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On a fractional nonlinear equation on a bounded domain of $\mathbb{R}^n$

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Abstract – We establish perturbation results for a Nirenberg type equation involving the fractional Laplacian on a bounded domain of $\mathbb{R}^n, n \geq 2$. Our method is based on the critical points at infinity theory of [6].

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Contents

1. Introduction

In this article we are interested in positive solutions of a fractional-Nirenberg equation on a regular bounded domain of $\mathbb{R}^n, n \geq 2$. More precisely, we are looking for a function $u\colon \Omega \to \mathbb{R}$ satisfying the following fractional nonlinear equation:

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$$
(1) \qquad \begin{cases} A_s u = K u^{\frac{n+2s}{n-2s}}, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}
$$

where $K\colon \overline{\Omega} \to \mathbb{R}$ is a given function and with $A_s, s \in (0,1)$ is the fractional Laplace operator defined by using the spectrum of the Laplace operator $(-\Delta)$ in Ω with zero Dirichlet boundary condition.

Recently, nonlinear equations involving A_s have attracted a great attention of several researchers since they appear in concrete applications in various scientific fields, see for example [21] and the references therein. The nonlocal property of A_s produces mathematical difficulties and many challenging to handle. Many traditional methods on local operators are not inherited or are only partially satisfied by the fractional order operators. To unlock this difficulty, Caffarelli-Silvestre [12], introduced the so-called "extension method" which makes a non local problem into local one in one more dimension. In this way, the classical approaches for local differential operators can be applied to the extended problems. This extension method has been applied successfully by several researchers to obtain interesting results on problems involving A_s. See [2], [3], [11], [17], [18], and [29].

Equation (1) has a variational structure with specific analytic difficulties. Indeed, due to the presence of the critical exponent, the assocaited variational problem fails to satisfy the Palais–Smale condition. This prevents to apply the standard variational methods to solve (1). Although some results were established on (1), see for example [29] and [3], the question of finding conditions on the domain Ω and the function K to obtain existence or non existence results remained open.

In the present article we prove a perturbation result for the problem (1) under the so-called "β-flatness" condition. Such a result was established in [14] and [4] for the prescribed fractional Q-curvature problem on the standard sphere $\mathbb{S}^n$, $n \geq 2$, under the hypothesis that the flatness order β of the prescribed function K lies in $(1, n)$. The method of [14] hinges on the perturbed method of Ambrosetti, Garcia Azorero, and Peral [5], and the method of [4] relies on the theory of critical points at infinity of A. Bahri [6]. The main advantage of this paper is the extension of the result of [14] and [4] for functions K satisfying the "β-flatness" condition with flatness order varies in the entire interval $(1, \infty)$.

Throughout this paper, we assume the following.

$(f)_\beta$ K is of class C^1 on $\overline{\Omega}$ and around each of its critical point y, K is expanded as follow:

$$
K(x) = K(y) + \sum_{k=1}^{n} b_k |(x-y)_k|^\beta + R(x-y),
$$

where $\beta = \beta(y)$ is the flatness order of K at y, $b_k = b_k(y) \in \mathbb{R} \setminus \{0\}$, for all $k = 1 \dots, n$, with

$$\begin{cases} \displaystyle\sum_{k=1}^{n} b_k(y) \neq 0, & \text{if } \beta(y) < n - 2s, \\ \displaystyle\sum_{k=1}^{n} b_k(y) - c_0 K(y) H(y, y) \neq 0, & \text{if } \beta(y) = n - 2s, \end{cases}$$

and

$$\sum_{j=0}^{[\beta]} |\nabla^j R(Z)||Z|^{j-\beta} = o(1), \quad \text{as } |Z| \text{ small.}$$

Here

$$c_0 = \int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^2)^{\frac{n+2s}{2}}} \left(\int_{\mathbb{R}^n} |x_1|^{n-2s} \frac{|x|^2 - 1}{(1 + |x|^2)^{n+1}} dx \right)^{-1}$$

and $H(.,.)$ is the regular part of the Green function associated to A_s, see [3], Section 3.

(b) For all $x \in \partial\Omega$,

$$\frac{\partial K}{\partial N}(x) < 0,$$

where N denotes the unit outward normal vector on $\partial\Omega$.

Let us set

$$\mathcal{C}^+_{<n-2s} = \Big\{ y \in \Omega \colon \nabla K(y) = 0, \beta(y) < n - 2s \text{ and } -\sum_{k=1}^{n} b_k(y) > 0 \Big\},$$

$$\mathcal{C}^+_{n-2s} = \Big\{ y \in \Omega \colon \nabla K(y) = 0, \beta(y) = n - 2s \text{ and } \\ -\sum_{k=1}^{n} b_k(y) + c_0 K(y) H(y, y) > 0 \Big\},$$

and

$$\mathcal{C}^+_{>n-2s} = \{ y \in \Omega \colon \nabla K(y) = 0 \}.$$

In the following, we denote $\chi(\Omega)$ the Euler–Poincaré characteristic of Ω and we denote $\mathcal{C}^+ := \mathcal{C}^+_{<n-2s} \cup \mathcal{C}^+_{n-2s} \cup \mathcal{C}^+_{>n-2s}$. We shall prove the following perturbation theorem.

THEOREM 1.1. *Let* $K : \overline{\Omega} \to \mathbb{R}$ *be a positive function satisfying* (b) *and* $(f)_\beta$ *with* $\beta \in (1, \infty)$. *If*

$$\sum_{y \in \mathcal{C}^+} (-1)^{n - \tilde{\imath}(y)} \neq \chi(\Omega),$$

then (1) *has s solution provided* $\|K - 1\|_{L^\infty(\Omega)}$ *is small.*

Here $\tilde{\imath}(y) = \sharp\{b_k(y), 1 \leq k \leq n,\ s.\ t,\ b_k(y) < 0\}$.

In view of the result of Theorem 1.1, a natural question arises: what is the situation where the above sum is equal to $\chi(\Omega)$ and a partial one (on some critical points of $\mathcal{C}^+$) is not equal to $\chi(\Omega)$? under which assumptions can one use such an information to obtain existence theorems? In the following we give an answer to these questions.

THEOREM 1.2. *Assume that* Ω *is a contractible bounded domain and* K *is a positive function satisfying* (b) *and* $(f)_\beta$ *with* $\beta \in (1, \infty)$. *If there exist an integer* ℓ *such that*

(i) $$n - \tilde{\imath}(y) \neq \ell \quad \textit{for all } y \in \mathcal{C}^+,$$

asd

(ii) $$\sum_{\substack{y \in \mathcal{C}^+ \\ n - \tilde{\imath}(y) \leq \ell - 1}} (-1)^{n - \mathrm{ind}(K, y)} \neq 1,$$

then (1) *has a solution provided* $\|K - 1\|_{L^\infty(\Omega)}$ *is small.*

We point out that the main contribution of Theorem 1.2 addresses the case where the sum in Theorem 1.1 (on all critical points of K in $\mathcal{C}^+$) equals $\chi(\Omega)$ but a partial one is different from $\chi(\Omega)$ (1 if Ω is contractible). The main issue being the possibility of using such information to prove existence of solution.

Concerning some situations where the result of Theorem 1.2 gives solutions to (1) without using the hypothesis of Theorem 1.1, we consider for example the case of Morse functions K having two maximum values at y_0 and y_1 in Ω. Therefore, $\tilde{\imath}(y_0) = \tilde{\imath}(y_1) = n$, since for $i = 0, 1$, we have $b_k(y_i) < 0$, for all $k = 1, \ldots, n$. If all the other critical points y of K are of index $\tilde{\imath}(y) \leq n - 2$, then the conditions (i) and (ii) of Theorem 1.2 are satisfied for $\ell = 1$. Observe that we can obtain solutions of (1) in this situation independently of the value of the sum in Theorem 1.1 on all elements of $\mathcal{C}^+$.

Our proof relies on the theory of critical points at infinity of Bahri [6]. Throughout a careful analysis of the loss of compactness, we identify the critical points at infinity of the associated variational structure. We then use topological arguments to prove our results. To prove Theorem 1.1, we deform the level sets of the Euler–Lagrange functional J by using the flow lines of a suitable pseudo-gradient (see Proposition 3.6). If we assume that (1) has no solution, the topological difference between these level sets will be given by the topological contribution of all critical points at infinity in $\mathcal{C}^+$. Therefore, by Lemma 4.1, this topology equals the one of Ω. This achieves a contradiction. In the proof of Theorem 1.2 we apply the deformation Lemma on a contraction $\Theta(Y_\ell^\infty)$ of Y_ℓ^∞; a set defined by using some critical points in $\mathcal{C}^+$. In order to avoid the critical points at infinity of p-masses, $p \geq 2$, we need to perform such a contraction below a certain level of J. For this, we need to assume that the domain Ω is contractible. Therefore, the question related to extend the result of Theorem 1.2 to any bounded domain remains open.

We point out that the result of Theorem 1.1 may be seen as the extension of the result of [26] on the classical local Yamabe-type problem (the case of $s = 1$) to the fractional setting.

Before concluding this section, we would like to mention that the fractional Nirenberg problem was first studied when the fractional Laplacian is defined as in a different way; for all $x \in \mathbb{R}^n$

$$(-\Delta)^s u(x) = \frac{1}{2} c(n,s) \int\limits_{\mathbb{R}^n} \frac{2u(x) - u(x+z) - u(x-z)}{|z|^{n+2s}} dz,$$

where

$$c(n,s) = \Big(\int\limits_{\mathbb{R}^n} \frac{1 - \cos(x_1)}{|x|^{n+2s}} dx \Big)^{-1}.$$

A detailed list of similarities and differences between operators A_s and $(-\Delta)^s$ is available in Section 2.3 of the lecture notes [1]. See also [22], [24], and [25].

