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RIEMANN'S *COMMENTATIO MATHEMATICA*, A REASSESSMENT

Alberto Cogliati

Abstract. — Starting with the publication of Riemann's *Gesammelte Mathematische Werke* in 1876, the *Commentatio Mathematica* has attracted considerable interest among mathematicians and historians. Nonetheless, there appears to be no consensus on the most appropriate approach to the interpretation of the paper, namely on its relationship with Riemann's *Habilitationsvortrag*.

This article represents a contribution to such an interesting debate. Special attention is paid to the *pars secunda* of Riemann's *Commentatio*. In particular, the focus is on the interpretation of a certain trinomial expression from which Riemann claimed that an understanding of his "curvature tensor" could be achieved.

Résumé (Un réexamen du *Commentatio Mathematica* de Riemann)

À commencer avec la publication des œuvres de Riemann en 1876, la *Commentatio Mathematica* de Riemann a suscité beaucoup d'intérêt parmi les mathématiciens et les historiens. Il semble pourtant qu'aucun consensus sur la bonne lecture de ce texte en rapport avec le *Habilitationsvorstrag* de Riemann ne se soit dégagé. Cet article contribue à ce débat intéressant. Nous prêtons particulièrement attention à la *pars secunda*, nous centrons notre interprétation autour d'une certaine expression trinomiale dont Riemann prétend qu'elle permet de comprendre le « tenseur de courbure ».

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1. INTRODUCTION

According to an established historiographical notion, Riemann's *Commentatio Mathematica* submitted in 1861 for a prize to the *Académie des Sciences* of Paris can be regarded as a further development and as a more explicit elaboration of the revolutionary ideas which Riemann himself set forth in 1854 on the occasion of his habilitation lecture [Riemann 1854] on the hypotheses laying at the basis of geometry.

Such an interpretation attitude was first propounded in 1876 by H. Weber, the editor with R. Dedekind of Riemann's *Werke*. At the beginning of his highly detailed commentary notes appended to Riemann's paper, he wrote that "These investigations are most intimately connected to the paper Ueber die Hypothesen welche der Geometrie zu Grunde liegen".[1] Accordingly, Weber devoted a large amount of his attempt at clarifying Riemann's ideas to a discussion of their geometrical content, by putting special emphasis upon the introduction of a special type of coordinates (later known as normal Riemannian coordinates) and of second order differentials which represented an indispensable technical means for the geometrical interpretation of the four-index quantities introduced by Riemann in his discussion of the equivalence problem of two quadratic differential forms.

In 1869, before Riemann's *Commentatio* was published for the first time, and thus independently of it, Elwin Bruno Christoffel and Rudolf Lipschitz who in reason are considered the founding fathers of what later became known as tensor calculus, tackled the very same problem with which Riemann had confronted himself in the *Commentatio*, providing a detailed analysis of the equivalence problem. They both obtained necessary and sufficient conditions guaranteeing the existence of a coordinate transformation which carries one quadratic differential form into another.

Their contributions [Christoffel 1869] and [Lipschitz 1869], both appeared in the same volume of *Crelle's Journal*, were characterized by distinct

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1 *Diese Untersuchungen hängen aufs Innigste zusammen mit der Abhandlung Ueber die Hypothesen welche der Geometrie zu Grunde liegen* [Riemann 1876, p. 384]. It is interesting to note that in the second edition of [Riemann 1876] Weber rephrased this statement as follows: "These investigations contain the analytical details of the results outlined in the paper Ueber die Hypothesen welche der Geometrie zu Grunde liegen." *Diese Untersuchungen enthalten die analytischen Ausführungen zu den in der Abhandlung "Ueber die Hypothesen welche der Geometrie zu Grunde liegen" angedeuteten Resultaten.* See [Riemann 1876, 2nd edition, p. 405].

approaches and quite dissimilar emphases upon the significance to be attributed to the investigations contained therein. While Christoffel's analysis can aptly be regarded as a chapter in the theory of algebraic invariants, Lipschitz's contribution revealed a closer link with Gauß' theory of curved surfaces and with Riemann's n-dimensional metric geometry.

Lipschitz was particularly explicit in singling out the main motivation at the basis of his research by providing, from the very start, a geometrical interpretation of a quadratic differential form as the square of the line element of a given (in general, n-dimensional) surface. He wrote: "Correspondingly, the analytic expression for the square of the line-element of a given surface in two independent variables, upon which Gauss has based his disquisitiones generales circa superficies curvas, is a quadratic form of two differentials. Gauss's researches on this subject have found an outstanding application in Riemann's posthumously published essay über die Hypothesen, welche der Geometrie zu Grunde liegen. Owing to this, the interest in the forms, and especially in those of second degree and in any number of differentials, has considerably increased. A key point in Riemann's investigations is represented by the conveyence of criteria guaranteing the existence of a change of variables which transforms one such form into another one which is sum of the square of the differentials of the new variables. The present work aims at providing a direct answer to this question in the case of second degree forms of n differentials whose determinant does not vanish identically under the imposed conditions. In this sense, it represents an extension of the theory of Gaussian curvature".[2]

After the publication of [Riemann 1876], in [Lipschitz 1876] Lipschitz came back to the topic which he had tackled in 1869 by interpreting those

2 *Demgemäss ist der analytische Ausdruck für das Quadrat des Linearelements einer gegebenen Fläche in zwei unabhängigen Veränderlichen, den Gauss den* disquisitiones generales circa superficies curvas *zu Grunde gelegt hat, eine quadratische Form von zwei Differentialen. Die hierauf bezüglichen Gaussischen Forschungen haben in dem aus Riemann's Nachlasse publicirten Aufsatz über die Hypothesen, welche der Geometrie zu Grunde liegen, eine merkwürdige Anwendung gefunden, und es ist damit das Interesse an der bezeichneten Gattung von Formen, und in's besondere an denen des zweiten Grades und von beliebig vielen Differentialen, bedeutend gewachsen. Einen Angelpunkt von Riemann's Untersuchungen bildet die Ermittelung der Criterien, von denen es abhängt, ob eine derartige Form durch Einführung eines neuen Systems von Variabeln in eine Form transformirt werden könne, welche das Aggregat der Quadrate von den Differentialen der neuen Variabeln ist.* [...] *Die gegenwärtige Arbeit verfolgt vornehmlich den Zweck, die gestellte Frage in Betreff der Formen zweiten Grades von n Differentialen, deren Determinante zufolge der getroffenen Einschränkung nicht identisch verschwindet, direct zu beantworten, und in diesem Sinne die Theorie des Gaussischen Krümmungsmasses auszudehnen* [Lipschitz 1869, p. 71 and p. 73].

results in the light of Riemann's *Commentatio* which he regarded as a completion of [Riemann 1854].[3]

In accordance with Weber's and Dedekind's historiographical position, in 1879 Richard Beez, a Gymnasium teacher in Plauen, referring to the recent publication of the *Commentatio*, wrote that the second part of this work "is of the utmost interest, since it essentially contains a complete theory of the so-called Riemannian curvature which at least one presumes to be present at that place. In this respect, this paper may be considered as the conclusion of the investigations carried out by Riemann seven years before in his habilitation thesis On the Hypotheses...".[4]

In 1900, Gregorio Ricci-Curbastro and Tullio Levi-Civita in a joint paper which represents a milestone for the development of tensor analysis[5] made explicit mention of Riemann's *Commentatio* when they introduced the *système covariant de Riemann*. Although they did not provide any geometrical interpretation of Riemann's symbols on that occasion, they were, however, quite clear in assessing the dominant role played by differential geometry in the conceptual development of their absolute differential calculus. Indeed, we read in their introduction: "L'algorithme du Calcul différentiel absolu, c'est-à-dire, l'instrument matériel des méthodes, sur lesquelles nous allons entretenir les lecteurs des Mathematische Annalen, se trouve tout entier dans une remarque due à M. Christoffel. Mais les méthodes-mêmes et les avantages qu'ils présentent, ont leur raison d'être et leur source dans les rapports intimes, qui les lient à la notion de variété à n dimensions, que nous devons aux génies de Gauss et de Riemann" [Ricci & Levi-Civita 1900, p. 127].

