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ON THE CIRCULATION OF ALGEBRAIC KNOWLEDGE IN THE IBERIAN PENÍNSULA: THE SOURCES OF PÉREZ DE MOYA'S *TRATADO DE ARITHMETICA* (1573)

Maria do Céu Peireira da Silva

Abstract. — This paper focuses on the algebraic contents of Juan Pérez de Moya's *Tratado de Arithmetica* [1573]. The author, who was the main disseminator of algebra in 16th century Spain, carefully reviewed his previous algebraic works, and implemented substantial improvements in them. The comparative study we have carried out with some of the best known algebraic works from the Renaissance has allowed us to identify the main algebraic sources of Moya. In particular, this article shows that Moya read carefully Tartaglia's *General Trattato* [1560], Nunes' *Libro de Algebra* [1567], and Stifel's *Arithmetica Integra* [1544], and it throws a new light on the algebraic culture in the Iberian peninsula during the last third of the sixteenth century.

Résumé (Sur la circulation des connaissances algébriques dans la Péninsule ibérique ; les sources du *Tratado de Arithmetica* de Pérez de Moya)

Cet article est centré sur le contenu algébrique du *Tratado de Arithmetica* [1573] de Juan Pérez de Moya. L'auteur, qui a été le principal vulgarisateur de l'algèbre en l'Espagne au xvi^e^ siècle, a examiné minutieusement ses précédents travaux algébriques, et a apporté des améliorations significatives en

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Mots clefs. — Algèbre, péninsule ibérique, Renaissance, Pérez de Moya, Tartaglia, Pedro Nunes, Stifel.

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eux. L'étude comparative que nous avons menée avec quelques-unes des plus connues œuvres algébriques de la Renaissance nous a permis d'identifier les principales sources algébriques de Moya. En particulier, ce document montre que Moya a lu attentivement le *General Trattato* [1560] de Tartaglia, le *Libro de Algebra* [1567] de Pedro Nunes et l'*Arithmetica Integra* [1544] de Stifel, et il apporte un nouvel éclairage à la culture de l'algèbre dans la péninsule ibérique au cours du dernier tiers du seizième siècle.

1. INTRODUCTION

In 1573, the Spanish mathematician Juan Pérez de Moya published in Alcalá de Henares the *Tratado de Mathematicas*, surely the most authoritative treatise of Mathematics printed in Portugal and the Hispanic kingdoms in the sixteenth century [1]. This work consists of three parts, the first of which, the *Tratado de Arithmetica*, is devoted to arithmetic and algebra. The seventh chapter of the first part consists of an extensive study of algebra [2]. Up to that moment, Pérez de Moya had published the *Compendio dela regla dela cosa, o Arte Mayor* [Pérez de Moya 1558], the first printed work of an Iberian author entirely devoted to algebra, and the *Arithmetica practica, y speculativa* [Pérez de Moya 1562], where algebra is studied in an extensive chapter [3]. The *Arithmetica practica, y speculativa* became a reference in Spain during the sixteenth century and the following centuries. We can not state the exact number of times it was printed, since half a dozen works of Moya have this same title and the same content, but they do not display date or printer mark [4]. Nevertheless, we can ensure that after 1562 it was reprinted, at least, twenty-five times: four times in the sixteenth century, eleven times in the seventeenth century, ten times in the eighteenth century, not to mention the *Arithmetica practica, y speculativa* edited in the twentieth century by Consolacion Baranda [5].

According to Docampo [2009, p. 124], the Catalan manuscript Ms. 71 of Sant Cugat (*Arxiu de la Corona d'Aragó*, Barcelona), dated to around 1520, contains the first known account of algebra in a vernacular Iberian language. At that time, Joan Andres's *Sumario breve de la pratica de la Arihtmetica* [Andres 1515] had been published in Spain and Gaspar Nicolas's

1 For a biography of Moya see [Valladares Reguero 1997].

2 [Pérez de Moya 1573, pp. 429-605].

3 [Pérez de Moya 1562, pp. 443-615].

4 See [Silva 2011, p. 23].

5 [Baranda 1998].

Tratado da Pratica Darismetyca [Nicolas 1519] in Portugal. The authors of these two mercantile arithmetics had heard of the existence of algebra, but both looked at it with suspicion [Silva 2011, p. 188]. In a previous paper, Docampo [2006, pp. 51- 52] says that the Catalan manuscript of Sant Cugat uses mostly the notations from Pacioli's *Summa*, and includes many topics from it, such as a theory of equations and algebraic fractions. Pacioli's *Summa* and Euclid's *Elements* were the main references for the Iberian authors in 16th century [Romero Vallhonesta 2012, p. 120].

By the time Pérez de Moya wrote the *Compendio dela regla dela cosa* [1558], Pacioli's *Summa* [Pacioli 1494] was well disseminated in the Iberian peninsula. Moreover, Marco Aurel had published the *Libro primero de Arithmetica Algebratica* [1552], which was the first Spanish work containing algebra, though its arithmetic part occupies half of the work. We stress that Moya and Aurel wrote in Spanish in order to reach a wide audience, not necessarily skilled in mathematics. The algebraic content of all these Spanish arithmetics is very similar: they state the characters representing the unknown and other notations to be used, study different types of equations and the rules to solve them, deal with square and cube roots of binomials and residuals, and apply these insights to solve a lot of problems. In addition they include the method of using a second unknown. Heeffer [2010, p. 60] believes that this method "is an exception in algebraic practice before 1560". With very small differences this is also the content of the algebraic *Libro Septimo* of Pérez de Moya's *Arithmetica practica, y speculativa* [1562].

Less known but also important for the history of Iberian algebra in the 16th century is the Portuguese merchant Bento Fernandes's *Tratado da arte de arismetica* [Silva 2008]. It is the earliest published treatise in Portuguese that contains algebra. Fernandes works with roots, powers and polynomials, and solves several types of equations, but he uses no symbolic notation except for natural numbers and fractions. The *Tratado da arte de arismetica* essentially drew on Italian abacus manuscripts from the early 15th century, but the possibility of an Arabic influence should not be excluded [Silva 2008, p. 211].

Leitão [2002, p. 410] dates to around 1535 or 1536 the original manuscript of Pedro Nunes's *Algebra*. He says that this text originally written in Portuguese was gradually extended until about 1564, then being translated into Spanish, and published [Nunes 1567]. Leitão also recalls [2010, p. 11] that shortly after being published and during all the 17th century Nunes' *Libro de Algebra* was very widespread; however, among those who knew Nunes' work, he does not mention any Spanish author from the 16th century. According to Angel Rodriguéz [1915, p. 173] Moya may have used "Nunes's not yet published work" when he wrote his *Compendio dela regla dela cosa*; but Rodriguéz does not point to convincing evidence. Nunes's *Libro de Algebra* includes all the algebraic material of the earlier Spanish algebras, adding geometrical constructions and proofs not presented in them. Labarthe [2010, pp. 24-25] stresses Nunes's double goal of extending usual arithmetic operations to the new algebraic objects (roots, dignities, second kind of fractions): to show how to carry out usual operations with these quantities, and to try to justify or at least to prove the validity of the operations done.

Stifel's *Arithmetica Integra* [1544], Cardano's *Ars Magna sive de regulis algebraicis* [1545], and Tartaglia's *General Trattato de Numeri, et Misure* [1556] (the sixth volume of which is devoted to algebra) were published before Moya's *Arithmetica practica, y speculativa.* Høyrup [2011, pp. 23-24] points out that Stifel "generalizes [arithmetic and algebra] in a way his predecessors had not done (both in his development of polynomial algebra and in his use of symbolism). He presents everything (or almost) developed or used by *some* Italian abacus algebraist and deploys it systematically in a way *none* of them (including Pacioli) had done". Nevertheless, as far as we know, before Pérez de Moya's *Tratado de Arithmetica* [1573] there is no evidence of Stifel's influence in any Iberian work concerning the notation to be used in the context of systems solving. According to Romero [2011, p. 105] that work of Pérez de Moya "contains an important step forward as opposed to Aurel's work, thanks to the fact that he kept the symbol *co.* for the first unknown and he assigned the letters *a, b, c,* etc. to the following unknowns" [6]. Romero also notes that "this type of assignation, which gives relevance to the first unknown, is also highlighted in the Stifel's comment of *Die Coss de Rudolff*", but she sees no evidence that Stifel's comment in *Die Coss* has inspired Pérez de Moya. In her article "Spanish 'Arte Mayor' in the Sixteenth century" Massa [2012] shows the stage of the development of this subject in the first four Spanish language books containing algebra. Therefore, Massa excludes from her selection both Pedro Nunes's *Algebra* and Moya's *Tratado de Arithmetica,* which are key texts for this work.

Our goal in this paper is to identify the main influence of Pedro Nunes's *Algebra* and Tartaglia's *General Trattato* in Moya's *Tratado de Arithmetica* and to show that Moya used Stifel's *Arithmetica integra* [1544] to improve and

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6 We could add that the *Tratado de Arithmetica* also contains advances on this subject when compared with the previous algebraic works of Moya.

simplify his algebraic notation, especially as regards the resolution of systems. This paper also highlights the importance that Moya gives to equations as algebraic objects, creating special symbols to indicate the equality and the powers of the unknown.

In his early algebraic works, Moya only mentions Euclid and, occasionally, Pacioli [7]. He does not refer to their works, but surely they are the *Elements* and the *Summa*, whose influence in Moya's texts is documented in [Silva 2011, pp. 184-192]. Moya never mentions either Aurel or the title of Aurel's work, but we admit that when he published the *Compendio dela regla dela cosa, o Arte Mayor* he already knew the *Arithmetica Algebratica*; as it is stressed in [Silva 2011, pp. 196-200], there are similarities in the algebraic contents of these two works [8]; nevertheless, there are also some striking differences concerning essentially the way the material is organized, the notation and the problems [9].

It is shown in [Silva 2008, p. 125] that Tartaglia was a reference for Moya when he dealt with geometric topics in his *Fragmentos Mathematicos* [1568]; [Silva 2013, p. 4-10] establishes the clear influence of Tartaglia's *General Trattato* in several passages of Moya's *Tratado de Geometria* [1573]. On the other hand, in his manuscript on navigation *Arte de Marear*, Moya mentions several times Pedro Nunes [Silva 2012, p. 374]. [Silva 2012, pp. 360, 364-365, 367] also points to the presence of several common features between Moya's *Arte de Marear* and Nunes's *Tratado em defensam da carta de marear* [10].

In his *Tratado de Arithmetica*, Moya mentions Stifel, Tartaglia, and Pedro Nunes [11], but, in general, he does not say what source he was following at each moment. However, as we will see below, it seems that Moya used ideas of these authors to improve his work. Moya also mentions Cardano's *Ars Magna*, published 28 years before his *Tratado de Arithmetica*, and indicates it to the reader interested in deepening his knowledge about equations, saying [12]:

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7 [Pérez de Moya 1558, p. 113]; [Pérez de Moya 1562, p. 605].

8 See also [Massa-Esteve 2012, p. 125].

9 [Baranda 1998, p. IX-XXV] shows analogies between the *Arithmetica practica, y speculativa* and the *Libro primero de Arithmetica Algebrática*. A summary of the contents of *Libro primero de Arithmetica Algebrática* is given in [Docampo Rey 2004, 549-556].

10 The full title of this work is: *Tratado que ho doutor Pero nunez Cosmographo del Rey nosso senhor fez em defensam da carta de marear: cõ o regim to da altura.*

11 [Pérez de Moya 1573, 395]; [Pérez de Moya 1573, 596].

12 "Otras varias, y diversas yguалaciones ay que dexo de poner, porque para sabios no sõ menester, y para principiantes no se entenderan. Quien quisiere ver algo, lea el decimo d'Arithmetica de Cardano". [Pérez de Moya 1573, p. 589]. It should be noted that "the tenth of Cardano's Arithmetic" is a reference to Cardano's *Ars Magna.*

> There are several other equations which I leave out, because they are not necessary for the learned and would not be understood by beginners. Those who wish to see something [about this subject] may read the tenth of Cardano's Arithmetic.