In the next section, we recall the general variational framework involving the local equivalent equation of (1). In Section 3, we state some useful estimates to identify the critical points at infinity of one mass and in Section 4 we prove theorems 1.1 and 1.2.

2. Some preliminaries

In this section we recall some preliminary results. First we state the local equivalent problem to (1). Let $C = \Omega \times [0, \infty)$ be the half cylinder with base Ω and

$$C_{0L}^\infty(C) := \{v \in C^\infty(\overline{C}), s.t.\ \ v = 0 \text{ on } \partial_L C\},$$

where $\partial_L C := \partial\Omega \times [0,\infty)$. Following [12] and [11], we know that for any u in the fractional Sobolev space $H_0^s(\Omega)$, there exists a unique s-harmonic function denoted $s-h(u)$ in the Sobolev space $H_{0L}^s(C)$ defined by the closure of $C_{0L}^\infty(C)$ with respect to the norm

$$|v|^2 = \int_C t^{1-2s}|\nabla v|^2 dxdt,$$

satisfying the equation

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v) = 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C := \partial\Omega \times [0,\infty), \\ v = u & \text{on } \Omega \times \{0\}, \end{cases}$$

It follows that A_s is expressed as

$$u \in H_0^s(\Omega) \longmapsto A_s(u) = \partial_\nu^s(s-h(u))/_{\Omega\times\{0\}},$$

where ν denotes the unit outward normal vector to C on $\Omega \times \{0\}$ and

$$\partial_\nu^s(s-h(u))(x,0) = -c_s \lim_{t\to 0^+} t\frac{\partial(s-h(u))}{\partial t}(x,t).$$

Here

$$c_s := \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}.$$

Therefore under appropriate regularity assumptions, a map $u\colon \Omega \to \mathbb{R}$ solves (1) if and only if the s-harmonic extension of u; $s-h(u)$ solves

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v) = 0 & \text{in } C, \\ v > 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C, \\ \partial_\nu^s(v) = K(x)v^{\frac{n+2s}{n-2s}} & \text{on } \Omega \times \{0\}. \end{cases} \tag{2}$$

For further details, we refer to [10], [13], [27], and [30].

Equation (2) has a variational nature. The functional is

$$J(u) = \frac{|u|^2}{\left(\int\limits_{\Omega} K(x)u(x,0)^{\frac{2n}{n-2s}}dx\right)^{\frac{n-2s}{n}}}, \quad u \in \mathcal{H} \setminus \{0\}.$$

Here

$$\mathcal{H} = \{u \in H^s_{0L}(C) \colon \operatorname{div}(t^{1-2s}\nabla u) = 0 \text{ in } C\}$$

and

$$|u|^2 = c_s^{-1} \int\limits_{\Omega\times\{0\}} \partial^s_\nu u(x,0)u(x,0)dx = \int\limits_{C} t^{1-2s}|\nabla u|^2 dxdt.$$

The solution of (2) can be constructed as a critical point of J in

$$\Sigma^+ = \{u \in \mathcal{H}, u \geq 0, |u| = c_s^{-1/2}\},$$

see [3]. The exponent $\frac{2n}{n-2s}$ is critical in the sense that the Sobolev trace embedding $\mathcal{H} \hookrightarrow L^{\frac{2n}{n-2s}}(\Omega)$ is just continuous and not compact. Therefore the functional J fails to satisfies the Palais–Smale condition. The failure of the Palais–Smale condition can be characterized along the ideas introduced in [28], [6], and [7] as follows. For $a \in \Omega$, $\lambda > 0$ and a suitable choice of a positive constant $\gamma = \gamma(s,n)$, the function

$$\delta_{(a,\lambda)}(x) = \gamma\left(\frac{\lambda}{1+\lambda^2|x-a|^2}\right)^{\frac{n-2s}{2}}, \quad x \in \mathbb{R}^n$$

solves the equation

$$A_s(u) = u^{\frac{n+2s}{n-2s}}, \quad u > 0 \text{ in } \mathbb{R}^n \text{ and } \lim_{|x|\to\infty} u(x) = 0.$$

See [15], [19], and [20]. Let $\tilde{\delta}_{(a,\lambda)}$ be the s-harmonic extension of $\delta_{(a,\lambda)}$. It satisfies

$$(3) \qquad \begin{cases} \operatorname{div}(t^{1-2s}\nabla\tilde{\delta}_{(a,\lambda)}) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial^s_\nu \tilde{\delta}_{(a,y)} = \delta_{(a,y)}^{\frac{n+2s}{n-2s}} & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Let $P\tilde{\delta}_{(a,\lambda)}$ be the unique solution in $H^s_{0L}(C)$ of

$$(4) \qquad \begin{cases} \operatorname{div}(t^{1-2s}\nabla v) = 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C, \\ \partial^s_\nu v = \delta_{(a,y)}^{\frac{n+2s}{n-2s}} & \text{on } \Omega \times \{0\}. \end{cases}$$

For $\varepsilon > 0$ and $p \in \mathbb{N}^*$ we set

$$V(p,\varepsilon) := \Big\{u \in \Sigma^+\colon \text{there exist } a_1,\dots,a_p \in \Omega,\ \lambda_1,\dots,\lambda_p > \varepsilon^{-1},\ \alpha_1,\dots,\alpha_p > 0, \text{ such that } \Big|u - \sum_{i=1}^{p} \alpha_i P\tilde{\delta}_{(a_i,\lambda_i)}\Big| < \varepsilon \text{ with } \lambda_i d(a_i,\partial\Omega) > \varepsilon^{-1},\ |\alpha_i^{\frac{4s}{n-2s}} K(a_i) J(u)^{\frac{n}{n-2s}} - 1| < \varepsilon \text{ for all } i \neq j\Big\}.$$

Here

$$\varepsilon_{ij} := \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\Big)^{-\frac{n-2s}{2}}.$$

Proposition 2.1. *Assume that J has no critical point in Σ^+. Let $(u_k)_k$ be a sequence in Σ^+ such that J is bounded and $\partial J(u_k)$ tends to zero, then there exist $p \in \mathbb{N}^*$, a positive sequence $(\varepsilon_k)_k \to 0$ and an extracted subsequence $(u_{k_r})_r$ of $(u_k)_k$ such that $u_{k_r} \in V(p,\varepsilon_{k_r})$, for all $r \in \mathbb{N}$.*

For $p \in \mathbb{N}^*$, $\varepsilon > 0$ small enough and $u \in V(p,\varepsilon)$, we introduce the minimization problem

$$\min\Big\{\Big|u - \sum_{i=1}^{p}\alpha_i P\tilde{\delta}_{(a_i,\lambda_i)}\Big|, a_i \in \Omega, \lambda_i > 0, \alpha_i > 0 \text{ for all } i = 1,\dots,p\Big\}.$$

Proposition 2.2. *For $p \in \mathbb{N}^*$, $\varepsilon > 0$ small enough, the above minimization problem has a unique solution $(\bar{\alpha},\bar{\lambda},\bar{a})$. Setting*

$$v = u - \sum_{i=1}^{p}\bar{\alpha}_i P\tilde{\delta}_{(\bar{a}_i,\bar{\lambda}_i)},$$

then v satisfies

$$(V_0) \qquad \langle v,\varphi\rangle = 0 \quad \textit{for } \varphi \in \Big\{P\tilde{\delta}_{(\bar{a}_i,\bar{\lambda}_i)}, \frac{\partial P\tilde{\delta}_{(\bar{a}_i,\bar{\lambda}_i)}}{\partial\lambda_i}, \frac{\partial P\tilde{\delta}_{(\bar{a}_i,\bar{\lambda}_i)}}{\partial a_i}, i = 1,\dots,p\Big\}.$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathcal{H}$ associated to the norm $|\cdot|$ defined by

$$\langle u,v\rangle = c_s^{-1}\int_{\Omega\times\{0\}} \partial_\nu^s u(x,0)v(x,0)dx.$$

The proof of Proposition 2.2 follows the similar argument of ([7], Appendix A).

We state now the definition of a critical point at infinity.

Definition 2.3. ([6]) A critical point at infinity of J in Σ^+ is a limit of a flow line $u(t)$ of the equation $\dot{u}(t) = -\partial J(u(t))$ which remains in $V(p, \varepsilon(t))$, for all $t \geq t_0$. Here t_0 is a time depending on the initial condition $u(0)$ and $\varepsilon(t)$ is a positive function tending to zero. Using the parametrization of Propositions 2.2, $u(t)$ can be written as

$$u(t) = \sum_{i=1}^{p} \alpha_i(t) P\tilde{\delta}_{a_i(t),\, \lambda_i(t)} + v(t),$$

Letting

$$\alpha_i = \lim_{t\to+\infty} \alpha_i(t) \quad \text{and} \quad y_i = \lim_{t\to+\infty} a_i(t),$$

then

$$\sum_{i=1}^{p} \alpha_i P\tilde{\delta}_{y_i,\infty} \quad \text{or} \quad (y_1, \dots, y_p)_\infty$$

denotes a such critical point at infinity.

Notice that our used topological arguments to prove Theorems 1.1 and 1.2 avoid the critical points at infinity of two masses and more. Thus, our next construction and analysis will be performed only in $V(1, \varepsilon)$; a small neighborhood of the critical points at infinity of J of one mass.