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3 Lipschitz wrote: "Riemann's mathematical works, whose publication represents an imperishable merit of R. Dedekind and H. Weber, contain the response in Latin to a problem posed by the Paris Academy in 1858 concerning heat conduction. This completes Riemann's paper Über die Hypothesen, welche der Geometrie zu Grunde liegen in the most remarkable way". *Die gesammelten mathematischen Werke Riemanns, deren Herausgabe ein bleibendes Verdienst der Herren R. Dedekind und H. Weber bildet, enthalten die in lateinischer Sprache verfasste Beantwortung einer im Jahre 1858 gestellten Aufgabe der Pariser Akademie über ein Problem der Wärme-vertheilung, durch welche Arbeit die Schrift Riemanns über die Hypothesen, welche der Geometrie zu Grunde liegen, auf das merkwürdigste ergänzt wird* [Lipschitz 1876, p. 316].

4 [...] *ist hauptsächlich deshalb von hohem Interesse, weil er im Grunde genommen eine vollständige Theorie des sogenannten Riemann'schen Krümmungsmasses enthält, welche man wohl an jener Stelle am wenigsten vermuthet. Insofern darf diese Arbeit als der Abschluss der bereits sieben Jahre zuvor von Riemann in der Habilitationsschrift Ueber die Hypothesen… angestellten Untersuchungen betrachtet werden.* [Beez 1879, p. 1].

5 In this respect, [Dell'Aglio 1996] should be consulted.

Only two years later, in 1902, Luigi Bianchi published a revised edition of his well known handbook on differential geometry [Bianchi 1902]. Bianchi explicitly mentioned Riemann's *Commentatio* in Chapter II, where he discussed the equivalence problem of two quadratic differential forms. He emphasized that such expressions were introduced by Riemann well before Christoffel in a posthumous memoir. Bianchi also conveyed a detailed discussion of the geometrical meaning of what he called Riemannian curvature (the sectional curvature) of an n dimensional space, by closely following the procedure which Riemann had set forth in words in [Riemann 1861, p. 382].

A thorough discussion of Riemann's prize paper was offered by Tullio Levi-Civita in his fundamental article [Levi-Civita 1917] on the notion of parallel displacement (*trasporto parallelo*[6]). After introducing his own geometrical interpretation of Christoffel's symbols in terms of a generalization of the notion of parallelism for manifolds embedded in a Euclidean space, Levi-Civita offered an attentive discussion (see [Levi-Civita 1917, pp. 35-38]) of Riemann's *Commentatio* in which he dealt with some difficult passages of Riemann's reasonings. There is hardly any doubt that in Levi-Civita's opinion the distinctive trait of Riemann's paper was represented by its highly innovative geometrical content.

Hermann Weyl in 1919 in his commentary notes to Riemann's *Habilitationsvortrag* explicitly referred to the *Commentatio* by asserting that in it explicit analytical computations pertaining to the "Riemannian theory of curvature[7]" were accomplished for the first time. In so doing, Weyl explicitly underlined the link between the Parisian memoir and the *Habilitationsvortrag*.

In the course of the last century, mathematicians and historians such as J. D. Struik and M. Kline provided an analysis of the *Commentatio* which essentially coincided with Weber's point that one can read the second part of Riemann's paper as conveying a geometrically oriented treatment of the equivalence problem for two n dimensional manifolds.[8]

More recently, however, a different attitude towards Riemann's *Commentatio* has made its way among historians of mathematics. [Farwell & Knee 1990] provided a detailed analysis of Riemann's paper which included a broad review of some works by different commentators who confronted themselves with Riemann's *Commentatio*. The core of Farwell and Knee's

6 On Levi-Civita's notion of parallel displacement see [Reich 1992] and [Bottazzini 1999].

7 *Riemannsche Krümmungstheorie*. [Riemann 1919, p. 34].

8 See [Struik 1933, pp. 175-176] and [Kline 1972, pp. 894-896].

historiographical thesis can be summarized as follows: Interpreting Riemann's *Commentatio* as furnishing the technical details missing in his *Habilitationsvortrag* and thus providing the explicit development of Riemann's 1854 geometrical ideas, is not due to Riemann, but to a prejudicial modern perspective which tends to conflate two research areas which should be kept apart: tensor analysis and differential metric geometry.

Farwell and Knee arrived at quite definite and peremptory conclusions: "(a) the 'Commentatio' does not explicitly contain any geometrical analysis and is not a conscious mathematical elaboration of the geometrical concepts described in the *Habilitationsvortrag* of 1854; (b) the next stage in the development of Riemann's geometry of *n*-dimensional manifolds following its conception in the 1854 *Habilationsvortrag* was tensorial and not geometrical." [Farwell & Knee 1990, p. 236]

As for the first point, they highlighted that Riemann did not consider the equivalence of forms in relation to geometry, but rather in the context of heat conduction. Farwell and Knee also added that Riemann's mathematical derivations in the *Commentatio* contain no reference to heat conduction, but equally they contain no reference to geometry and that the one allusion to geometry is an illustration, which is not linked to heat conduction and does not obviously serve as a "useful addition".

As for the second conclusion, Farwell and Knee admitted nonetheless that Riemann's *Commentatio* exerted some influence upon the development of differential geometry. However, this is their argument, such an influence was of an indirect type since it was essentially filtrated through the emergence of what they called tensor analysis.

Some years later, K. Reich in her book on the history of tensor calculus [Reich 1994] substantially concurred with Farwell and Knee's historiographical thesis by questioning the existence of a close relationship between the *Commentatio* and Riemann's *Hypothesen.* She observed that "[...] clearly Riemann's transformation of quadratic differential forms is neither closely connected with the prize paper, i.e., the question of heat conduction, nor is directly related to the "Hypothesen". Indeed, the discussion of the transformation of quadratic differential forms lacks any geometrical allusion [...]".[9]

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[9] [...] *steht Riemanns Transformation quadratischer Differentialausdrücke offensichtlich weder in engem Zusammenhang mit der Preisaufgabe, d.h. der Frage nach der Wärmeleitung, noch in direkter Beziehung zu den 'Hypothesen'. In der Tat fehlt der Darstellung der Transformation quadratischer Differentialausdrücke jeglicher geometrischer Hinweis* [...] [Reich 1994, p. 28].