Before taking up the main topic of the paper, we present below a brief description of the algebras written by Moya, pointing to the major changes he introduces from one work to the next. These topics will be an important help to draw comparisons with Stifel's *Arithmetica Integra*, Tartaglia's *General Trattato*, and Pedro Nunes' *Libro de Algebra*.

2. THE *ALGEBRAS* OF MOYA

The *Compendio dela regla dela cosa* [1558] is a little book of about 159 pages, focused on solving equations and their applications, in which Moya introduces terms and concepts needed for a complete understanding of the subject. The text develops along sixteen chapters. Moya begins with the clarification of the meaning of "the rule of algebra" (in his words, *regla de Algebra, regla dela cosa, del cos, reglas reales, arte mayor*). Then, he explains that his favourite algebraic symbols to represent the unknown and its powers (see Fig. 3 below) do not exist in the presses available to him, and therefore, instead of these, he has decided to adopt the symbols *n., co., ce., cv., cce. R. cecv., RR., ccce., ccv.* (see Fig. 5 below). He says: "Estos characteres me ha parecido poner, porque no auia otros en la imprenta" [13].

Moya introduces the radicals with index 2, 3 and 4 (he calls them *numero quadrado, numero cubico,* and *numero medial*), and he presents the rules to operate with them; nonetheless he does not teach any algorithm to obtain approximate values of these roots. Then, Moya includes notions of monomial and polynomial that are close to the modern ones, and provides the elementary operations with polynomials in the chapter entitled "On the four general rules of characters" (*Trata de las quatro reglas generales de caracteres*) [14]. Moya does not use the names monomial and polynomial; he uses the word *caracter* with the actual meaning of monomial. And he distinguishes them in "similar in kind" (*semejantes en especie*), if they are similar monomials, and "different" (*diferentes*), if they are not similar. He names the polynomials *partidas de caracteres,* or simply *partidas*" [15]. In the *Compendio*

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13 "These characters I am moved to adopt, because others are not to be had in the printing office" [Pérez de Moya 1558, p. 1].

14 Chapter VIII in [Pérez de Moya 1558, pp. 31- 46].

15 See, for example, [Pérez de Moya 1558, p. 33].

dela regla dela cosa [1558] Moya occasionally uses the word *quinomio*[16] with the meaning of polynomial with 5 terms. We note that the name *quinomio* appears in the *Diccionario de la Ciencia y la Técnica del Renacimiento*[17], and in that reference the first use of the word is attributed to Moya in his *Arithmetica practica* [1562]. We remark that Moya sometimes uses the word *trinomio* for a polynomial with 3 terms, and *quatrinomio* for a polynomial of 4 terms[18]. Pedro Nunes[19] uses *quadrinomio* and *quinomio* with the same meaning.

Moya teaches, in a methodical way, how to add, subtract, and multiply polynomials. Division is only discussed if the divisor is a monomial. This chapter also includes a short reference to the simplification of the quotient of two monomials, and two examples for determining the square root of polynomials (trinomial and quinomial), each of which is a perfect square. According to Moya, the preliminary concepts necessary to the study of equations (*lo que me ha parescido ser necessario, para operacion desta regla de la cosa*) include the definition and operations with binomials and residuals (*binomios* and *residuos*)[20]; these are mathematical expressions like $A + B$, and $A - B$ $(A, B > 0)$, at least one of them being a quadratic radical, and $A > B$ in the case of the difference. Moya devotes much of the *Compendio* to study the equations of the first and second degree, (the *ygualaciones*); he gives a classification, and the rules to solve them, in two different moments of his book (see below table 1, and table 2). Moreover, he deals with systems of equations of the first degree (*regla de la quantidade*), using a second unknown different from the "thing" (*cosa*).

He also deals with the universal root (*raiz universal*)[21], a concept which had already been introduced when he defined the operations with quadratic radicals[22]. Moya defines the sum of radicals using a radical whose radicand is an expression containing radicals (in modern notation this would be written as follows: $\sqrt{a} + \sqrt{b} = \sqrt{a + b + \sqrt{4ab}}$). Surprisingly,

16 See, for example, [Pérez de Moya 1558, p. 45].

17 See Mancho Duque (María Jesús) [2000–2013] (`http://dicter.usal.es/?palabra=quinomio&;tipo=0`, on the 26/01/2016).

18 See, for example, [Pérez de Moya 1573, p. 482], and [Pérez de Moya 1573, p. 490].

19 He says [1567, f. 136v]: "(...) porque trinomio es raiz de quinomio, y no de quadrinómio (...)"..

20 The term binomial doesn't have the actual meaning of the algebraic sum of two monomials.

21 For Moya the universal roots are generated by the sum or difference of *sordas rayzes*.

22 [Pérez de Moya 1558, p. 9].

the book ends with some topics on proportion and proportionality, which Moya applies to solve problems without using equations.

The algebraic contents of *Arithmetica practica, y speculativa* [1562] are studied in the *Libro Septimo.* Moya has revised the text of the *Compendio* before including it in the *Arithmetica.* This resulted in some differences in the initial text. We notice among them, more references to Euclid's *Elements*: besides *Elements* II, he mentions *Elements* VII, VIII, and IX [23]; occasionally, Moya refers to *Elements* X, but without specifying the numbers of the propositions concerning the subject that he is discussing [24]. In the *Compendio* Moya does not explain how to calculate the values of roots, but in the *Arithmetica* he give algorithms which make it possible to obtain the exact square and cubic roots, in the case of perfect square and cubes, and approximate values of them, in the other cases [25]. Moya proposes an algorithm to compute an approximate value of the cube root [26], that in modern symbology is $\sqrt[3]{a} \approx b + \frac{a-b^3}{3b(b+1)+1}$, b being the biggest integer whose cube does not exceed a. This rule contributed to make his name known within the scientific community of the time. Stevin refers to him in the *Practique de Arithmetique* [Stevin 1585, p. 30]; he compares the procedure given by Moya with the one given by Tartaglia, and he shows a preference for that of our author. Stevin says: "Nicolas Tartaglia traictant curieusement de ceste reste, n'ajouste poinct ce dernier 1 [referring to the 1 in the denominator, furthest to the right], comme nous (avec Iuan Peris de Moya) avons faict, &; à mon Avis, voulant donner reigle generale, il y a plus de raisons de ajouster, que de le laisser)". Furthermore Stevin appliesthis algorithm to determine an approximate value of $\sqrt[3]{600}$; and while he emphasizes that sometimes the rule used by Tartaglia [27] leads to values closer to the cube root, he prefers an approach by default, just as in the case of Moya.

In his *Arithmetica* Moya replaces some examples given in the *Compendio* by others with different features: a few involve easier calculations, so that

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23 For example, *Elements* II, 4 is cited in connection with the algorithm of the square root, with the sum of roots of index four, and with the square root of binomial in [Pérez de Moya 1562, pp. 457, 501 & 532]; *Elements* IX, 8 is cited in connection with the square root and with the cubic root in [Pérez de Moya 1562, pp. 454 e 482].

24 In the study of the composition and origin of binomial in [Pérez de Moya 1562, p. 525].

25 [Pérez de Moya 1562, pp. 456–467 & 481–489].

26 In *Liber Abaci,* Fibonacci included numerous examples of extracting the cube root, in a way that corresponds to the formula that we mention, and claimed for himself the discovery of the procedure [Allard 1997, p. 227, footnote 183].

27 See, for example, the computation of a cubic root in [Tartaglia 1556, p. 27r].

the reader may focus in the problem, the others are more abstract and increase the diversity of cases presented [28]; nevertheless, the methods used in both texts are the same.

As mentioned above, proportions and proportionality are explained at the end of the *Compendio*; in the *Arithmetica* this subject preceds the study of algebra. This option is logically correct, since the notion of proportion is necessary to define important algebraic concepts, like the "thing" (or *cosa* in Moya's words. See for example its definition, in the beginning of algebra [29]: "The second [character] is said cosa (thing). It is root, or side, of a square number: and this is the first of the numbers of a continuous proportion").

There are some differences, but also important similarities between these two works, such as the organization of the subjects, the titles of the chapters and even their numbering. Moya uses the same algebraic notation in both texts (he just changes the spelling *cv* and *rv* to *cu* and *ru*), and presents the issues in the same way. The *Tratado de Arithmetica* includes all the algebraic topics which the author had previously published; but with substantial improvements that show important reflexion and systematization. Moya is now much more careful when he quotes his sources, and often refers the Book X of Euclid's *Elements* to justify speculative passages of the text. This was not the case in the *Compendio* and in the *Arithmetica*. He appeals to [30] *Elements* X, 31, 32, 33, 34, 36 to study questions related with square root of binomials, and to *Elements* 86, 87, 88, 89, 90, 91 [31] in the case of residuals.

In the *Tratado* Moya takes care to establish the foundations of algebra; immediately after the definition of algebra he includes a set of propositions (*presupuestos, y comunes sentencias*) that did not exist in *Libro Septimo* of *Arithmetica practica, y speculative*: "1. If equal amounts are added to equal numbers or amounts, the sums or ensembles will be equals. 2. If equal amounts are subtracted to equal numbers or amounts, the sums or ensembles will be equals, the differences will be equals. 3. If equal amounts or quantities are multiplied by the same quantity, the products will be equals. 4. If the

28 See three examples given by Moya when he practices the equations of the type $ax^2 + bx = c$ [PerezdeMoya1562, pp. 586-589]; the replacement of problem 1. in [Pérez de Moya 1558, p. 590] for problem 1. in [Pérez de Moya 1562, 99], which is different from the following; the replacement of question 1. in [Pérez de Moya 1558, p. 102] for question 1. in [Pérez de Moya 1562, p. 593].

29 "*El segundo [character] se dize cosa. Es rayz, ò lado, de un numero quadrado: y este es el primero de los numerous de una continua proporcion.*" [Pérez de Moya 1562, p. 449].

30 [Pérez de Moya 1573, pp. 534-535].

31 [Pérez de Moya 1573, p. 538].

unity multiplies a number, the product will be the same number. 5. If the unity divides a number, the quotient will be the same number. 6. If equal quantities are divided by equal quantities, the quotients will be equal. 7. Quantities that are equal to a third, all are equal to each other. 8. Numbers or quantities that are proportional to each other will be equals" [32].

Moya corrects some errors commited in his previous algebraic works, such as replacing the wrong formula, $x = \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a} \pm \frac{b}{2a}}$ for solving the equations [33] $ax^2 + c = bx$; $a, b, c > 0$, by the correct one, $x = \frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$. Furthermore he provides the following contents not listed in the *Arithmetica practica, y speculativa*: elementary operations with radicals with index five [34]; the calculation of square and cubic roots [35]; the analysis of the number of solutions of the equation of the first and second degree, including cases currently referred as indeterminate and impossible, and the discussion of the possibility of the equations $ax^2 + c = bx$ [36]; and also the geometric demonstration of the rules to solve the complete equations of the second degree [37].