3. Useful estimates—critical points at infinity

We start this section by expanding the functional J and its gradient in the potential set $V(1, \varepsilon)$.

Proposition 3.1. *For $u = \alpha P\tilde{\delta}_{(a,\lambda)} + v \in V(1, \varepsilon)$,*

$$\begin{aligned} J(u) = \frac{\widetilde{S}^{2s/n}}{K(a)^{\frac{n-2s}{n}}} \Big[1 &+ \frac{c_2}{\widetilde{S}} \frac{H(a,a)}{\lambda^{n-2s}} - \frac{2}{\alpha K(a) \widetilde{S}} \int_\Omega K P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\, dx \\ &+ \frac{1}{\alpha^2 \widetilde{S}} \Big(|v|^2 - \frac{n+2s}{n-2s} \frac{1}{K(a)} \int_\Omega K P\tilde{\delta}_{(a,\lambda)}^{\frac{4s}{n-2s}} v^2 dx \Big) \Big] \\ &+ O\Big(\frac{1}{\lambda}\Big) + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big) + o(|v|^2), \end{aligned}$$

where

$$\widetilde{S}=\int\limits_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^n}\quad and\quad c_2=\int\limits_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{\frac{n+2s}{2}}}.$$

In the next we denote

$$f(v)=\frac{2}{\alpha K(a)\widetilde{S}}\int\limits_{\Omega}KP\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}v\,dx,$$

$$Q(v,v)=|v|^2-\frac{2(n+2s)}{(n-2s)}\frac{1}{K(a)}\int\limits_{\Omega}KP\tilde{\delta}_{(a,\lambda)}^{\frac{4s}{n-2s}}v^2dx.$$

Proof of Proposition 3.1. For $u=\alpha P\tilde{\delta}_{(a,\lambda)}+v\in V(1,\varepsilon)$,

$$J(u)=\frac{|u|^2}{\Big(\int\limits_{\Omega}Ku^{\frac{2n}{n-2s}}\Big)^{\frac{n-2s}{n}}}=\frac{N}{D^{\frac{n-2s}{n}}}.$$

Using the fact that v satisfies (V_0), we have that

$$N(u)=\alpha^2|P\tilde{\delta}_{(a,\lambda)}|^2+|v|^2. \tag{5}$$

We claim

$$|P\tilde{\delta}_{(a,\lambda)}|^2=\widetilde{S}-c_2\frac{H(a,a)}{\lambda^{n-2s}}+o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big). \tag{6}$$

Indeed,

$$\begin{aligned}|P\tilde{\delta}_{(a,\lambda)}|^2&=\int\limits_{\Omega}\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}P\tilde{\delta}_{(a,\lambda)}dx\\&=\int\limits_{\Omega}\tilde{\delta}_{(a,\lambda)}^{\frac{2n}{n-2s}}dx+\int\limits_{\Omega}\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}(P\tilde{\delta}_{(a,\lambda)}-\tilde{\delta}_{(a,\lambda)})dx.\end{aligned}$$

Using the fact that

$$P\tilde{\delta}_{(a,\lambda)}-\tilde{\delta}_{(a,\lambda)}=-\tilde{c}\frac{H(.,a)}{\lambda^{\frac{n-2s}{2}}}+O\Big(\frac{1}{\lambda^{\frac{n+2s}{2}}d^{n-2s+2}}\Big),$$

see [3], we get for $\eta > 0$ small enough

$$|P\tilde{\delta}_{(a,\lambda)}|^2 = \int\limits_{\Omega} \tilde{\delta}_{(a,\lambda)}^{\frac{2n}{n-2s}} dx - \tilde{c}\frac{H(a,a)}{\lambda^{\frac{n-2s}{2}}} \int\limits_{B(a,\eta)} \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} dx + O\Big(\frac{1}{\lambda^{\frac{n-2s}{2}}} \int\limits_{\mathbb{R}^n} |x-a|\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} dx\Big) + O\Big(\frac{1}{\lambda^n d^{n-2s}}\Big) + O\Big(\frac{1}{\lambda^{\frac{n+2s}{2}} d^{n+2s-2}} \int\limits_{\mathbb{R}^n} \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} dx\Big).$$

A direct computation shows that

$$\int\limits_{\Omega} \tilde{\delta}_{(a,\lambda)}^{\frac{2n}{n-2s}} dx = \int\limits_{\mathbb{R}^n} \tilde{\delta}_{(a,\lambda)}^{\frac{2n}{n-2s}} dx + O\Big(\frac{1}{(\lambda d)^n}\Big) = \tilde{S} + O\Big(\frac{1}{(\lambda d)^n}\Big),$$

$$\int\limits_{B(a,\eta)} \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} dx = \int\limits_{\mathbb{R}^n} \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} dx + O\Big(\frac{1}{\lambda^{\frac{n+2s}{2}}}\Big) = \frac{c_2}{\lambda^{\frac{n-2s}{2}}} + O\Big(\frac{1}{\lambda^{\frac{n+2s}{2}}}\Big),$$

$$\int\limits_{\mathbb{R}^n} |x-a|\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} dx = O\Big(\frac{1}{\lambda^{\frac{n-2s+2}{2}}}\Big).$$

Hence our claim follows. Therefore

$$N(u) = \alpha^2\tilde{S}\Big[1 - \frac{c_2}{\tilde{S}}\frac{H(a,a)}{\lambda^{n-2s}} + \frac{1}{\alpha^2\tilde{S}}|v|^2 + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big)\Big].$$

Now we compute $D^{\frac{n-2s}{n}}$. Observe that

$$D = \int\limits_{\Omega} K(\alpha P\tilde{\delta}_{(a,\lambda)})^{\frac{2n}{n-2s}} dx + \frac{2n}{n-2s}\int\limits_{\Omega} K(\alpha P\tilde{\delta}_{(a,\lambda)})^{\frac{n+2s}{n-2s}} v\, dx + \frac{n(n+2s)}{(n-2s)^2}\int\limits_{\Omega} K(\alpha P\tilde{\delta}_{(a,\lambda)})^{\frac{4s}{n-2s}} v^2 dx + O(|v|^{\min(3,\frac{2n}{n-2s})}).$$

Using the same computation of the proof of 6, we have

$$\begin{aligned}
&\int_{\Omega} K(\alpha P\tilde{\delta}_{(a,\lambda)})^{\frac{2n}{n-2s}}dx \\
&= \alpha^{\frac{2n}{n-2s}}\Big[\int_{\Omega} K(x)\tilde{\delta}_{(a,\lambda)}^{\frac{2n}{n-2s}}dx - \frac{2n}{n-2s}c_2\frac{K(a)H(a,a)}{\lambda^{n-2s}} + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big)\Big] \\
&= \alpha^{\frac{2n}{n-2s}}\Big[K(a)\widetilde{S} + O\Big(\frac{1}{\lambda}\Big) - \frac{2n}{n-2s}c_2\frac{K(a)H(a,a)}{\lambda^{n-2s}} + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big)\Big] \\
&= \alpha^{\frac{2n}{n-2s}}K(a)\widetilde{S}\Big[1 - \frac{2n}{n-2s}c_2\frac{H(a,a)}{\widetilde{S}\lambda^{n-2s}} + O\Big(\frac{1}{\lambda}\Big) + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big)\Big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
D = \alpha^{\frac{2n}{n-2s}}K(a)\widetilde{S}\Big[1 &- \frac{2n}{n-2s}c_2\frac{H(a,a)}{\widetilde{S}\lambda^{n-2s}} + O\Big(\frac{1}{\lambda}\Big) + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big) \\
&+ \frac{2n}{n-2s}\frac{1}{\alpha K(a)\widetilde{S}}\int_{\Omega} KP\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}v\,dx \\
&+ \frac{n(n+2s)}{(n-2s)^2}\frac{1}{\alpha^2 K(a)\widetilde{S}}\int_{\Omega} K(P\tilde{\delta}_{(a,\lambda)})^{\frac{4s}{n-2s}}v^2dx \\
&+ O(|v|^{\min(3,\frac{2n}{n-2s})})\Big].
\end{aligned}$$

It follows that

$$\begin{aligned}
D^{\frac{n-2s}{n}} = \alpha^2 K(a)^{\frac{n-2s}{n}}\widetilde{S}^{\frac{n-2s}{n}}\Big[1 &- 2c_2\frac{H(a,a)}{\widetilde{S}\lambda^{n-2s}} + \frac{2}{\alpha K(a)\widetilde{S}}\int_{\Omega} KP\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}v\,dx \\
&+ \frac{n+2s}{n-2s}\frac{1}{\alpha^2 K(a)\widetilde{S}}\int_{\Omega} K(P\tilde{\delta}_{(a,\lambda)})^{\frac{4s}{n-2s}}v^2dx \\
&+ O\Big(\frac{1}{\lambda}\Big) + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big) \\
&+ O(|v|^{\min(3,\frac{2n}{n-2s})})\Big].
\end{aligned}$$

This finishes the proof of Proposition 3.1. □

Next, we expand the gradient of J at $\lambda\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}$ and $\frac{1}{\lambda}\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k}$ where a_k is the k^{th} component of a.

