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PRIORITY, PARALLEL DISCOVERY, AND PRE-EMINENCE NAPIER, BÜRGI AND THE EARLY HISTORY OF THE LOGARITHM RELATION

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ABSTRACT. — There has never been any doubt as to the importance of the logarithm, a mathematical relation whose usefulness has persisted in different aspects to the present day. Within years of their introduction, logarithms became indispensable for mathematicians, astronomers, navigators, and geographers alike. The question of their origins, however, is more contentious. At least two scholars, the Scottish nobleman John Napier and the Swiss craftsman Jost Bürgi, simultaneously and independently produced proposals which embodied the logarithmic relation and, within years of one another, produced tables for its use. In light of this parallel discovery, we read, analyzed, and interpreted the texts of Napier and Bürgi to better understand and contextualize the two distinctly different approaches. As a result, here we compare and contrast the salient features of Napier's and Bürgi's endeavors and the construction of each man's tables of logarithms. Through these details, we will query the focus on the issue of priority and pre-eminence when discussing the historical development of logarithms, and pose critical questions about the phenomenon of parallel insights and what they can reveal about the mathematical environment at the time they arose.

RÉSUMÉ. — Il n'y a jamais eu de doute sur l'importance du logarithme; une relation mathématique dont l'utilité a persisté de différentes manières jusqu'à nos jours. Quelques années à peine après leur introduction, les logarithmes sont devenus indispensables aux mathématiciens, ainsi qu'aux astronomes, navigateurs et géographes. Cependant, la question de leur origine est délicate. Au moins deux savants, le noble écossais John Napier et l'artisan suisse Jost Bürgi, ont produit simultanément et indépendamment des propositions qui englobent la relation logarithmique, et à quelques années de distance ont

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produit des tables pour son utilisation. Partant de cette découverte parallèle, nous avons lu, analysé et interprété les textes de Napier et Bürgi pour mieux comprendre et contextualiser ces deux approches bien différentes. En guise de résultat, nous comparons les principales caractéristiques des efforts de Napier et Bürgi et la construction de leurs tables de logarithmes réciproques. À travers ces détails, nous nous posons la question de la priorité et de la pré-éminence dans la discussion du développement historique des logarithmes, ainsi que des questions critiques sur le phénomène d'intuitions parallèles et ce qu'elles peuvent dévoiler sur l'environnement mathématique de leur époque.

1. INTRODUCTION

The computationally powerful and conceptually brilliant logarithmic relation has followed an interesting course in the history of mathematics. Making its first official appearance in 1614 in Scotland in a work by John Napier (1550–1617), the logarithm relation was simultaneously being developed in Switzerland by Jost Bürgi (1552-1632). This overlap has intrigued historians and the question of priority has often dominated historical scholarship, with varying effects. Several scholars have been compelled to determine who rightfully has the "title to priority" [Cajori 1915, p. 93]; [González-Velasco 2011, p. 101], or even allude to a race that was "won" [Shell-Gellasch 2008, p. 6]. In other accounts, the effects have been more subtle (see, for instance, [Katz 1998, p. 416] [1]; [Naux 1966/1971, p. 12–13]; [Calinger 1995, p. 282] [2]), with historians acknowledging the achievements of Bürgi, but admitting that scant information concerning his works has prevented further discussion. Indeed, despite much attention to this topic, Napier's works have received thorough and repeated examination, while the contributions of Bürgi remain only poorly studied.

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[1]   In this case, Napier's work gets an entire section in a discussion of the logarithmic relation, whereas Bürgi gets but a single mention: "the Scot John Napier and the Swiss Jost Bürgi came up with the idea of producing an extensive table… Napier published his work first".

[2]   Calinger indicates a certain confusion concerning the connection, as he seems to imply that Bürgi in fact 'enthusiastically advanced' Napier's logarithms.

More broadly, throughout the history of mathematics, critical insights have frequently been made by more than one individual almost simultaneously.[3] These parallel insights prove themselves to be fertile episodes for the historian. What is the importance of chronology when one considers the emergence of a mathematical concept?[4] What is the relevance and repercussions of determining which individual can be identified as being 'the first' when it comes to mathematical insight and inquiry? What counts as 'publication' in the times before the emergence of professional academic societies and journals? These issues and others reveal that the question of priority is a delicate one and, in certain cases, overemphasizing it has resulted in some regrettable biases. We aim here then, in this contexte, to begin to redress the balance.

The late Renaissance brought with it endeavors which required ever new and improved computational techniques from mathematics. Demands from these areas—notably, observational astronomy and long-distance navigation and, not long after, geodesic science and the efforts to measure and represent the earth[5]—meant that much energy and scholarly effort was directed towards the art of computation. The central foundation for these fields was trigonometry, and its articulation and computation were the subjects of massive enterprise. For the most part, such applications required detailed and long computations—reducing the burden of calculation and, with it, the errors that inevitably crept into the results, became a prime objective. Techniques which could bypass lengthy

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[3] For example, Newton and Leibniz, regarding calculus; Tartaglia, Cardano, del Ferro, Ferrari, and Bombelli regarding the race to the solution for cubic equations; and Hadamard and de la Vallée Poussin regarding the prime number theorem, to name but a few of many instances. Sometimes key concepts and ideas are developed independently by practitioners in different cultures of inquiry which are also separated by many centuries, for instance Al-Samaw'āl (b. ca. 1130) and Pascal (b. 1623) and the 'triangle' of binomial coefficients; Indian scholar Mādhava (b. ca. 1350) and Newton (b. 1642) and others with the infinite series expansions of sine, cosine, and arctangent.

[4] In this sentiment, Stedall [2008, p. 106] remarks "It… [is] a useful reminder that mathematical progress in any but the simplest problems is rarely straightforwardly linear."

[5] [Gridgeman 1973, p. 50]

processes, or could replace long multiplications or divisions with equivalent additions and subtractions were explored. One method originating in the late sixteenth century that was used extensively to save computational effort was the technique called *prosthaphaeresis*. [6] This relation transformed long multiplication and division into addition and subtraction via suitable trigonometric relationships. [7] The technique of *prosthaphaeresis* did have its disadvantages though. It was based upon trigonometric tables that were not always accurate, and rounding errors accumulated in the process further decreased the accuracy of this technique.

Yet another mathematical insight that offered potential to facilitate computation was that concerning the relations between arithmetical and geometrical sequences. This relation had been recognized by mathematicians as early as Archimedes (287–212 BCE), [8] and more than a millennium after Nicolas Chuquet (c. 1430–1487) and Michael Stifel (c. 1487–1567) turned their attention to the relationship. Their examination was more probing due to the newly developing notation for representing exponents. Now the mathematical connection between a geometric and an arithmetic sequence could be made all the more apparent by symbolically capturing these sequences as successive powers of a given number and the powers themselves, respectively. Such symbolism brought more sharply into view various mathematical properties and relations.

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[6] A compound constructed from the Greek terms: *prosthesis* 'addition' and *aphaeresis* 'subtraction'.

[7] Such as $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$. This, and related rules, were recognized as early as the beginning of the sixteenth century by Johannes Werner in 1510 and perhaps in the eleventh century by Ibn Yunus [Berggren 2003] but this claim is dubious at best and is often only cited in older scholarship [Thoren 1988, p. 32]. Its application specifically for multiplication first appeared in print in 1588 in a work by Nicolai Reymers Ursus [Thoren 1988, p. 33]. It was further extended in use by the likes of Christopher Clavius, François Viète, and Tycho Brahe.

[8] See [Heath 1953]: "When numbers are in continuous proportion starting from the unit, and that some of these numbers are multiplied between them, the product will be in the same progression, far away from largest from the multiplied numbers of as many numbers as the smallest one of the multiplied numbers is from the unit in the progression, and far away from the unit of the sum minus one of the numbers from which the multiplied numbers are far away from the unit."

For instance, from more general formulations, Stifel[9] explored the relationship between powers of two (a geometric sequence; the lower sequence here) and the index associated with it (an arithmetic sequence; the upper sequence here):

$$-3 \ -2 \ -1 \ 0 \ 1 \ 2 \ 3 \ \ 4 \ \ 5 \ \ 6$$
$$\tfrac{1}{8} \ \ \tfrac{1}{4} \ \ \tfrac{1}{2} \ 1 \ 2 \ 4 \ 8 \ 16 \ 32 \ 64$$

Here, Stifel introduced the two progressions using the prepositions *supra* (above) and *infra* (below).[10] He further noted that each can be extended infinitely in both directions and that calculations with elements of the lower sequence in fact have associated – but simpler – operations when using the elements in the upper sequence.[11] To explain such operations, Stifel employed the term *exponens*, or exponent, for the elements of the upper sequence: "6 is the exponent of the number 64, and 3 is the exponent of the number 8" [Stifel 1544, p. 250].[12]

It seems Stifel recognized that relations of these sorts would be of great importance for the scientific community and they undoubtedly served as a foundation for the invention of logarithms soon after. He glimpsed the potential of extending his contribution, however, commenting "I might write a whole book concerning the marvellous things relating to numbers, but I must refrain and leave these things with eyes closed".[13] However, while Chuquet, Stifel, and others noticed that certain operations could be carried out more efficiently in one domain and then converted back to the original, their application towards substantially reducing computational complexities was not seen.

Scottish mathematician John Napier spent a lifetime investigating devices and methods for assisting and reducing computational effort. His numbering rods, fondly known as 'Napier bones', provided a mechanical

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9   As given in his *Arithmetica Integra* [1544].

10   For example: "Nam sicut supra unitatem ponuntur numeri integri, & infra unitatem finguntur minutiae unitatis, & sicut supra unum ponuntur integra, & infra unum ponuntur minuta seu fracta…" [Stifel 1544, p. 249].

11   [Calinger 1999, p. 431]

12   "… 6 est exponens numeri 64, & 3 est exponens numeri 8."

13   "Posset hic fere nouus liber integer scribi de mirabilibus numerorum, sed oportet ut me hic subduca, & clausis oculis abea" [Stifel 1544, p. 250]; as translated by [Smith 1958, p. 521].

means for facilitating calculation. [14] In this spirit, Napier developed the logarithmic relationship to assist with computation. He set to the task of compiling tabular entries it is supposed in about 1594. The ways in which Napier decided to construct and organize these tables necessitated the computation of millions of calculations from which to select his entries. In addition, he gave the original problem kinematic expression and worked the relationships his tables rested upon into a larger mathematical framework. Tables and accompanying documentation were finally completed and published in 1614.