Apparently, for Reich, this interpretation was particularly apt and instrumental in supporting her much broader historiographical perspective according to which differential and metric geometry played only a limited role in the development of tensor analysis. Indeed, in open contrast with common *vulgata*, according to Reich "the absolute differential calculus had its main roots not, as has been often said thus far, in differential geometry but rather in invariant theory."[10]

Given the wide variety of opinions and the different historiographical attitudes reviewed, it does not seem inappropriate to provide some further analysis which, without any claim for definitiveness, might nonetheless be a useful contribution to such an interesting debate about the place to be attributed to Riemann's *Commentatio* in history of mathematics. The core of the present contribution is a detailed discussion of the second part of Riemann's paper. Special attention is paid to the interpretation of a trinomial expression in terms of which Riemann claimed an understanding of his "curvature tensor" could be achieved.

2. ANALYSIS OF THE PAPER

As the title suggests, the *Commentatio Mathematica, qua respondere tentatur quaestioni ab Ill.ma Academia Parisiensi propositae...* was Riemann's response to a question concerning heat conduction which was posed for the first time in 1858 by the Academy of Sciences of Paris on the occasion of the *Grand Prix de Mathématiques.*[11] Riemann's paper, which was submitted in 1861, did not win the price, which was eventually withdrawn in 1868.

The question to which Riemann set out to provide an answer read as follows: *"Trouver quel doit être l'état calorifique d'un corps solide homogène indéfini pour qu'un système de courbes isothermes, à un istant donné, restent isothermes après un temps quelconque, de telle sorte que la température d'un point puisse s'exprimer en function du temps et de deux variables indépendantes."*

In order to answer the proposed question, Riemann divided his treatment into two steps. First, he observed, it is necessary to solve the problem of "which are the properties of a body which determine the heat flow

10 [...] *der absolute Differentialkalkül hat seine Hauptwurzeln nicht, wie man bisher häufig angenommen hat, in der Differentialgeometrie, sondern in der Invariantentheorie* [Reich 1994, p. 213].

11 A first version of the question concerning heat conduction in a ellipsoid was first proposed in 1855. In 1858 the prize committee composed by Liouville, Lamé, Chasles, Poinsot and Bertrand decided to modify the problem, since no answer had been received.

and the heat distribution such that there exists a system of lines which remain isothermal".[12] After that, in order to specify the general solution thus obtained to the request which is relevant to the question proposed by the Academy, one has to restrict oneself to the case of homogeneous media; in Riemann's words: "from the general solution of this problem, we choose those cases in which those properties remain everywhere the same, i.e., the body is homogeneous."[13]

As for the first step, it should be observed that Riemann considered a broader class of thermic bodies than that which the Academy singled out. Indeed, he assumed that the thermic properties of the body under consideration are described by a 3×3 matrix whose coefficients depend, in general, upon the spatial variables x_1, x_2, x_3. As a consequence of this, the heat equation considered by Riemann takes on the following generalized form:

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{3} a_{ij} \frac{\partial u}{\partial x_j} \right) = h \frac{\partial u}{\partial t}, \tag{1}$$

where the value of the function u at (x_1, x_2, x_3, t) is the temperature of the body at the point (x_1, x_2, x_3) and at time t. The quantities a_{ij} are the resulting conductibility coefficients (*conductibilitates resultantes*); h is the specific heat per unit of volume. Besides, Riemann explicitly restricted himself to the case in which the matrix of the coefficients a_{ij} is symmetric.

On the basis of these assumptions, Riemann considered an arbitrary change of coordinates. His purpose was that of deriving a simplified version of the heat equation which reflected the symmetry of the problem proposed by the Academy. Indeed, a judicious choice of coordinate system allowed him to express the temperature as a function of only two spatial variables.

Riemann deduced the effect of such a change of coordinates upon equation (1) in the following, ingenious way. He wrote: "This transformation of equation (I) can be most easily obtained, for this equation is the necessary and sufficient condition that, when δu indicates an infinitesimal variation of u, the integral

$$\delta \iiint \sum_{i,j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx_1 dx_2 dx_3 + \iiint 2h \frac{\partial u}{\partial t} \delta u dx_1 dx_2 dx_3, \tag{2}$$

[12] [...] *quales esse debeant proprietates corporis motum caloris determinantes et distributio caloris, ut detur systema linearum quae semper isothermae maneant* [Riemann 1861, p. 370].

[13] [...] *ex solutione generali hujus problematis eos casus seligamus, in quibus proprietates illae evadant ubique eaedem, sive corpus sit homogeneum* [Riemann 1861, p. 370].

over the volume of the body, depends upon the value of the variation δu on the surface only."[14]

More explicitly, Riemann replaced equation (1) with an integro-variational condition upon which it was easier to operate the change of coordinates under consideration.

Riemann's statement can be verified in a straightforward way by computing the variation of the first integral. Indeed, modulo boundary terms, one can easily obtain[15]:

$$\begin{aligned}\delta \iiint \sum_{i,j=1}^{3} a_{ij} u_i u_j dx^3 &= -\iiint \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \frac{\partial}{\partial u_k} \left[\sum_{i,j=1}^{3} a_{ij} u_i u_j \right] \delta u\, dx^3 = \\ &= -2 \iiint \left(\sum_{k=1}^{3} \frac{\partial}{\partial x_k} \sum_{j=1}^{3} a_{jk} u_j \right) \delta u\, dx^3 = \\ &= -\iiint 2h \frac{\partial u}{\partial t} \delta u\, dx^3,\end{aligned}$$

where in the last equality one has taken into account equation (1). It should also be observed that the factor 2 is a consequence of the symmetry of the matrix a_{ij}. In view of this observation, Riemann operated the change of coordinates upon the integral expression (2), thus obtaining the following laws of transformation. If s_1, s_2, s_3 indicate the new independent variables, the integral (2) is transformed into

$$\delta \iiint \sum_{i,j=1}^{3} b_{ij} \frac{\partial u}{\partial s_i} \frac{\partial u}{\partial s_j} ds_1 ds_2 ds_3 + \iiint 2k \frac{\partial u}{\partial t} \delta u ds_1 ds_2 ds_3, \tag{3}$$

provided that one poses:

$$\frac{\sum_{r,s=1}^{3} a_{rs} \frac{\partial s_i}{\partial x_r} \frac{\partial s_j}{\partial x_s}}{\det \left[\frac{\partial s_g}{\partial x_k} \right]} = b_{ij}, \qquad \frac{h}{\det \left[\frac{\partial s_g}{\partial x_k} \right]} = k. \tag{4}$$

14 *Haec trasformatio aequationis (I) facillime inde peti potest, quod haec aequatio conditio est necessaria et sufficiens, ut, designante δu variationem quamcunque infinite parvam ipsius u, integrale*

$$\delta \iiint \sum_{i,j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx_1 dx_2 dx_3 + \iiint 2h \frac{\partial u}{\partial t} \delta u dx_1 dx_2 dx_3$$

per corpus extensum, solum a valore variationis δu in superficie pendeat. [Riemann 1861, p. 371]. The reference to (I) is equivalent to (1). The translation provided in [Farwell & Knee 1990, p. 241] is misleading. By the expression *haec aequatio,* Riemann referred to the untransformed heat equation and not to the transformed one. There appears to be no other way of making sense of the mathematical context.