PROPOSITION 3.2. *For any $u = \alpha P\tilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$, we have the following three expansions*

(i) $$\Big\langle \partial J(u), \alpha\lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle = -(n-2s)c_2\alpha^2 J(u)\frac{H(a,a)}{\lambda^{n-2s}} + o\Big(\frac{1}{\lambda}\Big) + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big).$$

Furthermore, if a is close to a critical point y of K, then

(ii) $$\begin{aligned}\Big\langle \partial J(u), \alpha\lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle = &-(n-2s)c_2\alpha^2 J(u)\mathfrak{S}\\ &+ O(|a-y|^{\beta}) + o\Big(\frac{1}{\lambda^{\beta(y)}}\Big) + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big),\end{aligned}$$

where

$$\mathfrak{S} := \begin{cases} -\dfrac{c_1}{K(a)}\dfrac{\sum_{k=1}^{n} b_k(y)}{\lambda^{\beta(y)}} & \textit{if } \beta(y) < n-2s,\\[2ex] c_2\dfrac{H(y,y)}{\lambda^{n-2s}} - \dfrac{c_1}{K(a)}\dfrac{\sum_{k=1}^{n} b_k(y)}{\lambda^{\beta(y)}} & \textit{if } \beta(y) = n-2s,\\[2ex] c_2\dfrac{H(y,y)}{\lambda^{n-2s}} & \textit{if } \beta(y) > n-2s,\end{cases}$$

and

(iii) $$\begin{aligned}\Big\langle \partial J(u), \alpha\lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle = &-(n-2s)c_2\alpha^2 J(u)\frac{H(y,y)}{\lambda^{n-2s}}\\ &+ O\Big(\sum_{i=2}^{[\min(n,\beta)]}\frac{|a-y|^{\beta-i}}{\lambda^{i}}\Big)\\ &+ O\Big(\frac{1}{\lambda^{\min(n,\beta)}}\Big) + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big).\end{aligned}$$

Here

$$c_1 = \int_{\mathbb{R}^n} |x_1|^{\beta}\frac{|x|^2-1}{(1+|x|^2)^{n+1}} \quad \textit{and} \quad c_2 = \int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{\frac{n+2s}{2}}}.$$

PROOF. Let $u = \alpha P\tilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$. We have

$$\begin{aligned}
&\Big\langle \partial J(u), \alpha\lambda \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle \\
&\quad = 2J(u)\Big[\alpha^2\Big\langle P\tilde{\delta}_{(a,\lambda)}, \lambda\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle \\
&\qquad\qquad - \alpha^{\frac{2n}{n-2s}} J(u)^{\frac{n}{n-2s}} \int_\Omega K(x) P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \lambda \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda} dx\Big].
\end{aligned}$$

Using the fact that $J(u)^{\frac{n}{n-2s}}\alpha^{\frac{4s}{n-2s}}K(a) = 1 + o(1)$, we get

$$\begin{aligned}
&\Big\langle \partial J(u), \alpha\lambda \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle \\
&\quad = 2J(u)\alpha^2\Big[\Big\langle P\tilde{\delta}_{(a,\lambda)}, \lambda\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle \\
&\qquad\qquad - \frac{1}{K(a)} \int_\Omega K(x) P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \lambda \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda} dx\Big] \\
&\qquad + o\Big(\Big|\int_\Omega K(x) P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \lambda \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda} dx\Big|\Big).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\Big\langle P\tilde{\delta}_{(a,\lambda)}, \lambda\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle \\
&\quad = \int_\Omega \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \lambda \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda} dx \\
&\quad = \int_\Omega \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \lambda \frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial\lambda} dx + \int_\Omega \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \Big(\lambda \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda} - \lambda\frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big) dx.
\end{aligned}$$

Using the fact that

$$\int_\Omega \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial\lambda} dx = -\int_{\mathbb{R}^n\setminus\Omega} \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial\lambda} dx = O\Big(\frac{1}{(\lambda d)^n}\Big),$$

and

$$\lambda\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda} - \lambda\frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial\lambda} = \frac{n-2s}{2}\tilde{c}\frac{H(.,a)}{\lambda^{\frac{n-2s}{2}}} + O\Big(\frac{1}{\lambda^{\frac{n+2s}{2}} d^{n-2s+2}}\Big),$$

(see [3]), we get by expanding $H(\cdot, a)$ around a:

$$\Big\langle P\tilde{\delta}_{(a,\lambda)}, \lambda\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle = \frac{n-2s}{2}c_2\frac{H(a,a)}{\lambda^{n-2s}} + o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big);$$

$$\begin{aligned}\int_\Omega K(x)P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}\lambda\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}dx = &\int_\Omega K(x)\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}\lambda\frac{\partial\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}dx\\ &+ (n-2s)c_2K(a)\frac{H(a,a)}{\lambda^{n-2s}}\\ &+ o\Big(\frac{1}{(\lambda d)^{n-2s}}\Big).\end{aligned}$$

It remains only to compute

$$I := \int_\Omega K(x)\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}\lambda\frac{\partial\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}dx.$$

Since K is of class C^1, we expand K in a small neighborhood $B(a,\eta)$. We obtain

$$\begin{aligned}I = K(a)\int_{B(a,\eta)}\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}\lambda\frac{\partial\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}dx + \int_{B(a,\eta)} Dk(a)(x-a)\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}\lambda\frac{\partial\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}dx\\ + o\Big(\int_{\mathbb{R}^n}|x-a|\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}\lambda\frac{\partial\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}dx\Big) + O\Big(\frac{1}{(\lambda d)^n}\Big).\end{aligned}$$

A direct computation shows that

$$\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}\lambda\frac{\partial\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}(x) = \frac{n-2s}{2}\lambda^n\frac{1-\lambda^2|x-a|^2}{(1+\lambda^2|x-a|^2)^{n+1}}.$$

Therefore by an argument of symmetry, we have

$$\int_{B(a,\eta)} DK(x)(x-a)\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}\lambda\frac{\partial\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}dx = 0, \tag{7}$$

$$\int_{\mathbb{R}^n}\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}}\lambda\frac{\partial\tilde{\delta}_{(a,\lambda)}}{\partial\lambda}dx = 0. \tag{8}$$

Hence

$$I = o\Big(\frac{1}{\lambda}\Big) + O\Big(\frac{1}{(\lambda d)^n}\Big).$$

The proof of the estimate (i) follows.

If a is near a critical point y of K, we have under $(f)_\beta$-condition,

$$I = \int_{B(a,\eta)} \Big(K(y) + \sum_{k=1}^{n} b_k |(x-y)_k|^\beta\Big) \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \lambda \frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial \lambda} dx$$

$$+ o\Big(\int_{B(a,\eta)} |x-y|^\beta \tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \lambda \frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial \lambda} dx\Big) + O\Big(\frac{1}{(\lambda d)^n}\Big).$$

Using (8) and a change of variables $z = \lambda(x-a)$, we obtain

$$\begin{aligned} I &= \frac{n-2s}{2} \sum_{k=1}^{n} \frac{b_k}{\lambda^\beta} \int_{B(0,\lambda\eta)} |z_k + \lambda(a-y)_k|^\beta \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz \\ &\quad + o\Big(\frac{1}{\lambda^\beta} \int_{B(0,\lambda\eta)} |z|^\beta \frac{|1-|z|^2|}{(1+|z|^2)^{n+1}} dz\Big) + o(|a-y|^\beta) + O\Big(\frac{1}{(\lambda d)^n}\Big) \\ &= \frac{n-2s}{2} \sum_{k=1}^{n} \frac{b_k}{\lambda^\beta} \int_{B(0,\lambda\eta)} |z_k|^\beta \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz + O(|a-y|^\beta) \\ &\quad + o\Big(\frac{1}{\lambda^\beta} \int_{B(0,\lambda\eta)} |z|^\beta \frac{|1-|z|^2|}{(1+|z|^2)^{n+1}} dz\Big) + O\Big(\frac{1}{(\lambda d)^n}\Big) \\ &= O(|a-y|^\beta) + O\Big(\frac{1}{(\lambda d)^n}\Big) + \begin{cases} O\Big(\frac{1}{\lambda^n}\Big) & \text{if } \beta > n, \\ O\Big(\frac{\log \lambda}{\lambda^\beta}\Big) & \text{if } \beta = n, \\ -\frac{n-2s}{2} c_1 \sum_{k=1}^{n} \frac{b_k}{\lambda^\beta} & \text{if } \beta < n. \end{cases} \end{aligned}$$

This finishes the proof of the second expansion (ii).

To get the estimate (iii), we expend K around the concentration point a as follows:

$$K(x) = K(a) + \sum_{i=1}^{[\min(n,\beta)]} \frac{D^i K(a)(x-a)^i}{i!} + O(|x-a|^{\min(n,\beta)}).$$

Setting $z = \lambda(x-a)$,

$$I = \int_{B(0,\lambda\eta)} K(a) \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz + \sum_{i=1}^{[\min(n,\beta)]} \int_{B(0,\lambda\eta)} \frac{D^i K(a)(z)^i}{i!\lambda^i} \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz$$

$$+ O\Big(\frac{1}{\lambda^n}\Big) + O\Big(\frac{1}{\lambda^{\min(n,\beta)}}\Big).$$

Using the fact that

$$\int_{B(0,\lambda\eta)} K(a)\frac{1-|z|^2}{(1+|z|^2)^{n+1}}dz = O\Big(\frac{1}{\lambda^n}\Big),$$

$$\int_{B(0,\lambda\eta)} DK(a)(z)\frac{1-|z|^2}{(1+|z|^2)^{n+1}}dz = 0,$$

and under $(f)_\beta$-condition, $D^i K(a) = O(|a-y|^{\beta-i})$, the estimate (iii) follows. The proof of Proposition 3.2 is thereby completed. □

Proposition 3.3. *Let $u = \alpha P\tilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$. We have*

