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Jemma Lorenat

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Secrétariat :
Nathalie Christiaën
Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie, 75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99 / Fax : (33) 01 40 46 90 96
Mél : `rhmsmf@ihp.fr` / URL : `http//smf.emath.fr/`

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RADICAL, IDEAL AND EQUAL POWERS: NEW DEFINITIONS IN MODERN GEOMETRY 1814–1826

Jemma Lorenat

Abstract. — Alongside new practices, early nineteenth-century geometers in France and Germany developed a variety of new definitions with which to designate their objects of study. Accompanying arguments for choosing or creating new names or rejecting alternative designations reveal careful attention to epistemic values, potential user experiences, and evolving academic careers. To observe the effects of these arguments within the practice of geometry, we will focus on the nomenclature introduced in research articles by Louis Gaultier, Jean-Victor Poncelet, and Jakob Steiner to designate what were respectively called radical axes, ideal common chords, and lines of equal powers. During the 1820s each of these terms found currency in overlapping contexts, signifying both the geometrical value of the terminology and the proliferation of texts. We will show that naming was perceived as an important aspect in successfully introducing what these geometers declared to be modern geometry.

Résumé (Radicaux, idéaux et puissances égales : nouvelles définitions en géométrie moderne)

Parallèlement à de nouvelles pratiques, les géomètres du début du dix-neuvième siècle en France et en Allemagne développèrent une variété de nouvelles définitions pour désigner leurs objets d'étude. Les arguments donnés pour choisir ou créer de nouveaux noms ou pour rejeter des désignations alternatives révèlent l'attention portée aux valeurs épistémiques, aux expériences d'un utilisateur possible et aux évolutions des carrières académiques.

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J. Lorenat, Pitzer College, 1050 N. Mills Avenue, Claremont, CA 91711, États-Unis.

Courrier électronique : jemma_lorenat@pitzer.edu

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Pour observer les effets de ces arguments dans la pratique de la géométrie, nous nous concentrerons sur la nomenclature introduite dans des articles de recherche par Louis Gaultier, Jean-Victor Poncelet et Jakob Steiner pour se référer à ce qu'ils appelaient respectivement axes radicaux, cordes communes idéales et lignes d'égales puissances. Pendant les années 1820, chacun de ces termes fut utilisé dans des contextes en chevauchement, témoins à la fois la valeur de la terminologie pour la géométrie et de la prolifération des textes. Nous montrerons que nommer était perçu comme un aspect important dans l'introduction réussie de ce que ces géomètres voyaient comme la géométrie moderne.

1. INTRODUCTION

During the early nineteenth century, the language of geometry evolved and expanded. In 1827, an anonymous reviewer in the *Bulletin des sciences mathématiques, astronomiques, physiques et chimiques* (also referred to as the *Bulletin de Férussac* after the editor) praised the recent changes in pure geometry. Additions to vocabulary, he wrote, had clarified, condensed, and simplified the science. The reviewer recognized this simplification as analogous to the symbolic abbreviations used by analysts.

> Analysts perceiving that certain quite complicated functions are reproduced frequently in their calculations, have called them exponentials, logarithms, sines, tangents, factorial derivatives, etc.; they have created abbreviated signs to designate them, and their formulas have acquired greater clarity and conciseness. And thus for certain points, certain lines and certain circles whose consideration is frequently represented in geometric speculations, it is natural to do the same with respect to them, and to call them, following their properties, similitude centers, radical centers, polars, similitude axes, radical axes, circles of common power, etc. This attention must inevitably introduce analogous simplifications in the statement of theorems and in the solution of problems which belong to the science of magnitude.[1] [*Bulletin* 1827, p. 279]

[1] "Les analystes s'étant aperçus que certaines fonctions assez compliquées se reproduisaient fréquemment dans leurs calculs, les ont appelées exponentiels, logarithmes, sinus, tangentes, dérivées factorielles, etc. ; ils ont créé des signes abréviatifs pour les désigner, et leurs formules en ont acquis beaucoup de clarté et de concision. Puis donc qu'il est certains points, certaines droites et certains cercles dont la considération se représente fréquemment dans les spéculations de la géométrie, il est naturel d'en user de même à leur égard, et de les appeler, suivant leurs propriétés, centres de similitude, centres radicaux, polaires, axes de similitude, axes radicaux, cercles de commune puissance, etc. Cette attention doit introduire inévitablement des simplifications analogues dans l'énoncé des théorèmes et dans la solution des problèmes qui appartiennent à la science de l'étendue."

By focusing on their interrelated properties, points, lines, and circles received new names. Somewhat paradoxically, an increase in new terms seemed to simplify a geometry that increasingly focused on these previously unnamed relationships.

Geometers presented their emerging vocabulary as improving the practice of geometry. The success of new terms depended on the strength of these presentations, but also on the status of the publications in which they appeared and the existent contemporary vocabulary and research questions. Like solving problems or proving theorems, definitions served as a potential medium with which a geometer might illustrate the advantages of his approach and make a name for himself. However, unlike other geometrical contributions, new definitions could render old definitions obsolete. While a geometer might introduce his new solution as equivalent to past solutions, definitions needed to be better.

This paper considers a series of definitions proposed between 1814 and 1826: Louis Gaultier's radical axes, Jean-Victor Poncelet's ideal common chords, and Jakob Steiner's lines of equal powers. The definitions and their respective receptions will be traced chronologically as each geometer responded to his predecessors. As we will see, these definitions relied on numerical properties, figure-based constructions, and a careful elaboration of key terms. A brief outline here will illuminate the main features.

Gaultier defined radical axes in 1814.[2] The radical axis was the locus of points o such that for secants g_1k_1o and g_2k_2o drawn respectively to two given circles $\sqrt{g_1o \cdot k_1o} = \sqrt{g_2o \cdot k_2o}$.

Several years later, Poncelet defined ideal common chords. Any common chord MN to two conic sections possessed two defining properties. First, the direction of MN determined a unique diameter to each conic section, and the respective conjugate diameters AB and $A'B'$ intersected at the center O of the chord.[3] Secondly, for uniquely defined constant values p and p' associated with the respective conic sections, $p \cdot OA \cdot OB = p' \cdot OA' \cdot OB'$. An ideal common chord was a common chord with imaginary points of intersection with the given conic.

Shortly after Poncelet's definition appeared, but with limited knowledge of contemporary French geometry, Steiner defined lines of equal powers as the line PG with respect to given circles centered at M and m with radii R and r. The line of equal powers was the locus of points of

2 Gaultier's name often appeared as Gaultier de Tours in early nineteenth century citations, including this article. As there is no risk of ambiguity here, he will be referred to simply as Gaultier.

3 Conjugate diameters will be discussed at greater length in Section 4.2

equal powers to both circles so that $MP^2 - mP^2 = R^2 - r^2$. Steiner then proposed that the definition could be easily extended to conic sections.

These three definitions differed in construction, notation, and intended applications. Yet, for the case of two given non-concentric circles, each defined the same line. Since the definitions and the objects they designate overlap, why were several definitions provided and how did they coexist or replace one another? In answering these questions, this paper will argue that definitions served to assign values. Thus, tracing the emergence and proliferation of these new definitions will reveal competing epistemic priorities among contemporary geometers. These priorities were not stagnant, but rather shifted from introduction to adoption or rejection.

Supplementing this primary focus, we will also use the description and applications of these new definitions to better understand the multiple meanings behind qualitative claims. During the early nineteenth century, pure geometry was experiencing a revival as geometers questioned the merits of applying coordinate equations to solving geometry problems. Gaultier, Poncelet, and Steiner primarily researched and published in pure geometry and their new definitions reflected an ambition to make their method as powerful as analytic geometry. In presenting their contributions to geometry, each repeated how his results were both simple and general. Yet we will see that these terms were not universally understood, so their use in this context helps to draw out potential meanings.

With respect to mathematicians of the late nineteenth century, Catherine Goldstein has observed how following the use of often cited key words can reveal nuanced connotations. Even clichéd descriptions such as "simplicity, clarity, fruitfulness" could communicate allegiances.

> [...] they enjoy, as the word *Anschauung* in Felix Klein's entourage, the role of a banner, rallying mathematicians, types of explanations, methods. Employed together or no, rather rarely in association with other terms ("simple and general", a little more often "precise"), these words recurrently come to oppose the pair "rigorous" and "complicated" [...] [4] [Goldstein 2011, p. 145]

Similarly, the prevalent use of the words simple and general among early nineteenth century geometers suggest a common effort against perceived complication in the computations of analytic geometry and particularity

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[4] "[...] elles jouent, tout comme dans l'entourage de Felix Klein le mot *Anschauung*, le rôle d'une bannière, ralliant des mathématiciens, des types d'explication, des méthodes. Employés ensemble ou non, assez rarement en association avec d'autres termes ("simple et général" , un peu plus souvent "précis"), ces mots viennent s'opposer de manière récurrente au couple "rigoureux" et "compliqué" [...]"

in the figures of ancient geometry. By attending to the results presented or praised as simple and general, we will determine the changing manifestations of these terms as well as other adjectives that came to be positively associated with geometry.

The connotations of "generality" in early nineteenth century geometry have been explored independently by both Karine Chemla and Philippe Nabonnand ([Chemla 1998], [Chemla 2016], [Nabonnand 2011a]). Chemla and Nabonnand have each shown how claims of generality served as focal points in the new geometrical principles and methods introduced by Lazare Carnot, Poncelet, and Michel Chasles. We will further see how more conservative geometers like Gaultier also drew attention to multiple forms of generality, and how standards of generality impacted definitions.

Through more applicable, more representative, more systematic, more concise or more inclusive terms, each geometer framed his contributions as adding value to geometry. Although some argued that ideal common chords subsumed radical axes or lines of equal powers were redundant, all three terms survived through the nineteenth century. This paper focuses on how terminology provided new ways of talking about geometry that reflected and directed its changing shape. Each of the following three sections will be divided into four parts that coincide with the format of each geometer's text. The first part will summarize the motivations and contexts presented for introducing new definitions. This will be followed by an account of the definition as it first appeared. We will then analyze how the new definitions manifested the stated priorities for geometry. Finally, we will consider how contemporary geometers reacted to each new definition. The variability of definitions over time was observed by Joseph-Diez Gergonne in an article he wrote on the theory of definitions in 1818.

> It is thus, at most, only the first times that a new word offers itself to us, that we are obliged to recall, in an explicit manner, all the ideas that it expresses; thus we feel that it is only then that its usage causes us some confusion; but this confusion soon disappears by the effect of habit; and we soon find, to the contrary, a great comfort in the use of this same expression whose utility we had at first barely glimpsed. [5] [Gergonne 1818a, pp. 12–13]

5 "Ce ne peut donc être, au plus, que les premières fois qu'un mot nouveau vient s'offrir à nous, que nous sommes obligés de nous rappeler, d'une manière explicite, toutes les idées qu'il exprime; aussi éprouvons nous que c'est alors seulement que son usage nous cause quelque embarras; mais cet embarras disparaît bientôt par l'effet de l'habitude; et nous ne tardons pas à trouver, au contraire, un très-grand secours dans l'usage de cette même expression dont, au premier abord, nous avions à peine entrevu l'utilité."

Complementing Gergonne's description, we will see that the early life of a definition elicited new dialog that faded upon entering the standard lexicon.

2. GAULTIER'S RADICAL AXIS

Louis Gaultier (1776–1848) was a professor of descriptive geometry at the *Conservatoire des Arts et Métiers* in Paris when he published his article, "Mémoire sur les moyens généraux de construire graphiquement les cercles déterminées par trois conditions, et les sphères déterminés par quatre conditions," in the *Journal de l'École polytechnique* [Gaultier 1813]. Gaultier had attended the École polytechnique between 1798 and 1801, and as such was eligible for publishing in the school's journal. A version of this paper had been given at the *Institut de France* the year before. In his education and his current profession, Gaultier was committed to the teachings of his former teacher Gaspard Monge. [6] Indeed, he introduced his text by surveying "the mathematical works of the most distinguished modern geometers"– mostly comprised of recent graduates and faculty of the École polytechnique.

2.1. *Why radical axes?*

Admittedly struck by the overall rapid and satisfying progress, Gaultier particularly noted the attention given to analysis. He declared that the "generality and fruitfulness" of analytic concepts had influenced all other branches of mathematics. He praised these analytic works as "immortal" and "signaling a memorable epoch in the annals of science" [Gaultier 1813, p. 124].