15 For the sake of brevity, we have posed $u_i = \frac{\partial u}{\partial x_i}$ and $dx^3 = dx_1 dx_2 dx_3$.

We observe, in anachronistic (though reasonable) terms, that the transformation law for the coefficients a_{ij} coincides with that of tensor of rank 2 modulo the determinant of the jacobian matrix. In this respect, Riemann pointed out that the coefficients α_{rs}, β_{ij} of the adjoint matrices obey an easier transformation law, namely

$$\beta_{ij} = \sum_{kl} \alpha_{kl} \frac{\partial x_k}{\partial s_i} \frac{\partial x_l}{\partial s_j}, \qquad i, j = 1, 2, 3. \tag{5}$$

It was thus in terms of the coefficients of the adjoint matrices that his well known equivalence problem, which he had already formulated in his *Habilitationsvortrag*, found application in the present situation.

At this point Riemann clarified his strategy for answering the proposed question. First he set out to consider the problem for an inhomogeneous medium and to list the cases in which the temperature u depends upon the variables s_1, s_2 and t only. In this way, the coefficients b_{ij} and the specific heat k (and hence β_{ij}) could be determined. After that, Riemann observed, it is necessary to investigate whether the differential expression $\sum_{ij} \beta_{ij} ds_i ds_j$ is susceptible of being transformed into a given differential form $\sum_{ij} \alpha_{ij} dx_i dx_j$ where α_{ij} are constant coefficients as it is required by the fact that the thermic medium under consideration is homogeneous.

Anticipating the content of the second part of the *Commentatio*, Riemann observed that the last problem could be tackled with a method which is almost the same as the one employed by Gauss in his theory of curved surfaces.[16]

The remaining part of *pars prima* of the paper was precisely devoted to the resolution of the first problem which we have just mentioned. We will not go into details about this section.

The *pars secunda* deals with the already mentioned equivalence problem for two differential quadratic forms. More precisely, Riemann set out to provide necessary and sufficient conditions which guarantee the existence of a change of coordinates transforming a given differential quadratic form $\sum b_{ij} ds_i ds_j$ into an other one $\sum a_{rs} dx_r dx_s$ with given constant coefficients. It is important to note that Riemann considered in this section the more general case of differential quadratic forms in n variables.

16 In this respect, a minor mistake in the translation of Riemann's *Commentatio* provided in [Farwell & Knee 1990] should be reported. The lines "and we shall see below that this question can be examined using a method similar to that employed by Gauss in his theory of curves on surfaces", see [Farwell & Knee 1990, p. 242], should be replaced by "and we shall see below that this problem can be tackled with a method which is almost the same as the one employed by Gauss in his theory of curved surfaces".

These relations were deduced by Riemann in a purely analytical way by computing the integrability conditions for the set of equations

$$\sum_{k=1}^{n} \frac{\partial x_k}{\partial s_i}\frac{\partial x_k}{\partial s_j} = b_{ij}, \qquad i,j = 1,\ldots,n, \tag{6}$$

which can easily be derived from the requirement that $\sum_{ij} b_{ij} ds_i ds_j = \sum_r dx_r^2$.[17] Indeed, by differentiating equations (6) with respect to s_l, one obtains a set of n^3 equations which read as follows:

$$2\sum_{k=1}^{n} \frac{\partial^2 x_k}{\partial s_j \partial s_l}\frac{\partial x_k}{\partial s_i} = \frac{\partial b_{ij}}{\partial s_l} + \frac{\partial b_{il}}{\partial s_j} - \frac{\partial b_{jl}}{\partial s_i}. \tag{7}$$

At this point Riemann introduced a new notation by designating the right-hand side of the last equations with the symbols p_{ijl}[18]. By computing their cross-derivatives, he obtained the following relations

$$\frac{\partial p_{ijl}}{\partial s_t} - \frac{\partial p_{ijt}}{\partial s_l} = 2\sum_k \frac{\partial^2 x_k}{\partial s_j \partial s_l}\frac{\partial^2 x_k}{\partial s_i \partial s_t} - 2\sum k \frac{\partial^2 x_k}{\partial s_j \partial s_t}\frac{\partial^2 x_k}{\partial s_i \partial s_l}, \tag{8}$$

from which he could finally draw the conditions sought for:

$$\frac{\partial^2 b_{il}}{\partial s_j \partial s_t} + \frac{\partial^2 b_{it}}{\partial s_i \partial s_l} - \frac{\partial^2 b_{it}}{\partial s_j \partial s_l} - \frac{\partial^2 b_{jl}}{\partial s_i \partial s_t} + \frac{1}{2}\sum_{\nu,\mu} \left(p_{\nu jt} p_{\mu ij} - p_{\nu it} p_{\mu jl}\right) \frac{\beta_{\nu\mu}}{B} = 0, \tag{9}$$

where $\beta_{\nu\mu}$ indicates the cofactor of $b_{\nu\mu}$ and $B = \det(b_{ij})$.

Quite interestingly, Riemann thought that the four-index quantities appearing on the left hand side of equations (9) deserved special consideration, so much as to require the introduction of the new symbols (ij, lt).

It should be observed that the results obtained by Riemann thus far could in principle be considered enough to allow the fulfillment of the solution strategy which was described before. It is certainly true that the proof of the fact that the conditions obtained are also sufficient to guarantee the existence of the required transformation left something to be desired; nonetheless, when the structure of the paper is globally considered, there is no doubt that the treatment of the question could be limited to a purely analytical level.

Despite all that, Riemann did not content himself with providing an analytic approach of the subject only. He felt strongly the need to discuss explicitly a geometrical interpretation of the symbols (ij, lt). The purpose

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17 Riemann explicitly pointed out that "Expressionem $\sum a_{rs} dx_r dx_s$, si est, id quod supponimus, forma positiva ipsarum dx, semper in formam $\sum_r dx_r^2$ redigi posse constat".

18 Except for an inessential numerical factor, these quantities coincide with today's Christoffel symbols.

was that of achieving a better understanding of the real nature (*indoles*) of equations (9).

In order to do that, Riemann wrote down the following trinomial:

$$\delta\delta \sum_{ij} b_{ij} ds_i ds_j - 2d\delta \sum_{ij} b_{ij} ds_i \delta s_j + dd \sum_{ij} b_{ij} \delta s_i \delta s_j, \tag{10}$$

and simply declared that under certain conditions, to which the definition of second-order differentials are subject, this expression is equal to the following invariant expression involving the four-index symbols (ij, lt):

$$\sum_{ijlt} (ij, lt)(ds_i \delta s_j - ds_j \delta s_i)(ds_l \delta s_t - ds_t \delta s_l). \tag{11}$$

At this point Riemann moved to discuss the geometrical meaning underlying his investigations, by providing a geometric interpretation of the four-index symbols.