On the other hand, there are topics already addressed in the *Arithmetica practica, y speculativa* that Moya recovers and develops in the *Tratado de Arithmetica*. Among them we emphasize the following: the introduction of

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32 "1. Se a numeros, o a quãtidades yguales, se añadieren quãtidades yguales, las sũmas, o cõjunctos seran yguales. 2. Si de numeros, o quãtidades yguales, quitares quantidades yguales las restas que quedarem seran yguales. 3. Si numeros, o quãtidades yguales, se multiplicaren por una misma quãtidade, los produtos seran yguales. 4. Si la unidad multiplicare un qualquiera numero, el produto sera el mismo numero. 5. Si la unidad partiere un qualquiera numero, el quociente sera el mismo numero. 6. Si quãtidades yguales fueren partidas, por quãtidades yguales, los quociente seran yguales. 7. Las quãtidades que a una tercera fueren yguales, todas ellas entresi seran yguales. 8. Los numeros, o quãtidades que fueren proporcionales entre si, seran yguales". Proposition 8 looks like a mathematical mistake, but what Moya means is the following: if there is a proportion between four quantities, then que fraction of the first two is equal to the fraction of the last two. For instance, in page 491, he says: "there is such a proportion from 4.cu. to 1.R. as from 4.n. to one census (...) necessarily the first fraction will be $\frac{4.cu.}{1.R.}$ or $\frac{4.n.}{1.ce.}$, which is the purpose". In other words, from the proportion "such as $4x^3$ is to x^5 so 4 is to x^2", Moya concludes the equality "$\frac{4x^3}{x^5} = \frac{4}{x^2}$". [Pérez de Moya 1573, pp. 429-430].

33 [Pérez de Moya 1558, p. 70]; [Pérez de Moya 1562, p. 554].

34 [Pérez de Moya 1573, pp. 462-465].

35 [Pérez de Moya 1573, pp. 537-539].

36 [Pérez de Moya 1573, pp. 550-552].

37 [Pérez de Moya 1573, pp. 589-593].

significant improvements in terminology and notation for the unknowns previously adopted (see section 6); the study of operations with polynomials, particularly division and root extraction (see sections 3 and 4); the operations (addition, subtraction, multiplication, division and root extraction) relating to a second unknown different from *cosa* (the chapter is called "*En que se pone la regla que dizen de la quantidade, o segunda rayz, o cosa*" [38]) and the possibility of using as many unknowns as necessary to solve a problem (see section 5); the study of equations in which more than three non-similar monomials are equal to a number [39] (see section 3) (the chapter is entitled "En que se ponẽ otras ygualaciones, diversas de las que hasta aqui hemos dicho"); the attention he provides to the equations involving radicals, studying more cases and more varied than in *Libro Septimo* da *Arithmetica practica, y speculativa* [40] (see below). We also noticed an increased number of problems [41] in which Moya applies the rules for solving equations of the forms $ax^2 = b$, $ax^3 = b$, $ax^4 = b$, $a, b > 0$, $ax^2 + bx = c$, $ax^2 + c = bx$, $ax^2 = bx + c$; $a, b, c > 0$, and systems of equations of the first degree [42].

There are some differences between the organization of the algebraic contents in the *Tratado de Arithmetica* and in the *Libro Septimo* of *Arithmetica practica, y speculativa*. One concerns the point at which the study of universal roots is introduced; this section is at the end of the *Arithmetica practica, y speculativa*, but in the *Tratado* it precedes the study of square and cubic roots of binomials and residuals. The latter option is better because the concept of universal root intervenes in the definitions of those mathematical entities.

In his three algebraic works, Moya provides the study of second degree equations in two moments. First he distinguishes seven types of equations [43] (Table 1) - four of them with monomials of two different degrees (*ygualaciones simples de dos quantidades*), and the other three with monomials of three different degrees (*ygualaciones compuestas de três quantidades*), and he describes in rhetorical language the algorithms to solve them.

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38 [Pérez de Moya 1573, pp. 593-596].

39 [Pérez de Moya 1573, pp. 587-589].

40 Problems 14 and 15 in [Pérez de Moya 1562, pp. 569-570]; [Pérez de Moya 1573, pp. 542-543].

41 The numbers are, respectively, (8, 3), (4, 2) (5, 3), (8, 4), (6, 2), (5, 2); [Pérez de Moya 1573, pp. 558-583], [Pérez de Moya 1562, pp. 573-593].

42 In the *Tratado de Arithmetica* are proposed 6 problems and in *Libro Septimo* da *Arithmetica practica, y speculativa* are proposed only 3 problems.

43 See [Pérez de Moya 1558, pp. 67-73], [Pérez de Moya 1562, pp. 550-556], and [Pérez de Moya 1573, pp. 546-549].

TABLE 1. The forms of equations (*ygualaciones*) studied in Moya's algebraic texts, and the corresponding solutions. The sign (*) indicates the correct solution included in the *Tratado de Arithmetica* [Pérez de Moya 1573, pp. 546-549].

Equation	Type	Solution
Primera ygualacion simples	$ax^{n+1} = bx^n$, $a, b > 0$, $n \in \mathbb{N}_0$	$x = \frac{b}{a}$
Segunda ygualacion simples	$ax^{n+2} = bx^n$, $a, b > 0$, $n \in \mathbb{N}_0$	$x = \sqrt{\frac{b}{a}}$
Tercera ygualacion simples	$ax^{n+3} = bx^n$, $a, b > 0$, $n \in \mathbb{N}_0$	$x = \sqrt[3]{\frac{b}{a}}$
Quarta ygualacion simples	$ax^{n+4} = bx^n$, $a, b > 0$, $n \in \mathbb{N}_0$	$x = \sqrt[4]{\frac{b}{a}}$
Primera ygualacion compuesta	$ax^{n+2} + bx^{n+1} = cx^n$, $a, b, c > 0$, $n \in \mathbb{N}_0$	$x = \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} - \frac{b}{2a}$
Segunda ygualacion compuesta	$ax^{n+2} + cx^n = bx^{n+1}$, $a, b, c > 0$, $n \in \mathbb{N}_0$	$x = \frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$ (*)
Tercera ygualacion compuesta	$cx^n + bx^{n+1} = ax^{n+2}$, $a, b, c > 0$, $n \in \mathbb{N}_0$	$x = \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} + \frac{b}{2a}$

When Moya revisited this subject in his three texts, he condensed the earlier study in four types [44] (Table 2), saying that they were sufficient to solve a multitude of cases [45].

In this context, Moya gives a rule to solve all complete quadratic equations, which simplifies the task of calculating when the coefficient of x^2 is 1. These rules are similar to those of Nunes [46], the second time he studies the *conjugaciones compuestas*. Chuquet in his *Triparty* had already proposed an analogous summary [47], but the similarity between Moya's *Tratado de Arithmetica* and Nunes' *Libro de Algebra* suggests the influence of Nunes in Moya's work. Our author even uses the word denomination (*denominaciõ*) with the same meaning given by Nunes; and this does not occurr in Moya's previous works [48].

44 See [Pérez de Moya 1558, pp. 113-118]; [Pérez de Moya 1562, pp. 605-611]; [Pérez de Moya 1573, pp. 585-586].

45 [Pérez de Moya 1573, pp. 585-586].

46 [Pérez de Moya 1573, p. 548]; [Nunes 1567, pp. 142r-142v].

47 [Chuquet 1880, p. 748].

48 [Pérez de Moya 1573, pp. 585-586]; [Nunes 1567, pp. 148v-151v].

TABLE 2. New proposal to classify the equations provided in the *Tratado de Arithmetica* [Pérez de Moya 1573, pp. 585-586].

Equation	Type	Solution
Ygualacion simples	$ax^{n+k} = bx^k$, $a, b > 0$, $n, k \in \mathbb{N}$	$x = \frac{b}{a}, n = 1; x = \sqrt[n]{\frac{b}{a}}, n > 1$
Primera ygualacion compuesta	$ax^{2n+k} + bx^{n+k} = cx^k$, $a, b, c > 0$, $n, k \in \mathbb{N}$	$x = \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} - \frac{b}{2a}, n = 1;$ $x = \sqrt[n]{\sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} - \frac{b}{2a}}, n > 1;$
Segunda ygualacion compuesta	$ax^{2n+k} + cx^k = bx^{n+k}$, $a, b, c > 0$, $n, k \in \mathbb{N}$	$x = \frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}, n = 1;$ $x = \sqrt[n]{\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}}, n > 1;$
Tercera ygualacion compuesta	$cx^k + bx^{n+k} = ax^{2n+k}$, $a, b, c > 0$, $n, k \in \mathbb{N}$	$x = \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} + \frac{b}{2a}, n = 1;$ $x = \sqrt[n]{\sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} + \frac{b}{2a}}, n > 1;$

In the *Tratado de Arithmetica* Moya deals with equations involving radicals and algebraic fractions [49], and solves them by methods currently in use. The examples do not come from daily life problems and they are similar to those of Tartaglia's *General Trattato* [1560] and of Nunes's *Libro de Algebra* [1567] [50]. Moya also refers to extravagant equations (in his words *ygualaciones estravagantes*) [51], which are equations that become quadratic when both sides are divided by x; but oddly he does not recognize them as being part of the cases studied before.

49 Cap. XLIIII, Art. III: "En que se ponen reglas, y aviso para ygualar, quando en alguna parte de la ygualacion, ay algun genero de rayz quadrada, o cubica, o otra qualquiera" [Pérez de Moya 1573, p. 542], and Cap. XLIIII, Art. IIII: "En que se ponen avisos, y reglas generaes, para quando en las ygualaciones viene quebrados" [Pérez de Moya 1573, p, 543].

50 [Pérez de Moya 1573, pp. 542-543]; [Tartaglia 1560, pp. 12v-15]; [Nunes 1567, pp. 138r-142r].

51 Cap. LIX.: "En que se ponen otras ygualaciones, diversas de las que hasta aqui hemos dicho" [Pérez de Moya 1573, p. 589].

3. TARTAGLIA

According to Heeffer [2008, p. 115], at the beginning of the sixteenth century any serious work in algebra at least included an introductory explanation of the addition, subtraction and multiplication of the algebraic polynomials. But we think that the situation was not much different in the second half of this century: for example, we see that Tartaglia in his *General Trattato* does not propose an algorithm to divide polynomials [52]. And even the *Tratado de Arithmetica* and the *Libro de Algebra* only deal with the algebraic operations involving polynomials in the context of solving equations. Malet [2007, p. 10] observes that this happens in almost all books of algebra before Viète, and emphasizes that the exception is *La Practique de l'Arithmétique* [1585], in which "Stevin deals with polynomials and algebraic fractions (which he calls *fractions algébriques*) as fully legitimated mathematical objects".

It is interesting to realize that the study carried out on polynomials in the *Arithmetica practica y speculativa* and in the *General Trattato* is similar; Moya illustrates the algorithm to multiply polynomials with an example very similar to that of Tartaglia [53], and uses a scheme in which the calculations are arranged in the same way (see Fig. 1). In order to multiply $3x - 5$ by $4x^3 - 2x$, Moya begins by multiplying the first monomial on the right, -5, of the multiplier, $3x - 5$, by all the monomials of the multiplicand, $4x^3 - 2x$, typing the result in a first line [54]; then he applies this procedure to the second monomial of the multiplier, and places the product in another line. This scheme is a bit different from the one included in the *Tratado de Arithmetica* (see section 4).

As previously mentioned, Moya does not use the terms monomial and polynomial; he designates similar monomials by *caracteres semejantes en especie* (Tartaglia uses the term *dignità de una medesima specie* [55]), and those monomials which are not similar by *caracteres diferentes en especie* (Tartaglia uses the term *dignità de specie diverse* [56]).

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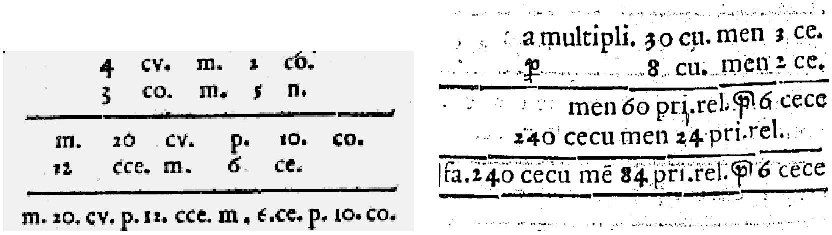
52 This also occurs in Aurel's *Arithmetica Algebratica*, in Moya's *Compendio dela regla dela cosa*, in Moya's *Arithmetica practica y speculativa* de Moya, all of them subsequent to 1500.