Simultaneously, the Swiss craftsman, Jost Bürgi [15] whose skills in instrumentation, particularly clock manufacture, were highly prized, also had the burdens of calculation impressed upon him. Working in the court of Landgrave Wilhelm IV in Kassel, [16] he was assigned the task of compiling and constructing astronomical charts. He eventually came under the employ of Emperor Rudolf II (Prague, 1603) where he worked with Kepler (1571–1630). This interaction would have exposed him firsthand to the challenges facing astronomers. Bürgi too devoted attention to the technique of *prosthaphaeresis*, supplying formal proofs for at least two of the identities. [17] He recognized the potential of this technique to simplify computations with very large numbers and his work on this subject must have also stimulated, by analogy, the broad idea that certain more involved arithmetical operations could be replaced by the simpler algorithms of addition and subtraction. [18] With this in mind, he set himself the task of compiling tables utilizing this insight, with a view to facilitate calculation.

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14   [Gridgeman 1973, 53–54].

15   No substantial personal information on Bürgi exists, other than where he was born, lived, and died. Some of the information about the almost three decades he spent in Prague is vague. For example, the *Dictionary of Scientific Biography* entry on Bürgi notes that, "He lived in Prague from about 1603 and became assistant to and computer for Kepler" [Nový 1970, p. 602]. The entry on Kepler, however, makes no mention of this.

16   Bürgi did not receive a formal education in his youth. Nový [1970] speculates that he completed his education while in the service of Duke Wilhelm IV while working at the duke's observatory. He did not appear to know Latin.

17   [Thoren 1988, p. 38]

18   To do so Bürgi would have worked on tables of sines. Nový [1970, p. 602] indicates that Bürgi completed such tables. They were not, however, ever published and no

There is speculation that Bürgi computed his tables at some point between 1603 and 1611 [19], but the earliest extant document we have dates to 1620.

In an environment where many practical endeavors were dependent on such intensive calculative techniques, it was no coincidence that alleviating some of the burden of computation was on the mind of more than one scholar. That such a relation was recognized, explored, and developed by at least two practitioners simultaneously is not at all surprising. The significant point is that wider conditions propelled multiple scholars, not just Napier, towards this insight. However, history has deemed Napier pre-eminent and he has thus been accorded priority. As a result, historical attention has favored Napier's work over all others.

Accordingly, our present investigation aims to consider and compare the approaches of these two scholars through their works: the *Mirifici Logarithmorum Canonis Descriptio* of Napier and the *Arithmetische und Geometrische Progress Tabulen* [20] of Bürgi. Modern scholarship on Napier is plentiful and there exist many modern analyses of his endeavors. Details on his work will be included largely for comparison purposes with the analysis of Bürgi's system, which will be treated much more thoroughly given the lack of primary source examination of his work. [21] After a technical examination

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manuscripts have since been found. Bürgi may have learned of the idea to relate arithmetic and geometric progressions from the work of Stifel [J. van Maanen, personal communication]. Naux described aspects of Bürgi's work as a reproduction of Stifel's pages" [1966/1971, p. 96] "D'abord, Bürgi a reproduit assez fidelement quelques pages de Stiffel…", yet instead of Stifel Bürgi mentioned the famous 16th century German reckoning master Simon Jacob, as well as Moritius Zonz in the "*Bericht*" (page 2; see section 3). Simon Jacob's treatment of series and the nature of exponents is the more likely influence on Bürgi's work [Smith 1958, p. 338]. Cantor [1900, p. 726] also noted that Jacob was perhaps the stronger influence and that little was known about Moritius Zonz.

[19] Several sources of speculation exist, such as [Cajori 1915, p. 103]. Moreover, [Folta & Nový 1968, p. 98] noted that Bürgi already had the idea of logarithms at the end of the 16th century, while he still resided in Kassel. [Grattan-Guinness 1997, p. 180–181] places Bürgi's work as early as 1590; [Boyer & Merzbach 2011, p. 290] as early as 1588.

[20] The complete title is *Arithmetische und Geometrische Progress Tabulen/sambt gründlichem unterricht/wie solche nützlich in allerley Rechnungen zu gebrauchen/und verstanden werden sol.* See section 3.

[21] A complete translation and commentary of his work is in preparation.

of the tables and the accompanying texts of these two scholars, we will highlight the distinctive features and underlying differences between the two approaches. This will allow us to explore the issue of priority and pre-eminence in this context and redress a perceived imbalance in historical accounts of the emergence of the logarithm relation in the early seventeenth century. In turn, we will reflect more generally about the historiographical issues relating to the phenomenon of parallel insights in mathematics. Our perspective casts critical light on those historians who have singled out as pertinent a claim to priority when analyzing episodes in the history of mathematics.

2. NAPIER

In 1614, Napier published his work *Mirifici logarithmorum canonis descriptio*, literally *A Description of the Wonderful Table of Logarithms* (henceforth referred to as the *Descriptio*). Napier's very title signaled his ambitions for this proposal—the provision of tables based on a new technique that would be nothing short of 'wonder-working' for practitioners. Napier conceived of a new technical term for his concept—'logarithmus'—a compound derived from two terms from ancient Greek: *logos* meaning here proportion, [22] and *arithmos* meaning number.

Even though Napier did not consistently nor continually use this term, it persisted. Within a decade, his initial formulation had been transformed by two of his contemporaries: Henry Briggs and Edmund Gunter. [23] Originally written in Latin, the work was translated into the vernacular, English,

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[22]   This disagrees with traditional interpretations, such as [Gridgeman 1973, p. 55], who translates it as 'ratio'. [Naux 1966/1971, p. 66] interprets this term as 'la raison de différence'. The ancient Greek term *logos* has many meanings, including (in a mathematical context) reckoning, computation, measure, relation, sum, rule. In modern mathematics, a ratio is the relation between two numbers of the same kind; a proportion is a more general relation, typically to express the equality between multiple ratios. We prefer the more general term 'proportion'. Napier does not seem to have explicitly described why he selected this term and the intended reading.

[23]   Briggs published his *Logarithmorum chilias prima* in 1617, constructing a table of logarithms of numbers of base 10; Gunter published his *Canon triangulorum* in 1620, which extended Briggs's work to logarithms of base 10 of trigonometrical functions.

two years later for the immediate use of seafarers and navigators. [24] What made Napier's tables so readily received by practitioners was the fact that his logarithm relationship was imbedded in tabulations of increasing sines of angles. Navigation and astronomy both involved the computation of trigonometrical functions. Particularly time-consuming was the multiplication of sines, and Napier sought to additionally reduce this computational effort by tabulating the sine then its logarithm.

The *Descriptio* contains Napier's tables, as well as a description of their features and worked examples. A second work was written entitled the *Mirifici logarithmorum canonis constructio*, literally *The Construction of the Wonderful Table of Logarithms* (henceforth the *Constructio*). This work addressed in more detail the ways in which the tables in the *Descriptio* had been computed and the motivation behind them. However, this work was not published until two years after Napier's death under the supervision of his son, Robert Napier, in 1619. This delay might have been in some part intentional, as Napier seemed concerned to await the reception by the scholarly community of his first work, "to await the judgement and criticism of the learned." [25]

2.1. *The* Descriptio

The *Descriptio* represented a monumental undertaking, and was the result of years of involved calculation. The work itself is comprised of two distinct sections: text (57 pages of text giving mathematical background, context, as well as a few examples) and 90 pages of tables. Taking Napier literally on his statement that he had been working on logarithms for twenty years would locate his initial attention to this task to about 1594.

The text of the *Descriptio* contains two books, each divided into chapters, with the following topics being addressed:

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[24]   The English translation is that of Wright [1616].
[25]   As cited from [Gridgeman 1973, p. 55].

Table 1
Napier's *Descriptio*

Book I	1	Definitions
	2	Propositions
	3	Description of the tables; in particular, each of the seven columns
	4	How to use the tables with worked examples
	5	How to use the tables for proportions with worked examples
Book II	1	Trigonometry—general background
	2	Application of logarithms to triangles
	3	Spherical Triangles
	4	Worked examples

The work opens with his preface where he gives his motivations:

> Seeing there is nothing (right well beloued students in the Mathematickes) that is so troublesome to Mathematicall practise, nor that doth more molest and hinder Calculators, than the Multiplications, Diuisions, Square and cubical Extractions of great numbers, which besides the tedious expence of time, are for the most part subiect to many slippery errors. I began therefore to consider in my minde, by what certaine and ready Art I might remoue those hindrances.
> Napier *Descriptio* (from the English translation of [Wright 1616, A5])

Furthermore, in this English edition, Napier endorses the translation into English by Wright because it makes, in his words, "his secret inuention… so much the better as it shall be the more common". Clearly, Napier was concerned with making his ideas as widely disseminated as possible.

In *Descriptio* I,1, Napier sets out the foundations for his logarithmic relation. Notably, Napier does not invoke the comparison between arithmetic and geometric sequences directly, but rather firmly grounds his conception in a kinematic scenario—for him, the make-up of the problem was based on the analysis of motion. Relations between arithmetic and geometric sequences are present, but are embedded in a physical problem that involves points, lines, distances covered, and changing velocities. His description focuses on a comparison between the distance traveled by two particles in equal increments of time, one with constant velocity the other with geometric deceleration. [26]

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[26] Whiteside [1961] calls it a "distance-speed model" and comments that "Napier's distance-speed model is medieval rather than modern and its now usual treatment by methods of the calculus is quite foreign to its kinematic nature" [220].
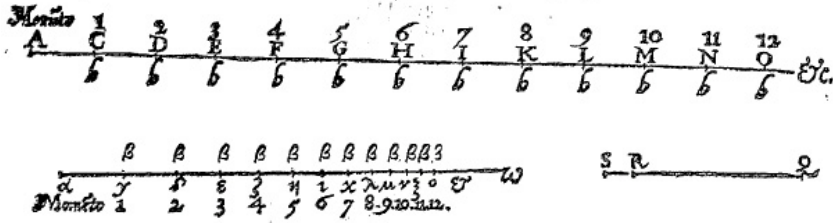
FIGURE 1. Napier's two parallel lines with moving particles.