(i)
$$\Big\langle \partial J(u), \frac{\alpha}{\lambda}\frac{\partial P\widetilde{\delta}_{a,\lambda}}{\partial a_k}\Big\rangle = -2\alpha^2 J(u)\Bigg(c_3\frac{\frac{\partial K}{\partial a_k}(a)}{K(a)\lambda} - c_2\frac{\frac{\partial H(a,a)}{\partial a_k}}{\lambda^{n+1-2s}}\Bigg) + o\Big(\frac{1}{\lambda}\Big) + o\Big(\frac{1}{(\lambda d)^{n+1-2s}}\Big),$$

Moreover, if a is close to a critical point y of K, we have

(ii)
$$\begin{aligned}\Big\langle \partial J(u), \frac{\alpha}{\lambda}\frac{\partial P\widetilde{\delta}_{a,\lambda}}{\partial a_k}\Big\rangle = &-2\alpha^2 J(u)c_3\frac{b_k}{K(a)}\beta(\mathrm{sign}(a-y)_k)|(a-y)_k|^{\beta-1}\\ &+ O\Big(\sum_{j=2}^{[\min(n,\beta)]}\frac{|a-y|^{\beta-j}}{\lambda^j}\Big)\\ &+ O\Big(\frac{1}{\lambda^{\min(n,\beta)}}\Big) + O\Big(\frac{1}{\lambda^{n+1-2s}}\Big).\end{aligned}$$

Furthermore, if $\beta(y) < n+1$ and $\lambda|a-y|$ is bounded, we have

(iii)
$$\begin{aligned}&\Big\langle \partial J(u), \frac{\alpha}{\lambda}\frac{\partial P\widetilde{\delta}_{a,\lambda}}{\partial a_k}\Big\rangle\\ &= -\alpha^2 J(u)(n-2s)\frac{b_k}{K(a)}\frac{1}{\lambda^\beta}\int_{\mathbb{R}^n}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz\\ &\quad + o\Big(\frac{1}{\lambda^\beta}\Big) + O\Big(\frac{1}{\lambda^{n+1-2s}}\Big).\end{aligned}$$

Here

$$c_3 = (n-2s)\int_{\mathbb{R}^n}\frac{|z|^2}{(1+|z|^2)^{n+1}}dz.$$

Proof. We argue as in the proof of Proposition 3.2. The proof of Proposition 3.3 follows from the computation of

$$I := \int_{\Omega} K(x)\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \frac{1}{\lambda} \frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial a_k} dx.$$

Since K is of class C^1, we have

$$I = \int_{B(a,\eta)} DK(a)(x-a)\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \frac{1}{\lambda} \frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial a_k} dx + o\Big(\int_{\mathbb{R}^n} |x-a|\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \frac{1}{\lambda} \frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial a_k} dx\Big) + O\Big(\frac{1}{(\lambda d)^n}\Big).$$

By elementary computation, we have

$$\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} \frac{1}{\lambda} \frac{\partial \tilde{\delta}_{(a,\lambda)}}{\partial a_k} dx = (n-2s)\frac{\lambda^{n+1}(x-a)_k}{(1+\lambda^2|x-a|^2)^{n+1}}.$$

Setting $z = \lambda(x-a)$, we have

$$I = \frac{\partial K}{\partial a_k}(a)\frac{1}{\lambda}\int_{\mathbb{R}^n} \frac{|z_k|^2}{(1+|z|^2)^{n+1}} dz + o\Big(\frac{1}{\lambda}\Big) + O\Big(\frac{1}{(\lambda d)^n}\Big).$$

The first estimate of the proposition follows.

Now, if a is close to a critical point y of K, by an expansion of K around a, we have

$$I = (n-2s)\frac{1}{\lambda}\int_{\mathbb{R}^n} \frac{DK(a)(z)z_k}{(1+|z|^2)^{n+1}} dz + O\Big(\sum_{r=2}^{[\min(n,\beta)]} \frac{1}{\lambda^r}\int_{\mathbb{R}^n} \frac{|D^r K(a)||z|^{r+1}}{(1+|z|^2)^{n+1}} dz\Big) + O\Big(\frac{1}{\lambda^{\min(n,\beta)}}\Big).$$

Observe that under $(f)_\beta$-condition, we have

$$|D^r K(a)| = O(|a-y|^{\beta-r}), \quad \text{for all } r = 1, \dots, [\min(n,\beta)],$$

and

$$\frac{\partial K}{\partial a_k}(a) = b_k \beta \text{ sign } ((a-y)_k)|(a-y)_k|^{\beta-1} + o(|a-y|^{\beta-1}).$$

Hence

$$I = b_k \beta c_3 \text{ sign } ((a-y)_k) \frac{|(a-y)_k|^{\beta-1}}{\lambda} + O\Big(\sum_{r=2}^{[\min(n,\beta)]} \frac{|a-y|^{\beta-r}}{\lambda^r}\Big) + O\Big(\frac{1}{\lambda^{\min(n,\beta)}}\Big) + o\Big(\frac{|a-y|^{\beta-1}}{\lambda}\Big).$$

The second estimate follows.

Now if $\beta < n+1$ and $\lambda|a-y|$ is bounded, then

$$I = (n-2s)\frac{b_k}{\lambda^\beta}\int_{\mathbb{R}^n} |(z+\lambda(a-y))_k| \frac{z_k}{(1+|z|^2)^{n+1}} dz + O\Big(\frac{1}{(\lambda d)^n}\Big) + o\Big(\frac{1}{\lambda^\beta}\Big).$$

This completes the proof of Proposition 3.3. □

The following proposition deals with the v-part. It gets rid of the contribution of v with respect to the concentration phenomenon.

Proposition 3.4. *Let $u = \alpha P\tilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$. The following minimization problem*

$$\min\{J(u+v),\ v \in \mathcal{H} \text{ satisfying } (V_0)\}$$

has a unique solution denoted $\bar{v} = \bar{v}(\alpha, a, \lambda)$. In addition, there exists a change of variables $v - \bar{v} \to V$ such that

$$J(\alpha P\tilde{\delta}_{(a,\lambda)} + v) = J(\alpha P\tilde{\delta}_{(a,\lambda)} + \bar{v}) + |V|^2.$$

Moreover,

$$|\bar{v}| \le M\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^\beta} + \frac{1}{\lambda^{n/2}}\Big) + M \begin{cases} \dfrac{1}{(\lambda d)^{\frac{n+2s}{2}}}, & \text{if } n > 6s,\\[2mm] \dfrac{\log(\lambda d)^{\frac{2}{3}}}{(\lambda d)^{\frac{n+2s}{2}}}, & \text{if } n = 6s,\\[2mm] \dfrac{1}{(\lambda d)^{n-2s}}, & \text{if } n < 6s. \end{cases}$$

Proof. Using a similar argument of [6], the quadratic form $Q(v,v)$ defined in Proposition 3.1 is definite and positive. Therefore, the above minimization problem has a unique solution $\bar{v}$ which satisfies $|\bar{v}| \le M\|f\|$. Here f is the linear form defined in Proposition 3.1.

For all $v \in F := \{v \in \mathcal{H}$ satisfying $(V_0)\}$, we have

$$f(v) = \frac{2}{\alpha K(a)\widetilde{S}}\Big(\int\limits_{B(a,d)} K(x) P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\,dx + \int\limits_{\Omega\setminus B(a,d)} K(x) P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\,dx \Big).$$

Observe that by Hölder inequality,

$$\Big| \int\limits_{\Omega\setminus B(a,d)} K(x) P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\,dx \Big| \le M|v| \frac{1}{(\lambda d)^{\frac{n+2s}{2}}}.$$

In order to compute

$$I := \int\limits_{B(a,d)} K(x) P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\,dx,$$

we distinguish two cases

Case 1: $a \in \big(\bigcup_{\nabla K(y)=0} B(y,\rho)\big)^c$. In this case, by an expansion of K at the first order around a, we obtain

$$\begin{aligned} I = {} & K(a) \int\limits_{B(a,d)} P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\,dx + \int\limits_{B(a,d)} \nabla K(a)(x-a) P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\,dx \\ & + o\Big(\int\limits_{\Omega} |x-a| P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} |v|\,dx \Big). \end{aligned}$$

By Holder inequality and elementary computation, we have

$$\Big| \int\limits_{B(a,d)} P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\,dx \Big| \le M|v| \begin{cases} O\Big(\dfrac{1}{(\lambda d)^{n-2s}}\Big) & \text{if } n < 6s, \\ O\Big(\dfrac{\log(\lambda d)^{\frac{2}{3}}}{(\lambda d)^{2n/3}}\Big) & \text{if } n = 6s, \\ O\Big(\dfrac{1}{(\lambda d)^{\frac{n+2s}{2}}}\Big) & \text{if } n > 6s. \end{cases} \tag{9}$$

Therefore,

$$|I| \le M|v| \frac{|\nabla K(a)|}{\lambda} + M|v| \begin{cases} O\Big(\dfrac{1}{(\lambda d)^{n-2s}}\Big) & \text{if } n < 6s, \\ O\Big(\dfrac{\log(\lambda d)^{\frac{2}{3}}}{(\lambda d)^{2n/3}}\Big) & \text{if } n = 6s, \\ O\Big(\dfrac{1}{(\lambda d)^{\frac{n+2s}{2}}}\Big) & \text{if } n > 6s. \end{cases}$$

CASE 2: $a \in B(y,\rho)$ WITH $\nabla K(y) = 0$. In this case, we use the $(f)_\beta$-expansion, we obtain

$$I = K(y) \int_{B(a,\eta)\subset B(y,\rho)} P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\,dx + \sum_{k=1}^{n} b_k \int_{B(a,\eta)} |(x-y)_k|^\beta P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} v\,dx$$
$$+ o\Big(\int_{B(a,\eta)} |x-a|^\beta P\tilde{\delta}_{(a,\lambda)}^{\frac{n+2s}{n-2s}} |v|\,dx\Big).$$

Using (9) and elementary computations, we find

$$|I| \le M|v|\Big(\frac{1}{\lambda^\beta} + \frac{1}{\lambda^{n/2}}\Big) + M|v| \begin{cases} O\Big(\dfrac{1}{(\lambda d)^{n-2s}}\Big) & \text{if } n < 6s, \\ O\Big(\dfrac{\log(\lambda d)^{\frac{2}{3}}}{(\lambda d)^{2n/3}}\Big) & \text{if } n = 6s, \\ O\Big(\dfrac{1}{(\lambda d)^{\frac{n+2s}{2}}}\Big) & \text{if } n > 6s. \end{cases}$$

This completes the proof of Proposition 3.4. □

We end this section by characterizing the critical points at infinity of J in $V(1,\varepsilon)$.