53 [Tartaglia 1560, p. 4v].

54 The only difference to his other works is related with the decimal representation of the numbers and, therefore, with the advancement of one position to the right, from line to line [Pérez de Moya 1562, pp. 56-57].

55 [Tartaglia 1560, p. 1v].

56 [Tartaglia 1560, p. 1v].


4 cv. m. 2 cô.
3 co. m. 5 n.
m. 20 cv. p. 10. co.
12 cce. m. 6 ce.
m. 20. cv. p. 12. cce. m. 6. ce. p. 10. co.

a multipli. 30 cu. men 3 ce.
p 8 cu. men 2 ce.
men 60 pri.rel. p 6 cece
240 cecu men 24 pri.rel.
fa. 240 cecu mē 84 pri.rel. p 6 cece

FIGURE 1. Displaying the product of (in our notation) $4x^3 - 2x$ by $3x - 5$, in [Pérez de Moya 1562, p. 516] on the left; and of $30x^3 - 3x^2$ by $8x^3 - 2x^2$, in [Tartaglia 1560, 4v] on the right.

The square root of polynomials is a subject that Moya takes up and expands in the *Tratado de Arithmetica.* We point out that in the *Compendio* he only studied cases in which the sign of the monomials were positive; Aurel does the same in *Libro primero de Arithmética Algebrática*[57]. In the *Tratado de Arithmetica,* Moya refers to the case in which the middle term of a trinomial is negative, like in $\sqrt{4x^4 - 12x^3 + 9x^2}$; and he uses the same geometrical argument as employed by Tartaglia, to demonstrate that the square root of a trinomial can only be a binomial. He says [58]: "(…) because multiplying a binomial by itself, one adds to it a square of both terms, (and) the double of the surface of one term by the other (…)".

In Moya's and Tartaglia's own words:

> (…) porque multiplicando un binomio por si mismo se le acrecienta un quadrado de ambos los terminos, del duplo de la superficie de un termino, por el otro. [Pérez de Moya 1573, p. 490]

> (…) perche a multiplicare el binomio in se medesimo gli interviene gli quadrati de ambedue le parti over termini di quelle, &; il doppio dell'una in altra di esse parti over termini (…) [Tartaglia 1560, pp. 13v-14r].

On this topic Moya offers a deeper study than Tartaglia, since it includes the determination of the cube root of a polynomial. Based on the case of the square root, our author states that to extract the cube root of a polynomial it must have $3n + 1$ terms:

> If you need to extract the cube root [Moya indicates cube root by *rrr*] of more than one character [i.e. a polynomial] you should note that it must have 4 or 7

57 [Pérez de Moya 1558, pp. 44-45]. In [Aurel 1552, pp. 75r-75v] the examples are $\sqrt{4x^4 + 12x^3 + 9x^2}$, and $\sqrt{25x^6 + 30x^5 + 29x^4 + 12x^3 + 4x^2}$.

58 [Pérez de Moya 1573, p. 490].

> or 10, etc. [terms] always increasing by three, just as in extracting square root it increases by two[59].

But taking the example $\sqrt[3]{x^6 + x^5 + x^4 + x^3}$ [in Moya's notation, *rrr. de cecu. p. R. p.cce. p. cu.*] Moya claims that the condition is not sufficient. Then, he establishes a necessary and sufficient condition for the existence of the cube root of a polynomial with four terms. It is the following: the monomials with higher and lower degrees must have cube roots, and furthermore the ratio between the second monomial and the cube root of the first monomial is equal to the ratio between the third monomial and the cube root of the fourth one (supposing that the polinomial is ordered according to the decreasing powers of the unknown quantity)[60].

> The cube root of $x^6 + x^5 + x^4 + x^3$ what will be? You will extract from it the characters of the extremes, and from x^6 will come x^2 and from x^3 will come x [;] add both of them, and it will be $x^2 + x$, and so will be the cube root of such four characters. But for this be true, you must see now if it is the same to divide x^4 (which is the third) by x (which was the cube root that came out of the fourth) as to divide x^5 (which is the second) by x^2 (which was the root that came out of the first) and so (as it is) you will say that this cube root is exact, and if not, that it is not[61].

Moya quotes the example[62] $\sqrt[3]{27x^6 + 54x^5 + 36x^4 + 8x^3}$, which agrees with the given conditions, and points out that the result is $3x^2 + 2x$. Tartaglia, Nunes and Stifel do not treat this subject. In his *Arithmetica* Stevin only states a necessary condition to the existence of the cube root of a quadrinomial[63]; this requires the calculation of the cube of the binomial obtained, to decide whether it is actually the cube root of the given polynomial.

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59 "Si ovieres de sacar rrr. de mas de un caracter, notaras que ha de ser de 4 o de 7, o de 10, &;c. creciẽdo siempre tres mas, asi como en el sacar rayz cuadrada crecen dos." [Pérez de Moya 1573, p. 490].

60 [Pérez de Moya 1573, p. 490].

61 "La *rrr.* de *cecu. p. R. p.cce. p. cu.* que sera? Sacarla has de los characteres de los extremos, y del *cecu.* vendra un *ce.* y del *cu.* vendra *co.* junta el uno cõ el otro, y sera *ce. p. co.* y tanta sera la *rrr.* de los dichos quatro characteres. Mas para que sea verdad, has de mirar agora si es lo mismo partir *cce.* (que es el tercero) por la *cosa* (que fue la *rrr.* que salio del quarto) como partir *R.* (que es el segũdo) por *ce.* (que fue la *rrr.* que salio del primero) y siendo assi (como lo es) diras que es cierta esta *rrr.* y sino, no." [Pérez de Moya 1573, p. 490].

62 [Pérez de Moya 1573, p. 490].

63 [Stevin 1585, pp. 58-59].

In his *Tratado de Arithmetica* Moya solves some equations made from several different monomials. He uses a strategy consisting of adding or subtracting the same number to both sides of the equation, and then extracting the roots of the results (whatever its order is) [64] . We quote the following example: $x^4 + 2x^3 + 3x^2 + 2x = 117{,}648$. This equation is also given in Tartaglia's *General Trattato*[65], and we notice that Moya gives exactly the same procedure to solve it. First, he adds 1 to both members to transform the equation in $(x^2 + x + 1)^2 = 343^2$, and then he reduces it to $x^2 + x + 1 = 343$ (he says nothing about the equation $x^2 + x + 1 = -343$), and solves it.

The study of such fourth degree equations is also included in Pedro Nunes's *Libro de Algebra*[66], and in the *Appendix* of Stifel's *Arithmetica Integra*[67], but with different examples.

4. PEDRO NUNES

In his *Tratado de Arithmetica* Moya provides the algorithms to operate with polynomials and algebraic fractions. Leaving aside the extraction of the root of the polynomials, which Nunes does not address, all the procedures are very similar to those given in Nunes's *Libro de Algebra.* The order in which these operations are studied is almost the same in both (only the simplification of fractions and the reduction of fractions to a common denominator have the reverse order), and the exemples are analogous.

Multiplication is a crucial example, because Moya replaces the algorithm given in his *Arithmetica practica y speculativa* (see section 3) by the one described in *Libro de Algebra* (see Fig. 2). Still he takes care to put similar terms under each other in the intermediate calculations, and to dispose the result according to the order of the powers of the unknown. Nunes does not do this.

In his *Arithmetica Integra*[68] Stifel considered the division of two polynomials as an essential operation in algebra, but Nunes's *Libro de Algebra* was the first Iberian work that includes an algorithm to divide polynomials. In

64 [Pérez de Moya 1573, p. 587].

65 [Tartaglia 1560, p. 16v].

66 In [Nunes 1567, pp. 136r-136v] the example is $9x^4 + 12x^3 + 10x^2 + 4x = 1155$.

67 In [Stifel 1544, p. 307] the example is $x^4 + 2x^3 + 3x^2 + 2x = 177{,}241$.

68 [Stifel 1544, p. 227v].

3 ce. m. 4 n.
2 cu. m. 1 co.

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6 R. m. 8 cu.
m. 3 cu. p. 4 co.

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6 R. m. 11 cu. p. 4 co.

	.15.		m̃	.4. co.		
	.3. ce.		m̃	5. co.		
	.45. ce.		m̃	.12. cu.		
			m̃	75. co.	p̃	.20. ce.
Sũma	.65 ce.	m̃	.75. co.	m̃	12. cu.	

FIGURE 2. Displaying the product of (in our notation) $Ex^2 - 4$ by $2x^3 - x$, in [Pérez de Moya 1573, 482] on the left; and of $15 - 4x$ by $3x^2 - 5x$ in [Nunes 1567, 28v] on the right.

the *General Trattato*[69] Tartaglia only explains the method of dividing a polynomial by a number, taking as an example the division of $24x^6 + 12x^2$ by 4. In the division of polynomials, he merely indicates the notation to be used. For example, he indicates the division of $13x^6+3-2x$ by $5x^3-7x^2$ in the following manner[70]: $\frac{13x^6+3-2x}{5x^3-7x^2}$. As Moya emphasizes[71] this representation is important when solving equations involving algebraic fractions. As is well known, the algorithm of division allows the factorization of polynomials, which is essential to solve equations of degree ≥ 3.

In his *Tratado de Arithmetica* Moya introduces the division of polynomials with a remark that also occurs in Nunes's *Libro de Algebra*[72]. And he gives the same algorithmic procedure to do it (Moya calls them *caracteres,* and Nunes calls them *dignidades*).

In the own words of Moya and Nunes:

> Si el partidor fuere cõpuesto de dos quantidades, partiras los mayores caracteres de los de la particiõ, por el mayor caracter de los del partidor, dexando en que pueda caber el otro character menor del

> Si el partidor fuere compuesto, partiremos las mayores dignidades de lo que se ha de partir por la mayor dignidad del partidor, dexandole en que pueda caber la otra dignidad del partidor, y lo que viniere mul-

69 [Tartaglia 1560, p. 5r].

70 [Tartaglia 1560, p. 5r].

71 [Pérez de Moya 1573, p. 487].

72 Pérez de Moya [1573, p. 483] says: "Entendida la razõ del multiplicar, es notoria la del partir"; [Nunes 1567, p. 30] says "Quiẽ sabe la razõ del multiplicar, no puede errar el partir".

partidor, y lo que viniere al quociente, multiplicalo por ambos caracteres del partidor y el producto restese de la particion, y prosigue haziendo lo mismo en cada una de las restas, hasta llegar a character de menor denominaciõ, que el mayor character de los del partidor, porque entõces el menor se põdra sobre el mismo partidor cõ una raya en medio a modo de quebrado [Pérez de Moya 1573, p. 486].

tiplicaremos por el partidor, y lo producido por essa multiplicación sacaremos de toda la suma que se parte y lo mismo obraremos en lo que restare, por el modo que tenemos quando partimos numero por numero. Y llegando a numero o dignidad en esta obra que sea de menor denominación, que el partidor, quedara essa quantidad en quebrado escreviendose sobre una raya, y el partidor debaxo dessa linea [Nunes 1567, p. 31r].

Moya gives only one example [73], which is the division of $10x^3 + 26x^2 + 17x + 14$ by $2x + 4$, that turns out to be very similar to that of Nunes [74]; and he presents the result in the same form: $5x^2 + 3x + \frac{5}{2} + \frac{4}{2x+4}$. Furthermore, Moya demonstrates that the division is correct using the full reverse operation, like Nunes does. Our author also provides a second "demonstration" [75], which does not exist in *Libro de Algebra*. It consists of substituting in the dividend, divisor, and quotient polynomials, x by a specific numerical value (in this case, $x = 2$), followed by the verification that the number resulting from the divison of the former two is equal to the number obtained in the latter. Of course this numerical verification is not a proof, but Moya uses it as such.