Napier imagined these two particles traveling along two parallel lines. The first line was of infinite length and the second of a fixed length (the length of the whole sine). The two particles start from the same point at the same time with same velocity. The first particle (b) [27] progresses uniformly as time elapses, and so covers equal distances. The second particle (β) progresses at a decreasing rate proportional to the distance remaining to a fixed terminal point, that is, its velocity is proportional to its distance from the end of this second finite line segment. [28]

Upper case letters of the alphabet delimit the equal increments of the first line (see Fig. 1) [29]. The second line of finite length is delimited by the Greek letters α and ω. This length is set as the length of the 'whole sine' (i.e., the radius, R); in Napier's case this is 10,000,000. The particle traveling along this line decelerates, traversing, in equal times, intervals decreasing proportionally to the particle's distance from the end (ω). Napier then describes how these moving particles and their distances covered are related to the sines and the logarithm:

> For the sake of an example: Let both figures from earlier on be repeated and let B be moved always and everywhere with equal velocity which β began to be moved, being initially at α. Then, at the first moment, let B proceed from A to C and in the same moment β from α to γ proportionally. The number expressing AC will be the logarithm of the line, or the sine $\gamma\omega$. Then, in the second moment

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27 This is referred to in his text as 'B'.

28 For further details on this see [Katz 1998, p. 417].

29 In fact, first the upper part of this diagram appears on p.1 of the 1614 edition. Then after some prose-text, the lower part appears on p. 2. But then the two figures are put together and repeated (this is pointed out by Wright).
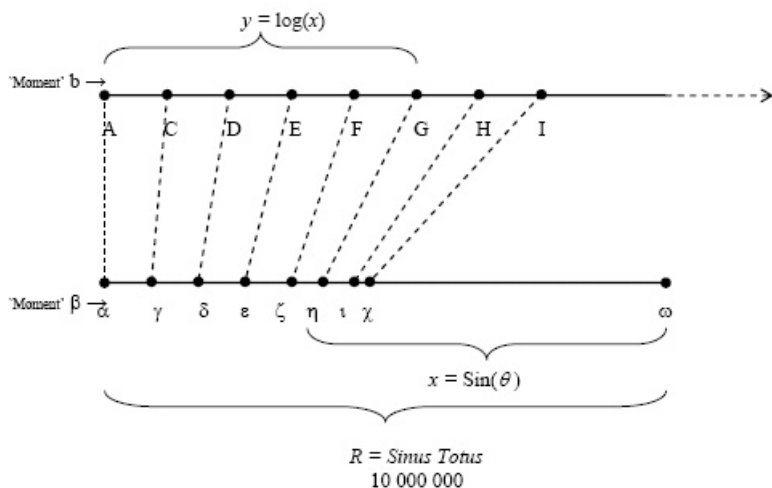
FIGURE 2. The relation between the two lines and the logs and sines.

let β be moved forward proportionally from γ to δ, the number expressing *AD* will be the logarithm of the sine δω. Therefore, in the third moment, let *B* move equally from *D* to *E*, and in the same moment β move from δ to ε. The number defining *AE* will be the logarithm of the sine εω. Also, in the fourth moment let *B* move forward to *F* and β to ζ. The number *AF* will be the logarithm of the sine ζω. And preserving the same order continually, the number *AG* will be (taken from the above definition) the logarithm of the sine ηω. *AH* will be the logarithm of the sine ιω. *AI* will be the logarithm of the sine χω, *AK* the logarithm of the sin λω and so forth infinitely.

Descriptio I, 1 (p. 4); second author's translation

The lines can be imagined as being divided into two parts: that part which has been traversed by the particle, and that which is still to be traversed. According to the description above, the untraversed distance on the second (finite) line is the sine. The traversed distance on the infinitely long (first) line is the logarithm of that sine (see Fig. 2). While the sines decrease, Napier's logs increase. Thus by Napier's account:

$$AC = \log_{nap}(\gamma\omega) \quad \text{where} \quad \gamma\omega = \mathrm{Sin}\theta_1$$
$$AD = \log_{nap}(\delta\omega) \quad \text{where} \quad \delta\omega = \mathrm{Sin}\theta_2$$
$$AE = \log_{nap}(\epsilon\omega) \quad \text{where} \quad \epsilon\omega = \mathrm{Sin}\theta_3$$
$$\text{and so on...}$$

or in other words, using modern notation: [30]

$$x = \mathrm{Sin}\theta$$
$$y = \log_{nap}(x)$$

The important corollary is that the logarithm of the whole sine, or R, is zero (Napier observes this in chapter 1 [1614, 'Corollary', p. 4]) and the logarithm of zero is infinite.

Embedded within this kinematic conception are arithmetic and geometric sequences. The increasing arithmetic sequence is:

$$0, AC, AD, AE, AF, \ldots = 0, b, 2b, 3b, 4b, \ldots$$

The decreasing geometric sequence is formed from the distances from the right endpoint to the particle, namely:

$$\alpha\omega, \gamma\omega, \delta\omega, \epsilon\omega, \ldots = R, aR, a^2R, a^3R, \ldots$$

where R (total sine = 10,000,000) is the length of the line and a is some number chosen to be less than but very close to 1. Napier set this number at $1 - 10^{-7}$. Therefore, Napier's geometric progression, first term $x_1 = R$ = 10,000,000 and common ratio $a = 1 - 10^{-7}$ is generated via:

$$x_{n+1} = x_n \cdot (1 - \tfrac{1}{10^7})$$

so that the first dozen terms and their corresponding logarithms are:

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[30] Care must be taken in general regarding notation. In Napier's case, the Greek letters γ, δ, etc., are used to indicate a line segment by means of its endpoints: i.e., $\gamma\omega$ etc.

Table 2
Napier's logarithms

x_{n+1}	$n = \log_{nap}(x_{n+1})$
10000000.0000000	0
9999999.0000000	1
9999998.0000001	2
9999997.0000003	3
9999996.0000006	4
9999995.0000010	5
9999994.0000015	6
9999993.0000021	7
9999992.0000028	8
9999991.0000036	9
9999990.0000045	10
9999989.0000055	11

This is not the final form of the tables however, as Napier wished to present the logarithmic relationship tabulated not in order of increasing natural numbers, but rather in increasing angles, given in degrees and minutes. Thus, for the sine of each angle, the appropriate logarithm was selected, as follows:

Angle (θ)	$x = \mathrm{Sin}\theta$	$\log_{nap}(x)$
90°00′	10 000 000	0
89°59′	10 000 000	1
89°58′	9 999 998	2
89°57′	9 999 996	4
89°56′	9 999 993	7
89°55′	9 999 989	11

Compare this to the last three columns (in reverse) of Napier's table (see Fig. 3).

Essentially, the tables given in the *Descriptio* are logarithms of sines, and by ingenious arrangement, related trigonometric functions. Although the above derivation represents the intent of Napier, in practice, computing values in this way and then extracting the appropriate ones for the corresponding sine would have required Napier to compute about ten million values to fourteen significant places. This, for obvious reasons, was

min	Sinus		Logarithmi	Differentiæ	logarithmi		Sinus	
0	10		Infinitum	Infinitum	0		10000000	60
1	2990		81425681	81425680	1		10000000	59
2	5818		74494313	74494311	2		9999998	58
3	8727		70439564	70439560	4		9999996	57
4	11636		67562745	67562739	7		9999993	56
5	14544		65331315	65331304	11		9999989	55
6	17453		63508092	63508083	16		9999986	54
7	20362		61966595	61966573	22		9999980	53
8	23271		60631284	60631256	28		9999974	52
9	26180		59453453	59453418	35		9999967	51
10	29088		58399857	58399814	43		9999955	50
11	31997		57446759	57446707	52		9999950	49
12	34906		56576646	56576584	62		9999940	48
13	37815		55776222	55776149	73		9999928	47
14	40724		55035148	55035064	84		9999917	46
15	43632		54345225	54345129	96		9999905	45
16	46541		53699843	53699734	109		9999892	44
17	49450		53093690	53093577	123		9999878	43
18	52359		52522019	52521881	138		9999863	42
19	55268		51981356	51981202	154		9999847	41
20	58177		51468431	51468361	170		9999831	40
21	61086		50980537	50980450	187		9999813	39
22	63995		50515342	50515137	205		9999795	38
23	66904		50070827	50070603	224		9999776	37
24	69813		49645239	49644995	244		9999756	36
25	72721		49237030	49236765	265		9999736	35
26	75630		48844826	48844539	287		9999714	34
27	78539		48467441	48467122	309		9999692	33
28	81448		48103763	48103431	332		9999668	32
29	84357		47752859	47752503	356		9999644	31
30	87265		47413852	47413471	381		9999619	30

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FIGURE 3. The first page of Napier's tables.

impractical, and Napier devised some ingenious interpolation techniques

and computation saving short-cuts. [31] He described these techniques in
the later published *Constructio*.

Napier then set about presenting these computed values in a tabular ar-
rangement to further facilitate their practicality. Indeed, the layout of the
tables has been carefully thought out and the tables contain clever symme-
tries; these are described by Napier in *Descriptio* I, 3. Each page consists of
seven columns. The first column lists the arcs (or angles) in degrees and
minutes, increasing minute by minute from 0° to 45°. Each page covers
30 minutes; 1° takes two pages to tabulate so that the tables are 90 pages
in length. This column is related to the seventh column on the far right,
which gives the angles decreasing, minute by minute from 90°. In this way,
the whole quadrant is tabulated, invoking the symmetry of the sine func-
tion. The second and sixth columns are headed *sinus* and are the results
of computing the sines of the angles given in the first and seventh column
respectively. These sines are taken from the sine tables of Erasmus Rein-
hold [Baron 1974, p. 611]. The third and the fifth columns are headed *log-
arithmi* and contain the logarithms of the sines in the preceding column.
The fourth column is called *differentiae*. This column contains the differ-
ences between the logarithms in the third and fifth columns.

What is particularly ingenious about this layout is that each of the
columns when read in a particular direction gives each of the trigonomet-
ric functions. Reading from the left-most column rightwards, the tables
present an angle, its sine, the logarithm of its sine, the logarithm of its
tangent, [32] the logarithm of its cosine, its cosine, [33] then the angle's com-
plement. [34] These relationships are essentially the same reading the table
from the right-most column leftwards, except that the middle column is
the logarithm of the *cotangent*.

---

[31] This explains the discrepancies in the computations presented above, and the ac-
tual numbers presented in Napier's tables. For details on his interpolations, see for
example [Baron 1974, p. 612].

[32] One can take the difference of the tabular value corresponding to the sine and
the cosine to find this entry, since $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

[33] As $\sin \theta = \cos(\theta + 90)$.

[34] As $\theta + \bar{\theta} = 90°$.

Napier provided his readers with some worked examples. For instance, the first example he gives is how to find the logarithm of a given sine:

> I seeke the *Logarithme* of the sine 694658. I finde that sine precisely in the second column, answering to the arch 44 degrees 0 min etc in the same line of the third column, there standeth ouer-against it, the *Logarithme* 364335 which I sought.
> I,4 (as translated by Wright):14

Thus, by simple table look-up:

$$\log_{nap}(694658) = 364335$$

Napier gives many other examples of increasing complexity, including the computation of mean proportionals (sometimes known as the geometric mean). Here, he details the original way in which it would have been computed, and points out that his technique using logarithms finds the answer 'earlier'.