Despite the prominent position and achievements of mathematicians in analysis, Gaultier found that for mathematical applications descriptive geometry was just as fruitful and general. He posited that the utility of descriptive geometry was grounded in its basis in "the most simple and most elementary principles." Moreover, he argued, the "essential merit" of descriptive geometry lay in its rigor and clarity. As evidence, Gaultier offered a list of recent geometers, all graduates of the École polytechnique,

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[6] The legacy of Gaspard Monge among students at the École polytechnique has been discussed in [Taton 1951], [Belhoste & Taton 1992], [Sakarovitch 2005], [Laurentin 2007].

who had recently contributed new results in pure geometry: "Dupuis, Lancret, Poinsot, Dupin, Cauchy, Français, Brianchon, &c." [Gaultier 1813, p. 125]).

Within the tradition of descriptive geometry, Gaultier proposed to find a general solution to the problem of constructing a circle given three conditions and the problem of constructing a sphere given four conditions. For the planar situation, these conditions could include passing through a given point, being tangent to a given line, being tangent to a given circle, having a given radius, being tangent to a given line at a given point, and having the center on a given line. Gaultier would show that there were thirty-three possible combinations of these givens, as he illustrated with the following table.

Cercles déterminés par trois conditions.

N.os	COMBINAISONS.	SOLUTIONS.	N.os	COMBINAISONS.	SOLUTIONS.
1.	*tp*..........		18.	*pcd*.........	
2.	*td*..........		19.	*lld*.........	2.
3.	*rdd*.........		20.	*cdd*.........	
4.	*ppp*.........	1.	21.	*rpc*.........	
5.	*ppd*.........		22.	*rll*.........	
6.	*pdd*.........		23.	*rcd*.........	
7.	*tr*..........		24.	*plc*.........	4.
8.	*tl*..........		25.	*pcc*.........	
9.	*tc*..........		26.	*lll*.........	
10.	*rpp*.........		27.	*lcd*.........	
11.	*rpl*.........		28.	*ccd*.........	
12.	*rpd*.........	2.	29.	*rlc*.........	
13.	*rld*.........		30.	*rcc*.........	
14.	*ppl*.........		31.	*llc*.........	8.
15.	*ppc*.........		32.	*lcc*.........	
16.	*pll*.........		33.	*ccc*.........	
17.	*pld*.........				

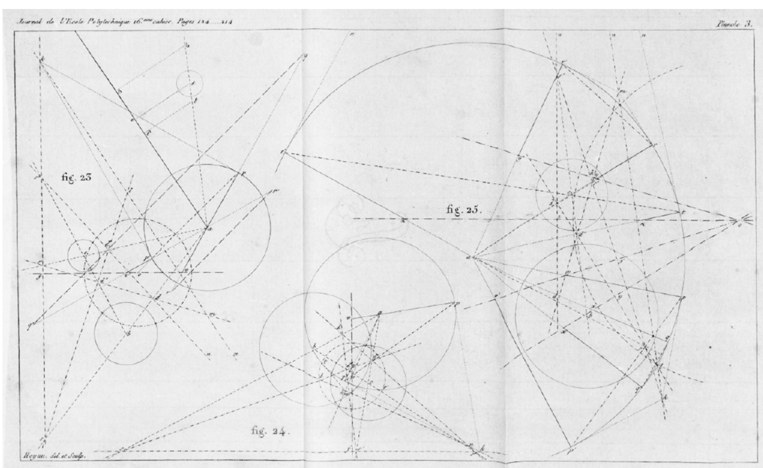
Gaultier's table showing numerous choices of three given properties to determine a circle using his own abbreviations [Gaultier 1813]

As the table indicates, "the most general case" of each combination resulted in either one, two, four, or eight possible circles that fulfilled the given criteria. Through his solutions, Gaultier promised to further indicate when the desired circle would be imaginary, which to Gaultier meant it would be impossible to construct.

As a comparison to his own method and solutions, Gaultier surveyed past approaches to finding a circle with three given conditions. While Gaultier contended that the earlier solutions secured the existence of these solutions, the graphic constructions provided had not always been practicable and thus were unsatisfactory.

> [...] but the considerations that they make use of, although very ingenious, render the graphic constructions too difficult [...] [7] [Gaultier 1813, p. 127]

He noted that the problem had been solved analytically by Newton in his *Arithmetica Universalis* (1707),[8] as well as Euler and Fuss in the *Annales de l'Académie de Pétersbourg* (1788). The analogous sphere problem had been solved geometrically by Fermat (1679) as well as Monge and his students at the École polytechnique. These latter researches had been published in the first volume, second number of the *Correspondance sur l'École impériale polytechnique*. In contrast to these earlier approaches, Gaultier promised to provide solutions based on the "most elementary principles" of planar and descriptive geometry that could be constructed by means of the ruler and compass alone. Indeed, Gaultier's text was richly illustrated with circles in various configurations, auxiliary lines, and completed constructions. Gaultier's figures documented how constructive steps led to a determinate circle or sphere.



Gaultier's Figures 23, 24, 25, showing how to find circle solutions [Gaultier 1813]

7 "[...] mais les considérations qu'ils ont mises en usage, quoique très-ingénieuses, rendaient les constructions graphiques trop difficiles [...]"

8 The French translation, *Arithmétique universelle* appeared in 1802.

Gaultier's attention to the graphic applicability of geometry extended to his own use of illustrated figures, including how the reader should view and employ them. For instance, he explained that he would often use one figure to demonstrate several propositions. Further, he intended to avoid complications by only illustrating the parts of circles that were used in the construction. Therefore certain circles would only appear as arcs or radii, such as the partial circle in his Figure 25 (shown above). Geometers or their illustrators frequently employed this same practice, but Gaultier appears extraordinary in making explicit his attention to figure appearance. He even added a footnote crediting the plates to M. Hoyau, a student who would become a well-respected mechanical engineer [Poncelet 1857, p. 402].

> The Plates of this memoir requiring a lot of precision, I entrusted their implementation to M. Hoyau, a student at the Conservatory, who had drawn them for the manuscript memoir, and constructed them immediately on copper.[9] [Gaultier 1813, p. 128]

He further guided his reader by providing textual summaries in the margins of the page. This level of consideration for his audience suggests that Gaultier wrote for students or those unfamiliar with many geometry texts.

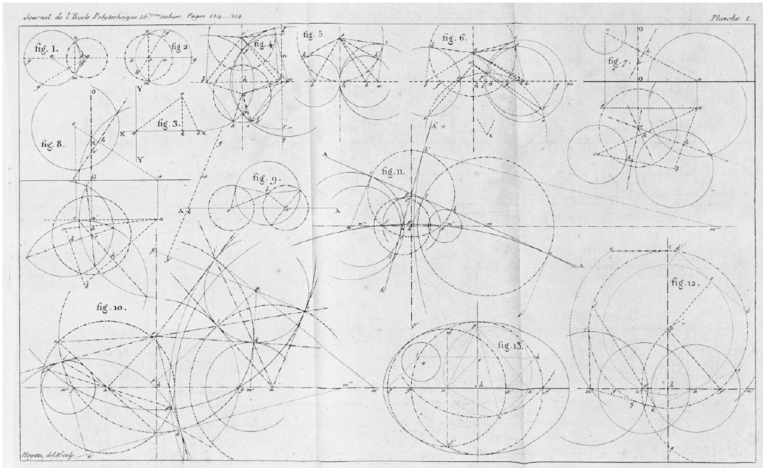
Gaultier next explained what he called "the notations" employed in his text. He began by citing conventions for labeling various geometric figures with one or two upper or lowercase letters. Alongside this symbolic notation, Gaultier employed new expressions to refer to frequently repeated phrases.

> One will find in the course of this work several expressions which must be considered as abbreviations of certain phrases that it has been necessary to repeat very frequently, as well as several particular notations depending on the theory of which it forms the subject; but they will be successively explained in the development of this theory: so it seemed to us superfluous to speak of it here.[10] [Gaultier 1813, pp. 129–130]

In keeping with Gaultier's commitment to elementary geometry, he introduced these expressions as only abbreviations. For Gaultier, definitions

9 "Les Planches de ce mémoire exigeant beaucoup de précision, j'en ai confié l'exécution à M. Hoyau, élève du Conservatoire, qui les avait dessinées pour le mémoire manuscrit, et qui les a construites immédiatement sur le cuivre."

10 "On trouvera dans le cours de l'ouvrage plusieurs expressions qui doivent être considérées comme l'abrégé de certaines phrases qu'il aurait fallu répéter trop fréquemment, ainsi que plusieurs notations particulières dépendantes de la théorie qui en fait le sujet; mais elles seront successivement expliquées dans le développement de cette théorie: en sorte qu'il nous a semblé superflu d'en parler ici."



Gaultier's first sheet of figures, including illustrations of radical circles, axes, and centers [Gaultier 1813]

played a similar role as figures, in merely helping to elucidate his constructive steps. Only one page later, Gaultier began his first chapter with a footnote again forewarning of his particular vocabulary.

> The theory that we develop in this 1st chapter, necessarily employs several particular denominations that serve to recall, more briefly than paraphrases, the new properties which are the subject here; but we have reduced this type of nomenclature, which will at first appear a little complicated, to only the terms that designate the most fundamental properties, and whose frequent use would have resulted in inevitable lengthiness.
>
> One will find most of these expressions, which are but few, stated and defined in the first numbers of this chapter. [11]

Though he admitted that the forthcoming nomenclature would at first appear a bit complicated, Gaultier promised that their use would ultimately keep the text shorter and presumably easier to read.

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11 "La théorie que nous développons dans ce I^{er} chapitre, a nécessité l'emploi de quelques dénominations particulières qui servent à rappeler, plus brièvement que des périphrases, les propriétés nouvelles qui en sont l'objet; mais nous avons réduit cette espèce de nomenclature, qui paraissait d'abord un peu compliquée, aux seuls termes qui désignent les propriétés les plus fondamentales, et dont l'emploi fréquent aurait entraîné dans des longueurs inévitables." [Gaultier 1813, pp. 130–131]
"On trouvera la plupart de ces expressions, qui d'ailleurs sont en petit nombre, énoncées et définies dans les premiers n.os de ce chapitre."

Even so, Gaultier admitted a change in focus. His conservative emphasis on graphical correspondence balanced with a commitment to generality reflecting more recent standards from analysis. Moreover, he called for a new emphasis on what should be considered the most fundamental properties of elementary geometry, while the succinctness of his new definitions maintained simplicity.

2.2. *Defining radical axes*

The paper was organized into three chapters: (1) Geometric properties of radical circles, spheres and their series; (2) Geometric properties of tangent circles and spheres; (3) Tables of most of the problems to which one can apply the principles developed in the preceding chapter and complete discussion of three problems. While Gaultier emphasized his contributions to problem solving, his work would become best known for his general definitions of radical centers, axes, and circles. We will focus on Gaultier's explanatory introduction and the first part of his first chapter, "Notions préliminaires sur les Cercles radicaux et les Sphères radicales," in which he defined radical circles, the radical axis between two circles, and the radical center between three circles.

Gaultier introduced his terms by providing constructions, then pointing out their features. He began by considering a circle A, which he called the *primitive circle.*[12] With reference to his Figures 1 and 2, from a coplanar point o (either exterior to the circle as in his Figure 1 or inside the circle as in his Figure 2) one could draw a secant or chord meeting the circumference of A at points g and k. A *radical circle* was one centered at o whose radius om was the length of the square root of $og \cdot ok$, which Gaultier wrote symbolically as $\sqrt{og \cdot ok}$, the constant product of the segments of the secant (in his Figure 1) or the chord (in his Figure 2) drawn from its center to the circumference of the primitive circle. Thus the adjective *radical* referred to the algebraic radical sign that identified the key property.

Gaultier continued by considering two configurations. If the point o lay outside of circle A then the radical circle O was a *reciprocal radical circle* because A was also a radical circle of O. If the point o lay within circle A then the radical circle O was a *simple radical circle* because A was not also a radical circle of O. So any radical circle was either of the reciprocal or simple type.

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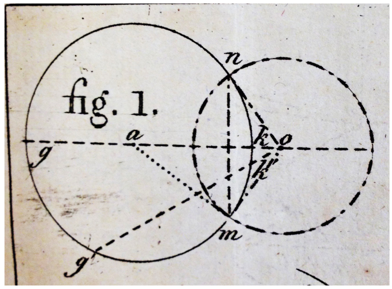
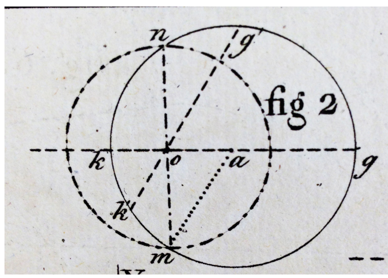
12 The choice of *primitive* here may be borrowed from Lazare Carnot, who introduced the terms *primitive* and *correlative* to distinguish between figures before and after "mutations" [Carnot 1803], xxxiv.