Moya also proposes a similar numerical substitution and verification in the division of characters in which he attributes numerical values to them [76]. Again he uses this procedure when he deals with the operative rules with the second unknown [77]. As Malet [2007, p. 17] remarks, "Stevin's most usual way to provide 'demonstrations' for the operations with polynomials is numerical substitution and verification". For example, to prove the validity of the division between the algebraic quantities $16x^4$ and $8x^3$, he makes use of [78] the numbers 1296 and 216, which are the measures of the straight lines representing those quantities, *HI* and *FG*, and verifies that its quotient is equal to the measure of the straight line

73 [Pérez de Moya 1573, p. 487].

74 The division of $12x^3 + 18x^2 + 27x + 17$ by $4x + 3$, in [Nunes 1567, p. 31v].

75 [Pérez de Moya 1573, p. 487].

76 [Pérez de Moya 1573, p. 484].

77 [Pérez de Moya 1573, p. 596].

78 [Stevin 1585, p. 54].

AC that represents $2x$. Aurel also deals with this "kind of proof", but not to divide polynomials [79].

Two chapters of the *Tratado de Arithmetica* are dedicated to algebraic fractions. Moya distinguishes the following two kinds of expressions with fractions: kP in which k is a numerical fraction and P a monomial (which he calls *quebrados del primer modo*, ou *quebrados que son parte, o partes de algun character*), and expressions such as $\frac{P}{Q}$, in which P and Q are monomials, and the degree of P is less than the degree of Q (which he calls *quebrados del segundo modo*, or, *quebrados de caracteres*).

Moya stresses that the operations with *quebrados del primer modo* reduces to those of rational numbers. For the others he develops a systematic study that includes the simplification (which he calls *abreviar caracteres*), the reduction to the same denominator (which he calls *reduzir quebrados diversos, a una misma denominacion*), and the operations of addition, subtraction, multiplication and division.

Nunes also distinguishes the same categories of fractions, which he calls *quebrados de primera intenciõ*, and *quebrados de segunda intenciõ*, and he deals with the same operations. The similarity between Moya and Nunes is clear when they work on simplifying fractions. The wording and the notation are the same in both authors, and also the examples given. Let us see how Moya and Nunes simplify a fraction (they both called it *quebrado*), working with the examples $\frac{4x^3}{x^5}$ and $\frac{20x^2}{x^3}$ [80]. Moya explains that the ratio (Moya and Nunes called it *proporcion*) between $4x^3$ and x^5 is equal to the ratio between 4 and x^2, and Nunes does exactly the same with $20x^2$ and x^3, and 20 and x. In their own words:

> Luego, tal proporcion ay de 4.*cu.* a 1.*R.* como de 4 *n.* a un *censo*, y si tal proporcion ay del primero numerador para el primero denominador, como del segundo numerador para el segundo: necessariamente tal quebrado sera $\frac{4cu.}{1R.}$ como $\frac{4n.}{1ce}$ que es el proposito. [Pérez de Moya 1573, p. 491].

> Luego tal proporcion aura de .20.*ce.* para .1.*cu.* como del numero *.20.* para .1.*co.* Pues si tal proporcion ay del primero numerador para el primero denominador, como del segundo numerador para el segundo denominador, necessariamente tal quebrado sera $\frac{.20.ce.}{.1.cu.}$ como $\frac{.20.}{.1.co.}$ y esto queriamos demonstrar. [Nunes 1567, pp. 37v-38r].

79 [Aurel 1552, p. 74v]. He does not handle the division of polynomials, and his remark about the division of monomials is absurd.

80 See [Pérez de Moya 1573, p. 491], and [Nunes 1567, pp. 37v-38r].

In the context of the algebra, Moya deals with whole numbers (which he calls *numeros* or *enteros*), fractional numbers (which he calls *quebrados* or *rotos*) and mixed numbers (which he calls *entero y quebrado*). The notion of irrational number or *surd* (to Moya *irracional* and *sordo* have the same meaning) appears associated to the meaning of a quadratic radical in which the radicand is not a perfect square. This idea was confusingly expressed in Book V of the *Tratado de Arithmetica*, where Moya considered 82 a surd number, because $\sqrt{82}$ is not exact. He says:

> You will say that the square root of eighty two, is nine, and it remains one. Therefore, this number eighty two is said surd [*sordo*] or irrational, and the number nine is said its surd [*sorda*], or irrational, or indiscreet root [81].

In Book VII of the same work, Moya distinguishes between a rational root (which he calls *racional, o discreta*) and a surd root (which he calls *sorda*); the first being denoted by a whole number (*entero*), a fractional number (*quebrado*) or a mixed number (*entero y quebrado*) [82]. But, astonishingly, he keeps calling irrational or surd a number like 3 which is not a perfect square. He says:

> If the number is irrational or surd, I mean, one which has no exact root, like three, or other similar [numbers], in order to represent it, or to write what it is, do not care about approximations, just say that it is the square root of three, and write it in this way, *r3*, and let this first example [*documento*] be general for any other kind of roots [83].

Moya used this procedure in dealing with square roots, or with roots of higher order. For example, referring to $\sqrt{3}+\sqrt{5}$ Moya says [84]: "If you have to add roots of irrational numbers (...)"; and referring to $\sqrt[3]{7}+\sqrt[3]{3}$ he says [85]: "If you have to add cube roots of irrational numbers (...)". Nevertheless, this confusion with the meanings of rational number, and surd or irrational number has no impact in the solutions of the problems, because

81 "Diras que la rayz quadrada de ochenta y dos, es nueve, y sobra uno. Por lo qual, este numero ochenta y dos, se dira sordo, o irracional, y el nueve se dira su rayz sorda, o irracional, o indiscreta." [Pérez de Moya 1573, p. 329].

82 [Pérez de Moya 1573, p. 442].

83 "Si el numero fuere irracional o sordo, quiero decir, que no tenga rayz justamente, como três, o otros semejãtes, para representarla, o escribir lo que es, no cures de aproximaciones, sino di que es rayz quadrada de tres, y escrivela deste modo, r3, y este documento primero sea general para otra qualquiera especie de rayzes." [Pérez de Moya 1573, p. 442].

84 [Pérez de Moya 1573, p. 446].

85 [Pérez de Moya 1573, p. 455].

Moya usually calls them, indistinctly, number. For example, when referring to the square root of 48, Moya says [86]:

> (...) since it is not possible to extract the exact [root] by numbers, say that the tetragonic side [*lado tetragonico*], or the root of this square 48, is r48, and as much [*tanto*] will be the value of the thing, and therefore as much [*tanto*] is the number that is requested [87].

Notice that Moya introduces the operations with numerical radicals because he thinks that they are needed in various sciences and arts, like geometry, cosmography and military arts, which use square roots and cubic roots; and, especially in algebra which depends on all sorts of roots. He says:

> Concerning these aforesaid roots, the square root is used in geometry, and in cosmography, and in the military arts, and to extract the mean proportional between two extremes, and in algebra [*arte mayor*]. The cube [root] is used to extract two mean proportional between two extremes, and to extract the quantity of the diameters of solid bodies, and other various things. And this, with all the other [roots] here generally treated, are used in the rule of the thing, or algebra [*arte mayor*], as will be seen in the course of this work [88].

Furthermore, Moya dedicates one chapter of the *Tratado de Arithmetica* to the study of the square root of astronomical fractions, which shows the importance of this subject in astronomy [89].

The calculation with unknowns *historically* accompanied the calculation with roots, probably because the algorithms involved are *technically* similar or related [90]. Moya teaches how to add, subtract, multiply, and divide quadratic radicals; then he generalises these operations to higher index radicals (goes up to radicals with index 5), with the same or different indexes.

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86 See also, for example [Pérez de Moya 1573, p. 553], where it is stated he following: "y el outro vendra 4 y medio, y este es el otro numero".

87 "(...) porque por numeros no se puede sacar justa, di que el lado tetragonico, o rayz deste quadrado 48, es r48, y tanto sera el valor de la cosa, y por consiguiente tanto es el numero que se pide". [Pérez de Moya 1573, p. 561].

88 "Destas rayzes que se há dicho, la quadrada sirve para geometria, y Cosmographia, y para el arte militar, y para sacar medio proporcional entre dos estremos, y para a arte mayor. La cubica sirve para sacar dos medios proporcionales entre dos estremos, y para sacar la quantidad de los diametros en los cuerpos solidos, y para otras varias cosas, Y esta, con todas las demas que aqui tratamos generalmente, sirven para la regla de la cosa, o arte mayor, como en el processo desta obra se vera." [Pérez de Moya 1573, 328].

89 [Pérez de Moya 1573, p. 395].

90 Antoni Malet made me aware of this.

As stated above (see section 2) Moya relies on the development of the binomial square to define $\sqrt{a}+\sqrt{b}$ as $\sqrt{a+b+\sqrt{4ab}}$ whenever $\sqrt{ab}$ is an irrational number, or as $\sqrt{a+b+2\sqrt{ab}}$ whenever $\sqrt{ab}$ is a rational number. These definitions were given in Pacioli's *Summa*, and were very popular at the time [91].

Nevertheless, in the *Tratado de Arithmetica* Moya provides another rule to add radicals, which in modern symbolism is $\sqrt{a}+\sqrt{b}=\left(\sqrt{\frac{a}{b}}+1\right)\times\sqrt{b}$. The same rule is given in Nunes's *Libro de Algebra*, and in Chuquet's *Triparty* [92]. This method is particularly advantageous to add the radicals $\sqrt{a}$ and $\sqrt{b}$ in which the square root $\sqrt{\frac{a}{b}}$ is exact (which Moya calls *numeros comunicantes*). In addition, one may generalize it to radicals of any kind, since they have equal indexes; both Moya and Nunes emphasize this [93]. Moya reproduces the arguments given by Nunes to demonstrate that the previous rule is correct:

> La demostracion es evidente, porque si se multiplica el numero menor por el quociente, resultara el mayor. Y porque tanto viene multiplicando un numero por otro, como por sus partes, como en otro lugar demostramos, siguese, que si multiplicarmos el numero por la unidad, y por el resto del quociente ambas dos multiplicaciones, seran tanto como el numero mayor. Y porque multiplicando el numero menor por la unidad, hazemos el mismo numero menor, multiplicando luego el numero mayor [94], por el resto del quociente se hara la diferencia, o ventaja que ay del mayor al menor, porque la diferencia, juntamente con el menor, cõstituye el mayor: lo qual tambien tiene lugar en cualquiera naturaleza de rayzes [Pérez de Moya 1573, pp. 448-449].

> La demonstracion es muy clara, porque si multiplicamos la quãtidad menor por el quociente, hazemos la quantidad mayor. Y porque tanto haze multiplicar una quantidad por otra, como por sus partes, siguese que si multiplicaremos la quãtidad menor por la unidad, y por el resto del quociente, haremos con estas dos multiplicaciones la misma quantidad mayor. Y porque multiplicando la quantidad menor por la unidad, hazemos la misma quantidad menor: Multiplicando luego la quãtidad menor por el resto del quociente, haremos la differencia que ay entre mayor y menor, porque essa differencia juntamente con la menor cõstituyen la mayor. Y que esto tenga lugar en las raizes no es difficil de entender [Nunes 1567, pp. 57v-58r].

91 [Pacioli 1494, p. 116v].

92 [Chuquet 1880, p. 65r].

93 [Pérez de Moya 1573, p. 448]; [Nunes 1567, pp. 58r-59v].