> Let the extremes 1000000 and 500000 bee given, and let the meane proportionall be sought: that commonly is found by multiplying the extreames given, one by another, and extracting the square root of the product. But we finde it earlier thus; We adde the Logarithme of the extreames 0 and 693147, the summe whereof is 693147 which we divide by 2 and the quotient 346573 shall be the Logar. of the middle proportionall desired. By which the middle proportionall 707107, and his arch 45 degrees are found as before… found by addition onely, and division by two.
> I, 5 (as translated by Wright):25

In order to find the mean proportional by traditional methods, Napier observes that one has to compute the product and then take the square root, that is:

$$\sqrt{1000000 \times 500000} = \sqrt{500000000000} \approx 707106.78$$

Instead, Napier proposes:

$$\log_{nap}(1000000) + \log_{nap}(500000) = 0 + 693147 = 693147$$
$$693147 \div 2 = 346573 \quad (6s.f.)$$

So the mean proportional is 707106   as required.

Therefore, rather than an involved procedure to extract a square root, one needs only perform addition once and division by two, a saving which Napier himself points out.

Napier's tables were logarithms, but with additional convenience to practitioners. The tables were specifically designed to be immediately useful to those who needed the relation to facilitate arithmetical operations primarily with trigonometric functions. This, however, had the consequence that the table could not easily assist with arithmetic operations outside of a trigonometric context. Napier's ambition to avoid tedious and long computations resulted in a method which he gave a kinematic basis. It was, as Katz observed, an "imaginative idea of using geometry to construct a table for the improvement of arithmetic" [1998, p. 417]. But questions persist in the study of Napier and his logarithm. Most intriguing is where Napier's initial motivation to consider the grounding of the problem to be the motion of particles arose. Why did his particles have such a relationship, such that their distances were inversely proportional? Why was his second particle undergoing uniform deceleration, so that his geometric sequence decreased? Can the fact that he tabulated the logarithms of sines and not the logarithms of numbers be explained only with recourse to application, or was it because of something deeper? These questions are key, and although they are not addressed here they are important insofar as they highlight the contrast in Napier's intention, motivation, and approach with those of Bürgi.

3. BÜRGI

Jost Bürgi collected his work on the logarithmic relationship into the *Arithmetische und Geometrische Progress Tabulen/sambt gründlichem unterricht/wie solche nützlich in allerley Rechnungen zu gebrauchen/und verstanden werden sol*[35] (henceforth referred to as *Progress Tabulen*), which was issued in 1620 in Prague; as yet, this is his only surviving work available to modern scholars. Only a few copies of the *Progress Tabulen* are known to exist,

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[35] *Arithmetic and Geometric Progression Tables/with thorough instruction/how these can be usefully applied in various calculations/and how they are to be understood.*

though not all of these are considered complete because they do not contain the handwritten "*Bericht*".[36]

3.1. *The Manuscript*

The copy of Bürgi's *Progress Tabulen* used for this investigation is housed in the Universitätsbibliothek Graz (University Library Graz, Austria) and contains the shelf mark *Graz, University Library, I 18600-18601.* The manuscript[37] is bound together with Krabbe, Johannes: *Newes Astrolabium*; Frankfurt: Becker, 1609.

The manuscript was previously owned by Oswald Müller and Paul Guldin. Paul Guldin, a Jesuit and trained mathematician, amassed in his lifetime (1577-1643) a large collection (some 300 titles) of 16th and 17th century volumes, manuscripts, and correspondence, which are now part of the Special Collections of the University Library in Graz. Possibly because it was owned by just one other individual before Guldin, the Bürgi manuscript is in very good condition. There is evidence of age and moisture spotting (e.g., page 1) and two pages (15 and 16) are missing a bottom corner. No portion of the handwritten text appears to have been affected by damage. Lastly, the manuscript contains no commentary or notes, other than the text intended for the manuscript. "Justis Byrg[i]y" is handwritten in red ink above the first phrase of the title on the printed cover page. Also, the initials, "J B", are written above the phrase "Die ganze Rote Zahl" ("The greatest red number") on the same page, though these initials are of a different hand than "Justis Byrg[i]y".

The manuscript comprises three parts: the printed title page, printed tables of logarithms (by Paul Sessen ("bey Paul Sessen") in Prague, 1620) and a handwritten *"Kurzer Bericht"* ("Short Report") or *"Bericht"* ("Report"), which includes a two-page foreword. The tables are printed and written on paper with pages of size 19 cm by 16 cm and bound using flexible parchment. The tables fill 58 unnumbered pages, with labels centered at the bottom of every first and eighth page (most likely formed

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[36] See, for example, the holdings at BSB-Münich, whose copies do not include the handwritten "*Bericht*" (H. Tichy, personal communication, May 9, 2009).

[37] Here we refer to the *Progress Tabulen* as a 'manuscript', although it is a hybrid containing a printed part and a handwritten part.

as a quire – four sheets of paper that formed eight pages when folded or stitched together). For example, the ninth page is labelled "B" and the eleventh page is "B 2". The odd-numbered pages of the 21-page handwritten *"Bericht"* are so indicated, beginning with 1 and ending with 21. The foreword does not contain page numbers. It is understood that the entire manuscript described here was issued in 1620.

The use of red and black ink is evident in the printed cover page, printed tables, and handwritten *"Bericht"*. The first and second-to-last line of the title page (see Fig. 4) are written in red ink, along with the appropriate values of logarithms and "Die ganze Rote Zahl/230270022", and two different instances of either Bürgi's name or initials. The tables themselves are printed with the logarithms' values (the top row and left-most column) in red and the antilogarithms in black. Finally, red ink is used throughout the *"Bericht"*, whenever elements of an arithmetic sequence or the red numbers from the tables are used, or operations on the red numbers are performed. (There are, of course, a few exceptions to this due to irregularities.)

The language used in the printed title page and the handwritten foreword and *"Bericht"* is most likely Early New High German, which was generally dominant from around 1350 to 1650. The small amount of non-numerical text that appears on the printed title page (other than the first line) is consistent with Fraktur, a blackletter typeface and handwriting style used in German and other European languages from the 16th century until 1950. Much of the first line of text of the printed title page is reminiscent of some form of a minuscule script, characterized by uniform and rounded shapes. Curiously, there is a slight mixture of Fraktur along with this crisper typeface, as is seen in the "-sche" endings and the word "und".

There are notable uses of Latin (or at least a hybrid use) for mathematical terms in both the handwritten foreword and the handwritten *"Bericht"*. The change in this handwritten script for Latin terms found in the *"Bericht"*, however, is more distinct. Terms such as *Fundamenta* (*"Bericht"*, page 1); *Radicem Quadratum* (page 10); and hybrid terms such as *Medio Proportional* (page 14) appear throughout the text. In each instance the terms are written in a more distinct script, characterized by
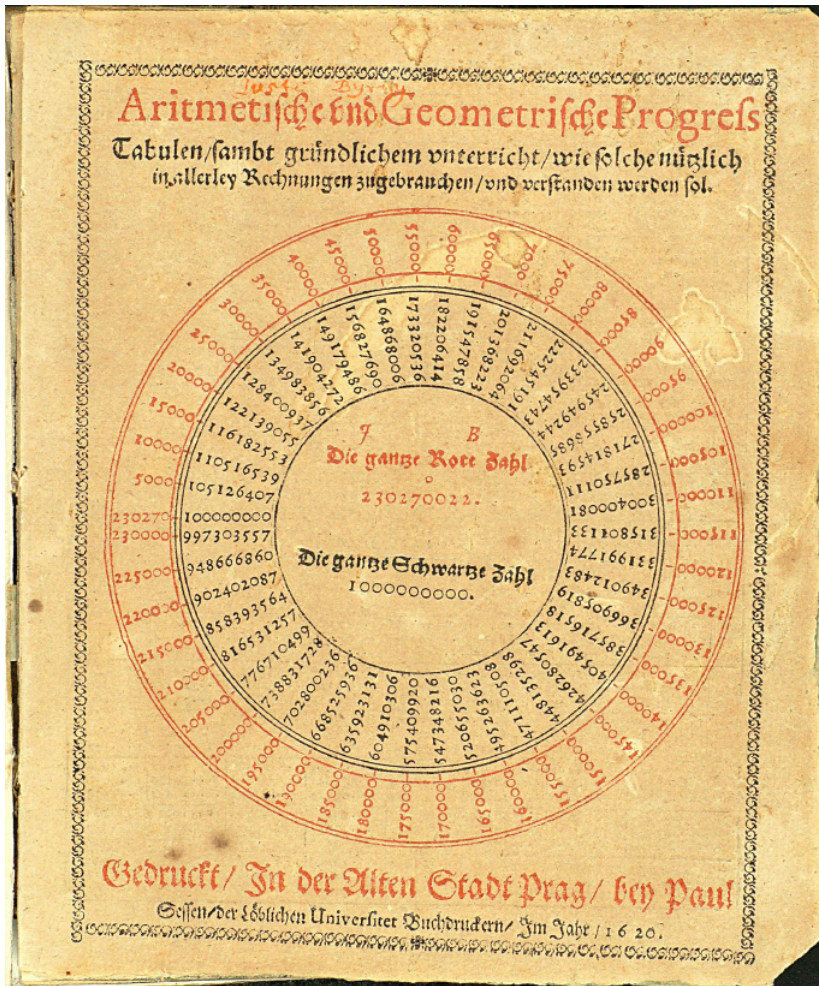
FIGURE 4. Title page of Bürgi's *Progress Tabulen.*

disconnected and more pronounced letters. Perhaps the most frequently used of these Latin mathematical terms is the hybrid *Medio Proportional,* which first appears in the *"Bericht"* as "mitler Proportional" (page 3).

Finally, the scribe of this copy of the manuscript could in fact be Jost Bürgi. Three key features of the manuscript lead us to this conclusion. First, several passages are written in the first person, as if Bürgi is taking the

opportunity to be more instructive than is found within the calculations of the examples. [38]

A second feature that is actually one of omission is that no colophon is found anywhere in the manuscript. The omission of a colophon, coupled with the fact that the red initials "J B" on the title page could be the rubrication of the scribe completing this copy [39], provide evidence for our claim.