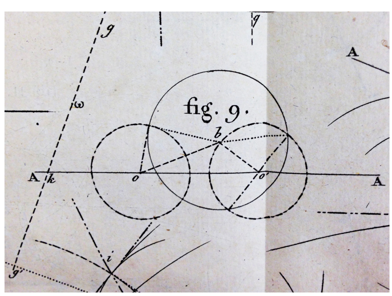


Gaultier's Figure 1 [Gaultier 1813]



Gaultier's Figure 2 [Gaultier 1813]

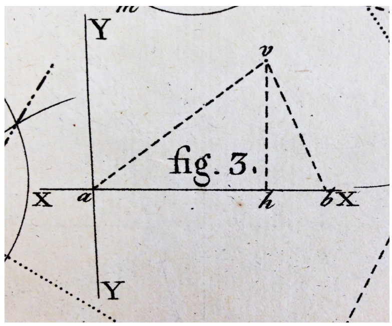
Gaultier showed how his definitions could be extended beyond the circle and included the limiting cases where the circle had no radius and became a point or had an infinite radius and became a line (illustrated in his Figure 9). In this manner, Gaultier could apply his definition to nearly every configuration of points and planar circles, barring the case where the point *o* fell on the circumference of the given circle. He thus developed a



Gaultier's Figure 9 [Gaultier 1813]

uniform vocabulary around the concept of radical. To designate all types of radical circles, he introduced the expression *radical relations* to signify the situation when at least one of the circles A and O was a radical circle with respect to the other.

Gaultier's commitment to graphical geometry did not preclude the use of calculations. Referring to his Figure 3, he created "coordinates" originating from the center of a given circle a. He proceeded by defining the

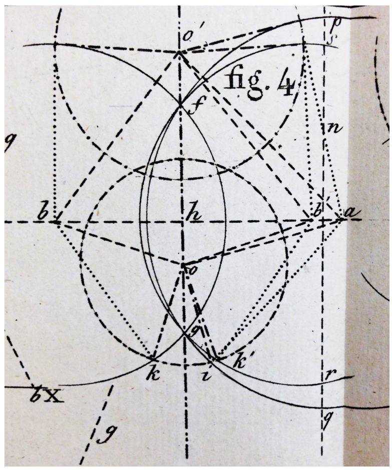


Gaultier's Figure 3 [Gaultier 1813]

variable m as the distance between the two given centers a and b, the point v as the center of the radical circle V, and the point h as the foot of the perpendicular from v to ab. Then he defined two new variables, $ah = x$ and $vh = y$, to conclude that $hb = m - x$. Arithmetical calculations with these assigned variables and right triangle relations, enabled Gaultier to determine that $\overline{va}^2 - \overline{vb}^2 = 2mx - m^2$. He noted that in this first degree equation the left hand side would remain constant because of the radical relation, and so the value of x would also be constant. He concluded that the locus of centers of circles with a radical relation to two given circles would be perpendicular to ab.

Gaultier then considered his Figures 4, 5 and 6 where two given circles A and B, still centered at a and b, shared a series of radical circles of the same type. With a few calculations based on the product of the radii of radical circles, the mean and extreme ratio, and the equation of the locus of centers, he showed that the centers of the radical circles in the series would all fall on the same line perpendicular to the line ab. This line Gaultier designated as the *radical axis* of circles A and B, which he further abbreviated as "axe rad. AB" [Gaultier 1813, p. 139]. Here too, he considered limiting cases and proved that the definition of radical axes continued to hold whether the given circles were graphically circles, points or straight lines.

Having demonstrated that his definition resulted in a unique straight line, Gaultier continued by showing how this line could be constructed for two given circles. He progressed through three distinct cases. First, when the given circles A, B intersected at points f and g (shown in Gaultier's Figure 4), the line fg, their common secant, determined by the intersection points would be their radical axis.

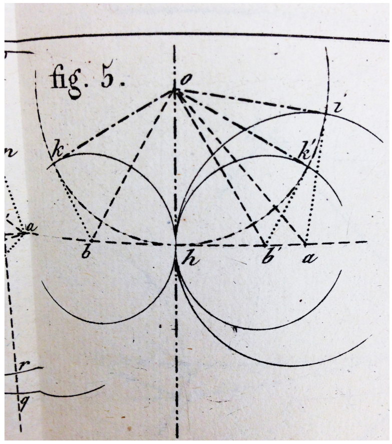


Gaultier's Figure 4 [Gaultier 1813]

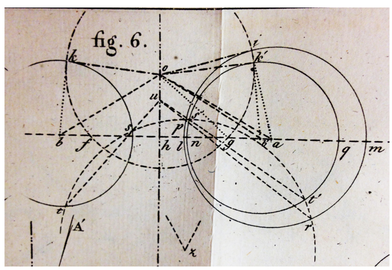
When the circles A and B were tangent at a point h (shown in his Figure 5), then the radical axis would be their common tangent at the point of tangency. In this case, all the common radical circles would be reciprocal circles.

Finally, if the circles A and B (shown in his Figure 6) were either exterior circles or one enveloped the other, then they had no common points and their radical axis would be exterior to both circles A and B. To construct the radical axis in this case, one could draw an arbitrary circle Z meeting A and B at points p, r and s, t respectively. By extending the intercepted chords pr and st, their point of intersection u would lie on the radical axis of A and B. Gaultier observed that this final construction proved a "remarkable property" of two given circles. When intersected by an indefinite number of arbitrary circles Z', Z'', Z''' the points u', u'', u''' as defined above would be collinear because situated on the radical axis of A and B.

Thus, depending on the configuration of the given circles, the radical axis could be a common secant, a common tangent, or a newly defined line exterior to both circles. So while Gaultier defined radical circles with a



Gaultier's Figure 5 [Gaultier 1813]



Gaultier's Figure 6 [Gaultier 1813]

single locus description, the construction process required three separate cases.

Gaultier concluded his lengthy "preliminary notions" by advising his readers to exercise wisdom when applying these properties to solving specific problems.

> Finally it depends on the cleverness of those who want to make use of the properties that we have just developed, to reduce questions of intersections of straight lines and circles, either directly, or by employing properties common to two new conditions which completely determine it. [13][Gaultier 1813, p. 170]

13 "Enfin il dépend de la sagacité de celui qui veut faire usage des propriétés que nous venons de développer, de ramener les questions aux intersections de lignes droites et de cercles, soit directement, soit en employant les propriétés communes aux deux nouvelles conditions qui complètent la détermination."

Gaultier illustrated how this could be done in solving the three tangent circle problem. The case where the three given circles A, B, C are all tangent to a fourth circle in the same way (all interior or all exterior) exemplifies Gaultier's solutions. To show that the tangents were all of the same kind, Gaultier wrote $A(sBsC)$.

The three given circles A, B, C when considered pairwise corresponded to six similitude points. Gaultier had shown earlier in his paper that these points lay on four lines, which he designated as "F lines" (183). Further, the three circles shared a radical center o. In his solution, Gaultier began by raising perpendiculars of, og, oh, oi from the radical center to each of the F lines. He reminded the reader that each perpendicular contained the centers of all circles that would satisfy the given tangency conditions. Then, focusing on the situation where the center o lay outside of all three circles, Gaultier drew the radical axis to A and the circle O, defined as the radical reciprocal circle to A, B, C. This axis defined four new points by intersections with the four F lines. Each of these points was the center of a circle radical to both A and O. Gaultier selected one of these points, k as the intersection with of and drew tangents kp, kp' to circle A. Then points p and p' would be the points of tangency with the two circles satisfying $A(sBsC)$. Finally, the two circles would be centered at the intersections of of with ap and ap'. Gaultier indicated that the remaining six circles could be obtained "in the same manner" and the generality of his definitions reduced the need for case distinctions. The proof of these solutions rested on the many properties of centers, axes, and circles developed throughout the text. As Gaultier stated in his introduction, radical axes had been designed with this application in mind. They would consequently be adopted in similar problem solving scenarios.

2.3. *Radical axes as general, simple and graphic*

Gaultier had focused on making his definitions and the first part of his text accessible and comprehensive. He explained his figures and use of notation in advance and relegated all definitions to one section. Further, each definition was accompanied by several pictures illustrating potential configurations.

This appearance reiterated how Gaultier classified his text as elementary geometry, but this was not the elementary geometry of Euclid. Instead of simply attaching descriptions to points and lines, Gaultier extended the variety of named objects. Moreover, by using the nouns "axis" and "center" Gaultier focused attention on the relational qualities of these designated objects. A point and a line could exist independently, but axes and centers depended on other figures. An axis suggested collinearity and a center suggested concurrence, which recalled the featured properties of these objects.

The choice of the word "radical" pointed to the intimate connection between symbolic manipulation and configurations within geometry. Gaultier insisted on the constructive qualities of his problem solving, but his text reveals that this did not exclude algebraic techniques in his derivations of loci. The radical part of radical axes was invisible in the constructive procedure and only apparent with the use of equations and calculations. These inclusions suggest that for Gaultier geometric problems necessitated graphical solutions, but coordinates and equations could be adopted unproblematically in definitions and proofs.

In his introduction, Gaultier had declared that the success of both analysis and descriptive geometry was grounded in the "generality of their considerations." For analysis, this generality was evident in its widespread application within other branches of mathematics. For descriptive geometry, this generality emerged from simple and elementary principles and resulted in diverse useful applications. This appreciation of the respective contributions of analysis and descriptive geometry reflected the position of Monge.

> It is desirable that these two sciences be cultivated together: descriptive geometry brings its characteristic evidence to the most complicated analytic operations, and in turn, analysis brings to geometry the generality that is proper to it.[14] [Monge 1798a, p. 18]

Gaultier strove to achieve this balance through his general quantitative definition of radical axes that could then be constructed in particular graphic applications.

In Gaultier's paper, these particular graphic applications included "all one hundred seven different problems" derived from the thirty-three planar and seventy-four solid combinations. In fact, Gaultier would only explicitly state the constructive procedures for three given circles and suggested that "one can regard most of the other problems as particular cases" of three circles [Gaultier 1813, p. 201]. His solution contained only one textual discussion of separate cases, depending on whether the

[14] "Il serait à désirer que ces deux sciences fussent cultivées ensemble: la Géométrie descriptive porterait dans les opérations analytiques les plus compliquées l'évidence qui est son caractère, et, à son tour, l'Analyse porterait dans la Géométrie la généralité qui lui est propre."

first constructed point lay within the given circles or not. Additional cases were implied through his invocation of multiple figures, thus illustrating distinct configurations. Simplicity at the level of solutions was thus made possible by examining particular cases at the level of definitions in his preliminary section. As we have seen, constructing a radical axis could be done in one of three ways dependent on the two given circles. Moreover, Gaultier noted certain configurations in which the construction would be impossible because the solution was imaginary. For Gaultier, his solution was general even though not applicable in certain situations, such as three given concentric circles, where radical axes could not be constructed. With Gaultier's criteria, in all possible constructions descriptive geometry was as general as analysis, imaginary points had no graphic correspondence and so were not part of geometry.

2.4. *Applications and interpretations of radical axes*

Gaultier presented his paper as offering new solutions to the tangent circle problem. In contrast, he de-emphasized the novelty of his definitions. His caution here suggests concern for conservative readers who would appreciate graphic solutions but might be wary of obscure terminology. However, the immediate reception of Gaultier's work revealed that his definitions were far more attractive to the mathematical community.

Geometers frequently cited Gaultier in the *Annales* ([Gergonne 1814b], [Durrande 1820], [Poncelet 1820], [Steiner & Gergonne 1827], [Plücker 1827]). The diverse applications show that radical axes could be adopted without committing to Gaultier's graphical approach to geometry or his subsequent solutions. Indeed, the year after Gaultier's text appeared, Gergonne applied Gaultier's *radical axis* and *radical center* in his own analytic proof of the three circle tangent problem [Gergonne 1814b, p. 351]. At the same time, Gaultier's attention to graphic applications won favor among more conservative geometers like Jean Baptiste Durrande, who intended to "vindicate ancient geometry from the reproach of impotence" [Durrande 1820, p. 4]. Durrande employed only "the most elementary notions" to provide a purely elementary proof of Gergonne's tangent circle solution, in which he also employed radical axes.

As in the two above cases, part of the success of Gaultier's terms may be ascribed to the popularity of the problem of finding a circle tangent to three given circles (frequently called Apollonius problem or the Taktionsprobleme) throughout the nineteenth century. Gaultier himself described his text as motivated by finding graphic solutions to this and similar problems, with definitions serving merely as abbreviations in this process.

He touted the simplicity of his solutions, which depended on a more expressive vocabulary. For Gaultier, a simple solution was one in which the construction of each step was apparent and practical. Radical axes conveyed specific quantitative properties, but also an effective ruler and compass construction. However, Gaultier's solutions and proofs were not seen as conclusive or even worth mentioning by later geometers. In subsequent papers on the Apollonius problem authors cited Gaultier's introduction of radical axes and centers without mention of his original constructive solutions ([Gergonne 1814b], [Plücker 1827]).