Moya makes a brief introduction to radicals in which he refers to the elements that are present in a radical, the index (which he calls *denominacion*), and the radicand (which he calls *numero*), and then he explains how to reduce radicals to the same index. He uses the same explanations and terminology as given in Nunes's *Libro de Algebra*[95], even replacing the word *genero*, which he used in the *Compendio*, by the word *denominacion*. Furthermore, when Moya deals with the rationalizing of radical denominators, he provides justifications which are very similar to those of Nunes:

> La razon desto es, porque multiplicando las primeras quantidades, o disminuyendo las ygualmente, queda la misma proporcion entre lo que se parte, y el partidor, por lo qual el quociente sera el mismo que tuvieramos si no hizieramos las tales multiplicaciones, o diminuciones. [Pérez de Moya 1573, p. 522].

> La razon desta obra es muy clara, porque multiplicado las primeras quantidades ygualmente, queda la misma proporciõ entre lo que se parte y el partidor, por lo qual el quociente será el mismo que tuuieramos sino hizieramos las tales multuplicaciones. [Nunes 1567, p. 60r].

Moya also deals with binomials and residuals, defining operations which are similar to those that he provides for polynomials. He distinguishes six types of binomials (and six types of residuals), which in usual notation can be characterized as follows: first binomial (in Moya's words, *binomio primero*): $a+\sqrt{b}$ such that a^2-b is a square number (for example, $4+\sqrt{7}$); second binomial (*binomio segundo*): $\sqrt{a}+b$ such that $\frac{a-b^2}{a}$ can be written as a ratio of squares (for example, $\sqrt{112}+7$); third binomial (*binomio tercero*): $\sqrt{a}+\sqrt{b}$ such that $\frac{a}{b}$ can not be written as a ratio of squares, and whenever $a>b$, $\frac{a-b}{a}$ can be written as a ratio of squares (for example, $\sqrt{32}+\sqrt{14}$); fourth binomial (*binomio quarto*): $a+\sqrt{b}$ such that a^2-b is not a square (for example, $5+\sqrt{12}$); fifth binomial (*binomio quinto*): $\sqrt{a}+b$, such that $a>b^2$ and $\frac{a-b^2}{a}$ can not be written as a ratio of squares (for example, $\sqrt{20}+3$); sixth binomial (*binomio sexto*): $\sqrt{a}+\sqrt{b}$ such that $a>b$, and $\frac{a-b}{a}$ can not be written as a ratio of squares (for example, $\sqrt{20}+\sqrt{8}$). Aurel also gives this classification, but the condition that characterizes each binomial does not always coincide with that of Moya; moreover, the examples are all different in the two texts[96].

95 [Pérez de Moya 1573, p. 466], and [Nunes 1567, pp. 46r-46v].

96 See [Silva 2011, pp. 119-121].

Concerning the study of equations, we see that the classification of the equations given by Moya is different from those of Tartaglia and Nunes [97]; but both the description of the procedures and the terms used are very similar. It is interesting to note that Moya uses the term *en balança*, as also Tartaglia and Nunes (Tartaglia [1560, p. 12r] say *in bilancia*; Nunes [1567, p. 1r] says *en balança*) to signify the equilibrium of the two members of the equation.

Moya gives geometric demonstrations of the rules to the three complete second degree equations that are analogous to those given in Nunes's text [98]. Both Nunes and Moya handle each type of equation with generality, not specifying the values of the coefficients. Consequently, both algebraists indicate the segments of the supporting figures by the letters labelling the extreme points, but make no assumptions concerning their measures. Tartaglia's procedure is different: each type of equation is represented by a specific equation, with numerical coefficients, and therefore the geometric proofs use numerical measures. For example, to demonstrate the rule for solving $x^2 + bx = c$, Tartaglia deals with the equation $x^2 + 24x = 340$ [99].

Moya often uses the kind of *argument* used by Nunes. Just to take an example, they both invoke *experience* (in Moya's words *y assi lo muestra la experiencia*, in Nunes's words *E la experiencia assi lo dize*) (see below the transcriptions), which means *verification*: the solution of the equation can be easily confirmed. In the example below, Moya solves the equation $5x = 30$, and then using *experience* checks the solution by giving the unknown the value of 6. Nunes does the same taking the example $4x^2 = 20x$. Let us read the own words of Moya and Nunes:

> (...) and comes six, to the value of the thing, and so the experience shows (Moya says, *y assi lo muestra la experiencia* [100]). Because multiplying six (which is the value of the thing) by the five things, gives six, which is as much as the smallest character. [Pérez de Moya 1573, p. 547].

> (...) and comes .5. to the value of the thing. And the experience says this. (Nunes says, *E la experiencia assi lo dize* [101]): because .20. multiplied by .5. which is the value of the thing, gives 100. [Nunes 1567, p. 1r].

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97 Tartaglia [1560, pp. 5r-9r] and Nunes [1567, pp. 1r-2v] consider three simple equations, $ax = b$, $ax^2 = b$ and $ax^2 = bx$, and three composite equations, $ax^2 + bx = c$, $bx + c = ax^2$ and $ax^2 + c = bx$.

98 For example, in [Pérez de Moya 1573, p. 590], and in [Nunes 1567, p. 7v].

99 See, for example, [Tartaglia 1560, p. 8r] and [Pérez de Moya 1573, p. 590].

These geometric demonstrations of the rules to the complete second degree equations are included in very important algebraic texts from the 15th and 16th centuries, for example, in Pacioli's *Summa*, in Cardano's *Ars Magna*, and in Tartaglia's *General Trattato* [102]. But Nunes' *Libro de Algebra* is the first printed work of an Iberian author where they are presented.

5. STIFEL

The subjects related to the existence of an efficient notation (more in section 6) are very important when operating by algebra, particularly concerning the study of systems of linear equations. Franci and Rigatelli [103] notice that "Antonio [de Mazzinghi] is the first vernacular author who uses two unknows to solve problems". Actually, Fibonacci already does so [104]. Stifel and Moya, in their works, drew attention to the issue of the notation in the context of systems solving (which nowadays we call linear systems).

In the *Compendio dela regla de la cosa*, and in the *Arithmetica Practica y Speculativa*, Moya deals with the rule of the thing (*regla de la cosa*), and with the rule of the quantity (*regla de la quantidad*), but in addition to the symbol for the main unknown quantity (the *co.*) he only uses the symbol *1q.* or *q.* to represent a new unknown quantity. We also see this option in the *Libro de Algebra*, in which Nunes does not use more than two unknowns (the *co.*, and the *quantidade*). Even when a problem required a second unknown [105], the *quantidade* was used just to write down the conditions and then disapeared, the resolution being reduced to the use of a single unknown, the *co* [106].

In the *Tratado de Arithmetica*, Moya considers the possibility of using as many unknowns as necessary to solve a proposed problem. He says:

> In many questions it happens that the position of the thing is not enough [*no bastar sola la posicion de la cosa*] for what is wanted as last answer to the question, and some other position or positions are needed. (...) To tell the positions one from the other it is necessary to put them in such a way that we can distinguish them and understand which one was the first, the second, &;c [107].

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102 [Cardano 1545, p. 33 and foll.]; [Tartaglia 1560, p. 7r ad foll.].

103 [Franci & Rigatelli 1985, p. 42].

104 I thank Professor Norbert Schappacher for his observation.

105 [Nunes 1567, pp. 224-225].

106 [Nunes 1567, pp. 169v, 224v-225r].

107 "Acontece en muchas questiones no bastar sola la posicion de la cosa, para lo que cõviene a la ultima respuesta de la demanda, y es necessario poner otra posiciõ, o otras. Y por si aviẽdo puesto al principio una cosa, para buscar algun numero fuesse (como dicho avemos) necessario poner otra cosa segundariamente, o otra tercera, o

And he introduces an adequate notation to this purpose. In addition to *1.co.*, for the first unknown quantity, he indicates *1.a.*, *1.b*, *1.c*, *1.d*, or, alternatively[108], *1.q*, *1.b.co*, *1.c.co*, *1.d.co*. Their names are in accordance with the order in which they are introduced in the formulation of the problem: the first position (which he calls *primera posicion*, and also *primera cosa* or *primera rayz*); the second position (which he calls *segunda posicion*, and also *quantidade*, or *segunda rayz*); the third position (*tercera posicion*, and also *tercera cosa*, or *tercera rayz*), etc. Besides, Moya stresses that it is possible to use more unknows, whenever it is necessary:

> From what has been said, it is understood that the first position placed in order to make a question should be put like this, 1.co., and it is called first root [*primera rayz*]. And the second position should be put like this, 1.a. or like this, 1.q., and it is called second root [*segunda rayz*], or quantity, or one a. of thing [*una a. de cosa*]. The third position should be put like this, 1.b. or like this, 1.b.co., which means 1.b. of thing [*1.b. de cosa*], and it is called third root [*tercera rayz*]. The fourth position should be put like this, 1.c. or like this 1.c.co., and it is called thing fouth, or fourth root [*cosa quarta, o quarta rayz*]. And the fifth position, or posture [*postura*], should be put like this, 1.d. and it is called fifth root, or fifth position [*quinta rayz, o quinta posición*]. And so we proceed in the order of a.b.c. and other signs when [the letters] are are not enough, until infinity[109].

This option became important to solve problems that can be resolved as systems of linear equations were more than two unknowns are involved. Moya uses a procedure that turns out to be very similar to the one used nowadays[110].

Let us see how Moya uses the unknowns *co.*, *a.*, and *b.* to solve the problem concerning three students who want to buy a book costing 17 *reales* and none of them has enough money. To have the required amount, each one asked the other two for a part of his money. The goal is to find the part that

quarta, &;c. Para que unas posiciones se differencien de otras, es menester ponerlas de modo que se distinguan y entiendan qual fue primera posicion, y qual segunda, &;c." [Pérez de Moya 1573, p. 593].

108 [Pérez de Moya 1573, pp. 593-[694]].

109 "De lo que hemos dicho, queda entendido que la primera posicion que se pone para hazer alguna demanda se pone assi, 1.co., y llamase primera rayz. Y la segunda posicion se pone assi, 1.a. o assi 1 q. y llamase segunda rayz, o quantidade, o una a. de cosa. La tercera posicione se pone assi 1.b. o assi 1 b.co., que quiere dezir 1 b.de cosa, y llamase tercera rayz. La quarta posicion, se pone assi 1 c. o assi 1 c.co. y llamase cosa quarta, o quarta rayz. Y la quinta posicion, o postura, se pone assi 1 d. y llamase quinta rayz, o quinta posicion. Y deste modo se procede por la orden del a.b.c. y otras señales quando ellas no bastan en infinito." [Pérez de Moya 1573, pp. 593-594].

110 [Pérez de Moya 1573, pp. 601, 603].

each one of the students shall contribute to buy the book. Let us see how Moya states the problem:

> Three students wanted to buy a book (costing 17 *reales*). The first said to the others two, give me the half of your money, and with the money that I have, I'll pay the book. The second student asked the other two the third part of their money, and the third one asked the other two the quarter part. With how much money had each student? [111]

Moya assumes that the first student has x of *reales* [Moya says *1co.* de *reales*]. Then, he needs $17 - x$ [he writes $17n.m.1co.$], which is the half of the money of the other two, to buy the book. So, the second and the third together have $34-2x$ [he writes $34n.m.2co.$], and altogether have $34-x$ [he writes $34n.m.1co.$]. Moya says *keep this*. Then, he assumes that the second student has a *reales*. So the first and the third students together have $34-x-a$ [he writes $34n.m.1co.$], and the second plus the third part of the first and third have $11\frac{1}{3} - \frac{1}{3}x + \frac{2}{3}a$ [he writes $11\frac{1}{3}n.m.\frac{1}{3}co.p.\frac{2}{3}a$]. Then, Moya equals this quantity to 17, $11\frac{1}{3}-\frac{1}{3}x+\frac{2}{3}a = 17$ [he writes $11\frac{1}{3}n.m.\frac{1}{3}co.p.\frac{2}{3}a.yg.a17n.$], and expresses the quantity a in terms of the *co*, obtaining $\frac{2}{3}a = 5\frac{2}{3}+\frac{1}{3}x$. [he writes $\frac{2}{3}a.yg$ a $5\frac{2}{3}n.p.\frac{1}{3}co.$]. Moya then assumes that the third student has b (*reales*), so the first and the second students together have $34 - x - b$ [he writes $34n.m.1.co.m.1b.$]. He repeats the reasoning, obtaining the first and the second together, $34 - x - b$, and the fourth part of the first and the second, $8\frac{1}{2} - \frac{1}{4}x - \frac{1}{4}b$. Then he adds b, equals the sum to 17, and expresses b in terms of the *co*. Having a and b expressed in terms of the *co*, Moya adds the three quantities, equals the sum to $34 - 2x$, and obtains the value of x, which is 5. Then he substitutes this value in the expressions of a and b, obtaining 11 and 13.