Indeed, after having considered the expression

$$-\frac{1}{2} \frac{\sum_{ijlt}(ij, lt)\left(ds_i \delta s_j - ds_j \delta s_i\right)(ds_l \delta s_t - ds_t \delta s_l)}{\sum_{ij} b_{ij} ds_i ds_j \sum_{ij} b_{ij} \delta s_i \delta s_j - \left(\sum_{ij} b_{ij} ds_i \delta s_j\right)^2}, \tag{12}$$

and having observed that such expression is invariant under a linear transformation of the variations $ds_i, \delta s_j$, he wrote:

> "These investigations can be illustrated by means of a certain geometrical interpretation, which, though based upon unusual concepts, is nonetheless worth pointing out. The expression $\sqrt{\sum b_{ij} ds_i ds_j}$ can be regarded as the line element in a more general space of dimension n which goes beyond our intuition. Indeed, if in this space all shortest lines are drawn from the point $(s_1, s_2, \ldots, s_n)$ in such a way that in their initial elements the variations of s_i behave as $\alpha ds_1 + \beta \delta s_1 : \alpha ds_2 + \beta \delta s_2 : \cdots : \alpha ds_n + \beta \delta s_n$, α and β denoting two arbitrary quantities, then these lines will form a surface which can be represented in the ordinary space of our intuition. In this way, (III) [(12)] will be a measure of the curvature of this surface at point $(s_1, s_2, \ldots, s_n)$".[19]

In modern terms, Riemann derived the formula for the sectional curvature of an n-dimensional manifold which in the case of a 2-dimensional

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19 *Disquisitiones haece interpretatione quadam geometrica illustrari possunt, quae quamquam conceptibus inusitatis nitatur, tamen obiter eam addigitavisse juvabit. Expressio* $\sqrt{\sum b_{ij} ds_i ds_j}$ *spectari potest tanquam elementum lineare in spatio generaliore* n *dimensionum nostrum intuitum trascendente. Quodsi in hoc spatio a puncto* $(s_1, s_2, \ldots, s_n)$ *ducantur omnes lineae brevissimae, in quarum elementis initialibus variationes ipsarum* s *sunt ut* $\alpha ds_1 + \beta \delta s_1 : \alpha ds_2 + \beta \delta s_2 : \cdots : \alpha ds_n + \beta \delta s_n$, *denotantibus* α *et* β *quantitates quaslibet, hae lineae superficiem constituent, quam in spatium vulgare nostro intuitui subjectum evolvere licet. Quo pacto* (III) [(12)] *erit mensura curvaturae hujus superficiei in puncto* $(s_1, s_2, \ldots, s_n)$ [Riemann 1861, p. 381].

surface coincides with the classical expression for the Gaussian curvature. To be sure, here Riemann established a direct link with [Riemann 1854, §2-3] where he had already referred to Gauss's curvature of embedded geodesic bi-dimensional submanifolds[20].

3. A CRITICAL POINT

In his detailed commentary notes to Riemann's paper, Weber devoted a lot of of attention to clarifying the technical content of this second part. Nonetheless, some points of Riemann's treatment remain obscure and in need of explanation. In particular, the true significance of the trinomial (10) poses some interpretative difficulty.

For instance, in 1917 Levi-Civita observed that, even if one admits that the second order differentials of the variables are subject to the invariant definition provided by Weber[21], i.e., $2\sum_k b_{ik} d^2 s_k = -\sum_{jl} p_{ijl} ds_j ds_l$, it is difficult to provide justification of the assertion "(10)=(11)". Indeed, a simple substitution in (10) of the explicit expressions of the second order differential makes it clear that the trinomial (10) is identically equal to zero and thus it cannot coincide with the quadrilinear form (11), except in the case of zero curvature.

Let us provide an explicit computation of the term $\delta \sum_{ij} b_{ij} ds_i ds_j$. By supposing, as Weber did, that the second-order differentials $d\delta s_i$ are defined by the invariant expression

$$\sum_{i=1}^{n} b_{ij} \delta d s_i = -\frac{1}{2} \sum_{k,l=1}^{n} p_{jkl} ds_k \delta s_l,$$

we obtain the following relations:

$$\begin{aligned} \delta \sum_{ij} b_{ij} ds_i ds_j &= \sum_{ijk} \frac{\partial b_{ij}}{\partial s_k} \delta s_k ds_i ds_j - \frac{1}{2} \sum_{ijk} p_{jki} ds_k \delta s_i ds_j - \frac{1}{2} \sum_{ijk} p_{ikj} ds_k \delta s_j ds_i \\ &= \sum_{ijk} \left(\frac{\partial b_{ij}}{\partial s_k} - \frac{1}{2} p_{jik} - \frac{1}{2} p_{ijk} \right) \delta s_k ds_i ds_j, \end{aligned}$$

which, upon substitution of the expressions for p_{ijk}, leads to

$$\sum_{ijk} \left(\frac{\partial b_{ij}}{\partial s_k} - \frac{1}{2} \frac{\partial b_{ji}}{\partial s_k} - \frac{1}{2} \frac{\partial b_{jk}}{\partial s_i} + \frac{1}{2} \frac{\partial b_{ik}}{\partial s_j} - \frac{1}{2} \frac{\partial b_{ij}}{\partial s_k} - \frac{1}{2} \frac{\partial b_{ik}}{\partial s_j} + \frac{1}{2} \frac{\partial b_{jk}}{\partial s_i} \right) \delta s_k ds_i ds_j.$$

---

[20] For a clear technical treatment of the point, one can consult [Bianchi 1902, §158].

[21] See [Riemann 1876, pp. 387-388].

It is easy to see that, in accordance with Levi-Civita's claim (see the quotation below), as a consequence of the symmetry of the quadratic form b_{ij}, this expression is identically equal to zero. Analogously, one can easily prove that $\delta \sum_{ij} b_{ij} ds_i \delta s_j \equiv 0$ and $d \sum_{ij} b_{ij} \delta s_i \delta s_j \equiv 0$, as well.

In this respect, Levi-Civita pointed out:

> *Things being such, the significance to be attributed to Riemann's trinomial*
>
> $$R = \delta^2 ds^2 - 2d\delta\Phi + d^2\delta s^2,$$
>
> *does not seem to be ambiguous. Indeed, in virtue of (48)*[22], *this significance necessarily implies that* $R = 0$.
> *On the contrary, Riemann asserts: "Haec expressio [i.e., R] invenietur* $= J$*" [J assuming the value (45)*[23]*]. In the course of his explanations, Weber lingers on the way of introducing second-order differentials. However, after having drawn their explicit expression, he simply affirms: "woraus man leicht den Ausdruck erhält* $R = J$*." [whence one can easily deduce the expression* $R = J$*.]*
> *Probably, there is just a misprint in Riemann's expression for R which hides the intended concept. I pride myself of having essentially reconstructed this concept, however, I have not been able to adjust the formula. If this can be done, it would be very nice to do complete justice to Riemann's genius also on this account.*[24]

Levi-Civita's observation is not conclusive, since it seems to misinterpret Riemann's reasoning. According to Levi-Civita, conditions such as $2\sum_k b_{ik} d^2 s_k = -\sum_{jl} p_{ijl} ds_j ds_l$ should be thought of as being imposed on the first-order variations of $\sum_{ij} b_{ij} ds_i ds_j$, $\sum_{ij} b_{ij} ds_i \delta s_j$ and $\sum_{ij} b_{ij} \delta s_i \delta s_j$. In

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22 By formula (48) Levi-Civita referred to what he called Ricci's lemma according to which $\delta ds^2 = d\delta s^2 = d\Phi = \delta\Phi = 0$, where $ds^2 = \sum g_{ij} dx_i dx_j$ and $\Phi = \sum_{ij} g_{ij} dx_i \delta x_j$, g_{ij} being the coefficients of the metric.

