. Schechter – Z. Q. Wang – W. Zou, *New Linking Theorem and Sign-Changing Solutions*, Comm. Partial Differential Equations 29 (2004), no. 3-4, pp. 471–488.

[30] M. Schechter – W. Zou, *Infinitely many solutions to perturbed elliptic equations,* J. Funct. Anal. 228 (2005), no. 1, pp. 1–38.

[31] M. Schechter – W. Zou, *Sign-changing critical points from linking type theorems*, Trans. Amer. Math. Soc. 358 (2006), no. 12, pp. 5293–5318.

[32] M. Struwe, *Variational methods,* Applications to nonlinear partial differential equations and Hamiltonian systems, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 34, Springer-Verlag, Berlin, 1996.

[33] P. T. Thuy – N. M. Tri, *Nontrivial solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations*, NoDEA Nonlinear Differential Equations Appl. 19 (2012), no. 3, pp. 279–298.

[34] P. T. Thuy – N. M. Tri, *Long time behavior of solutions to semilinear parabolic equations involving strongly degenerate elliptic differential operators,* NoDEA Nonlinear Differential Equations Appl. 20 (2013), no. 3, pp. 1213–1224.

[35] N. M. Tri, *Critical Sobolev exponent for hypoelliptic operators,* Acta Math. Vietnam **23** (1998), no. 1, pp. 83–94.

[36] N. M. Tri, *Semilinear degenerate elliptic differential equations, local and global theories,* Lambert Academic Publishing, Saarbrücken, 2010.

[37] N. M. Tri, *Recent progress in the theory of semilinear equations involving degenerate elliptic differential operators,* Publishing House for Science and Technology of the Vietnam Academy of Science and Technology, Hanoi, 2014.

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