Curiously, we found that in his texts Moya does not apply more than three unknowns for solving problems [112].

Romero Vallhonesta [113] follows the resolution of a similar problem treated in Aurel's *Libro primero de Arithmetica Algebratica* [1552, p. 108v].

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111 "Tres estudiãtes querian cõprar un libro (que valia 17 reales) y dixo el primero a los otros dos, dadme la mitad de lo que ambos teneys; y com lo que yo tengo pagare este libro. El segundo pidio a los otros dos el tercio. Y el terceiro pidio a los otros el quarto, piedese que reales tenia cada uno por si." [Pérez de Moya 1573, p. 603].

112 [Pérez de Moya 1573, pp. 596-605].

113 [Romero Vallhonesta 2012, pp. 129-131].

Stifel, in his *Arithmetica Integra*[114], adopts a similar notation, respectively, 1*A*, 1*B*, 1*C*, 1*D*, 1*E*, and $1A\varkappa$, $1B\varkappa$, $1C\varkappa$, $1D\varkappa$, $1E\varkappa$; and there he interrupts the practice of representing by q the second unknown[115]. Buteo in his *Logistica*[116] also employed *A*, *B*, *C*, … and 1*A*, 1*B*, 1*C*, … to represent two or more unkhowns; he never mentions either the symbols 1*Aco*, 1*Bco*, 1*Cco*, 1*Dco*, 1*Eco* (like Moya) or $1A\varkappa$, $1B\varkappa$, $1C\varkappa$, $1D\varkappa$, $1E\varkappa$ (like Stifel). In the *liber quartus* of Gosselin's *De arte magna*[117] the notation is very similar to that used by Buteo, but when Gosselin deals with *De quantitate surde*[118] he only employs he letters L and q for the unknowns.

Moya was acquainted with the *Arithmetica Integra*[119], where Stifel exemplifies the use of as many unknowns as necessary to solve a system; and we notice that in his *Tratado de Arithmetica* Moya applies the same notation to solve problems with the same features[120].

6. INNOVATIONS IN THE *TRATADO DE ARITHMETICA*

6.1. *The* unknown quantity *and* their powers

Moya defines *algebra* as a way of determining an unknown number submitted to a geometrical proportion[121]. In his own words[122]:

> Algebra, es un modo de hallar algun numero dudoso demãdado subjecto a alguna proporcionalidad, por lo qual por otro nõbre le dizen cuẽta hecha por progressiones de proporcionalidad Geometrica.

Moya's algebraic works, in particular his *Tratado de Arithmetica*, shows that he was aware of the notation issues; in particular with the use of the unknown (which he sometimes calls *numero dudoso*) in the handling of equations. In his *Compendio dela regla de la cosa*, and in his *Arithmetica Practica y Speculativa*, Moya indicates ten symbols with which he represents the number (it is the first symbol on the left in Fig. 3, and he calls it

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114 [Stifel 1544, p. 251v].

115 [Paradís & Malet 1989, p. 141].

116 See [Buteo 1559, pp. 189-196].

117 See [Gosselin 1577, pp. 80r-84r].

118 [Gosselin 1577, pp. 84r-86v].

119 See, for example, [Pérez de Moya 1573, p. 19], and [Pérez de Moya 1573, p. 395], where he says "Pone Michael Stiphelio dos ordenes de numeros (…)".

120 See [Stifel 1544, pp. 254, 296v-297, 300].

121 In the *Compendio*, Moya designates the algebra by *regla dela cosa* or just *regla*, and says that it is a tool that helps the solving of problems. Chuquet designates it by *rigle*.

122 [Pérez de Moya 1573, pp. 429].

FIGURE 3. The number, and the first nine powers of the unkown from Moya's Compendio [1558, p. 1] and from Moya's Arithmetica Practica y speculaticva [1562, p. 449].

numero), and the first nine powers of the unkown (see Fig. 3); he calls them, respectively, *cosa, censo, cubo, censo de censo, primero relato, censo y cubo, segundo relato, censo de censo de censo, cubo de cubo.*

Moya says that these characters are chosen for shortness, as a simplification of the language, and he recommended its use to solve problems. These ten symbols are different from the Italian ones used in Pacioli's *Summa*; they are very similar to the German symbols, but not exactly the same [123]. Moreover, they do not coincide with the letters of any alphabet existing at the time [124]. Notice that with the exception of the symbols for power exponent 5 and 7, they are analogous to those used by Stifel and Aurel [125].

In the *Tratado de Arithmetica*, Moya extends this set of symbols until the 29th power of the unknown (Fig. 4), and he stresses, once again, the limitations concerning their printing. The rule given by Moya to construct these symbols distinguishes the cases in which the exponent is a prime number from those in which it is not. Stifel does the same in his *Arithmetica Integra* [126]. However, when the power of the unknown quantity is a prime number greater than three, Moya and Stifel use different symbols; Moya uses *1R, 2R, 3R, 4R,* … (in modern notation, $x^5, x^7, x^{11}, x^{13}, \ldots$), and Stifel employs for the same purpose the symbols β, $b\beta$, $c\beta$, $d\beta$, …. Moya's option is similar to that of Tartaglia [127]—the Italian mathematician uses *rel, 2.rel, 3.rel, 4.rel*—and has the advantage over that of Stifel to enable an effective generalization [128].

Together with the symbols just mentioned, in the *Tratado de Arithmetica* Moya lists the numerical value of each one, when the numerical value of the *cosa* is two (see Fig. 4). Up to 2^{24} the numbers coincide with those

123 See in [Cajori 1928, p. 339] the more usual notation among German writers for the powers of the unknown.

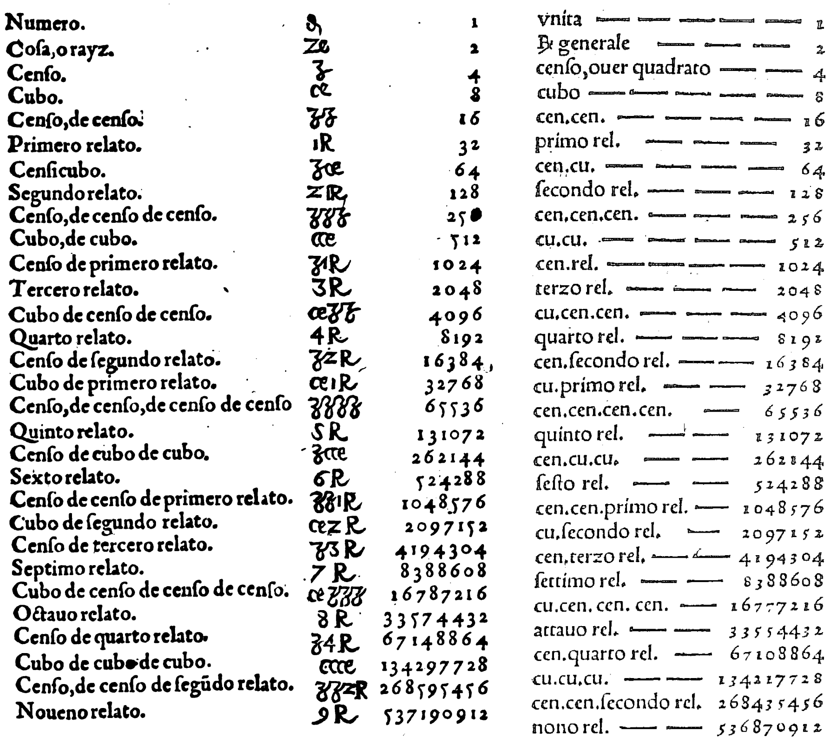
124 [Rodriguéz 1915, p. 174].

125 See [Stifel 1544, p. 235], and [Aurel 1552, p. 70v].

126 [Stifel 1544, p. 236].

127 [Tartaglia 1560, p. 1r].

128 See, for example [Pérez de Moya 1573, p. 478].



Numero.	1
Cosa, o rayz.	2
Censo.	4
Cubo.	8
Censo, de censo.	16
Primero relato.	32
Censicubo.	64
Segundo relato.	128
Censo, de censo de censo.	25[illegible]
Cubo, de cubo.	512
Censo de primero relato.	1024
Tercero relato.	2048
Cubo de censo de censo.	4096
Quarto relato.	8192
Censo de segundo relato.	16384
Cubo de primero relato.	32768
Censo, de censo, de censo de censo	65536
Quinto relato.	131072
Censo de cubo de cubo.	262144
Sexto relato.	524288
Censo de censo de primero relato.	1048576
Cubo de segundo relato.	2097152
Censo de tercero relato.	4194304
Septimo relato.	8388608
Cubo de censo de censo de censo.	16787216
Octauo relato.	33574432
Censo de quarto relato.	67148864
Cubo de cubo de cubo.	134297728
Censo, de censo de segũdo relato.	268595456
Noueno relato.	537190912

vnita	1
℞ generale	2
censo, ouer quadrato	4
cubo	8
cen.cen.	16
primo rel.	32
cen.cu.	64
secondo rel.	128
cen.cen.cen.	256
cu.cu.	512
cen.rel.	1024
terzo rel.	2048
cu.cen.cen.	4096
quarto rel.	8192
cen.secondo rel.	16384
cu.primo rel.	32768
cen.cen.cen.cen.	65536
quinto rel.	131072
cen.cu.cu.	262144
sesto rel.	524288
cen.cen.primo rel.	1048576
cu.secondo rel.	2097152
cen.terzo rel.	4194304
settimo rel.	8388608
cu.cen. cen. cen.	16777216
attauo rel.	33554432
cen.quarto rel.	67108864
cu.cu.cu.	134217728
cen.cen.secondo rel.	268435456
nono rel.	536870912

FIGURE 4. The powers of the unknown until 2^{29}, and the corresponding numerical values assuming the base 2; in [Pérez de Moya 1573, p. 432], on the left, and in [Tartaglia 1556, p. 73r], on the right.

presented in the *General Trattato*[129]; but unlike Tartaglia, Moya misses the calculation of 2^{24}, and obtains the following powers based on this wrong value. This seems to mean that Moya did the calculations by itself, because Tartaglia makes no such mistake, and Pacioli only miscalculates[130] 2^{29}.

Moya preserves the names for the unknown quantity and its powers (*cosa, censo, cubo, censo de censo* ...) which have origin in Arabic algebras[131]. They were usually adopted among the Italian writers, and they are used

129 [Tartaglia 1556, p. 73r].

130 [Pacioli 1494, p. 143r].

131 [Høyrup 2007, p. 7, Footnote 4].

in some Iberian works [132]. Except for the first, all the other quantities are often referred as dignities (Moya calls them *dignidades* [133]), and its construction is based on a geometrical progression, whose first element is the number (Moya calls it *numero*):

> Note, just as we say that putting some numbers in any Geometric progression, starting with the unit, the third number is always a square, and the fourth is a cube, and the fifth is a square of square [*censo de censo*], and the sixth is first relatus [*primero relato*], and the seventh a square of cube [*censicubo*], &;. so the first number after the unit, will be a root of the third, and of the fourth, and of the fifth, and of all the other oners. And therefore we say, that the thing is a root of all the characters placed after it [134].