Radical axes also appeared in cases where Gaultier's vocabulary was explicitly rejected. When Olry Terquem defined the "ligne disomologue" of two circles in his 1829 textbook on elementary geometry, he mentioned then dismissed the term radical axes.

> It is also called *radical axis*; we do not know the reason for this name.[15] [Terquem 1829, p. 178]

However, in a note accompanying his discussion of the Apollonius problem, Terquem acknowledged that the solution had been "greatly simplified by the introduction of radical axes" and credited "Gautier" (who appears as "Gauthier" in Terquem's list of cited authors). Terquem professed to "follow the same path" but replaced "radical axis" with "ligne disomologue" without further comment [Terquem 1829, p. 424]. Terquem's dismissal suggests that radical axes were well known enough by this time that they required only a brief mention before being replaced.

Poncelet also presented an alternative to radical axes with his *ideal chords*. Unlike Terquem, Poncelet would explain at length how ideal chords surpassed radical axes with respect to generality, naturalness, and simplicity.

3. PONCELET'S IDEAL CHORDS

Poncelet's ideal chords have featured both in historical studies of his principle of continuity and within the context of his approach to imaginary points. While Nabonnand and Jean-Pierre Friedelmeyer have focused on how Poncelet developed ideal chords, we will instead emphasize how Poncelet publicly presented ideal chords and the subsequent reception.[16]

15 "On l'a aussi appelée *axe radical*; nous ignorons la raison de cette dénomination."

16 More extended analyses of Poncelet's development and application of the principle of continuity and ideal chords can be found in [Gray 2005], [Friedelmeyer 2011],

Poncelet had also been a student at the École polytechnique and, as is well-known, he developed what he would call "modern pure geometry" while a prisoner of war in Saratov between 1812 and 1814 [Poncelet 1864]. Upon returning to France, Poncelet used articles and reports to advertise his forthcoming monograph [Poncelet 1822]. Like Gaultier, Poncelet argued that pure geometry could arrive at solutions as general as those found using the coordinate equations of analytic geometry. In 1817, while working as a military engineer in Metz, Poncelet presented several new problems, solutions, and theorems as evidence of the fruitfulness of his modern pure geometry. He explained that he had reached his results through purely geometric methods, but did not have the space within the bounds of this article (formatted as a letter to the editor) to delve into the underlying proofs and method.

Poncelet first detailed his principles and theory in a report to the *Académie des sciences* on May 1, 1820. About one month later, Augustin Louis Cauchy on behalf of himself, François Arago and Poisson composed a review of this report for members of the *Académie des sciences*. Having learned of its existence, Gergonne obtained a copy of the review, which he published in his *Annales* in 1821 supplemented by his extensive footnote commentary [Poncelet 1820]. Cauchy's text with Gergonne's footnotes was thus the first published introduction of Poncelet's ideal chords. In fact, Poncelet's original text as presented to the *Académie* was not published until 1864 [Poncelet 1864]. While many geometers were first introduced to Poncelet's ideal chords through Cauchy's review, here we will follow the definition as presented in the second chapter of [Poncelet 1822], which contained an exposition on ideal chords nearly identical to that in his 1820 memoir. Poncelet's *Traité* also included a transcription of Cauchy's review, preceding Poncelet's own introduction, but he appears to have made changes based on Cauchy's suggestions.

3.1. *Why ideal chords?*

As in 1817, Poncelet began his memoir for the *Académie* by comparing pure geometry to the analysis of coordinates. Poncelet claimed that the latter was uniform, everything "reduces to known principles" of calculation that the geometer employs to develop the consequences in a "more or less elegant" manner. He contrasted this with pure geometry, which lacked any principles that could lead directly and immediately toward the particular

[Nabonnand 2011b], [Nabonnand 2016]. Here we will provide only the details necessary for understanding the defining features and constructibility of ideal chords.

object in view. Instead, he asserted, geometers often resorted to trial and error, filling in gaps along the way to clear an easier route. Poncelet thus proposed to introduce a "rapid and sure way" toward geometrical resources within pure geometry that achieved the same uniformity of procedure as analysis. Poncelet grounded the generality of geometry on his principle of continuity, which established the properties that remained invariant as a figure was deformed under projection. Thus, while certain properties or parts of the figure might disappear, certain preserved properties played the same role that they had in the original, primitive figure. In this way, a single construction or proof that relied on invariant properties could be extended to all configurations of the given figures.

The principle of continuity applied to points of intersection and thus enabled imaginary points and points at infinity to be included in pure geometry, just as in analytic geometry. However, unlike in analytic geometry, Poncelet claimed that his research rested on the purely geometric objects of ancient geometry.

> To expose the theory of real and ideal chords, we will admit several known properties of conic sections; but we will expressly suppose that these properties have been deduced in a purely geometric manner from considerations of the oblique cone with a circular base, as did the ancients, and more particularly Apollonius. [17] [Poncelet 1820, p. 367]

While designating his research as "modern pure geometry," Poncelet emphasized how he continued an established tradition. [18]

Poncelet also situated his contributions with respect to recent publications including those of Gaultier. As Poncelet explained in his introduction, he had been in Russia when Gaultier presented and then published his memoir. Though the two geometers had common results on "the most elementary and easiest case" of the theory of planar tangent circles, Poncelet had gone farther.

> Moreover, in my work I had arrived at properties that M. Gaultier had not made known, undoubtably because they were useless for his research; these are

17 "Pour exposer la théorie des cordes *réelles* et *idéales*, nous admettrons quelques propriétés connues des sections coniques; mais nous supposerons expressément que ces propriétés aient été déduites, d'une manière purement géométrique, de la considération du cône oblique à base circulaire, comme l'ont fait les Anciens, et plus particulièrement Apollonius."

18 This tension has been discussed in [Nabonnand 2015], and can also be seen in the historical references of Gaultier and Steiner.

> precisely the properties that would conduct me, in 1813, to most of the consequences that I propose to develop here, consequences which seem to me to supply ordinary geometry with the resources which it did not possess before, and which can be compared, up to a certain point, with those that algebraic analysis furnishes. [19] [Poncelet 1822, v]

Poncelet provided an initial distinction between Gaultier's research on circles and his own on conic sections. While Gaultier remained focused on elementary geometry, Poncelet aimed to show that ordinary geometry could achieve the same resources as algebraic analysis without losing its figure-based character. Ideal chords served to achieve this balance.

3.2. *Defining ideal chords*

The introduction and first chapter of Poncelet's *Traité* included his critique of ordinary and analytic geometry and his presentation of the principle of continuity. He then introduced ideal chords in his second chapter, titled "Notions préliminaires sur les sécantes et les cordes idéales des sections coniques." His discussion rested on the concept of conjugate diameters of conic sections, which had been first defined in Apollonius' *Conics*. As understood by nineteenth century geometers, a given line defined a unique diameter parallel to it. If one then drew tangent lines from the points of intersection of that diameter with the conic section, the diameter to the conic section parallel to those tangent lines was defined as the conjugate diameter in the direction of the given line.

Poncelet began by stating two properties that held for any secant mn to a conic section C as shown in his Figure 6.

First, the middle or center O of the intercepted chord MN lay at the intersection of mn and the diameter AB conjugate to the parallel diameter mn. Secondly, the point O' where the tangent lines from the extremities of the chord M, N intercepted each other was the harmonic conjugate of O with respect to the conjugate diameter. Thus Poncelet found that $\frac{O'A}{O'B} = \frac{OA}{OB}$. He proposed that these properties would continue to hold in all possible cases, even when the line did not intersect the curve.

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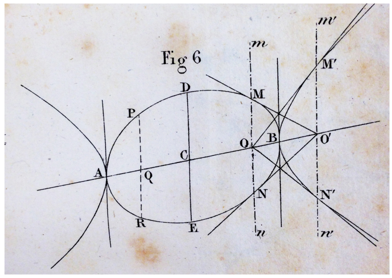
19 "Au reste, j'étais parvenu, dans mon travail, à des propriétés que M. Gaultier ne fait pas connaître, sans doute parce qu'elles étaient inutiles à son objet; ce sont précisément ces propriétés qui me conduisirent, dès 1813, à la plupart des conséquences que je me propose de développer ici, conséquences qui me semblent procurer à la Géométrie ordinaire des ressources qu'elle ne possédait pas auparavant, et qui peuvent être comparées, jusqu'à un certain point, à celles que fournit elle-même l'Analyse algébrique."



Poncelet's Figure 6 [Poncelet 1822]

In the case where a given line $m'n'$ did not meet the curve, then its center was the point O', where the line intersected the conjugate diameter AB. Tangents to the curve from this point O' met at the points M and N on the curve. So the defined chord MN shared the same conjugate diameter with and was parallel to the given line $m'n'$. Poncelet thus concluded that these two properties of the points O and O' were independent of the "reality or non-reality" of the intersection points M and N.

Poncelet recognized that other geometers had named the point O' the *pole* with respect to the line mn, and accordingly the line mn would be the *polar* of the point O'. However, Poncelet explained he did not want to create new terms. Instead one could continue to regard the line mn as a secant of the curve, even when they stopped meeting. Accordingly,

> [...] we will say, in order to conserve the analogy between ideas and language, that these points of intersection with the curve are *imaginary*, and consequently the corresponding chord is itself an *ideal secant* of this curve; and we distinguish it thus from all entirely inconstructible straight lines in its path and direction, which will retain, moreover, the already admitted denomination of *imaginary line*.[20] [Poncelet 1822, p. 27]

By invoking analogy, Poncelet extended the concept of secants. The terms "secant" and "imaginary line" had precedent within mathematics, even though the imaginary, as we saw in the work of Gaultier, might be excluded from geometry. Moreover, Poncelet's vocabulary re-oriented the idea of secants such that their actual intersection points were no longer

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20 "[...] nous dirons, afin de conserver l'analogie entre les idées et le langage, que ses points d'intersection avec la courbe, et par conséquent la corde correspondante, sont *imaginaires*, qu'elle est elle-même *sécante idéale* de cette courbe; et nous la distinguerons ainsi de toute ligne droite entièrement inconstructible dans son cours et sa direction, laquelle conservera d'ailleurs la dénomination, déjà admise, de *droite imaginaire*."

essential. Instead, the properties of the real or ideal center, O or O', served to define the essential features of the real or ideal secant. Poncelet thus attempted to convince his readers that these ideal objects remained constructible and functioned like ordinary chords and secants.

Poncelet continued by examining further properties of this center point O or O'. The ideal secant $m'n'$ defined a real conjugate diameter AB to the given conic. Then on $m'n'$ there would be two points M' and N' that satisfied $\overline{O'M'}^2 = \overline{O'N'}^2 = p.O'A.O'B$. Poncelet had earlier defined the constant p as $p = \frac{OA.OB}{\overline{OM}^2}$, where AB is the conjugate diameter in the direction of the chord MN and O the midpoint. Poncelet explained this was a "very ancient known property about conic sections" [Poncelet 1822, p. 18]. The segment $M'N'$ was evenly divided at the point O' and lay on the ideal secant $m'n'$. These properties defined $M'N'$ as an ideal chord.

Then by constructing all the ideal chords parallel to $M'N'$, the endpoints would form a second conic section sharing the same diameter AB, the same constant p and a conjugate diameter of the same size and direction DE as the given conic. This new conic was *supplementary* to the given conic with respect to the secant $m'n'$. Specifically, Poncelet showed that the supplementary conic to a hyperbola would be an ellipse, to a parabola would be parabola, and to a point would be two intersecting lines. The case of the ellipse and hyperbola is illustrated in his Figure 6.

Cementing this analogy between real and ideal, Poncelet showed that ideal chords in a given conic would become real chords in the supplementary conic, and reciprocally. In this way, the relationship between given and supplementary conics served to further blur the distinction between real and ideal objects. This was precisely Poncelet's intent in conceiving of ideal objects that possessed both real and imaginary points.

> [...] the epithet *ideal* will serve to designate the particular mode of existence of an object that, in becoming, to the contrary, real in the transformation of the primitive figure, will no longer depend in an absolute and real manner on other objects which define it graphically, because these objects will become imaginary. [21] [Poncelet 1822, p. 28]

Poncelet described this transformation as occurring through a progressive and *continuous* movement. While the transformation between real and

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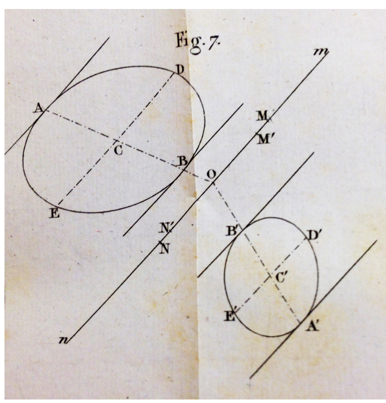
21 "[...] l'épithète idéal servirait à désigner le mode particulier d'existence d'un objet qui, en demeurant au contraire réel dans la transformation de la figure primitive, cesserait cependant de dépendre d'une manière absolue et réelle d'autres objets qui le définissent graphiquement, parce que ces objets seraient devenus imaginaires."

imaginary appeared novel, Poncelet compared it to the different modes of existence in the use of infinitely small and infinitely large.