THEOREM 3.5. *Under the assumptions* (b) *and* $(f)_\beta$, $\beta \in (1,\infty)$, *the critical points at infinity of J in $V(1,\varepsilon)$ are*

$$(y)_\infty := \frac{1}{K(y)^{\frac{n-2s}{n}}} P\tilde{\delta}_{(y,\infty)},\ y \in \mathcal{C}^+.$$

The index of $(y)_\infty$ *is* $i(y)_\infty = n - \tilde{i}(y)$.

To prove Theorem 3.5, we introduce the following proposition which describes the loss of compactness and the concentration phenomenon of the problem in $V(1,\varepsilon)$

PROPOSITION 3.6. *Under the assumptions of Theorem 3.5, there exist a bounded pseudo-gradient W of J and a positive constant c such that for any $u = \alpha P\tilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$, we have*

(i) $$\langle \partial J(u), W(u)\rangle \le -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^{n-2s}} + \frac{1}{(\lambda d)^{n+1-2s}}\Big);$$

(ii)
$$\begin{aligned}&\langle \partial J(u+\bar{v}), W(u)+\frac{\partial \bar{v}}{\partial(\alpha,a,\lambda)}(W(u))\rangle\\&\leq -c\Big(\frac{|\nabla K(a)|}{\lambda}+\frac{1}{\lambda^{n-2s}}+\frac{1}{(\lambda d)^{n+1-2s}}\Big);\end{aligned}$$

(iii) *the distance* $d(t)=d(a(t),\partial\Omega)$ *increases if it is small enough.*

(iv) *the only case where* $\lambda(t)$ *increases and goes to* $+\infty$ *is when* $a(t)$ *converges to a critical point in* $\mathcal{C}^+$.

Proof. Under the assumption (b), there exists $d_0>0$ small enough such that for any $a\in\Omega$ with $d(a,\partial\Omega)\leq d_0$, we have

$$\frac{\partial K}{\partial N_a}(a)\leq -c,$$

where N_a is the unit outward normal vector at a of

$$\partial\Omega_a := \{x\in\Omega, d(x,\partial\Omega)=d(a,\partial\Omega)\}.$$

For $u=\alpha P\tilde{\delta}_{(a,\lambda)}\in V(1,\varepsilon)$, we distinguish three cases.

Case 1: $d(a,\partial\Omega)\leq d_0$. In this case, we move the point a inward of the domain Ω with respect to the equation

$$\dot{a}=-\frac{N_a}{\lambda}.$$

The associated vector field is

$$W_1(u)=-\alpha\frac{1}{\lambda}\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a}N_a.$$

Using the first expansion of Proposition 3.3, we have

$$\begin{aligned}&\langle\partial J(u),W_1(u)\rangle\\&=2\frac{\alpha^2}{K(a)}J(u)\Big(c_3\frac{\frac{\partial K}{\partial N_a}(a)}{\lambda}-c_2K(a)\frac{\frac{\partial H(a,a)}{\partial N_a}}{\lambda^{n+1-2s}}\Big)+o\Big(\frac{1}{\lambda}\Big)+o\Big(\frac{1}{(\lambda d)^{n-2s+1}}\Big).\end{aligned}$$

Using the assumption (b) and the fact that

$$\frac{\partial H(a,a)}{\partial N_a}\sim\frac{1}{d(a,\partial\Omega)^{n-2s+1}},$$

see the corresponding statement [23], we get

$$\langle \partial J(u), W_1(u) \rangle \leq -c\Big(\frac{1}{\lambda}+\frac{1}{(\lambda d)^{n-2s+1}}\Big) \leq -c\Big(\frac{|\nabla K(a)|}{\lambda}+\frac{1}{\lambda^{n-2s}}+\frac{1}{(\lambda d)^{n-2s+1}}\Big).$$

Thus W_1 satisfies the conditions (i), (iii), and (iv). Concerning (ii), it follows from (i) and the estimate of $|\bar{v}|$ given in Proposition 3.4.

Let γ_0 be a positive constant such that if $|\nabla K(a)| \leq \gamma_0$ then there exists a critical point y of K such that $a \in B(y,\rho)$ a neighborhood of y where the $(f)_\beta$-expansion is valid.

Case 2: $d(a,\partial\Omega) \geq d_0$ and $|\nabla K(a)| \geq \gamma_0$. In this case, we move the concentration point a according the equation

$$\dot{a} = \frac{1}{\lambda|\nabla K(a)|}\nabla K(a).$$

The corresponding vector field is

$$W_2(u) = \alpha\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a}\dot{a}.$$

It satisfies the first expansion of Proposition 3.3,

$$\begin{aligned}\langle \partial J(u), W_2(u) \rangle &= -2\frac{\alpha^2}{K(a)}J(u)c_3\frac{|\nabla K(a)|}{\lambda} + O\Big(\frac{1}{\lambda^{n-2s+1}}\Big)\\ &\leq -c\frac{\gamma_0}{\lambda}\\ &\leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^{n-2s}} + \frac{1}{(\lambda d)^{n-2s+1}}\Big).\end{aligned}$$

Therefore W_2 satisfies the requirements of Proposition 3.6.

Case 3: $d(a,\partial\Omega) \geq d_0$ and $|\nabla K(a)| \leq \gamma_0$. In this case, there exists y a critical point y of K such that $a \in B(y,\rho)$. The construction of the required pseudo-gradient in this region depends on the value of the β-flatness order $\beta(y)$. We distinguish three subcases.

Subcase 3.1: $\beta = \beta(y) > n-2s$. We increase the concentration λ according the equation $\dot{\lambda} = \lambda$ if a is very close to y or we move each component a_k of the concentration point a according to

$$\dot{a}_k = \frac{b_k \text{ sign } (a-y)_k}{\lambda}.$$

Let $\eta > 0$ small enough and φ be a cut-off function such that $\varphi(t) = 1$ if $|t| \leq \eta$ and $\varphi(t) = 0$ if $|t| \geq 2\eta$. We set

$$W_3^1(u) = \varphi(\lambda^{n-2s-1}|a-y|^{\beta-1})\alpha \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial \lambda}\dot{\lambda} + (1-\varphi)(\lambda^{n-2s-1}|a-y|^{\beta-1}) \sum_{k=1}^{n} \alpha \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k}\dot{a_k}.$$

We claim that

$$\langle \partial J(u), W_3^1(u)\rangle \leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^{n-2s}}\Big). \tag{10}$$

Indeed, if $\lambda^{n-2s-1}|a-y|^{\beta-1} < 2\eta$, we use the third expansion of Proposition 3.2. Observe that for any $i = 2, \dots, [\min(n,\beta)]$, we have

$$\frac{|a-y|^{\beta-i}}{\lambda^i} = o\Big(\frac{1}{\lambda^{n-2s}}\Big) \quad \text{as } \lambda \to +\infty,$$

since

$$\frac{|a-y|^{\beta-i}}{\lambda^i}\lambda^{n-2s} \leq (2\eta)^{\frac{\beta-i}{\beta-1}} \frac{1}{\lambda^{\frac{i-1}{\beta-1}(\beta-(n-2s))}}.$$

Moreover,

$$\frac{1}{\lambda^{\min(n,\beta)}} = o\Big(\frac{1}{\lambda^{n-2s}}\Big) \quad \text{as } \lambda \to +\infty,$$

since $\beta > n-2s$. Using the fact that $H(\cdot,\cdot)$ is positive, we obtain from the third expansion of Proposition 3.2

$$\Big\langle \partial J(u), \alpha \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial \lambda}\dot{\lambda}\Big\rangle \leq \frac{-c}{\lambda^{n-2s}} \leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^{n-2s}}\Big),$$

since $|\nabla K(a)| \sim |a-y|^{\beta-1}$.

Now if $\lambda^{n-2s-1}|a-y|^{\beta-1} \geq \eta$, we use the second expansion of Proposition 3.3. Observe that for any $i = 2, \dots, [\min(n,\beta)]$,

$$\frac{|a-y|^{\beta-i}}{\lambda^i} = o\Big(\frac{|a-y|^{\beta-1}}{\lambda}\Big) \quad \text{as } \lambda \to +\infty,$$

since

$$\frac{|a-y|^{\beta-i}}{\lambda^i}\frac{\lambda}{|a-y|^{\beta-1}} \leq \Big(\frac{1}{\eta}\Big)^{\frac{i-1}{\beta-1}} \frac{1}{\lambda^{(i-1)(1-\frac{n-2s-1}{\beta-1})}}.$$

In addition,

$$\frac{1}{\lambda^{\min(n,\beta)}} = o\Big(\frac{|a-y|^{\beta-1}}{\lambda}\Big) \quad \text{as } \lambda \to +\infty,$$

since

$$\frac{1}{\lambda^{n-2s}} = O\Big(\frac{|a-y|^{\beta-1}}{\lambda}\Big).$$

Therefore, by the second expansion of Proposition 3.3 we get

$$\Big\langle \partial J(u), \sum_{k=1}^{n} \alpha \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k} \dot{a_k} \Big\rangle \leq -c\frac{|a-y|^{\beta-1}}{\lambda} \leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^{n-2s}}\Big).$$

Hence (10) follows and the requirements of Proposition 3.6 hold for W_3^1.