23 Formula (45) reads as follows: $\sum_{jlhk} a_{jk,lh} dx_j dx_l \delta x_h \delta x_k$, $a_{jk,lh}$ being the so called Riemann's symbols of second kind, see [Levi-Civita 1917, p. 31].

24 *In questa condizione di cose non sembra ambiguo il significato da attribuire al trinomio considerato da Riemann*

$$R = \delta^2 ds^2 - 2d\delta\Phi + d^2\delta s^2,$$

e un tale significato in virtù della (48), implica necessariamente $R = 0$.
Riemann afferma invece: "Haec expressio [cioè R] invenietur $= J$*" [avendo J il valore (45)]. Weber, nei suoi chiarimenti si diffonde sul modo di introdurre i differenziali secondi, ma, dopo averne ricavata l'espressione esplicita, dice semplicemente: "woraus man leicht den Ausdruck erhält* $R = J$*."*
Probabilmente, nella R di Riemann, c'è soltanto un qualche vizio di scrittura, che ne vela il concetto. Mi lusingo di aver sostanzialmente ricostruito tale concetto, ma non potei aggiustare il simbolo. Se la cosa è fattibile, sarà bene rendere, anche su questo punto, piena giustizia al genio di Riemann [Levi-Civita 1917, p. 36]. It should be observed that Levi-Civita was referring to the 1st edition of [Riemann 1876]. In the 1892 edition of [Riemann 1876], Weber had provided a full revision of his commentary remarks over this point which heavily relied upon [Lipschitz 1876] and [Beez 1879], see [Riemann 1876, 2nd edition, p. 411].

accordance with this, the vanishing of (10) is a simple consequence of the fact that every variation of zero is equal to zero, as well.

Riemann's genuine interpretation appears to have been a different one. Conditions such as $2\sum_k b_{ik}d^2s_k = -\sum_{jl} p_{ijl}ds_jds_l$ make sense only if they are applied to (10) after computation of the second-order variations. In Levi-Civita's defense, one can observe that Riemann's original text is a little ambiguous on this point because of the absolute ablative construction "determinatis variationibus ordinis secundi" [Riemann 1861, p. 381] which naturally implies anteriority temporal relationship with respect to the verb of the principal sentence "formetur expressio".

In actual fact, precisely by following this interpretative approach, Lipschitz [Lipschitz 1876] and, after him, Beez [Beez 1879] provided explicit verification of the relevant assertion "(10)=(11)", thus vindicating, well before Levi-Civita, the mathematical genius of Riemann. The calculations involved, though straightforward, are rather long (more than one page would be required) and tedious; we shall simply refer to [Beez 1879, Section II] which should be praised for its clarity and thoroughness.

But a difficulty remains with their reconstruction. As Beez pointed out, in order to carry out the proof of Riemann's assertion in full detail, one has to replace his trinomial with another slightly modified expression, namely:

$$\delta\delta\sum b_{ij}ds_ids_j - d\delta\sum b_{ij}ds_i\delta s_j - \delta d\sum b_{ij}ds_i\delta s_j + dd\sum b_{ij}\delta s_i\delta s_j.$$

The reason for this is explained by Beez as follows. He wrote:

> "We can easily convince ourselves that by closely following Riemann's instructions one does not achieve the desired goal. Indeed, if, starting from the expression (10), one computes all those terms which contain the second-order variations of the coefficients b_{ij} and if one denotes with Δ_2 the set of all the remaining terms, one obtains
>
> $$\begin{aligned}\delta\delta\sum_{ij} b_{ij}ds_ids_j - 2d\delta\sum_{ij} b_{ij}ds_i\delta s_j + dd\sum_{ij} b_{ij}\delta s_i\delta s_j = \\ = \sum\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l}ds_ids_j\delta s_k\delta s_l - 2\sum\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l}ds_i\delta s_jds_l\delta s_k + \\ + \sum\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l}\delta s_i\delta s_jds_lds_k + \Delta_2.\end{aligned}$$
>
> Since the symbol $\sum$ is extended to all the indices i, j, k, l which run through the series $1, 2, \ldots, n$, if the indices j and k and the indeces i and k and j and l are

exchanged in the second and in the third summation respectively, one obtains

$$\delta\delta\sum_{ij} b_{ij}ds_ids_j - 2d\delta\sum_{ij} b_{ij}ds_i\delta s_j + dd\sum_{ij} b_{ij}\delta s_i\delta s_j =$$

$$(**) \qquad = \sum\left\{\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l} - 2\frac{\partial^2 b_{ik}}{\partial s_j\partial s_l} - \frac{\partial^2 b_{kl}}{\partial s_i\partial s_j}\right\} ds_ids_j\delta s_k\delta s_l + \Delta_2;$$

however, the first term in the equation $(kjli) = 0$ reads

$$\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l} - \frac{\partial^2 b_{ik}}{\partial s_j\partial s_l} - \frac{\partial^2 b_{jl}}{\partial s_i\partial s_k} + \frac{\partial^2 b_{kl}}{\partial s_i\partial s_j}.$$

Thus, in order to obtain it, one has to replace in Riemann's expression (10) the term

$$2d\delta\sum b_{ij}ds_ids_j$$

with

$$d\delta\sum b_{ij}ds_i\delta s_j + \delta d\sum b_{ij}\delta s_ids_j."[25]$$

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[25] *Man überzeugt sich nun leicht, dass man bei stricter Befolgung der Riemann'schen Vorschrift nicht zu dem gewünschten Ziele gelangt. Denn bildet man zunächst nach dem Schema (10) diejenigen Ausdrücke, welche die zweiten Variationen der Coefficienten* b_{ij} *enthalten, und bezeichnet das Aggregat aller noch übrigen Glieder mit* Δ_2*, so kommt*

$$\delta\delta\sum_{ij} b_{ij}ds_ids_j - 2d\delta\sum_{ij} b_{ij}ds_i\delta s_j + dd\sum_{ij} b_{ij}\delta s_i\delta s_j =$$

$$= \sum\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l}ds_ids_j\delta s_k\delta s_l - 2\sum\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l}ds_i\delta s_jds_l\delta s_k +$$

$$+ \sum\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l}\delta s_i\delta s_jds_lds_k + \Delta_2.$$

Da sich das Zeichen $\sum$ *auf alle Indices* $i, j, k, l,$ *welche sämmtlich die Reihe* $1, 2, \dots, n$ *durchlaufen, bezieht, so ergibt sich, wenn man in der zweiten Summe* j *und* k*, in der dritten* i *und* k*,* j *und* l *vertauscht,*

$$\delta\delta\sum_{ij} b_{ij}ds_ids_j - 2d\delta\sum_{ij} b_{ij}ds_i\delta s_j + dd\sum_{ij} b_{ij}\delta s_i\delta s_j =$$

$$= \sum\left\{\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l} - 2\frac{\partial^2 b_{ik}}{\partial s_j\partial s_l} - \frac{\partial^2 b_{kl}}{\partial s_i\partial s_j}\right\} ds_ids_j\delta s_k\delta s_l + \Delta_2;$$

das erste Glied aber in der Gleichung $(kjli) = 0$ *lautet*

$$\frac{\partial^2 b_{ij}}{\partial s_k\partial s_l} - \frac{\partial^2 b_{ik}}{\partial s_j\partial s_l} - \frac{\partial^2 b_{jl}}{\partial s_i\partial s_k} + \frac{\partial^2 b_{kl}}{\partial s_i\partial s_j}.$$

Um dieses zu erhalten, hat man in dem Riemann'schen Schema (10) statt

$$2d\delta\sum b_{ij}ds_ids_j$$

zu schreiben

$$d\delta\sum b_{ij}ds_i\delta s_j + \delta d\sum b_{ij}\delta s_ids_j.$$

See [Beez 1879, p. 9-10], with slight modification of the notation employed.