Poncelet turned to the properties of common chords or secants to two conics. First, if two conics, C and C', shared a real or ideal chord MN, then the conjugate diameters to the direction of MN would meet in the same point, the chord's center O. Secondly, for respective diameters AB and $A'B'$ with associated constant values p and p', the product $p.OA.OB$ would be equal to $p'.OA'.OB'$. These two properties were necessary and sufficient for identifying common chords and enabled Poncelet to describe a single general construction for the real or ideal common chord common to two given conics.

Poncelet explained that for two given conics, C and C' there would be an infinite number of points O and lines mn satisfying the first property of concurrence. To show this, one could draw parallel tangents to the two given conics in any direction. Then by drawing diameters AB and $A'B'$ through the tangent points, their point of intersection would designate the center point O and the line mn parallel to the tangents and passing through O would be the corresponding line. Further, Poncelet added that all such O points would form a conic section passing through the centers of the conics C, C'.



Poncelet's Figure 7 [Poncelet 1822]

Then to satisfy the second property, since the values p and p' were determined by the value of the half-chord OM, the chord MN of the conic C had to be equal in length to the chord $M'N'$ of the conic C'. Poncelet demonstrated this equality as guaranteed by his principle of continuity. He showed that there existed positions of mn such that $MN < M'N'$ and

$MN > M'N'$. By a continuous deformation, he concluded, there must exist a position of the line mn such that $MN = M'N'$. Further, the position of mn was unique, since each M and M' defined by the fact that $\overline{OM}^2 = p.OA.OB$, would generate a curve intersecting at M and M', and when these points coincided, $OM = OM'$. So this unique segment $MN = M'N'$ was the real or ideal common chord of the two conics, as shown in his Figure 7.

3.3. *Ideal chords as natural, useful and general*

Poncelet lauded the graphic, figure-based qualities of pure geometry. He described his vocabulary as "natural" and complementing existent definitions. In this respect, he contrasted his ideal common chords with the much more limited radical axes, which presented

> [...] only a particular character of the object defined, applicable only in the case of the circle, and consequently lose from view the general and purely graphic dependence that links this object to other parts of the figure. [22]

More "particular and insignificant" terms, among which Poncelet included radical and polar axes, resulted in new, specialized vocabulary. In order to remain general, Poncelet used his principle of continuity under projection to determine properties of real and ideal chords. While Poncelet described his research as graphic, he did not address the fact that ideal chords did not look at all like previous conceptions of chords because they did not intersect their conic sections. Instead, he focused on the invariant properties of well-known ordinary chords that continued to hold for ideal chords. Poncelet further emphasized this continuity in their common noun. He explained that the adjectives *real* and *ideal* only pertained to the particular situation with respect to other objects. As the use of supplementary conics demonstrated, a chord could be at once real and ideal depending on the choice of conic section.

Like Gaultier, Poncelet also saw the potential of new definitions as abbreviations. But while Gaultier had assured his reader that these properties involved no more than familiar constructed objects, Poncelet claimed that his definitions at once abbreviated and extended geometry.

> These definitions have, over all those that one could substitute for them, the advantage of being able to extend directly to any points, lines and surfaces; they

22 "[...] qu'un caractère particulier de l'objet défini, applicable seulement aux cas du cercle, et de faire perdre de vue, par conséquent, la dépendance générale et purement graphique qui lie cet objet aux autres parties de la figure." [Poncelet 1822, p. 41]

> are also neither indifferent nor useless in themselves; they serve to abridge discourse and to extend the object of geometrical conceptions; finally they permit to establish a point of contact, if not always real at least fictive, between figures that appear, at first sight, to have no relationship between them, and to discover effortlessly the relations and properties that are common to them. [23] [Poncelet 1822, p. 28]

Poncelet demonstrated the utility of his definitions by solving problems in ordinary constructive geometry, including the Apollonius problem, but also problems that were previously limited to analytic geometry because of imaginary points of intersection or points at infinity.

He had published several solutions to the problem in the *Annales* at the request of Gergonne in 1821 [Poncelet 1821a]. Unlike in his *Académie submission*, here Poncelet used radical axes and radical centers in his constructions. Though the wording could have been altered by Gergonne, it seems more likely that Poncelet reverted to the already common "radical axes" for the more particular case of circles in order to appeal to a broader audience. He concluded by summarizing the merits of his constructions, equating radical axes with common chords.

> The preceding constructions have the advantage of being very simple, because they only require drawing straight lines and they dispense with constructing the common chords or radical axes that belong to the three proposed circles, combined two by two. [24] [Poncelet 1821a, p. 321]

The following year, Poncelet proved the validity of these constructions in his *Traité* using common secants instead of radical axes, though he continued to refer to radical centers [Poncelet 1822, pp. 141–144]. [25]

Because of the generality of common secants and the principle of continuity, Poncelet asserted that the constructions could be applied to any configuration of circles and conic sections in general.

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[23] "Ces définitions ont, sur toutes celles qu'on pourrait leur substituer, l'avantage de pouvoir s'étendre directement à des points, des lignes et des surfaces quelconques ; elles ne sont d'ailleurs ni indifférentes, ni inutiles en elles-mêmes ; elles servent à abréger le discours et à étendre l'objet des conceptions géométriques ; enfin elles permettent d'établir un point de contact, sinon toujours réel, au moins fictif, entre des figures qui paraissent, au premier aspect, n'avoir aucun rapport entre elles, et de découvrir sans peine les relations et les propriétés qui leur sont communes."

[24] "Les constructions qui precedent ont l'avantage d'être fort simples, puisqu'elles n'exigent que le tracé de lignes droites et qu'elles dispensent de construire les cordes communes ou les axes radicaux qui appartiennent aux trois cercles proposés, combinés deux à deux."

[25] A detailed description of Poncelet's solutions to the Apollonius problem can be found in [Friedelmeyer 2016].

> Moreover, as these properties intimately relate to the theory of common secants and tangents, and because they can be extended, as we shall see in what follows, to conic sections in general, by aid of the principle of projection posed in the first section, their examination essentially returns to the object of this work. [26] [Poncelet 1822, pp. 141–142]

Consequently, Poncelet's definitions resulted in increased problem solving capability and each solution required fewer cases.

Because of this extension, Poncelet argued that this generalization of the "language and conceptions of ordinary geometry" now aligned with that of analytic geometry and algebraic analysis. In fact, for Poncelet, a general language was the only means with which geometry could follow the progress of algebraic analysis.

> To want to banish expressions based on exact and rigorous, though sometimes purely figurative, relations in order to substitute insignificant names that only recall particular or unusual characteristics of the object defined; to avoid using in geometric reasoning expressions and notions which qualify and recall non-existence, this will be truly to refuse to rational Geometry the only means that it has of following the progress of algebraic Analysis, and of interpreting in a satisfying manner the consequences of often bizarre results that it reaches. [27] [Poncelet 1822, p. 35]

When applied to geometry, algebraic analysis had introduced imaginary points and distances. Rational geometry would lag behind unless these "often bizarre" objects received a constructive interpretation within the language of geometry. As we will see, ideal objects also provided a useful, new set of terms for analytic geometers.

3.4. *Applications and interpretations of ideal chords*

As mentioned above, the first review of Poncelet's text preceded publication. Though the *Académie* report was ultimately favorable, Cauchy

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26 "D'ailleurs, comme ces propriétés se rattachent d'une matière intime à la théorie des sécantes et tangentes communes, et qu'elles peuvent s'étendre, ainsi que nous le verrons par la suite, aux sections coniques en général, à l'aide des principes de projection posés dans la première section, leur examen rentre essentiellement dans l'objet de cet ouvrage."

27 "Vouloir repousser des expressions fondées sur des rapports exacts et rigoureux, quoique parfois purement figurés, pour y substituer des noms insignifiants et qui ne rappellent que des caractères particuliers ou insolites de l'objet défini; éviter de se servir dans le raisonnement géométrique des expressions et des notions qui qualifient la non-existence et la rappellent, ce serait véritablement refuser à la Géométrie rationnelle les seuls moyens qu'elle ait de suivre les progrès de l'Analyse algébrique, et d'interpréter d'une manière satisfaisante les conséquences des résultats souvent bizarres, auxquels elle parvient."

began by criticizing Poncelet's principle of continuity as no more than "strong induction" [Poncelet 1820, p. 73].[28] He admitted that Poncelet appeared to have used continuity unproblematically in his study of conic sections, but cautioned against its general application. Cauchy even provided a counterexample of how the principle of continuity failed if applied to definite integrals in measuring lengths, areas, and volumes. Nevertheless, he admired and recommended Poncelet's exposition. In particular, he considered Poncelet's development of ideal chords as "worthy of remark" [Poncelet 1820, pp. 73–74].

In these remarks, Cauchy decided to present ideal chords analytically, rather than in the form used by Poncelet. While differing fundamentally with Poncelet's efforts to advance pure geometry, Cauchy's explanation captured many of the object's key features and emphasized the ease of translation between Poncelet's geometry and coordinate representation. Indeed, as Cauchy noted, though he used analysis, "one could reach the same goal through geometrical considerations".[29] [Poncelet 1820, p. 79]

Cauchy considered the situation of a general conic section and a general line. The points of intersection of any line always could be found analytically, and in case the line did not intersect the curve, then the distance between the imaginary points of intersection would be of the form $2C\sqrt{-1}$. However, because the length of the chord would be the real quantity $2C$, the middle of the intercepted chord would still have real coordinates. Cauchy explained that to study properties of this imaginary chord, one could substitute a "fictitious" chord of the form $2C$, lying on the same line as $2C\sqrt{-1}$ and sharing the same midpoint. He defined this fictitious chord as Poncelet's ideal chord. Because the ideal chord had a real length, $2C$, ideal chords were constructible in geometry.

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[28] Cauchy's criticism of Poncelet gained critical historical attention, in part because of Cauchy's later contributions to continuity within analysis. However, the report was written on behalf of a three person committee, and the contributions of each individual member are unknown. For instance, Poisson expressed very similar doubts about Euler's use of real and imaginary quantities.

> These formulas are due to Euler, who found them by a sort of induction based on the passage from real to imaginary quantities; an induction that one can employ well as a means of discovery, but the results of which must be confirmed by direct and rigorous methods. (Ces formules sont dues à Euler, qui les a trouvées par une sorte d'induction fondée sur le passage des quantités réelles aux imaginaires; induction qu'on peut bien employer comme un moyen de découverte, mais dont les résultats ont besoin d'être confirmés par des méthodes directes et rigoureuses). [Poisson 1813, p. 219]

[29] Cauchy, on behalf of the *Académie*, would again rewrite Poncelet's geometry in a second report in 1825 [Poncelet & Cauchy 1825].

Cauchy concluded that because two conic sections could meet in four, two or no points, in general they shared six common chords through four real points or two common chords where one or both were ideal. The ideal or real common chords of any given conic sections could be interchanged through projection. The ease with which Cauchy rewrote Poncelet shows that ideal chords might be useful in both analytic and ordinary geometry. While Cauchy did not adhere to Poncelet's commitment to pure geometry, he found merit in Poncelet's principles and definitions and validated Poncelet's claim that they enabled "the means of simplifying and generalizing" solutions of numerous problems [Poncelet 1820, p. 77].

Mitigating Cauchy's praise, Gergonne objected to ideal chords in several accompanying and distracting footnotes. He described Poncelet's new definition as "subject to numerous exceptions" and requiring "ingenious ideas" [Poncelet 1820, p. 80]. Thus ideal chords lacked both generality and simplicity. Cauchy's summary showed that ideal chords followed from the coordinate representation of conic sections as second degree equations. Gergonne did not share Poncelet's determination to import this generality from analysis to geometry. He claimed that the use of ideal chords would compromise the advantages and superiority of geometry over other sciences. That is, a chord should look like a chord.

Both Cauchy and Gergonne observed the similarity between ideal chords and radical axes. Gergonne, however, compared the latter favorably, questioning why ideal chords were even necessary.

> In the case of two circles, for example, is it not better to define the radical axis, the locus of points at which the tangents to these two circles are of the same length, than to say that this is the common chord to these two circles? [30] [Poncelet 1820, p. 80]

Cauchy simply noted that this fact of tangents of equal length to two circles through a point was already known by geometers and the lines of this type had been named radical axes by Gaultier. Cauchy and Gergonne agreed that radical axes had been defined for circles, but Gergonne suggested that this made ideal chords unnecessary. On the other hand, Cauchy presented the property of equal tangents only as one of many possessed by the more applicable concept of ideal chords.