According to [Silva 2008] Bento Fernandes's Tratado da arte de arismetica (1555) is the first Iberian printed text that uses the names cited by Moya (Fernandes calls them *cousa, censo* or *çeço, cubo, censo de censo, censo de cubo, cubo de cubo*). Nevertheless, their meaning only coincide for the first four powers of the unknown, because Moya uses a multiplicative generator system to construct the names of the powers, while Fernandes uses an additive system [135].

Although, in this case, the additive system would be preferable [136], the multiplicative system based on Italian notation was most commonly used

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132 For example in [Roca 1565, p. 253r] and in Ms. 71, de Sant Cugat [Docampo Rey 2004, p. 439].

133 The *numero* operates as the unity in the arithmetic, and it is not considered a *dignidade*. [Pérez de Moya 1573, p. 430].

134 "Nota, asi como dezimos que poniẽdo algunos numeros en qualquiera progression Geometrica, começando de la unidad, siempre el tercero numero es quadrado, y el quarto es cubo, y el quinto es censo de censo, y el sexto primero relato, y el septimo censicubo, &;. assi el primer numero que siguiere tras la unidad, sera rayz del tercero, y del quarto y del quinto, y de todos los demas. Y por esto dezimos, que la cosa, es rayz de todos los characteres que despues de si se pusieren." [Pérez de Moya 1573, p. 431].

135 See [Silva 2008, pp. 197-198]. These systems differ in the following: in the multiplicative system the juxtaposition of x^a and x^b means, in modern symbolic notation, (x^b), or, x^{ba}, while in the additive it means x^{a+b}. Therefore, while in the first case *censo e cubo* denotes x^6, in the second it denotes x^5. The multiplicative system does not allow, for example, to construct the powers of the unknown with a prime exponent greater or equal to 5, using this procedure. As a result, those who use it need special terms to assign those powers.

136 The use of the multiplicative principle compelled Moya to indicate special names for the characters representing the powers whose exponent was not a multiple of only 2 or 3, such as the *primo relato* for x^5; the *segundo relato* for x^7, and so on; each name being related with the decomposition of the exponent of the corresponding unknown power. For example, x^8 is called *censo de censo de censo* because it denotes a number three times squared.

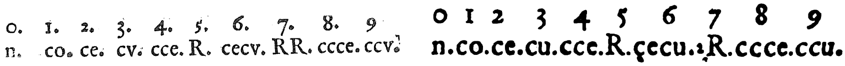

FIGURE 5. The powers of the unknown up to the exponent 9, in [Pérez de Moya 1558, p. 35], and in [Pérez de Moya 1562, p. 512], on the left; and in [Pérez de Moya 1573, p. 478], on the right.

by the authors from the 16th century [137]. As far as we know, the Portuguese merchant Bento Fernandes is the only Iberian author that uses an additive system.

Moya's symbols are, with some differences, those used by Tartaglia and Nunes (see Table 3). But there are very important differences concerning the powers with exponents four, eight and nine, because Moya uses symbols utterly different from the ones discussed so far.

Instead of *ce.ce*, *ce.ce.ce*, and *cu.cu.*, he uses *cce.*, *ccce.*, and *ccu.* (see Fig. 5).

As shown in the figures 4 and 5, if the exponent is a prime number greater than three, Moya associates to it the symbol nR, where n indicates the order of the first prime counting from 5, inclusive; for example x^{13} is denoted $4R$. On the other hand, if the power of the unknown is not a prime (all cases presented have as prime factors 2 and 3), Moya first decomposes it on the product of two factors, and takes the symbol corresponding to each factor. Then, as he is dealing with a multiplicative generator system, he simply juxtaposes these symbols, removing the first vowel, whenever it is repeated. For example, if the power is 8, as $8 = 2 \times 4$, x^8 is represented by x^2x^4 (notice that the absence of a symbol between x^2 and x^4 does not mean its product). Therefore, Moya takes the symbol *ccce.*, which results of the juxtaposition of *ce.* and *cce.*, removing the first vowel, and keeping the last one. Using the same procedure to 24 he gets *cccecu.*.

This notation, which we have not found in the major algebraic texts of Moya's time, can be interpreted as an attempt to take the brevity of the German symbols to the combinations of letters, which are typical of the Italian notations.

6.2. *A sign to the equality*

According to the evidence gathered by F. Cajori, "In the printed books before Recorde, equality was usually expressed rhetorically by such words as *aequales*, *aequantur*, *esgale*, *faciunt*, *ghelijck*, or *gleich*, and sometimes by the

137 [Docampo Rey 2004, p. 438].

Table 3. Correspondence of the symbols used in [Tartaglia 1560, p. 1], [Nunes 1567, p. 24v] and [Pérez de Moya 1573, pp. 432-435, 473] to designate the unknown and its powers [138].

	Tartaglia [1560]		Nunes [1567]		Pérez de Moya [1573]	
numero	*n.*	numero	–	–	*n.*	numero
x	℞ ou *co.*	cosa	*Co.*	Cosa	*co.*	cosa
x^2	*ce.*	censo	*Ce.*	censo	*ce.*	censo
x^3	*cu.*	cubo	*Cu.*	Cubo	*cu.*	cubo
x^4	*cece.*	censo de censo	*Ce.ce.*	censo de censo	*cce.*	censo de censo
x^5	*rel.*	relato	*Re.pž*	relato primo	*R.* ou *1R.*	primero relato
x^6	*ce.cu.*	censo de cubo	*Ce.cu. Cu.ce.*	censo de cubo	*cecu.*	censo y cubo
x^7	*2.rel.*	2.relato	–	–	*2R.*	segundo relato
x^8	*ce.ce.ce.*	censo de censo de censo	–	–	*ccce.*	censo de censo de censo
x^9	*cu.cu.*	cubo de cubo	–	–	*ccu.*	cubo de cubo

abbreviated form *aeq*" [139]. In 1557 Recorde [140] used a sign for the expression of equality which is analogous to our present sign for *equal*. But according to Cajori, this symbol did not again appear in print until 1618 [141].

In his *Tratado de Arithmetica*, Moya emphasizes the importance of having a symbol that expresses the equality between the two members of an equation. And he created, for this purpose, the sign –∩–. We did not find this symbol in any Renaissance algebra previous to Moya's *Tratado de Arithmetica*, but we think that his choice does not appear to have been arbitrary. Actually, our author uses it in 1568 in his manuscript text *Arte de Marear* to represent the astrological sign balance, whose meaning is connoted with equilibrium. Unfortunately, as Moya notes, the symbol –∩– does not exist

139 [Cajori 1928, p. 297].

140 See "The Arte. The rule of equation, commonly called Algebra Rule", in [Recorde 1557, p. 2r].

141 [Cajori 1928, p. 298].

in the Spanish printing office of the time, and so he could not use it in the equations; for this purpose he used the syncope *yg*:

> Note, this figure —∩— means equal, and, since it does not exist in the press, I put this one *yg* in stead of it. And what precedes it will be one part of the equality, and what follows it will be the other [part] [142]. [Pérez de Moya 1573, p. 540].

This symbol appears in Pérez de Mesa's manuscript [1598], *Libro y tratado del arismetica y arte mayor y algunas partes de astrología y matematicas.* On this subject, Romero Vallhonesta [2011, p. 113] says in the concluding remarks: "It is probably not by chance that Pérez de Mesa introduces the sign Ω for the equality when he considers the equations, consciously or not, as a new object in the algebra". We can now say that in 1598 this symbol was not an innovation in algebraic notation, because it was introduced by Pérez de Moya in his *Tratado de Arithmetica* [1573] twenty-five years before. [143]

7. CONCLUDING REMARKS

Pérez de Moya was acquainted with some of the most important algebraic works of his time, such as Stifel's *Arithmetica Integra,* Tartaglia's *General Trattato,* and Pedro Nunes's *Libro de Algebra,* when he wrote his *Tratado de Arithmetica.* These texts had a strong impact in Moya's algebra; probably influenced by their reading, he returned to his previous algebraic texts to incorporate several improvements, with which he intended to simplify and expand the study of algebraic equations. These improvements especially occurred in operations with polynomials and radicals, as well as in the search of algebraic notation that would facilitate the writing and solving of the equations.

We have proved that among these works the main source of Moya is Nunes's *Libro de Algebra.* This is, in a way, surprising as the influence of Nunes's algebra in any other Iberian works of the sixteenth century is not known. By the way, we recall that in the Historical-Bibliographical Notes (*Anotações histórico-bibliográficas*) to the edition of *Libro de Algebra,* Joaquim de Carvalho [1950, p. 436] attributes to Guillaume Gosselin the first direct citation of this work of Nunes, which Gosselin mentioned in his *De arte*

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142 "Nota, esta figura —∩— quiere dezir ygual, y en su lugar por no aver outra en la emprenta pongo esta yg. Y lo que tuviere antes de si sera la una parte de la ygualacion, y lo que tuviere despues de si sera la outra." [Pérez de Moya 1573, p. 540].

143 Notice that Romero Vallhonesta uses the sign Ω to represent the one used by Pérez de Moya and Pérez de Mesa.

magna, seu occulta parte numerorum [1577]. And although Leitão [2010, p. 11] claims that "the Libro de algebra enjoyed a huge spread in the decades that followed its publication", he does not indicate any Iberian author whose work reveals the influence of Nunes.

As shown here, in his *Tratado de Arithmetica* Moya works extensively with polynomials. Furthermore, we notice that Nunes's *Libro de Algebra* is the first Iberian printed work in which the algorithm for the division of polynomials is explained. In the sixth part of his *General Trattato* Tartaglia made a slight approach to this subject; he gives one example of the division of a trinomial by a binomial, but without explaining how to operate with it, merely translating it as a fraction.

The algebraic computation of polynomials and algebraic fractions in Moya's *Tratado de Arithmetica* has obvious similarities with those presented in Nunes's *Libro de Algebra*. These similarities cover the language used to describe the algorithm for dividing polynomials, the characteristics of the example, and even the proof of the result. In addition, Moya explains the equivalence of two algebraic fractions in a very similar way to Nunes. Regarding the study of equations Moya, like Nunes, indicates two rules for solving each one of the quadratic *compound conjugation*s (one of which is of Nunes' authorship); and offers analogous geometrical proofs of these rules, quoting the same propositions of Euclid's *Elements*.

While Nunes's *Libro de Algebra* has been the main source of Moya for the study of polynomials and equations, the *Tratado de Arithmetica* also reflects the influence of Tartaglia's *General Trattato*. Among the most important similarities, we highlight the procedure used to determine the square root of a trinomial in which the middle term is negative, together with the proof given, and also the method for solving a polynomial equation of the fourth degree. Both Tartaglia and Moya practice this method in an example which is not included in Nunes's *Libro de Algebra*.

Furthermore, we pointed out similarities with Stifel's *Arithmetica integra* with regard to the algebraic notation.

Despite the influence that Stifel, Tartaglia and Nunes had on Moya's algebraic works, we showed that the *Tratado de Arithmetica* is a genuine text, with original contributions. They disclose an author interested in simplifying and improving the algebraic notation established by Italians and Germans.

Moya created a sign to represent the equality between the two members of an equation, and dared to simplify the existing writing for the symbols indicating the powers of the unknown with a compound exponent. Furthermore, he discusses the existence of the cube root of a quadrinomial and establishes a necessary and sufficient condition for it. These topics do not exist in Nunes' *Libro de Algebra*, or in Tartaglia's *General Trattato*, or Stifel's *Arithmetica integra*. In his *La Practique de l'Arithmétique* [1585], Stevin deals with the cube root of a quadrinomial, but only gives a necessary condition for its existence.

To conclude, we emphasize that the *Tratado de Arithmetica* is the first Iberian printed work that refers to the possibility of using as many unknowns as necessary to solve a problem, and which indicates the appropriate notation for it.

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