From a modern point of view, the difficulty pointed out by Beez could be overcome by observing that, since the second order terms of Riemann's curvature tensor are contracted with $ds_i\, ds_j \delta s_k \delta s_\ell$, their expression is equivalent to the second order terms between curly brackets in Eq. $(**)$. Nonetheless, Beez's observation suggests a plausible heuristic path to the establishing of the Equation (10) $=$ (11) which Riemann might have pursued.[26] Indeed, the conditions imposed by Riemann on the second-order differentials (see [Riemann 1861, p. 381]) can be considered as specialized to a normal coordinate system.[27] Indeed, the first condition

$$\delta' \sum_{ij} b_{ij} ds_i \delta s_j - \delta \sum_{ij} b_{ij} ds_i \delta' s_j - d \sum_{ij} b_{ij} \delta s_i \delta' s_j = 0 \tag{13}$$

is easily seen to imply that

$$\sum_{ijk} p_{ijk} \delta' s_k ds_i \delta s_j - 2 \sum_{ij} b_{ij} d\delta s_i \delta' s_j = 0. \tag{14}$$

As for the other two:

(15)

$$\delta' \sum_{ij} b_{ij} ds_i ds_j - 2d \sum_{ij} b_{ij} ds_i \delta' s_j = 0, \quad \delta' \sum_{ij} b_{ij} \delta s_i \delta s_j - 2\delta \sum_{ij} b_{ij} \delta s_i \delta' s_j = 0,$$

as was observed in [Beez 1879, p. 9], are not independent of (13). Indeed, they can be deduced from (13) by replacing δ with d (one thus obtains the first one) and d with δ (one thus obtains the second one).

Now we suppose that Riemann decided to work in a normal coordinate system. At the origin of this system one has $p_{ijk} = 0$ and, consequently, $d\delta s = 0$.

If this conjecture is accepted, one should infer that Riemann limited himself to checking out the validity of his claim at the origin of such a special coordinate system where the four-index symbol reduces to the following expression involving second-order partial derivatives of b_{ij} only:

$$\frac{\partial^2 b_{il}}{\partial s_j \partial s_t} + \frac{\partial^2 b_{it}}{\partial s_i \partial s_l} - \frac{\partial^2 b_{it}}{\partial s_j \partial s_l} - \frac{\partial^2 b_{jl}}{\partial s_i \partial s_t}.$$

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26 A different account was provided in [Spivak 1979, pp. 278-284]. Though interesting and very accurate, it is doubtful whether Spivak's reconstruction has anything to do with Riemann's original line of thought. It should be noted that in Spivak's account too one needs to introduce further hypotheses concerning commutation properties of the vector fields under consideration.

27 For an analytical definition of this coordinate system, see [Riemann 1876, pp. 384-386].

Indeed, by supposing that all of the second order differentials d^2s, δds and $\delta^2 s$ vanish at the origin of the normal coordinate system[28], the trinomial (10) reduces to:

$$\sum_{ijkl}\left[\frac{\partial^2 b_{ij}}{\partial s_l \partial s_k} - 2\frac{\partial^2 b_{il}}{\partial s_j \partial s_k} + \frac{\partial^2 b_{kl}}{\partial s_i \partial s_j}\right] ds_i ds_j \delta s_k \delta s_l. \tag{16}$$

Furthermore, if the well known properties of normal coordinates are invoked, namely that, at the origin of the system, one has $\frac{\partial^2 b_{ij}}{\partial s_k \partial s_l} = \frac{\partial^2 b_{kl}}{\partial s_i \partial s_j}$, this last expression is easily found to be equal to

$$\sum_{ijkl}(ik, jl)\, ds_i ds_j \delta s_k \delta s_l.^{29} \tag{17}$$

Our conjecture suggests a very simple explanation of the heuristic path which Riemann probably pursued. Since, as we have seen, normal coordinates greatly simplify the computations involved, it is highly probable that Riemann first deduced the identity of (10)=(11) by working in this special coordinate system. Apparently, he thought that he could extend the identity (10)=(11) to an arbitrary coordinate system by invoking the invariance of (11).

Accordingly, the employment of normal coordinates seems to acquire a distinguished historical significance. Indeed, by making recourse to such a technical device, Riemann established an implicit connection with his *Habilitationsvortrag* in which normal coordinates made their first appearance.[30] Accordingly, Weber's historiographical interpretation which, as we saw, consists in regarding the *Commentatio Mathematica* as intimately connected to Riemann's previous investigations in the realm of metric geometry is reinforced.

4. FINAL REMARKS

As was mentioned before (sec. 1), some historians have called into question the extent of the geometrical contents of Riemann's *Commentatio*. Let us look at their argumentations in order to provide necessary clarification and, when needed, punctual confutation.

[Farwell & Knee 1990] emphasized the need for a proper contextualization of Riemann's paper. It is certainly important to bear in mind that

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28 Note that this is a consequence of (14).

29 If one supposes that the summation indeces in (11) are restriceted to $i < j$ and $k < l$, we have (11)=(17).

30 See [Riemann 1854, p. 261].

Riemann's curvature tensor made its first appearance in the course of a paper dealing mainly with a problem, the heat conduction, which is to be ascribed to the realm of mathematical physics. Nonetheless, this seems to have little to do with a supposed hierarchy to be established between the tensor analytic and the geometrical nature of the paper under consideration. Even more so, since, as is clear from the historical review of the contributions of mathematicians such as Christoffel, Lipschitz, Ricci-Curbastro and Levi-Civita provided in the introduction to the present article, tensor analysis and differential geometry hardly constituted separate research areas, until the beginning of the 20th century at least.

On the contrary, an attentive reading of the text of the *Commentatio* suggests that geometrical considerations played a very important role.

Let us consider for instance the following claim by Knee and Farwell:

> "Riemann did not consider the equivalence of forms in relation to geometry, but rather in the context of heat conduction. Riemann's mathematical derivations in the second part of the "Commentatio" contain no reference to heat conduction, but equally they contain no reference to geometry. The one allusion to geometry is an illustration, which is not linked to heat conduction and does not obviously therefore serve as a "useful addition".[31]

It is true that Riemann's mathematical derivations of the conditions for the equivalence of two differential forms do not contain any direct reference to the problem of heat conduction, the link between the two of them being purely formal, not physical. Nonetheless, it seems clear to us that Riemann's recourse to considerations of geometrical nature reflected his need to provide a conceptual significance to the four-index symbols which he had just introduced. In this respect, it is useful to emphasize that a proper translation from the Latin suggests that Riemman's intention was to point out (*addigitavisse*) a certain geometrical interpretation (*interpretatione quadam geometrica*) of his derivations and not to add a useful geometrical example.