30 "Dans le cas de deux cercles, par exemple, ne vaut-il pas mieux définir l'axe radical, le lieu des points pour lesquels les tangentes aux deux cercles sont de même longueur, que de dire que c'est la corde commune à ces deux cercles ?"

A second review followed the publication of Poncelet's *Traité* in 1823. Writing for the *Bulletin de Férussac*, Terquem unequivocally praised Poncelet's contributions, writing that "Poncelet has rendered a great service to scholars" [Terquem 1823]. His review pointed to the particular role played by ideal chords.

> The author calls ideal chord or secant the fixed line of which we spoke above, in the case where it is entirely outside of the cone, and when one brings all the lines of intersection into a single plane, and makes them turn around the line like a hinge. M. Poncelet makes great use of these ideal secants, which are the radical axes of M. Gauthier [sic].[31] [Terquem 1823, p. 6]

Though Terquem recognized the importance of ideal chords for Poncelet, he implied they were synonymous to radical axes.

Terquem clearly read Poncelet's *Traité*, but geometers reading and writing for the *Annales* responded to Poncelet's ideal chords as filtered by the sentiment of the *Académie*'s initial review. While ideal chords soon appeared in contemporary research, geometers remained skeptical about the principle of continuity.

In 1826 Charles Sturm published a two-part article on second order lines ([Sturm 1826a], [Sturm 1826b]). Within his research in coordinate geometry, Sturm described the midpoint of a chord as "a point always real and assignable" even when the chords extremities became imaginary. Moreover, Sturm claimed these properties could be extended to straight lines that intersected curves of any degree. As proof, he stated that all the necessary relations followed from the theory of transversals. Sturm revealed he was familiar with Poncelet's work though perhaps only through Cauchy, and had a similarly apprehensive view.

> One must not then confound them [the above relations] with the considerations of M. Poncelet on the law of continuity. The distinction has already been made with care by M. Cauchy in his report inserted in the XIth volume of the *Annales* (page 69) and placed afterward at the front of the *Traité des Propriétés projectives des figures*.[32] [Sturm 1826a, p. 279]

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31 "L'auteur appelle corde ou sécante idéale la droite fixe dont nous avons parlé cidessus, dans le cas où elle est entièrement hors du cône, et lorsqu'on ramène toutes les lignes d'intersections dans un seul plan, et les fesant tourner autour de la droite comme charnière. M. Poncelet tire un parti très-avantageux de ces sécantes idéales, qui sont les axes radicaux de M Gauthier."

32 "On ne doit pas d'ailleurs les confondre avec les considérations de M. Poncelet sur la loi de continuité. La distinction en a été déjà faite, avec soin, par M. Cauchy, dans son rapport inséré au tome XI[e] des *Annales* (pag. 69) et placé depuis en tête du *Traité des Propriétés projectives des figures*."

Sturm distanced himself from Poncelet, but then employed ideal chords without commentary or credit in the second half of his paper.

> One knows that three circles traced on the same plane and having a common chord, real or ideal, can be envisaged as three second order lines which have the same points of intersection, two of which situated at infinity.[33] [Sturm 1826b, p. 184]

Sturm thus knew and seemed to approve of Poncelet's vocabulary and results, though he masked them under the common anonymous attribution "one knows."

Though initially critical, Gergonne first employed ideal chords in a posed problem on lines of order m and their real or ideal secants in 1827. Rather than explain the adjective ideal, in a footnote he defined common secants.

> Here one understands uniquely by common secant to two lines of mth order a line that includes m of their intersections, real or imaginary.[34] [Gergonne 1827a, p. 35]

In this instance at least, Poncelet's goal of integrating ideal secants had succeeded as secants themselves were here independent of case distinctions.

Gergonne also found new applications of ideal within a non-coordinate setting, by using the term ideals as part of his new definition of the degree or class of a curve or surface [Gergonne 1827c, p. 151]. To accommodate this usage, Gergonne even introduced the plural "idéaux," which he explained in a footnote. However, Gergonne modified Poncelet's original meaning by employing ideal to describe intersection points. In Poncelet's vocabulary ideal could describe lines and planes, but points were either imaginary or real. Gergonne's alteration corresponded to his commitment to the principle of duality in which properties of points corresponded to properties of lines. Thus the ideal tangent lines of curves of mth class corresponded to the ideal intersection points of curves of mth degree. In turn, when Étienne Bobillier used the words degree and class in his article, he reiterated the definitions espoused by Gergonne [Bobillier 1827].

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33 "On sait que trois cercles tracés sur un même plan et ayant une corde commune, réelle ou idéale, peuvent être envisagés comme trois lignes du second ordre qui ont les même points d'intersections, dont deux situés à l'infini."

34 "On entends uniquement ici par *sécante commune* à deux lignes du m.ième ordre une droite qui renferme m de leurs intersections, réelles ou imaginaires."

As noted above, the German analytic geometer Plücker had employed radical axes in 1827. In this paper on solutions to the Apollonius problem, he equated radical axes to real or ideal common chords [Plücker 1827, p. 29].[35] In his 1828 monograph, *Analytisch-geometrische Entwicklungen*, Plücker credited both Poncelet and Gaultier, but chose instead to introduce the expression "Chordal":

> Poncelet has named the line "*ideal chord*", which we call Chordal, in the case where the respective circles do not intersect each other; more common is the expression introduced by Gaultier : "*radical axis*". Our Chordal point of two circles is called : "radical point of two circles"[36] [Plücker 1828a, p. 49]

Plücker seemed to suggest that "Chordal" was merely a German translation of the alternate French expressions. By contrast, he continued to write "ideal" when referencing objects that contained real and imaginary points.

Other rejections of Poncelet's vocabulary were more explicit. Michel Chasles began his article "Mémoire sur les propriétés des systèmes de sections coniques, situées dans un même plan" by asking to be excused for not citing Poncelet fully [Chasles 1828a]. Chasles asserted that his findings had first been reached without knowledge of Poncelet's contributions, and he was currently without the necessary resources to add in all the references. Chasles continued by citing Gaultier and his contributions to the study of circles. Accordingly, Chasles employed Gaultier's vocabulary. He constructively defined the *radical axis* of two circles with respect to similitude centers and polars and then remarked that this line "will be the common chord to two circles if it cuts them" [Chasles 1828a, p. 279]. This *if* ruled out the possibility of ideal common chords.

Chasles explained that he found Poncelet's ideals misleading.

> These fixed lines that, when the two curves meet, are common chords of them, have been called by M. Poncelet, in the contrary case, *ideal* common chords, which could make one think that then these lines disappear, whereas they have a real existence in every case, and there is nothing imaginary except the intersections of the two curves. On the other hand, the expression *radical axes* cannot be used here where it is not a question of circles. Consequently, until we have found for these lines, which play a very important role in the

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35 For a summary of Plücker's solutions to the Apollonius problem see [Lorenat 2016].

36 "Herr Poncelet hat die Linie, welche wir Chordale nennen, im Falle, dass die bezüglichen Kreise sich nicht schneiden, 'Corde ideale' genannt; gebräuchlicher ist der von H. Gaultier eingeführte Ausdruck : 'axe radical'. Unser Chordal Punct zweier Kreise heisst : 'point radical de deux cercles'."

> theory of conic sections, a more suitable expression, I will call them *axes of symptose*, because of their properties relative to the tangents from their different points. [37] [Chasles 1828a, p. 285]

Unlike Poncelet, Chasles continued to use radical axes in the case of circles, and only used his new term, *axis of symptose*, when describing conic sections in general. Other than his choice of words, Chalses mirrored many of Poncelet's arguments. Chasles derived the "real existence" of axes of symptose from algebraic analysis and claimed their applications "extended the domain of geometry." Chasles even referred to Poncelet's "very clear" discussion of axes of symptose in his *Traité*, as if the term was actually present there.

The article was published in April and a few months later Chasles wrote to Poncelet, apparently in response to a letter. In his response Chasles reiterated his critique of ideals as "much too inconvenient to apply to a real thing" and liable to result in errors since the word sometimes represented a real thing and sometimes an imaginary thing. [38] For Chasles, there should be "less vagueness and greater precision in this science of truth" [Chasles 1828b].

So while Poncelet's ideal chords were well received by analytic geometers, within "pure geometry" the new definitions were ignored, misunderstood, or rejected as confusing. Certainly, ideal chords proved useful in discussing imaginary points of intersection, but Poncelet's claims for the natural advantages of his word choice seemed to have little effect on his readers.

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37 "Ces droites fixes qui, lorsque deux courbes se coupent, en sont des cordes communes, ont été appelées par M. Poncelet, dans le cas contraire, des cordes communes *idéales*, ce qui pourrait faire penser qu'alors ces droites disparaissent, tandis qu'elles ont une existence réelle dans tous les cas, et qu'il n'y a alors d'imaginaires que les intersections des deux courbes. D'un autre côté, la dénomination d'*axes radicaux* ne saurait convenir ici où il n'est plus question de cercles. En conséquence, en attendant qu'on ait trouvé pour ces droites, qui jouent un rôle très-important dans la théorie des sections coniques, une dénomination plus convenable, je les appelerai des *axes de symptose*, à raison de leurs propriétés relatives aux tangentes issues de leurs différens points."

38 Chasles himself was not immune to these same accusations. During the second half of the nineteenth century, Charles Hermite criticized the geometric language adopted by Chasles, as discussed in [Goldstein 2011].

4. STEINER'S LINE OF EQUAL POWERS

Outside of France, Poncelet's *Traité* received a positive review in the first volume of the Berlin mathematics journal, *Journal für die reine und angewandte Mathematik*. The reviewer, probably the editor August Leopold Crelle, noted that Poncelet's text contained "fruitful methods" and was "of the highest interest to geometry." The review also served to establish that the work of Steiner, who had just published his first article in that same volume, was independent from that of Poncelet.

> The work in the above treatise is therefore related because the author of the treatise, Herr Steiner, and Herr Poncelet, both, without knowing one another, worked on the same subject and have found several identical results.[39] [*Crelle* 1826, p. 96]

The reviewer concluded in promising that Steiner would soon complete a substantial, interesting work on the same topic.

4.1. *Why lines of equal powers?*

In 1826 Steiner was employed as a private teacher in Berlin and looking for a more permanent position [Lange 1899]. He began publishing his geometrical research in the first volume of Crelle's *Journal für die reine und angewandte Mathematik* in 1826. In this issue, Steiner wrote six of the thirteen articles that had been classified as geometry, including "Einige geometrische Betrachtungen" [Steiner 1826b]. In contrast to the ambiguous title, this article centered on new results on relationships between circles.

> The following paragraphs of introductory remarks contain the groundwork of geometric research concerning the intersections of circles. These permit us to develop the solutions of almost all problems about intersection and contact of circles, and in fact in most cases very simply. As well a certain connection often becomes visible through them between several problems that at first sight seem to have no commonality.[40]

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39 "Mit der obigen Abhandlung steht das Werk deshalb in Beziehung weil der Verfasser der Abhandlung, Herr Steiner, und Herr Poncelet, beide, ohne von einander zu wissen, an dem nämlichen Gegenstande gearbeitet und mehrere gleiche Resultate gefunden haben."

40 "Die in den nachstehenden Paragraphen angefangenen Betrachtungen enthalten die Grundlage der geometrischen Untersuchung über das Schneiden der Kreise. Es lassen sich daraus die Auflösungen fast aller Aufgaben über das Schneiden und Berühren der Kreise entwickeln, und zwar in den meisten Fällen sehr einfach, auch wird durch sie oft zwischen mehreren Aufgaben, welche auf den ersten Anblick keine Gemeinschaft zu haben scheinen, ein gewisser Zusammenhang sichtbar." [Steiner 1826b, p. 161]

His research on this subject had developed in response to four sets of problems, which he stated with a variety of descriptions and proper names: finding a circle tangent to three given circles, the Malfatti problem (finding three mutually tangent circles tangent to the sides of a given triangle), the fifteenth theorem of the fourth book of Pappus's *Collectiones mathematicae* (concerning ratios between tangent circles inscribed in a semi-circle), and various porisms and "purely geometrical considerations" of second degree curves and surfaces. As had Gaultier and Poncelet, Steiner professed that he had reached his results independently of recent publications. As Crelle had noted in his review of Poncelet's *Traité*, Steiner had recently learned of Poncelet's similar findings. [41] However, Steiner explained that he had conducted his own research based only on the much older theories of François Viète and Pappus.