Subcase 3.2: $\beta = \beta(y) = n-2s$. Let

$$\dot{\lambda} = \Big(-\sum_{k=1}^{n} b_k(y) + c_0 K(y) H(y,y)\Big)\lambda$$

and let

$$\begin{aligned} W_3^2(u) = {} & \varphi_1(\lambda|a-y|)\alpha\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial \lambda}\dot{\lambda} \\ & + \varphi_2(\lambda|a-y|)\alpha\sum_{k=1}^{n} b_k \int_{\mathbb{R}^n} \frac{x_k|x_k+\lambda(a-y)_k|^{\beta}}{(1+|x|^2)^{n+1}}dx\frac{1}{\lambda}\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k} \\ & + \varphi_3(\lambda|a-y|)\alpha\sum_{k=1}^{n} b_k \operatorname{sign}\,((a-y)_k)\frac{1}{\lambda}\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k}. \end{aligned}$$

Here φ_1, φ_2 and φ_3 are three cut-off functions such that $\varphi_1(t) = 1$ if $|t| \leq \eta$, $\varphi_1(t) = 0$ if $|t| \geq 2\eta$, $\varphi_2(t) = 1$ if $2\eta \leq t \leq \frac{1}{\eta}$, $\varphi_2(t) = 0$ if $t \in (-\infty,\eta) \cup \big(\frac{2}{\eta},\infty\big)$, $\varphi_3(t) = 1$ if $t \geq \frac{2}{\eta}$ and $\varphi_3(t) = 0$ if $t \leq \frac{1}{\eta}$. We claim that

$$\langle \partial J(u), W_3^2(u)\rangle \leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^{n-2s}}\Big). \tag{11}$$

Indeed, if $\lambda|a-y| < 2\eta$, by the second expansion of Proposition 3.2, then

$$\begin{aligned} & \Big\langle \partial J(u), \alpha\frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial \lambda}\dot{\lambda}\Big\rangle \\ & \quad \leq -(n-2s)\alpha^2 J(u)\frac{c_1}{K(a)}\Big(-\sum_{k=1}^{n} b_k + c_2c_1^{-1}K(a)H(y,y)\Big)^2\frac{1}{\lambda^{n-2s}} \\ & \qquad + O(|a-y|^{n-2s}) + o\Big(\frac{1}{\lambda^{n-2s}}\Big). \end{aligned}$$

Recall that under $(f)_\beta$-condition we have,

$$-\sum_{k=1}^{n} b_k + c_0 K(a) H(y, y) \neq 0,$$

where $c_0 = c_2 c_1^{-1}$. Thus,

$$\left\langle \partial J(u), \alpha \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial \lambda} \dot{\lambda} \right\rangle \leq -\frac{c}{\lambda^{n-2s}} + O(|a-y|^{n-2s}).$$

Using the fact that

$$|a-y|^{n-2s} = o\Big(\frac{1}{\lambda^{n-2s}}\Big), \quad \text{as } \eta \text{ small},$$

we get

$$\left\langle \partial J(u), \alpha \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial \lambda} \dot{\lambda} \right\rangle \leq \frac{-c}{\lambda^{n-2s}} \leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^{n-2s}}\Big).$$

Now if $\eta \leq \lambda|a-y| \leq \frac{2}{\eta}$, we use the third expansion of Proposition 3.3. We have

$$\begin{aligned}
&\left\langle \partial J(u), \alpha \sum_{k=1}^{n} b_k \int_{\mathbb{R}^n} \frac{x_k |x_k + \lambda(a-y)_k|^\beta}{(1+|x|^2)^{n+1}} dx \frac{1}{\lambda} \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k} \right\rangle \\
&\quad \leq -\frac{c}{\lambda^{n-2s}} \Big(\int_{\mathbb{R}^n} |x_{k_a} + \lambda(a-y)_{k_a}|^\beta \frac{x_{k_a}}{(1+|x|^2)^{n+1}} dx \Big)^2 + o\Big(\frac{1}{\lambda^{n-2s}}\Big).
\end{aligned}$$

Here k_a is the index such that $|(a-y)_{k_a}| = \max_{1 \leq k \leq n} |(a-y)_k|$.

Using the fact that $\eta \leq \lambda|a-y|$, we derive

$$\Big(\int_{\mathbb{R}^n} |x_{k_a} + \lambda(a-y)_{k_a}|^\beta \frac{x_{k_a}}{(1+|x|^2)^{n+1}} dx \Big)^2 \geq c_\eta > 0.$$

Hence

$$\begin{aligned}
&\left\langle \partial J(u), \alpha \sum_{k=1}^{n} b_k \int_{\mathbb{R}^n} \frac{x_k |x_k + \lambda(a-y)_k|^\beta}{(1+|x|^2)^{n+1}} dx \frac{1}{\lambda} \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k} \right\rangle \\
&\quad \leq -\frac{c}{\lambda^{n-2s}} \\
&\quad \leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^{n-2s}}\Big).
\end{aligned}$$

If $\lambda|a-y| \geq \frac{1}{\eta}$, by the second expansion of Proposition 3.3. We have

$$\Big\langle \partial J(u), \alpha \sum_{k=1}^{n} b_k \operatorname{sign}((a-y)_k) \frac{1}{\lambda} \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k} \Big\rangle$$
$$\leq -c \sum_{k=1}^{n} b_k^2 \frac{|(a-y)_k|^{\beta-1}}{\lambda} + O\Big(\sum_{i=2}^{[\beta]} \frac{|a-y|^{\beta-i}}{\lambda^i}\Big) + O\Big(\frac{1}{\lambda^\beta}\Big).$$

Using the fact that

$$\frac{|a-y|^{\beta-i}}{\lambda^i} = o\Big(\frac{|a-y|^{\beta-1}}{\lambda}\Big), \quad \text{as } \eta \text{ small},$$
$$\frac{1}{\lambda^\beta} = o\Big(\frac{|a-y|^{\beta-1}}{\lambda}\Big), \quad \text{as } \eta \text{ small},$$

we get

$$\Big\langle \partial J(u), \alpha \sum_{k=1}^{n} b_k \operatorname{sign}((a-y)_k) \frac{1}{\lambda} \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k} \Big\rangle \leq -c \frac{|a-y|^{\beta-1}}{\lambda}$$
$$\leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^\beta}\Big).$$

Hence (11) follows and W_3^2 satisfies the requirements of Proposition 3.6.

Subcase 3.3: $\beta = \beta(y) \in (1, n-2s)$. Let

$$\dot{\lambda} = \Big(-\sum_{k=1}^{n} b_k\Big)\lambda$$

and let

$$W_3^3(u) = \varphi_1(\lambda|a-y|)\alpha \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial \lambda}\dot{\lambda}$$
$$+ \varphi_2(\lambda|a-y|)\alpha \sum_{k=1}^{n} b_k \int_{\mathbb{R}^n} \frac{x_k|x_k + \lambda(a-y)_k|^\beta}{(1+|x|^2)^{n+1}} dx \frac{1}{\lambda} \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k}$$
$$+ \varphi_3(\lambda|a-y|)\alpha \sum_{k=1}^{n} b_k \operatorname{sign}((a-y)_k) \frac{1}{\lambda} \frac{\partial P\tilde{\delta}_{(a,\lambda)}}{\partial a_k}.$$

where φ_1, φ_2, and φ_3 are cut-off functions defined in the second subcase. We have

$$\langle \partial J(u), W_3^3(u) \rangle \leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^\beta}\Big) \tag{12}$$
$$\leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^{n-2s}}\Big).$$

The proof of (12) proceed exactly as the one of the second subcase. Therefore W_3^3 satisfies the conditions of Proposition 3.6.