One can even venture to say that in Riemann's opinion, geometry represented the most proper and natural setting in order to attach an intrinsic meaning to the integral conditions which he had derived in a purely analytical manner. Even more so, if one accepts our interpretation concerning his employment of normal coordinates; from this standpoint, the contents of his *Habilitationsvortrag* represent essential prerequisites to a proper understanding of the *Commentatio*. Incidentally, one can add that, owing to the very limited circulation of Riemann's geometrical ideas, the jury of the

31 [Farwell & Knee 1990, p. 237].

French Academy charged with the evaluation Riemann's paper must have found it difficult to fathom the technical details of the work under examination. Therefore, it comes as no surprise that, despite its mathematical deepness and fecundity, the *Commentatio* was not awarded the award of the prize.

In view of these considerations, it seems clear that every attempt to confine Riemann's *Commentatio* to a specific area of mathematical research runs the risk to provide an incomplete picture of it, since, as is well known, a broader look at Riemann's entire mathematical production reveals a complex net of reciprocal influences among most diverse research interests such as analysis, mathematical physics and geometry[32].

It thus seems that Riemann's immediate followers, such as Weber, Dedekind, Levi-Civita were rather close to the truth when they tried to read the *Commentatio* as providing some technical details of the insights of Riemann's *Habilitationsvortrag*. Recent interpretations such as those put forth in [Farwell & Knee 1990] and [Reich 1994] seem to underestimate the geometrical content of the *Commentatio*, well beyond what Riemann's intention appears to have been.

Finally, a few words concerning the role of the *Commentatio* in the development of differential geometry are in order. It is undoubtedly true that milestones for the history of the discipline such as [Christoffel 1869] and [Lipschitz 1869] did not exhibit any direct connection with [Riemann 1861], despite the fact that they dealt with the very same problem discussed therein. In this respect, more influential were the general ideas put forth in Riemann's *Habilitationsvortrag*, as was explicitly admitted by Christoffel himself in [Christoffel 1869, p. 70].

Nonetheless, one should not forget that the publication of [Riemann 1876] and consequently of the *Commentatio* had the merit of fostering renewed interest in researches on the notion of curvature of an n-dimensional manifold ([Lipschitz 1876] and [Beez 1879] represented important examples) which aimed at deepening the understanding of some of Riemann's sketchy statements.

Furthermore, it may be useful to recall that still after the advent of General Relativity, Riemann's *Commentatio* continued to exert its prolific influence upon the geometrical investigations carried out in [Levi-Civita 1917]. It was precisely the need to clarify Riemann's notion of curvature which drove Levi-Civita to devise an alternative theoretical path which finally led him to interpret Christoffel's symbols and the curvature tensor in terms of the parallel displacement of vectors associated to a differential manifold.

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32 For this, see [Bottazzini & Tazzioli 1995].

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REFERENCES

Beez (R.)
[1879] Ueber das Riemann'sche Krümmungsmaß höherer Mannigfaltigkeiten, *Zeitschrift für Mathematik und Physik*, 24 (1879), pp. 1–17 & 65–82 (Schluss der Abhandlung).

Bianchi (L.)
[1902] *Lezioni di Geometria Differenziale*, vol. I, Pisa: Spoerri, 1902.

Bottazzini (U.) & Tazzioli (R.)
[1995] *Naturphilosophie* and its role in Riemann's mathematics, *Revue d'histoire des mathématiques*, 1 (1995), pp. 3–38.

Bottazzini (U.)
[1999] Ricci and Levi-Civita: from differential invariants to general relativity, in Gray (J.), ed., *The Symbolic Universe: Geometry and Physics 1890–1930*, Oxford: Oxford Univ. Press, 1999, pp. 241–259.

Christoffel (E. B.)
[1869] Über die Transformation homogener Differentialausdrücke zweiten Grades, *Journal für die reine und angewandte Mathematik*, 70 (1869), pp. 46–70.

Dell'Aglio (L.)
[1996] On the genesis of the concept of covariant differentiation, *Revue d'histoire des mathématiques*, 2 (1996), pp. 215–264.

Farwell (R.) & Knee (C.)
[1990] The Missing Link: Riemann's "Commentatio," Differential Geometry and Tensor Analysis, *Historia Mathematica*, 17 (1990), pp. 223–255.

Kline (M.)
[1972] *Mathematical Thought from Ancient to Modern Times*, Oxford: Oxford Univ. Press, 1972.

Levi-Civita (T.)
[1954–1973] *Opere Matematiche*, Bologna: Zanichelli, 1954–1973.
[1917] Nozione di parallelismo in una varietà qualunque e conseguente specificazione geometrica della curvatura riemanniana, *Rendiconti del Circolo Matematico di Palermo*, 42 (1917), pp. 173–205; ; also in [Levi-Civita 1954–1973, IV, 1-39]. All the references refer to Levi-Civita [1954–1973].

Lipschitz (R.)
[1869] Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen, *Journal für die reine und angewandte Mathematik*, 70 (1869), pp. 71–102.
[1876] Bemerkungen zu dem Prinzip des kleinsten Zwanges, *Journal für die reine und angewandte Mathematik*, 82 (1876), pp. 316–342.

Reich (K.)
[1992] Levi-Civitasche Parallelverschiebung, affiner Zusammenhang, Übertragungsprinzip: 1916/17–1922/23, *Archive for History of Exact Sciences*, 44 (1992), pp. 77–105.
[1994] *Die Entwicklung des Tensorkalküls*, Boston: Birkhäuser, 1994.

Ricci (G.) & Levi-Civita (T.)
[1900] Méthodes de calcul différentiel et leurs applications, *Mathematische Annalen*, 54 (1900), pp. 125–201.

Riemann (Georg Friedrich Bernhard)
[1876] *Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, Teubner, 1876; ; 2nd ed. 1892; repr. New York: Springer, 1991. (When not explicitly mentioned, all the references refer to the 1876 edition).
[1854] Ueber die Hypothesen, welche der Geometrie zu Grunde liegen, in Weber (Heinrich), ed., *Gesammelte mathematische Werke*, Leipzig: Teubner, 1854, pp. 254–269.
[1861] Commentatio mathematica, qua respondere tentatur quaestioni ab Ill^ma Academia Parisiensi propositae, in Weber (Heinrich), ed., *Gesammelte mathematische Werke*, Leipzig: Teubner, 1861, pp. 370–383.
[1919] *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. Neu herausgegeben und erläutert von H. Weyl*, Berlin: Julius Springer, 1919.

Spivak (Michael)
[1979] *A Comprehensive Introduction to Differential Geometry*, Berkeley: Publish or Perish, 2nd edition, 1979.

Struik (J. D.)
[1933] Outline of a history of differential geometry II, *Isis*, 20 (1933), pp. 161–191.

Note added in proof. — When the present paper was under press, author's proofs of an article intitled "The mystery of Riemann's curvature" by O. Darrigol were made available online on the web site of *Historia Mathematica*. There the author, by resorting to the formalism of modern tensor calculus, provides further evidence of the geometrically oriented character of Riemann's *Commentatio*. In particular, handwritten private computations by Riemann are published here for the first time which seem to confirm the use of normal coordinates as an important heuristic tool in Riemann's hands.