> For insurance that the author of this work did not know previously what the French had done in this regard, he hopes to rely on the witness not only of those of his acquaintances who, by daily interaction with him, observed the origin and development of his work, but also, for those who know the subject, the more comprehensive, more general mode of development in the research, from which not only all those matters but also a great quantity of new results issue forth of themselves. [42] [Steiner 1826b, p. 162]

Steiner promised his research on tangent circles would generalize to circles intersecting at given angles rather than in tangent points, to analogous results for spherical surfaces and spheres.

In order to address his three motivating problems, Steiner developed a systematic vocabulary to describe the relationships between circles, straight lines, and points in the plane. He divided this exposition into three sections with explanatory titles: "On the powers of circles that lie in the same plane," "On the similitude points and similitude lines of circles that lie in the same plane," and "On the common powers of circles that lie in the same plane." We focus on the first section, which was divided into five parts. Each part considered a new set of objects beginning with a configuration

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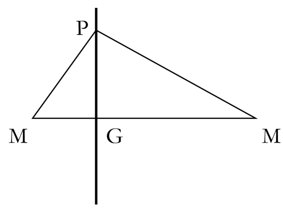
41 In the next few years, Steiner would begin to adopt many of the practices from Poncelet's modern pure geometry. His relationship to French geometers is discussed in [Lorenat 2016].

42 "Für die Versicherung, dass der Verfasser Dasjenige, was die Franzosen in dieser Hinsicht getan, vorher nicht gekannt habe, hofft er, werden nicht allein diejenigen seiner Bekannten, welche, bei täglichem Umgange mit ihm, die Entstehung und Entwicklung seiner Arbeiten beobachteten, sondern dem Sachkenner wird auch schon die umfassendere, allgemeinere Entwicklungsweise in den Untersuchungen, aus welcher nicht nur alle jene Betrachtungen, sondern auch eine grosse Menge neuer Resultate von selbst hervorgehen, ein Zeugnis ablegen."

of two points. Steiner organized this part of his text much like Gaultier, using multiple cases with accompanying figures. However, while Gaultier celebrated recent advances in modern and descriptive geometry, Steiner grounded his derivations in Euclidean geometry.

4.2. *Defining lines of equal powers*

Steiner began with the simplest case: two coplanar points M, m situated as pictured in his Figure 8. If the lines Mm and PG remained perpendicular, Steiner concluded that all points P constructed on a perpendicular to the line Mm from the point G conserved the fixed equality $MP^2 - mP^2 = MG^2 - mG^2$. Conversely, the locus of points P whose distance from each given point M and m when squared was equal to a given quantity would be a perpendicular line PG to Mm.

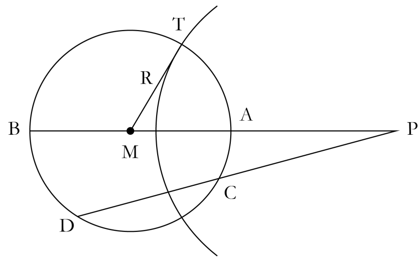


Steiner's Figure 8 (reproduced following [Steiner 1826b]

Steiner then turned to a circle centered at M and a coplanar point P. He vaguely remarked by way of introduction that "in the school books on geometry, one finds this relationship" [Steiner 1826b, p. 164].[43] In Steiner's Figure 9, two secants drawn to circle M and passing through point P would cut the circle at points A, B and C, D respectively. Then the intercepted secants determined four segments such that the product $PA \times PB$ was equal to $PC \times PD$. This product Steiner called "the power of a circle with respect to a point" or *reciprocally* "the power of a point with respect to a circle." In an accompanying footnote, he pointed to the "ancient" use of power of a hyperbola as a precedent in this choice of vocabulary.[44]

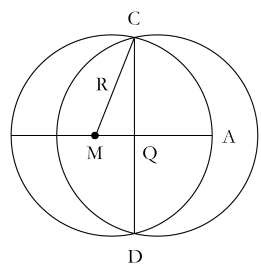
43 Steiner's manuscript more precisely attributes this result to Euclid III.36 [Steiner 1931, p. 28].

44 The power of a hyperbola is the area of the rhombus described on the major and minor axes. An explicit definition of the power of a hyperbola as "la moitié du quarré du demi-axe" can be found in the twenty-seventh volume of the *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers* [*Dictionnaire* 1778, p. 793].



Steiner's Figure 9 [Steiner 1826b]

Steiner elaborated three possible cases. Each case showed the relationship of the power of the point to lines associated with the given circle as well as how to calculate its value. If the point P lay outside of the circle, as in Steiner's Figure 9, the power of P would be equivalent to the square of the tangent segment to circle M from P. This distance could also be calculated as PM^2 minus the square of the radius, MA. Thus $PT^2 = PM^2 - R^2$.



Steiner's Figure 10 [Steiner 1826b]

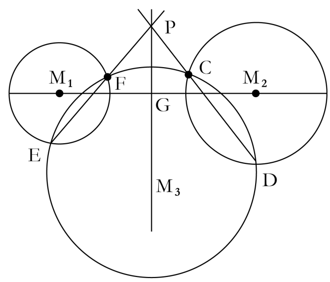
If point Q lay inside the circle, as in Steiner's Figure 10, then the power of Q with respect to M would be what Steiner described as "the square of half the shortest chord one can draw through Q to the circumference of M," that is, the square of the segment QC. Equivalently, the length of QC was the square of the circle's radius minus the distance MQ^2. So, $QC^2 = R^2 - MQ^2$. The symmetry between the equations for PT^2 and QC^2 showed how the tangent segment functioned like the circle's semi-chord.

Finally, the power of a point would be zero when the point lay on the respective circle's circumference. Steiner gave no illustration of this final case.

In the following part, Steiner combined his results by considering two circles of given size and position and denoted by their centers at points M and m. He combined equations derived from the powers of the centers to determine that the locus of points of equal powers to both M and m would form what he called the "line of equal powers." This line PG lay perpendicular to the line containing their centers, Mm.

Steiner illustrated only the case of intersecting circles, although he did not label circle m. In his Figure 10, the line of equal powers contained the common chord of M and m. In the case of two tangent circles, Steiner explained that the line of equal powers would also be the shared tangent line. From this property, he determined that the line of equal powers was also the locus of points with equal tangents to circles M and m.

Steiner then introduced points of equal powers of three circles M_1, M_2, M_3 as the intersection of the three lines of equal powers determined by each pair of circles. Only following this definition did Steiner explain a general construction for the line of equal powers for two circles in any possible configuration (shown in his Figure 13).



Steiner's Figure 13 [Steiner 1826b]

Steiner concluded his analysis of lines of equal powers by suggesting that the same properties "are also found in a similar manner for every second degree curve" and "analogous theorems can be found for all second degree surfaces" [Steiner 1826b, p. 169]. He gave no hint about how these results might be extended. In his application to the Malfatti problem at the end of the first half of his article, he applied his solutions to cases where the three circles lay on a sphere and for any three coplanar second degree curves. Thus, presumably Steiner had indeed determined analogous relations beyond planar circles, but the constructions or definitions of lines of equal powers were not specified. Steiner then considered the case of four spheres. Though not familiar with Gaultier's solutions, Steiner did cite an earlier generalization to finding four spheres by a Berlin mathematician that had appeared in Gergonne's *Annales* volume I page 196, volume II page 287, and volume X page 298. [45] Steiner observed that the case of four spheres was not always determinate, and instead suggested a version that could always be solved in which the circles would be inscribed by four planar surfaces rather than a single tetrahedron.

4.3. *Lines of equal powers as systematic, simple and applicable*

Steiner exhibited a distinctive and consistent presentation style. In each part he began by describing a new configuration, ascertaining some of its properties and deriving a new definition. Like the line of equal powers, all of his definitions included constructive and quantitative aspects. Following a demonstration, Steiner often stated the reverse conclusion without providing a justification. This step enabled progressing from properties determined by a given construction to constructions determined by given properties. His few references were to elementary texts, and his neologisms supported his initial claim of independent derivations as well as an expectation of a similarly educated audience.

Steiner's decision to create his own vocabulary may have been a result of his unfamiliarity with recent literature, as he hinted in his introduction. Nevertheless, he used the descriptor "power" to create a systematic vocabulary. By considering first two points, then one point and one circle, then two circles, then three circles, and finally an arbitrary set of circles, Steiner methodically applied Euclidean results to increasingly complicated configurations. In this way, he applied power to define points of equal powers, common power between coplanar circles, the power position of a point or a circle, and power circles. Thus the line of equal powers served as an intermediary definition finally leading to power circles. [46]

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45 These precise citations somewhat undermine Steiner's claim to be unfamiliar with the French literature, except that the first two reference are only to statements of the problem and the final reference is to a Berlin mathematician who also contributed to the first volume of Crelle's *Journal*, Daniel Charles Ludolph Lehmus (though his name is printed rather amusingly as the much more German-looking Lechmütz in the *Annales*).

46 Steiner's *Potenzkreise* were translated into French as "cercles de commune puissance" [Steiner & Gergonne 1827]. Steiner defined power circles with respect to a given pair of circles. Any pair of circles determined two similitude points, most frequently illustrated in the case of two exterior circles as the respective intersection points between their "inner" (intersecting between the two circles) and outer (intersecting on the side of the smaller circle) common tangents. He used this concept to define common power between two coplanar circles M_1, M_2 as the power of a similitude point with respect to any circle orthogonal to both M_1 and M_2 (these orthogonal circles are the series of radical circles noted by Gaultier). Each pair of circles thus had a common power corresponding to each of their two similitude points. Steiner then

Steiner suggested that these power circles could then be used to solve his three motivating problems (though he did not include a solution to the Apollonius problem nor a proof of his solution to the Malfatti problem). Steiner would emphasize this unified conception of geometry in the introduction to his *Systematische Entwickelung* a few years later, in which he aimed to present an "organically connected whole" [Steiner 1832a, p. 3].

The numerical properties designated by power and radical are almost identical as the former is simply the square of the latter. However, as Steiner noted, the use of the power relation dated back to ancient geometry. Thus, like Poncelet, Steiner was not so much creating new vocabulary as extending established terms. While the definition of the power of a hyperbola was known, Steiner applied power to define a reciprocal relation between points and circles. Moreover, Steiner mirrored the Euclidean hierarchy of points before lines by progressing from power of a point to power on a line. While Gaultier focused on circles and Poncelet generalized to conic sections, Steiner defined power by making a point the fundamental object of study. This is especially apparent in Steiner's definition of the radical axis as a locus of points of equal powers as compared to Gaultier's definition in terms of centers of a series of radical circles.

With his professed ignorance of contemporary geometry, it is not surprising that Steiner's article contained much that had been previously explored under different names in French. Yet, his system generated new results. Steiner's readers considered the common power of coplanar circles as a new concept and applied it to solving problems. Steiner had emphasized exactly these problem-solving capabilities in his introduction and his solution to the Malfatti problem seemed to offer the first satisfying figure-based solution. [47] Finally, Steiner claimed, though did not demonstrate, that his definitions and solutions could be generalized to conic sections and diverse configurations. With these contributions and promises, he succeeded in drawing the attention of a French audience.

4.4. *Applications and interpretations of lines of equal powers*

Soon after its publication, Gergonne rewrote Steiner's article for the *Annales* under the title "Géométrie Pure. Théorie générale des contacts et des intersections des cercles" [Steiner & Gergonne 1827]. Gergonne

defined a power circle of two given circles as a circle centered at one center of similitude, such that the radius squared equaled the common power of the two circles with respect to that similitude center.

47 On Steiner's solution to the Malfatti problem and its reception, see [Lorenat 2012].

did not merely translate Steiner's text, but also modified the exposition, references, and vocabulary. By 1827 Gergonne had employed radical axes and ideal chords in his publications, and his adaptation of Steiner reveals an understanding of Gaultier and Poncelet. As Gergonne noted, Steiner had presented the elementary theories of "similitude centers, axes and planes, radical axes, planes and centers, and finally poles, polars, polar planes and conjugate polars" to elegantly solve difficult problems. While the above theories had appeared often in the *Annales*, "the author at once had extended and simplified in a very notable manner" [Steiner & Gergonne 1827, p. 286].[48] On top of this, Gergonne promised to add greater clarity and brevity by abbreviating Steiner's treatment "for any intelligent reader." Gergonne updated Steiner's vocabulary, removed illustrated figures, added contemporary (French) references, and established a standard for what could be properly assumed as well known for *Annales* readers. Notably, this mode of exposition resulted in a so-called abridged Steiner, which was in fact substantially longer than the original German piece. Gergonne was the medium by which Steiner's work first became known to a broader French audience.

Gergonne changed the order of Steiner's sections so that Steiner's lines of equal powers appeared toward the middle of the text. Gergonne began his second section by stating known properties of a coplanar circle and point: "One knows that for any secant drawn from a coplanar point through a circle, the product of the distance from the point to the two intersections of the secant with the circumference will be a constant quantity, independent of the secant's direction" [Steiner & Gergonne 1827, p. 294].[49] Gergonne was impressed by the reciprocal relationship here and credited Steiner with designating this constant product as either the *power of a point with respect to a circle* or the *power of a circle with respect to a point*.