Lastly, another copy of this work that has been the focus of prior modern scholarship [Bruins 1980; Gieswald 1856; Gronau 1996; Lutstorf 2005] is the manuscript which was housed at some point (and may still be held, though several attempts at communicating with the library have gone unanswered) in the Danziger [40] Stadt-Bibliothek and transcribed by Gieswald in 1856. There are several discrepancies between the copy used by Gieswald and the copy housed in Graz. For example, several of the passages that provide further explanation (in the first person) to examples (e.g., *"Bericht"*, pages 16 and 18) do not appear in the Gieswald transcription. Consequently, the copy housed in Graz is probably the older of the two. The copy used by Gieswald also contains the initials "J B" on the printed title page, along with two antilogarithm corrections, which Lutstorf and Walter [1992] indicate as possibly being written by Bürgi. Thus, it is quite probable that the two *Progress Tabulen* copies that include the handwritten *"Bericht"* were written by Bürgi.

3.2. *The Content*

3.2.1. *Foreword to the "Good-hearted Reader"*

Bürgi announced clearly at the outset the intention behind his special tables, namely to "remove the difficulties from multiplication, division, and extraction of roots" [41] (Foreword, page 1). He continued, stating that: "through all time I looked for and worked to invent the general tables, with which man would like to perform all of the afore-mentioned things"

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[38] Examples of this can be seen in the *"Bericht"* on pages 3, 5, 7, and 16, to name just a few.

[39] Jeep [2001, p. 609] notes in his entry on 'Paleography' that upon completing a manuscript, a scribe would paint his initials in red.

[40] The municipal library of what is today the Polish city of Gdańsk.

[41] Each translated excerpt from the *Progress Tabulen* is the work of the authors.

(Foreword, page 1). In particular, Bürgi's key motivation was to construct special tables that could be used for a variety of calculations, rather than needing collections of various tables each of which were specific to a particular operation. Indeed, Bürgi notes that having separate tables for multiplication, division, square roots, and cube roots is "not alone irksome, but also laborious and cumbersome" (Foreword, page 1). It is here in the Foreword that Bürgi states he is able to create one table for a multiplicity of calculations by considering the two progressions: one arithmetic; the other, geometric. He closes the Foreword by noting that he will most likely work with the tables for years to come, and promises another work for those readers who desire a deeper understanding of the tables. Sadly, this grand explanation, the *"Unterricht"*, or literally, the *"Instruction"*, promised in both the title of the *Progress Tabulen* and in the Foreword was never delivered. (At least, no copy of the *"Unterricht"* is known to have survived.) Instead, the *"Bericht"* contains only a brief introduction to the relationship between an arithmetic and geometric progression (with eight examples of calculations using the whole numbers and the non-negative powers of 2) and some 26 examples of calculations using the tables he computed (Table 3).

Bürgi delivers on his promise to construct special tables that can be used for a variety of calculations. Table 3 identifies each type of calculation found in the *"Bericht"* and for each calculation (multiplication, division, square root, cube root, fourth root, fifth root, and middle proportionals), Bürgi's examples increase in complexity. The range of examples includes: (1) straight-forward use of the tables (e.g., black numbers and an operation are given; the corresponding red numbers are found and associated "simpler" operation performed; resulting black numbers are determined from the table); (2) interpolation (e.g., resulting red values that do not appear in the table that require linear interpolation between two that do appear); (3) adding or subtracting 230270.022 (e.g., a resultant red number larger than 230270.022 requiring that the "greatest red number" be subtracted before determining the associated black number); and (4) a combination of a subset of the first three types.

Table 3
Distribution of examples in the *Progress Tabulen*.

Page Number	Topic	Content
p. 1	Introduction	Arithmetic and geometric progressions (2^n)
p. 2–4	Definition of operations with examples using 2^n	8 examples
p. 4–5	Introduction to Tables	2 examples
p. 5–6	Non-tabulated values	1 example
p. 7	Multiplication	2 examples
p.8	Division	2 examples
p.8–10	Rule of three	3 examples
p. 10–11	Square Roots	2 examples
p. 11–12	Cube Roots	3 examples
p. 13	Fourth Root	1 example
p. 13–14	Fifth Root	1 examples
p. 14–18	Middle Proportionals (MP)	6 examples
p. 19–21	Middle Proportionals	1 example finding two MPs
		1 example finding three MPs
		1 example finding four MPs

3.2.2. *The* "Bericht"

The handwritten *"Bericht"* provides 21 pages of worked examples and explanations. Bürgi begins by introducing the reader to two types of numbers: red numbers that are elements of an arithmetic progression (which Bürgi uses for 'sequence') and black numbers that are elements of a corresponding geometric progression. In his introduction, he states that everyone will come to consider his tables as comprised of *Fundamenta* (or, 'Fundamentals') because they represented improvement on the previous practice of needing multiple tables to perform all manner of calculations. Now, one would only need Bürgi's tables.

In the *"Bericht"* Bürgi provides examples that show the application of the relationship between the two progressions ("Begriff der Eigenschafft diser Zweien Progressen",(*"Bericht"*, page 1)). At the bottom of the first page of

FIGURE 5. Arithmetic and Geometric Progression (powers of 2).

the *"Bericht"* (see Fig. 5) the reader is presented with the theoretical motivation that underlies Bürgi's tables, the juxtaposition of an arithmetic progression (the natural numbers, 0 to 12) and a geometric progression (the first 13 powers of 2).

After Bürgi establishes the primacy of relating the two progressions, he presents eight examples of performing a variety of calculations (e.g., multiplication, division, extracting square roots) on the black numbers using corresponding, yet simpler operations (e.g., addition, subtraction, halving) on the associated red numbers. He takes great care to highlight the relationship between two sequences throughout the examples in his text, as well as in his tables (see Figs. 6 and 8). In addition, he emphasizes the usability of his design by including examples that work "both ways"; that is, given black numbers, retrieve their corresponding red numbers and vice versa. Among the first eight examples is a single example of a *Regula detri* (*"Bericht"*, page 2). [42] The example shown in Fig. 6 (*"Bericht"*, page 3) highlights another significant feature of Bürgi's text. Note the careful alignment of the corresponding red numbers for each of the black numbers in the example. Here we find his deliberate juxtaposition of the colored numbers within the layout of the examples he uses to illustrate his relation.

Here, the problem posed by Bürgi reads:

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[42] *Regula detri* means the "rule of three" or, it is medieval Latin for "equating elements by two equal fractions", which is often considered an example of the rule of three, in that if a/b = c/d, and any three of a, b, c, d are known, the fourth can be determined.
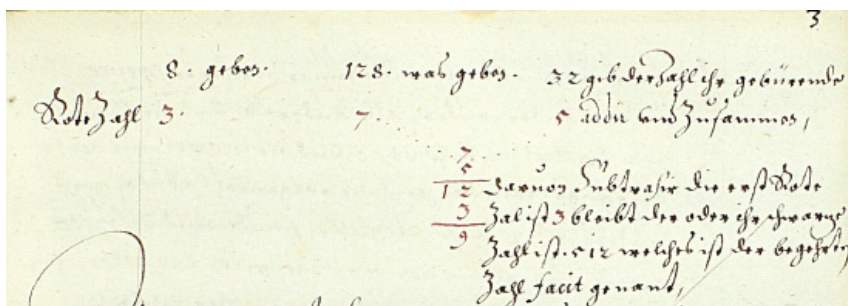
FIGURE 6. Regula detri example (*"Bericht"*, page 3).

8.geben.	128.was geben.	32 gib der Zahl ihr gebürrende
Rote Zahl 3.(in red ink)	7. (also in red ink)	5 (in red ink) addir und zusammen.

Restating this *Regula detri* problem in modern terms, the task is to compute *a* from

$$\frac{8}{128} = \frac{32}{a}$$

(*"Bericht"*, page 3). In this example, the black numbers 8, 128, and 32 are given, along with their red numbers (again, aligned just beneath the black numbers). A product of 128 and 32, and then the result of that number divided by 8, is sought. However, using only the red numbers in a vertically-oriented calculation, Bürgi adds the corresponding red numbers for 128 and 32 (7 and 5, respectively); subtracts the red number for 8 (which is 3) from this sum; and using the red number 9 to find the corresponding black number (512), the result is obtained. Thus,

$$\frac{8}{128} = \frac{32}{512}.$$

Bürgi uses color and alignment consistently throughout the presentation of examples based upon the tables he constructed in the *"Bericht"*. To calculate the "*vierte Proportional*" (*"Bericht"*, page 8) or the fourth proportional using the method of *Regula detri*, Bürgi gives the first, the second, and the third black numbers taken directly from his tables. Then, as before, he gives the corresponding red numbers by aligning them beneath the black numbers. The alignment is not limited to black number and red number correspondence, however. As Bürgi completes calculations, he ensures that he aligns his computations on the red numbers vertically.
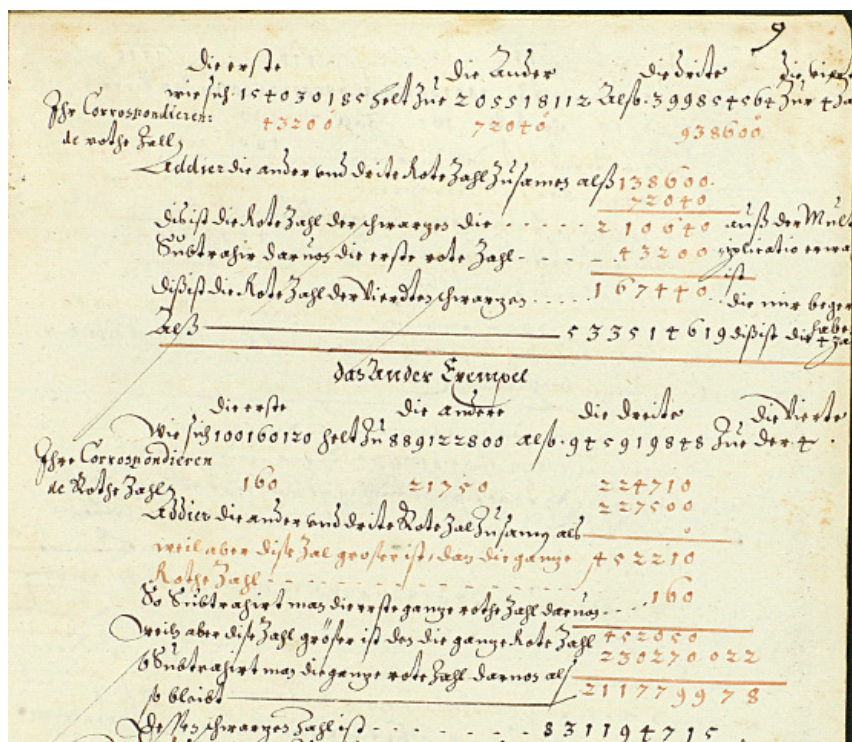
FIGURE 7. Calculating the fourth proportional, given three numbers (*"Bericht"*, page 9).