The required pseudo-gradient W in $V(1,\varepsilon)$ is defined by convex combinations of all above vector fields. This completes the proof of Proposition 3.6. $\square$

Proof of Theorem 3.5. Following the construction of Proposition 3.6, we observe that the only region where the concentrations $\lambda(t)$ increase only the flow lines of W are those where the points $a(t)$ converge to a critical point y in $\mathcal{C}^+$. Thus the critical points at infinity of J in $V(1,\varepsilon)$ are in one to one correspondence with the elements of $\mathcal{C}^+$. Concerning the index of a critical point at infinity, it follows from the expansion of $J(\alpha P\tilde{\delta}_{(a,\lambda)}+\bar{v})$ when a is close to $y\in\mathcal{C}^+$. Arguing as in the proof of Proposition 1 of [9], there is a change of variables

$$(a,\lambda)\longmapsto(a',\lambda')$$

such that

$$J(\alpha P\tilde{\delta}_{(a,\lambda)}+\bar{v})=\frac{S}{K(a)^{\frac{n-2s}{n}}}\Big(1+\frac{1}{\lambda^{\min(\beta,n-2s)}}\Big).$$

Observe that under $(f)_\beta$-condition, the generalized Morse index of K at y equals to $\tilde{\imath}(y)$. Since by the above expansion J behaves near $P\tilde{\delta}_{(y,\infty)}$ as $\frac{1}{K}$, the index of J at $P\tilde{\delta}_{(y,\infty)}$ equals to $n-\tilde{\imath}(y)$. The proof of Theorem 3.5 is thereby completed. $\square$

4. Proof of the theorems

Let J_1 be the Euler–Lagrange functional associated to (1) for $K=1$.

$$J_1(u)=\frac{|u|^2}{\Big(\int\limits_\Omega u^{\frac{2n}{n-2s}}\Big)^{\frac{n-2s}{n}}},\, u\in\mathcal{H}\setminus\{0\}.$$

Let

$$\tilde{S}=\int\limits_{\mathbb{R}^n}\delta_{(a,\lambda)}^{\frac{2n}{n-2s}}\,dx.$$

It is easy to see that $\tilde{S}$ is independent of a and λ and satisfies

$$\tilde{S}^{2s/n}=\inf_{u\in\Sigma^+}J_1(u).$$

For any $c\in\mathbb{R}$, we denote

$$J_{1c}=\{u\in\Sigma^+,\,J_1(u)\le c\}.$$

We introduce the following two Lemmas.

Lemma 4.1. *For* $\alpha > 0$ *small enough,* $J_{1\,\tilde{S}^{2s/n}+\alpha}$ *and* Ω *have the same type of homotopy.*

Proof. For any $\lambda > 0$. By the expansion of Proposition 3.1, there exists $\lambda_0 >> 1$ such that for any $a \in \Omega$ we have

$$J_1\Big(\frac{P\tilde{\delta}_{(a,\lambda_0)}}{|P\tilde{\delta}_{(a,\lambda_0)}|}\Big) \le \tilde{S}^{2s/n} + \alpha.$$

Therefore, the map

$$\begin{aligned} f_{\lambda_0}\colon \Omega &\longrightarrow J_{1\,\tilde{S}^{2s/n}+\alpha}, \\ a &\longmapsto \frac{P\tilde{\delta}_{(a,\lambda_0)}}{|P\tilde{\delta}_{(a,\lambda_0)}|}. \end{aligned}$$

is well defined and continuous.

In order to prove that f_{λ_0} is an homotopy equivalence, we first claim that there exits $\alpha_0 > 0$ small enough such that

$$J_{1\,\tilde{S}^{2s/n}+\alpha_0} \subset V(1,\varepsilon). \tag{13}$$

Indeed, we argue by contradiction. Assume that for any $k \in \mathbb{N}^*$ there exits $(u_k)_k$ in $J_{1\,\tilde{S}^{2s/n}+\frac{1}{k}}$ such that $u_k \notin V(1,\varepsilon)$. Therefore, $(u_k)_k$ is a minimizing sequence and hence it satisfies $\lim_{k\to+\infty} \partial J_1(u_k) = 0$. The result of Proposition 2.1 shows that $(u_k)_k$ converges in Σ^+ to w satisfying $J_1(w) = \inf_{u\in\Sigma^+} J_1(u)$. This is absurd since in [16] it is proved that $\inf_{u\in\Sigma^+} J_1(u)$ in not achieved. Hence (13) is valid.

Using now the fact that for any $u \in V(1,\varepsilon)$, u can be expressed uniquely as $u = \alpha P\tilde{\delta}_{(a,\lambda)} + v$, see Proposition 2.2, we define the projection

$$\begin{aligned} p\colon V(1,\varepsilon) &\longrightarrow \Omega, \\ u = \alpha P\tilde{\delta}_{(a,\lambda)} + v &\longmapsto a. \end{aligned}$$

Let $R = p \circ \imath$ where $\imath\colon J_{1\,\tilde{S}^{2s/n}+\alpha_0} \hookrightarrow V(1,\varepsilon)$. R is continuous and satisfies $R \circ f_{\lambda_0} = id_{\Omega}$. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let* $\alpha > 0$ *small enough. There exits* $\varepsilon > 0$ *which depend on* α *such that if* $\|K-1\|_{L^\infty(\bar{\Omega})} \le \varepsilon$ *then*

$$J_{\tilde{S}^{2s/n}+\frac{\alpha}{4}} \subset J_{1\,\tilde{S}^{2s/n}+\frac{\alpha}{2}} \subset J_{\tilde{S}^{2s/n}+\frac{3\alpha}{4}}. \tag{14}$$

Proof. Let $u \in \Sigma^+$.m We have

$$J(u) = \frac{1}{\left(\int_\Omega u^{\frac{2n}{n-2s}} + \int_\Omega (K-1)u^{\frac{2n}{n-2s}}\right)^{\frac{n-2s}{n}}}$$

(15)
$$= J_1(u)\frac{1}{\left(1 + \left(\int_\Omega u^{\frac{2n}{n-2s}}\right)^{-1} \int_\Omega (K-1)u^{\frac{2n}{n-2s}}\right)^{\frac{n-2s}{n}}}$$

$$= J_1(u)(1 + O(\|K-1\|_{L^\infty(\bar{\Omega})})).$$

Hence Lemma 4.2 follows. □

We now state the proof of our existence theorems.

4.1 – *Proof of Theorems* 1.1

We argue by contradiction, we assume that J has no critical points in Σ^+. Let $\alpha_0 > 0$ such that $\widetilde{S}^{2s/n} + \alpha_0 < (2\widetilde{S})^{2s/n}$. Under the assumption that $\|K-1\|_{L^\infty(\bar{\Omega})}$ is small, all critical points at infinity of J of p masses, $p \geq 2$, are above the level $\widetilde{S}^{2s/n} + \alpha_0$ and all critical points at infinity of J of of one mass are below $\widetilde{S}^{2s/n} + \frac{\alpha_0}{4}$. This is a consequence of the expansion of J_1 in $V(p, \varepsilon)$, (see [3], Proposition 3.1) and the estimate (15). Using this fact and the deformation theorem of [8], we have

$$J_{\widetilde{S}^{2s/n}+\frac{\alpha_0}{4}} \simeq \bigcup_{y\in\mathcal{C}^+} W_u(y)_\infty,$$

where $\simeq$ denotes retract by deformation and $W_u(y)_\infty$ denotes the unstable manifold of the critical point at infinity $(y)_\infty$. Moreover,

$$J_{\widetilde{S}^{2s/n}+\frac{3\alpha_0}{4}} \simeq J_{\widetilde{S}^{2s/n}+\frac{\alpha_0}{4}}.$$

Therefore, by Lemma 4.2 we obtain

(16)
$$J_{1\,\widetilde{S}^{2s/n}+\frac{\alpha_0}{2}} \simeq J_{\widetilde{S}^{2s/n}+\frac{\alpha_0}{4}}.$$

It follows that

$$J_{1\,\widetilde{S}^{2s/n}+\frac{\alpha_0}{2}} \simeq \bigcup_{y\in\mathcal{C}^+} W_u(y)_\infty.$$

Computing the Euler–Poincaré characteristic of each side, we get

$$\chi(J_{1\,\widetilde{S}^{2s/n}+\frac{\alpha_0}{2}}) = \sum_{y\in\mathcal{C}^+}(-1)^{n-\tilde{\imath}(y)}.$$

The result of Lemma 4.1 achieves the proof of Theorem 1.1.

4.2 – *Proof of Theorem* 1.2

We argue by contradiction. We assume that J has no critical points in Σ^+. Let

$$Y_\ell^\infty = \bigcup_{\substack{y \in \mathcal{C}^+ \\ n - \tilde{\imath}(y) \le \ell - 1}} W_u(y)_\infty .$$

Y_ℓ^∞ is a stratified set of top dimension $\ell - 1$. Without loss of generality, we may assume that is of dimension $\ell - 1$. Observe that, Y_ℓ^∞ lies in $J_{\tilde{S}^{2s/n} + \frac{\alpha_0}{4}}$ provided $\|K - 1\|_{L^\infty(\bar{\Omega})}$ is small. Using the fact that Ω is contractible, we derive from (16) and Lemma 4.1 that $J_{\tilde{S}^{2s/n} + \frac{\alpha_0}{4}}$ is a contractible set. Let $\Theta(Y_\ell^\infty)$ be a contraction of dimension ℓ of Y_ℓ^∞ in $J_{\tilde{S}^{2s/n} + \frac{\alpha_0}{4}}$. We use the flow lines of $(-\partial J)$ to deform $\Theta(Y_\ell^\infty)$. By dimension argument, we may assume that the deformation of $\Theta(Y_\ell^\infty)$ avoids all the critical points at infinity of index larger than ℓ. Thus

$$\Theta(Y_\ell^\infty) \simeq \bigcup_{\substack{y \in \mathcal{C}^+ \\ n - \tilde{\imath}(y) \le \ell}} W_u^\infty(y)_\infty .$$

We use now the fact that $n - \tilde{\imath}(y) \neq \ell$, for any $y \in \mathcal{C}^+$, we obtain

$$\Theta(Y_\ell^\infty) \simeq \bigcup_{\substack{y \in \mathcal{C}^+ \\ n - \tilde{\imath}(y) \le \ell - 1}} W_u^\infty(y)_\infty .$$

Therefore, by applying an Euler–Poincaré characteristic argument, we get

$$1 = \sum_{\substack{y \in \mathcal{C}^+ \\ n - \tilde{\imath}(y) \le \ell - 1}} (-1)^{n - \tilde{\imath}(y)} .$$

This contradict the assumption of Theorem 1.2. This completes the proof of Theorem 1.2.

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