Using Steiner's vocabulary, Gergonne stated that by definition, all points of equal power with respect to two circles would lie on a straight line perpendicular to the line containing their centers. Gergonne acknowledged Steiner's designation of the *line of equal power*, but decided "we will continue to call it their *radical axis*, after M. Gaultier." Gergonne elaborated four possible cases of two circle relationship: interior with no

48 "[...] l'auteur a tout à fois, étendues et simplifiées d'une manière assez notable."

49 "On sait que si, par une point situé comme on le voudra sur le plan d'un cercle, on mène à ce cercle une sécante arbitraire ; le produit des distances de ce point aux deux intersections de la sécante avec la circonférence sera une quantité constante, indépendante de la direction de cette sécante."

common points, exterior with no common points, intersecting or tangent (interior or exterior) at a point. In the latter two cases, the radical axis would be respectively the common chord or the common tangent.

Gergonne further strayed from Steiner's exposition by considering two fixed circles and a circle of variable size and position. As one of the three circles varied, the radical center would traverse the radical axis of the two fixed circles. He declared that this result provided a means to construct the radical axis of two non-intersecting circles with the aid of an arbitrary third auxiliary circle intersecting them both. Further, in all cases one could also construct the radical center of three given circles. By allowing one of the three circles to move and grow in the plane, Gergonne was thus able to simultaneously consider multiple positional cases and generalize from an initial particular configuration. Neither Steiner nor Gaultier had invoked a moving circle and this variable motion appears strikingly like Poncelet's principle of continuity. Instead, Gaultier and Steiner had achieved the same result through distinct, static cases, followed by verification of the desired proportions. While Gergonne may not have been drawing from Poncelet's principle, his use of motion suggested similar considerations.

In his conclusion to Steiner's article, Gergonne assessed the importance of limiting cases in the work of Steiner.

> We are all the more surprised that M. Steiner has not understood it in this sense, as precisely one of the principle advantages of his beautiful constructions, for problems of contact, is to effortlessly conform to cases where all or some of the circles or spheres given become points, lines or planes. [50] [Steiner & Gergonne 1827, p. 314]

In his own constructions, Steiner had not explained how to generalize from a sphere to a plane, from a circle to a conic section nor from a curve to a point or a line. The process was simply assumed as possible or left unmentioned. Gergonne continued by praising modern over ancient geometry. For example, ancient geometers had not had the "freedom" to consider a straight line as part of the circumference of a circle of infinite radius. Gergonne thus seemed to adopt the modern ancient distinction as suggested by Poncelet in 1817 [Poncelet 1817c]. Though Steiner had associated his geometry with the more ancient style of Pappus and Viète, Gergonne presented Steiner as practicing modern geometry.

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50 "On doit être d'autant plus surpris que M. Steiner ne l'ait pas entendu dans ce sens que précisément un des principaux avantages de ses belles constructions, pour les problèmes de contact, est de se plier sans effort aux cas où tous ou partie des cercles ou des sphères donnés deviennent des points, des droites ou des plans."

In keeping with standard French usage, Gergonne had chosen Gaultier's radical axes instead of lines of equal powers. However, when describing the new concept of circles of common power (Steiner's Potenzkreise) he returned to Steiner's original vocabulary. This mixing of radicals and equal powers disrupted Steiner's systematicity. Steiner's genetic procedure was similarly lost as the path from equal power to circles of common power became obscured. These alternatives suggest two competing modes of simplicity: either by adding new concepts alongside known terms or beginning from scratch with a consistent naming system. The simpler decision depended on problem-solving, but also the audience. In this instance, radical axes had enough of a history to remain standard in French, while German geometers used equal powers.

Soon after Gergonne's interpretation of Steiner's paper appeared it was reviewed and summarized in the *Bulletin de Férussac.* [51] Here too, the solution to the problem of Malfatti was advertised as finally receiving a satisfactory purely geometric solution, and the content extended beyond Gergonne's translation by actually providing a summary solution. In describing Steiner's exposition, the circle of common power was emphasized as a fruitful innovation in problem solving.

> Steiner's means of solution rest uniquely on the known theory of similitude axes and centers, of that of radical axes and centers, of that of poles and polars, and finally of that which the author calls *circles of common power* of two given circles, which shows all the fruitfulness and all the importance of these various theories and justifies the particular attention that several French geometers have given them already for several years. [52] [*Bulletin* 1827, p. 277]

The review continued by explaining Steiner's circle of common power, in order to provide "an idea of the procedures of M. Steiner." In this summary Gergonne's vocabulary and notation were employed, and the reviewer showed how the circle of common power could be used to sketch a solution to the Malfatti problem.

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51 Results from the Gergonne-Steiner article were also cited by N. J. Didiez in his *Cours complèt de géométrie* published in 1828 [Didiez 1828]. Didiez's textbook is a rare counterexample of quickly incorporating new journal results into elementary geometry texts in this period.

52 "Les moyens de solution de M. Steiner se tirent uniquement de la théorie connue des centres et axes de similitude, de celle des axes et centers radicaux, de celle des pôles et polaires, et enfin de celle de ce que l'auteur appelle *cercles de commune puissance* de deux cercles donnés, ce qui montre toute la fécondité et toute l'importance de ces diverses théories, et justifie l'attention particulière qui leur a été donnée par plusieurs géomètres français depuis déjà plusieurs années."

The reviewer concluded by considering some philosophical issues concerning the relationship between analysis and geometry. In particular, he noted the simplification of both methods effected by the development of new concepts and terminology, as we quoted in our introduction. Such a comment appeared to reinforce the importance given to Steiner's development of new vocabulary, such as circles of common power. Although the review had begun by praising Steiner's problem solving, the emphasis ultimately returned to his new tools and their potential for future use.

Steiner returned to lines of equal powers in his 1833 book, *Eines festen Kreises, als Lehrgegenstand auf höheren Unterrichts-Anstalten und zur practischen Benutzung* [Steiner 1833]. This book marked the culmination of Steiner's research on circles and was reprinted in German and then translated into French and English. [53] In the French abridged translation, lines of equal power were translated as radical axes [Lévy 1908]. In the English translation, *Geometric constructions with a ruler given a fixed circle with its center*, the translator Marion Elizabeth Stark remarked in an endnote that "For English, American, and French readers a more familiar term for the line of equal powers is "radical axis", a term introduced by L. Gaultier" [Steiner 1950, p. 80]. This observation, over one-hundred years later, points to the significance of linguistic boundaries in shaping technical vocabulary.

5. CONCLUSIONS

The critiques and rejections of radical axes, ideal chords, and lines of equal powers display competing epistemological values and personal legacies. Despite these conflicting individual interests, definition formation remained a social enterprise. More so than solutions or theorems, definitions succeeded through their adoption by other mathematicians. Recognizing this, Gaultier, Poncelet, and Steiner took care in presenting the strengths of their definitions.

All three geometers aimed for greater simplicity, and achieved this in part by introducing new definitions at the beginning of the text to be applied to problems at the end. Though the definition might be complicated, it simplified the statements of solutions, theorems and proofs. Each geometer also agreed that geometry should be general and based in the study

[53] Both Steiner's original article [Steiner 1826b] and his book on circles [Steiner 1833] were republished as part of Ostwald's Klassiker der exakten Wissenschaften and in Steiner's complete works ([Steiner 1895], [Steiner 1901]).

of figures. Thus, the above definitions began with a summary of key properties and concluded with one or more constructions. However, simplicity and generality carried varied meanings, which became increasingly apparent as geometers focused on differentiating their definitions from those of predecessors and contemporaries.

Gaultier proposed that radical axes served to better solve long-standing problems by focusing on the relationship between any configuration of given circles, rather than the independent property of an individual circle. In particular, Gaultier used radical axes in his solution of the Apollonius problem thereby avoiding many case distinctions and providing one general solution. He further showed how the direct construction of radical axes preserved the graphic nature of geometry. Indeed, Gaultier interpreted geometry as bounded by what could be constructed. At the same time, he based radical axes in the algebraic notion of radicals and a quantitative relationship that remained constant despite the varieties of constructions.

Poncelet had a wider conception of geometry and ideal chords revealed one way in which geometry could be successfully extended. Unlike radical axes, ideal chords accounted for imaginary or infinite points of intersection. Further, properties of real common chords could be extended analogously to their ideal counterparts, thereby eliminating exceptional cases. Poncelet explained how the name "ideal chord" reflected this extension of ordinary geometry, rather than introducing new notions to accommodate particular relations. For Poncelet, analysis would surpass geometry unless geometers adapted vocabulary to interpret analytically derived objects. In this way, Poncelet promised that the practice of geometry would become uniform.

Steiner offered an alternative route to making geometry more uniform. Motivated, like Gaultier, by providing simple and general solutions to geometry problems, he built up a systematic vocabulary that he claimed could be developed from points to conic sections. Steiner would later advertise the organic and connected nature of a similar vocabulary structure in his *Systematische Entwickelung* of 1832. Unlike his French predecessors, in 1826 Steiner did not mention analytic geometry and only compared his research with practices that preceded the adoption of coordinate-equations. Lines of equal powers directly connected ancient vocabulary with contemporary interests in reciprocity and the potential for applications to second degree curves and surfaces.

Thus, even within the domain of pure geometry, there was little consensus as to what this geometry encompassed. New definitions and their associated attributes served to increase the heterogeneity of geometry. This multiplicity contrasts with the image of early nineteenth century geometry as divided clearly between analytic and synthetic methods. [54]

While definitions could be used to motivate new directions in geometry, novelty could also be an impediment. Gergonne noted the value and difficulty of language in describing the proliferation of the principle of duality in 1827.

> Thus it will be necessary, to present the new theory in the most advantageous light, to first create a language to fit it, if we may say so, but this language, we agree, will be difficult to make well and it will perhaps be even more difficult still, when it is made, to obtain a favorable welcome for it on the part of geometers. [55] [Gergonne 1827b, p. 276]

This skepticism mirrors Gaultier's caution in describing radical axes as mere abbreviations. The fact that the definitions introduced by Gaultier, Poncelet and Steiner became part of geometry stands as a testament to their perceived usefulness. The underlying rationale for employing a definition may have been for greater generality, simplicity, fruitfulness, uniformity, clarity or brevity, but textual evidence for such arguments is missing. Instead, geometers appeared to incorporate new definitions when they provided convenient designations for frequently invoked configurations. Diverse arguments emerged in introducing or rejecting definitions, but when accepted, definitions were used without comment beyond a citation.

As debates in favor of one or another definition receded, the new terms shifted the course of geometric research. Yet the success of a definition did not always signal the success of its underlying values and definitions changed geometry in ways not intended by their creators. Gergonne employed radical axes for circles, ideal chords for conic sections and the concept of common power when discussing certain points or circles. Sturm, Plücker, and Bobillier quickly adopted ideal chords, not to extend the purview of ordinary geometry, but to designate the imaginary points

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[54] This division first emerges in the historical literature of the late nineteenth century and can be seen, for instance, in [Klein 1926], [Coolidge 1940], [Kline 1972].

[55] "Il sera donc nécessaire, pour présenter la nouvelle théorie sous le jour le plus avantageux, de créer d'abord une langue à sa taille, s'il est permis de s'exprimer ainsi; mais cette langue, nous en convenons, sera difficile à bien faire, et il sera peut-être plus difficile encore, lorsqu'elle sera faite, de lui obtenir un accueil favorable de la part de géomètres."

of intersection that were already part of analytic geometry. Translators considered radical axes and lines of equal powers as French and German names for the same object, though each represented just one element from two distinct systems.

The continued existence of all three concepts into the twentieth century suggests their complementary advantages. [56] As Poncelet argued when describing the value of his definitions alongside existent vocabulary:

> [...] it seems extremely advantageous, for geometric language, to be able to designate the same object by several words, when these words correspond to different mental views, or recall distinct properties of the object. [57] [Poncelet 1822, p. 155]

Likewise, the respective definitions of Gaultier, Poncelet and Steiner each emphasized different relationships based upon how the definition had first appeared. Though the constructed radical axis, ideal common chord, or line of equal powers might look the same, the names continue to denote different ways of seeing geometry.

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[56] Notably, in the early nineteenth century, there were at least three additional alternative names–Chasles "axe de symptose", Terquem's "ligne disomologue", and Plücker's "Chordale".

[57] "[...] il nous semble extrêmement avantageux, pour la langue géométrique, de pouvoir désigner un même objet par plusieurs mots, quand ces mots correspondent à des vues différentes de l'esprit, ou rappellent des propriétés distinctes de cet objet."

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