Thus, the statement of each example is oriented horizontally so that the correspondence between the black numbers and their associated red numbers are easily seen and all calculations are shown vertically. Figure 7 shows two worked examples for calculating the "*vierte Proportional*" (*"Bericht"*, page 9).[43]

In the first example given on page 9, Bürgi seeks to find the fourth value, given the first three numbers in a proportion. He begins by listing three

43 The second half of this excerpt beginning with "Das Ander Exempel" (in this case, "The Second Example") does not appear in Gieswald [1856].

nine-digit numbers (154030185, 205518112, 399854564) and their corre-
sponding red numbers (43200, 72040, 138600 [44]). Then, Bürgi sets to cal-
culate on the red numbers only. He begins by adding the second and third
red numbers:

138600 + 72040 =
210640.

Next, Bürgi subtracts the first red number:

210640 - 43200 =
167440.

Finally, using the resulting red number, Bürgi identifies the "*vierte Propor-
tional*", 533514619, by locating the black number from the table corre-
sponding to 167440.

A second noteworthy example of Bürgi's use of color and alignment is
evident in an example of the extraction of the fourth root, the *"ZZR″"* (see
Fig. 8) or given as *"Radicem Zonsi Zonsicum"*[45] on (page 3 of the *"Bericht"*).
Using the same value (e.g., 56120370 or 561203700) from previous exam-
ples (another technique used by Bürgi), he seeks to calculate the fourth
root of 56120370.

To find the fourth root, Bürgi must first determine two things: how
many whole number digits will exist in the root and how many multiples
of his 'greatest red number' (230270022) [46] must be incorporated into
the calculations involving red numbers. Bürgi determines the former by
placing a dot above the ones digit of the number (56120370) and then
placing a dot over every fourth digit until less than four digits remain
(since the fourth root is desired). Thus, Bürgi has placed a dot over the

---

[44]   The red number corresponding to 399854564 is first given incorrectly in the
manuscript as 938600 (*"Bericht"*, page 9). On line five, Bürgi gives the correct red num-
ber as 138600. It is worth noting here that such errors may be identified as copyist
errors and not necessarily original to Bürgi.

[45]   Many would argue that the manuscript should read "*Zensi Zensicum Radicem*" (to
properly indicate the fourth root), but the second letter of each of the first two words
is clearly an "o".

[46]   For an explanation of this number, see section 3.2.1 below. Briefly, the loga-
rithm of a number that was 10 times another corresponds to adding (or subtracting)
230270.022 to (from) its logarithm in Bürgi's system.
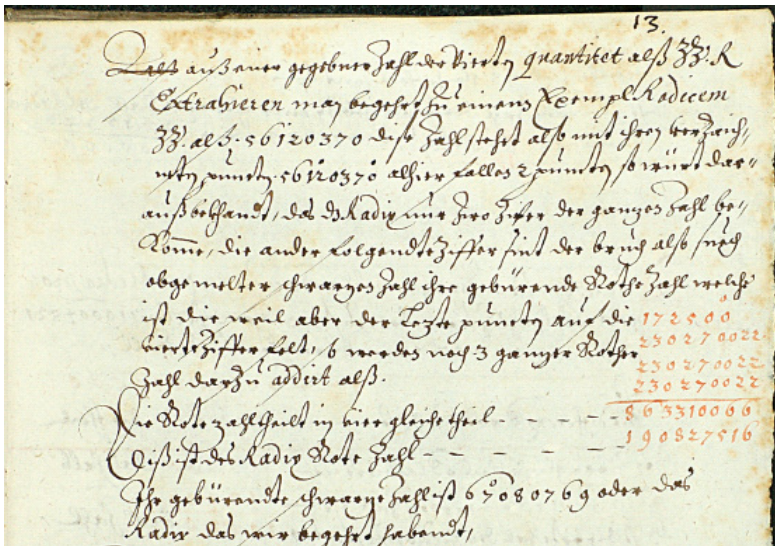
FIGURE 8. Calculating the fourth root of 56120370 (*"Bericht"*, page 13).

0 (in the ones place) and the 2 (in the ten-thousands place). This means the fourth root of 56120370 will have two whole-number digits.

To determine the latter, how many multiples of 230270022 are required, Bürgi simply counts the number of digits remaining after the dot-placing process. Thus, the red-number calculations will require three multiples of the greatest red number: first, 172500 is located as the corresponding red number for 56120370. Then, aligning and adding:

```
172500[000]
230270022
230270022
230270002
863310066
```

Since Bürgi seeks the fourth root, however, one-fourth of this resulting value (or, 215827516) is the red number that must then be located in the table.[47] Finally, using linear interpolation, along with the fact that

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[47]   Unfortunately, the final red number value that appears here (*"Bericht"*, page 13) is incorrect in the Graz copy. The final red number given (190827516) is most likely the result of dividing 763310066 by 4, instead of dividing 863310066 by 4.
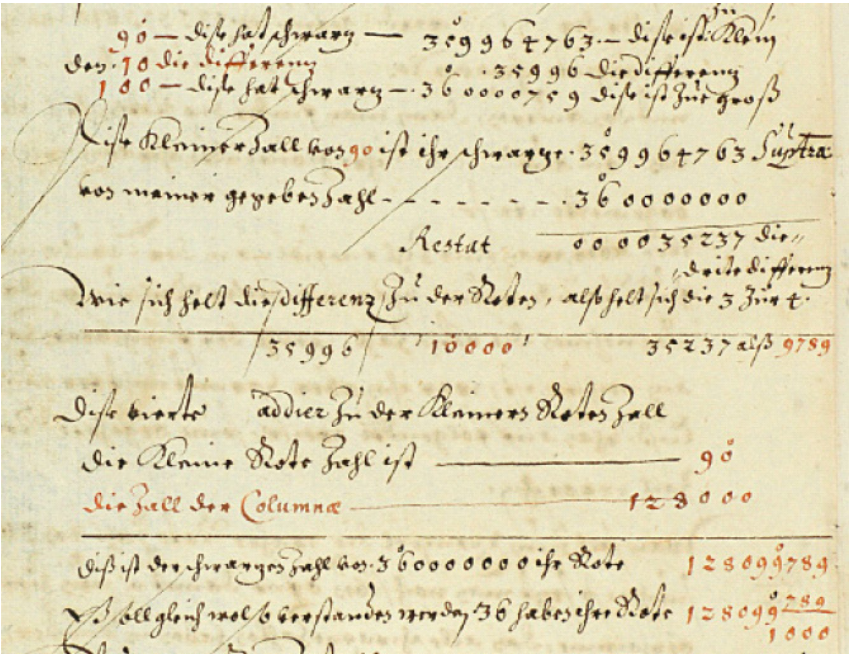
FIGURE 9. Interpolation example (*"Bericht"*, page 6).

the fourth root contains two whole-number digits, the fourth root is
86.5526026. [48]

The first instance of interpolation ("*Bericht*", page 6) is not unlike the
presentation of other examples, with regard to careful alignment and use
of color. The example also includes, however, a brief explanation of how
to interpolate to find the red number corresponding to the black number
36.0000000. [49]

In the example in Figure 9, Bürgi selects the two values from the table
closest to 36.0000000, and lists the black numbers and their corresponding
red numbers:

---

[48] This is a correction on the value given in the manuscript, 67.080769 (*"Bericht"*,
page 13).
[49] This value appears as 36 0000000, with a small "o" situated above the 6. This was
Bürgi's version of the decimal point, first introduced by Stevin [1585].

The red number [1280]90 corresponds to the black number 35.9964763
The red number [128]100 corresponds to the black number 36.0000759

Next, Bürgi performs a simple linear interpolation for the non-tabulated value 36.0000000:

First, 36.0000759 - 35.9964763 = .0035996 and
36.0000000 - 35.9964763 = .0035237.

Finally, solving the proportion

$$\frac{35996}{10000} = \frac{35237}{x}$$

Bürgi determines the value 9789 (page 6, line 14). To complete the interpolation, Bürgi adds 128000, 90, and 9.789 for a result of 128099.789 as the red number (logarithm) corresponding to 36.0000000.

When appropriate, Bürgi provides examples from this range for each type of calculation exhibited in the *"Bericht"*. The examples that are perhaps the most notable are the *"Medio Proportional"* calculations. In these examples, Bürgi establishes the use of black numbers from the table that are not always nine-digit numbers. In other words, the powers of 10 implicit in Bürgi's table are highlighted in the sequence of examples.

As we have seen in calculating the fourth root (Fig. 8), there are many advantages to basing the red numbers on multiples of 10 and using a scale of 10^8 for the black numbers. Examining the example in Fig. 10, (*"Bericht"*, page 16) in more detail provides another method in which Bürgi compensates for performing calculations with different numbers of digits.

In the example, Bürgi seeks the mean proportional (or, geometric mean) between 303419 and 304939818. To utilize the greatest red number in this example, Bürgi simply determines the difference in the number of digits. A six-digit and nine-digit number differ by three digits, therefore to add the corresponding red numbers together, the six-digit number must be increased 1000-fold, which requires Bürgi to add 230270022 three times in order to proceed with the red number calculations. Thus, Bürgi determines the corresponding red numbers from the table: 111000 and 111500. Adding the two red numbers yields 222500[000]. Next, Bürgi adds the three multiples of 230270022, and the resulting sum is 913310066. Since the geometric mean calculation requires the square root of the
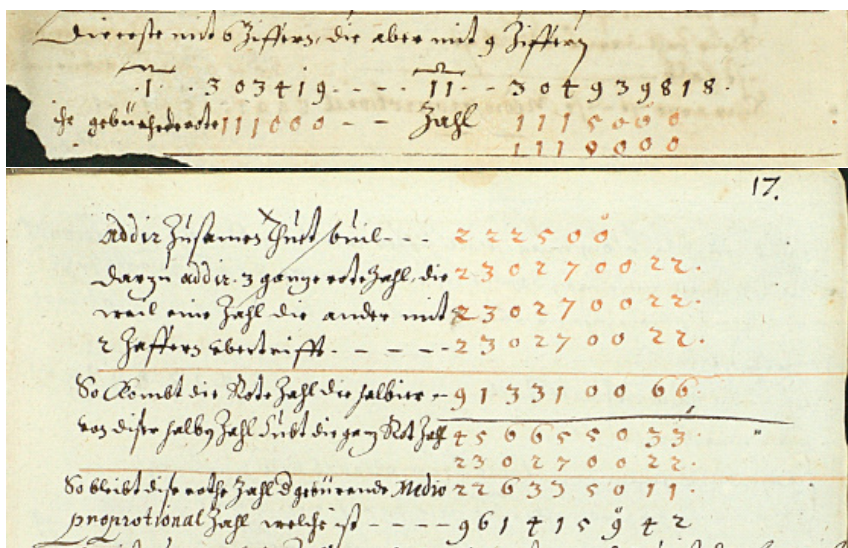
FIGURE 10. Calculating the middle proportional between a 6-digit
and 9-digit number (*"Bericht"*, pages 16–17).

product (represented here by the sum of the two red numbers and sub-
sequent multiples of 230270022), the next step is to divide 913310066 in
half, which is 456655033. This number, however, exceeds the greatest red
number found in the table and 230270022 must be subtracted so that the
resulting red number (226385011) can then be used to interpolate the
corresponding black number. [50]

The mathematical notation and symbolism used by Bürgi in the *Progress
Tabulen* illustrates his ability to present his work in ways that were relevant
to those who would read and use the tables and accompanying examples.
Most notable among Bürgi's use of several mathematical symbols is his ver-
sion of the decimal point that appears for the first time on the printed title
page of the *Progress Tabulen*. His decimal point, a small "o" or zero character
written above the digit meant as the ones place of a whole number part of
a number. On the title page of the manuscript, we find Bürgi's reference to

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[50]   There is an error in the Graz copy of the *Progress Tabulen* here: 226335011 is given
instead of 226385011. The resulting mean proportional is also incorrect (*"Bericht"*,
page 17). The value should be 9618967.441.

the greatest red number, 230270.022 corresponding to the greatest black number, 1,000,000,000. [51]

In addition to Bürgi's use of a decimal point, he also writes several resulting black numbers in fraction form (e.g., (*"Bericht"*, pages 8, 14); see Fig. 11). In each of these instances, the whole number part of the number is written very closely to the fractional part and in doing so, the ones digit of the whole number part is aligned with the "1" in the fraction's denominator. In all likelihood, Bürgi adopted this fraction-writing convention so as not to confuse the final digit of the whole number with any digit of the fraction.

Bürgi calculates two fourth roots (*"Bericht"*, pages 3, 13; see Fig. 12) and one fifth root (*"Bericht"*, pages 13; see Fig. 12) in the *"Bericht"*. In the first instance of calculating the fourth root, Bürgi uses the term *Radicem Zonsi Zonsicum* (*"Bericht"*, page 3). For the second, however, Bürgi utilizes the abbreviation *ZZR* and *Radicem ZZ* (page 13, lines 1–3). [52] When he provides the example of calculating the fifth root, however, his notation is merely *Radicem Ss,* [53] (*"Bericht"*, pages 13-14).

A final illustration of Bürgi's use of mathematical notation is found at the end of a *Medio Proportional* example (see Fig. 13). After Bürgi identifies two proportionals B and C between 119004521 (A) and 893423483 (D), he writes: "Wie sich halt A zu B: also halt sich B zur C: und C zur D:", or, 'As A is to B, in the same way B is to C and C to D:" (Fig. 13). Thus, he uses a description that is part rhetorical and part symbolic.

3.2.3. *The Tables*

In contrast to the handwritten *"Bericht"*, the tables in Bürgi's *Progress Tabulen* are printed. The arrangement of his tables contrasts markedly with Napier's. The most significant and immediate feature is that Bürgi's tables give antilogarithms, or powers of the base 1.0001, multiplied by

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[51]   Using the calculations discussed in the description of the tables, $1000000000 \approx 999999999.7 = (10^8)(1.0001)^{230270.022/10}$.

[52]   Related to the cossic notation for powers: ZZ for 'square-square'. Additionally, we emphasize that this quote is from the manuscript (see Fig. 12). One would expect to see *radix"*; instead, Bürgi has used the accusative *radicem.*

[53]   Again, from the cossic notation for powers: Ss for 'sursolid', or fifth power.

FIGURE 11. Sample fraction notation (*"Bericht"*, pages 8, 14).

10^8.[54] Thus, as tables of antilogarithms the arguments are the logarithms themselves and the base numbers are retrieved in the body of the table. There are several techniques Bürgi employs to make his tables more usable and comprehensible. As previously mentioned, throughout the *Bericht,* to emphasize the difference between the antilogarithms and logarithms Bürgi has consistently used color to demarcate the two. The body

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[54] The fact that Bürgi produced a table of antilogarithms was noted as being an important "marketing" device [Folta & Nový 1968, p. 98]. Also, Bürgi would not have used the term "base".

FIGURE 12. Bürgi's fourth- and fifth-root notation ("*Bericht*", pages 3, 13).

FIGURE 13. Middle proportional (*Medio Proportional*) notation ("*Bericht*", page 19).

of the table, the antilogarithms, are printed in black and the logarithms themselves which are arranged along the top and left edge are printed in red. So as not to overcrowd the tables, for each new page of the tables, only the first row of the body of the table always includes all nine digits for each entry. The red numbers increase by 10 for each row; however, there is also an implied scale factor of 10. By a careful cross-indexing of the left-hand column and the top row, the exact logarithm (red number) and its corresponding base number can be retrieved.

For ease of reading, the columns are divided into 17 clusters of three rows each. Additionally – and perhaps also for ease of reading – black numbers are not always given in their entirety (i.e., all nine digits). Instead, nine-digit numbers are initially given once among every eight, nine, or ten table entries in which the fourth digit of the sequence of antilogarithms changes. As the tables progress, complete nine-digit numbers appear more frequently until finally, beginning on the 45th page of the tables, the frequency of complete nine-digit numbers diminishes, sometimes as little as once every 16 entries.

Despite the fact that Bürgi presented the theoretical motivation for his tables via the comparison of an arithmetic and geometric series that was based on the powers of two, a different numerical parameter actually underlies his tables. Bürgi was aware that a geometric progression with a common ratio of 2 (or any value much larger than 1) would produce terms that became too large too quickly to be useful. Thus, selecting a common ratio of 1.0001 for constructing his tables produced values more amenable to interpolation. This common ratio choice created a smaller gap between any two successive black numbers, enabling Bürgi to use linear interpolation to determine close approximations for a black number (or red, if using the tables in that direction) corresponding to any red (black) number resulting from calculations.

The first value in the body of the table (in black) is 100,000,000 (see Fig. 14). Its corresponding logarithm (red number) is 0. Modern reconstructions, e.g., [Katz 1998] show that subsequent values can be generated via (where B is the antilogarithm (the black number) and R is the logarithm (the red number)):

$$B = (10^8)(1.0001)^{R/10},$$

so that the black numbers form a geometric progression with ratio $r = 1.0001$. Bürgi does not reveal any such details that underlie the construction of his tables. Straight forward indexing techniques have been used to tabulate the logarithmic values. To use the left- and top-edge of the table, simple addition provides the logarithm value (or, 10 times the logarithm value). For example, to find the logarithm of 101907877, we would add the column value (1500) to the row value (390), and divide by 10. Thus, the

FIGURE 14. Locating a logarithm value.

logarithm of 101907877 is 1890/10 or 189. To confirm this using the modern calculation, $101907877 = (10^8)(1.0001)^{189}$.

The first 57 of the 58 pages of tables contain eight columns and 51 rows which produce 408 entries per page, for a total of 23,256 entries. Except for the first entry of the first page, each final column entry is also the first entry of the next column. This means that of the 23,256 entries, 22,801 are distinct. Finally, page 58 of the tables includes 233 additional entries, of which 229 are distinct. Thus, Bürgi's tables are composed of 23,489 entries, 23,030 of which represent distinct calculations.
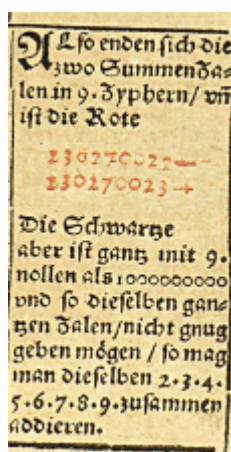
FIGURE 15. Last page of Bürgi's tables.

Although the construction of the tables appears to be driven by scale factors (10 and (10^8)) for improved in precision without the need for decimals, it is also clear from Bürgi's note at the conclusion of the tables that he saw the tables as being easily used for any number desired. On the 58th sheet of the tables Bürgi states (see Fig. 15):

Also enden sich die zwo Summen Zalen in 9. Zyphern/vñ ist die Rote
230270022-
23027023+
Die Schwarze aber ist ganz, mit 9 nollen als 1000000000 vnd so dieselben ganzen
Zalen/nicht gnug geben mögen/so mag man dieselben 2. 3. 4. 5. 6. 7. 8. 9. zusammen
addieren.

So ends the sum of two numbers in 9 digits / and the Red (numbers)
230270022 -
230270023 +
The black [numbers], however, with only 9 zeroes as in 1000000000 may not be enough / you can add the same as 2, 3, 4, 5, 6, 7, 8, 9 (-digit numbers), together. (Last page of Bürgi's tables; translation by first author)

Thus, Bürgi, with this note at the end of the tables, declares that the "entire" red number (or, rather, the greatest logarithm he has calculated) is between 230270.022 and 230270.023, and that the black numbers can either be taken as multiples of or as parts thereof.

In summary, Bürgi based his system on a geometric sequence, with ratio 1.0001 and first term 10^8. He tabulated from 100000000 to 999999999 with

$$\log_{bur}(10^8) = 0$$
$$\text{and}$$
$$\log_{bur}(999999999) = 230270.022$$

and, to finally advance to the ten-fold value 1000000000 Bürgi knew that $\log_{bur}(1000000010) = 230270.023$. Furthermore, the logarithm of a number that was ten-fold different was equivalent to adding or subtracting 230270.022 to its logarithm. This factor of 10^8 was presumably to maintain precision while using integers.

In addition to his tables, Bürgi included a graphical table which summarized his system—a syncopated presentation of his tables (see Fig. 4). This appears on the title page and consists of two concentric circles of values giving the red numbers in increments of 5000 and their corresponding black numbers. All but the last value follows this pattern; the final value is the largest red number 230270. Therefore, with this graphical rendering on a single sheet and the appropriate interpolation pattern, users could compute using Bürgi's system. Whether or not it was in fact used in place of the tables, this graphical image is emblematic of Bürgi's system in several ways: most importantly it highlights the cyclical nature of his system, it conveys the actual numerical relations, and it captures the red-black numerical relation.

Even with the unfortunate omission or loss of Bürgi's promised *"Unterricht"*, much can be learnt from a detailed analysis of the *Progress Tabulen* with respect to his conception of the logarithm. From his brief *"Bericht"* we know the motivation for the construction of the tables. Furthermore, the presentation of the examples highlights important details of Bürgi's use of notation, his concern for the ease of use of the tables (particularly highlighted by the use of color), and his implied instructional techniques. However, Bürgi remained silent as to any deeper conceptual foundations for what led to his choices for the construction of the tables. Nor did Bürgi provide any reason why he elected to construct his tables as he did, given that the scientific community he worked within was concerned with astronomical calculations that depended upon trigonometric values. Furthermore, we can only speculate that the fact that Bürgi promised a "user's

guide" [Gronau 1996, p. 1] may mean that he intended users of his tables to include non-astronomers.

4. THE ISSUE OF PRIORITY

Eminent historian Florian Cajori opened his study on the history of the logarithm with the comment "Few inventors have a clearer title to priority than has Napier to the invention of logarithms". [55] Cajori's emphasis was on Napier's claim to the logarithm concept, and his focus henceforth was a systematic examination of Napier's contemporaries to categorically dismiss any challenges that they might pose to this priority. Indeed, the emergence of tables of logarithms beautifully exemplifies a practice that is insidious throughout many accounts in the history of mathematics—the evaluation of the claim of an individual to an idea or insight and their rights to priority.

Even in the period near contemporaneous with the introduction of these ideas, scholars were already assessing the claims of one of the pair to priority. Kepler himself, who worked closely with Bürgi, comments:

> *qui etiam apices logistici Iusto Byrgio multis annis ante editionem Neperianam viam praeiverunt ad hos ipsissimos Logarithmos. Etsi homo cunctator et secretorum suorum custos foetum in partu destituit, non ad usus publicos educavit*

> These logistic points which showed Iustus Byrgi many years before the edition of Napier the way to these very same logarithms. Even so, this man, a procrastinator and guardian of his secrets, abandoned his baby in childbirth and did not nurture it for public use.
> Kepler, *Tabulae Rudolphinae* [1627, p. 48]. [56]

The issue of priority still grounded discussions in centuries following. In 1859, the French mathematician and politician Arago claimed: "I doubt whether it were possible to cite a single scientific discovery of any importance which has not excited discussions of priority". [57] More recently, scholars have conveyed sentiments such as the following: "Speaking of priority

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[55]   [Cajori 1915, p. 93].

[56]   From [Gronau 1996, p. 6].

[57]   *Biographies of Distinguished Scientific Men*, First Series, Boston, 1859, p. 383; as cited by [Cajori 1915, p. 93].

disputes, Napier does not hold the title to logs free and clear… But publication won out on this one, and Napier has retained his title to priority" [58] and even "long before John Napier, Stifel seems to have invented logarithms independently." [59]

When there is more than one individual wrapped up in a 'priority dispute', scholarship has tended to focus on the ways in which these individual insights were similar, neglecting important points of contrast. We should consider that such an emphasis on the similarities between accounts risks reducing our capacity to appreciate the details and nuances of each of these individual accounts. Examining the details of Napier and Bürgi's work and circumstances brings this point to light vividly. The two men were working in different communities in distinct locations, and, as the scant evidence suggests, simultaneously. The nature of their insight and design required years of intensive calculation efforts and was not the result of some instantaneous insight which could be written up quickly.

Their proposals differ in many significant ways. Napier's conception was given kinematic expression from the outset. [60] His comparisons rested on the motions of particles subject to both uniform velocity and deceleration over infinite and fixed distances. His work was designed and engineered very much with his audience in mind. His results were not presented as a table of logarithms, but rather as a table of uniformly increasing angles, with their corresponding sines and logarithms thereof. This was so that they would be immediately useful to those practitioners whose calculations were largely trigonometric. He devised ingenious interpolation schemes to save on computation efforts while allowing him to tabulate with a high degree of detail. Napier's work was quickly transformed into a practical tool which was commercially distributed and exploited.

Bürgi on the other hand grounded his description in arithmetic alone, defining both an arithmetic and a geometric sequence and carefully showed the ways in which they were related and how his system took

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58   [Shell-Gellasch 2008, p. 6].

59   [Pesic 2010, p. 506].

60   The question as to whether Napier's conception was kinematic, or whether it was simply presented that way is a fascinating one, but is beyond the scope of this present study.

advantage of this. He did not attempt to contextualize the relation more broadly. His proposal was more mathematically general, in that it did not tabulate according to trigonometric relations, and could be directly applied to any arithmetical operation. By design, his tables were essentially antilogarithmic. Bürgi computed each entry directly, using only basic rounding to furnish his table entries.

Napier's logarithms are essentially embedded in what is a table of sines. Bürgi's logarithms are the very argument of tabulation. Napier coined terminology for his concept, both the term logarithm that has persisted to the present day, as well as the appellations 'artificial' and 'natural' numbers for the elements of the related sequences. In contrast, Bürgi used color to delimit his relations in place of technical terminology, and simply referred to the two sets of numbers as 'red numbers' (logarithms) and 'black numbers' (ordinary integers). The distinction between the two was primarily visual, and not through technical terms. Bürgi based his system on an increasing geometric sequence, with common ratio 1.0001 and first term 10^8. Napier based his system on a decreasing geometric sequence. He selected $R = 10,000,000$ to accommodate its trigonometric orientation, and progressed in increments of $1 - 10^{-7}$.

Napier had both works on his logarithm relation published. Bürgi only one, though another was promised. One can only guess at the difference in reception had this more 'theoretical' work made its way to public consumption. Furthermore the dissemination and acceptance of results by the mathematical community depends also on the way results are communicated. [61] Napier's first work was published in 1614 and was type set and printed. Bürgi's was finally issued in 1620, and although his tables were printed his manuscript remains to the present day only in handwritten form. [62] Napier was part of the busier academic community, and had his work translated into the vernacular by Wright almost immediately. His

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[61]   In a broader context, [Bretelle-Establet 2010] edits a volume which considers the role of documents, collections, and archives, and their availability and dissemination, in the reception and development of ideas, both for the original cultures of inquiry as well as historians, in the context of Asia.

[62]   Or, as a printed German transcription from the original Gieswald acquired [1856].

work was also quickly adopted by wider commercial interests. Bürgi wrote in German which was arguably less accessible to the wider community. Napier seems to have responded more quickly to the reaction of his contemporaries and the suggestions that they had for improving the system. He promptly made various accommodations so that his system would be easier to use; contemporaries of his helped devise and initiate these changes. For instance, a year after the publication of the tables, Napier shared with Briggs that 0 (zero) should be assigned to the logarithm of unity (1)—as opposed to the logarithm of the whole sine (10,000,000) that Napier initially proposed. [63] In their discussions, Briggs wanted to retain the logarithm of the whole sine as 0, but he eventually stated that "0 should be the logarithm of unity" and that such a decision "was by far the most convenient" [Huxley 1970, p. 462]. Inspired by Napier's contributions, further scholars focused their efforts on developing and refining tables of logarithms, including Dutchmen Adriaan Vlacq (1600–1667) and Ezechiel DeDecker (ca. 1603–ca. 1647) and Englishmen John Speidell (fl. ca. 1619) and Edward Wright (1559–1615). For example, DeDecker and Vlacq published a "complete ten-place table of logarithms from 1 to 100,000" [Boyer & Merzbach 2011, p. 289] and three years after Briggs extended his own table in *Arithmetica logarithmica* (1624).

All of these contrasts reveal how progressive, yet halting the concept of the logarithm was at this initial stage. Napier and Bürgi both initially experienced positive receptions for their efforts, but one of them encountered inducements from his academic community that furthered his work. It is of little surprise, then, that his name has been the more pre-eminent.

More generally, the question of priority must be dealt with carefully by historians. When we focus on one individual's right to claim an insight, we lose a great deal of the richness from whence this idea originated. In fact, parallel insights intimate a certain fecundity in the broader intellectual scene. They earmark that particular moment in the history of mathematics as rich. Of importance is the singularity of a breakthrough when it occurs, but even more so are the conditions and incremental insights that

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[63]   Consequently, a new table of logarithms needed to be calculated. Since Napier was in poor health by this time, Briggs undertook the task [Suzuki 2009, p. 220].

preceded this point, without which such a breakthrough would not have been possible. A breakthrough made multiple times suggests an intensity of activity in the mathematical scene.

Furthermore, seeking similarities in past mathematics may encourage a reductionist attitude when evaluating the efforts of emerging mathematicians and nascent ideas. For example, seventeenth century English mathematician John Collins comments about his predecessor, mathematician and cartographer Edward Wright:

> [Wright] happened upon the logarithms and he did not know it, he made a table of logarithms… before logarithms were invented and printed, but did not know he had donne it.
> As cited in Cajori [1915, p. 99]

This reflection reveals the absurdity of evaluating the introduction of ideas in this way; in a similar spirit, Cajori notes:

> That such a table should turn out to be a table of logarithms is not as strange as it may seem. Any set of numbers in arithmetical progression placed parallel with a set of positive numbers in geometrical progression defines some system of logarithms… [Cajori 1915, p. 99]

Indeed, Cajori brings to mind the challenges related to the retrospective appraisal of ideas in history. To find traces of an idea in the work of mathematicians past is too easily done from a future standpoint, and it risks anachronism or ignoring the context in which these ideas were understood. Indeed, to speak of Napier's "clear title to priority" to logarithms obscures the fact that his proposal remained always within a particular context. As we have shown, the logarithm was an evolving concept in the history of mathematics, and its scope and articulation were developed by many different individuals.

It is unquestionably valuable to bring to light the resemblances between proposals and accounts of the various scholars, but emphasizing these similarities can distract us from the more insightful issue of how these scholars experimented with and articulated, both imperfectly and successfully, the concept themselves. These parallel insights reveal an abundant legacy of the challenges and motivations facing scholars and the ways in which each of them responded to these. They reveal the various idiosyncratic and creative ways in which scholars responded to the intellectual environment

around them, and the unique methods or products they developed as a result of these stimuli. Parallel insights are immensely rich for the historian; that legacy is too significant to be obscured by the determination to single out one individual as ultimately pre-eminent over the others. When the history of mathematics lapses into an account of various incremental insights and the individuals associated with those, the narrative simply becomes a sequencing of successes. The richness and movement of the mathematical landscape becomes characterized as a linear one that is initiated and maintained by key individuals. In this way, reflecting on the question of priority has elevated the role of the individual over the collective. History has demonstrated, by contrast, that intellection is so very much a collaborative process.

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