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FERMAT'S METHOD OF QUADRATURE

Jaume Paradís, Josep Pla & Pelegrí Viader

Abstract. — The *Treatise on Quadrature* of Fermat (c. 1659), besides containing the first known proof of the computation of the area under a higher parabola, $\int x^{+m/n}dx$, or under a higher hyperbola, $\int x^{-m/n}dx$—with the appropriate limits of integration in each case—has a second part which was mostly unnoticed by Fermat's contemporaries. This second part of the *Treatise* is obscure and difficult to read. In it Fermat reduced the quadrature of a great number of algebraic curves in implicit form to the quadrature of known curves: the higher parabolas and hyperbolas of the first part of the paper. Others, he reduced to the quadrature of the circle. We shall see how the clever use of two procedures, quite novel at the time: the change of variables and a particular case of the formula of integration by parts, provide Fermat with the necessary tools to square—quite easily—as well-known curves as the folium of Descartes, the cissoid of Diocles or the witch of Agnesi.

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RÉSUMÉ (La méthode de Fermat pour les quadratures). — Le *Traité des quadratures* de Fermat (vers 1659), contient, outre la première démonstration connue du calcul de l'aire sous une parabole supérieure, $\int x^{+m/n} dx$, ou sous une hyperbole supérieure, $\int x^{-m/n} dx$—avec les limites d'intégration correspondants a chaque cas—, une seconde partie qui est passée presque inaperçue aux yeux de ses contemporains. Cette partie du *Traité* est obscure et difficile à lire. Fermat y réduit la quadrature d'un grand nombre de courbes algébriques données sous forme implicite à la quadrature connue de certaines courbes: les paraboles et hyperboles de la première partie de son article. D'autres quadratures sont obtenues par réduction à la quadrature du cercle. Nous verrons comment l'usage intelligent de deux procédés, assez nouveaux à l'èpoque, le changement de variables et un cas particulier de la formule d'intégration par parties, en fait un outil pour quarrer—assez facilement—des courbes aussi fameuses que le folium de Descartes, la cissoïde de Dioclès et la cubique (sorcière) d'Agnesi.

1. INTRODUCTION

One of the last papers of Fermat is devoted to the quadrature (in the sense of finding the area of a plane region enclosed by a curve and some other lines) of a wide family of algebraic curves, among which the best known and more widely treated by historians are the "higher parabolas", that is curves with equations of the form $y = x^{m/n}$, with m, n integers and $m/n > 0$ and "higher hyperbolas", with the same equation but with $m/n < 0, \neq -1$. This is done in the first part of Fermat's paper. The second part is concerned with the reduction of the quadrature of some curves to the quadrature of others. The paper was written around 1659 and has quite a lengthy title:

> On the transformation and alteration of local equations for the purpose of variously comparing curvilinear figures among themselves or to rectilinear figures, to which is attached the use of geometric proportions in squaring an infinite number of parabolas and hyperbolas. [Translation by Mahoney, Mahoney 1994, p. 245]. [1]

This long title, understandably enough, has been abridged to *Treatise on Quadratures* [*ibid*].

1 *De æquationum localium transmutatione et emendatione ad multimodam curvilineorum inter se vel cum rectilineis comparationem, cui annectitur proportionis geometricæ in quadrandis infinitis parabolis et hyperbolis usus* [Fermat c. 1659, p. 255].

It was published in 1679 as part of the collected works of Fermat edited by his son, Clément-Samuel [Fermat 1679, pp. 44–57].

Prior to the *Treatise*, Fermat had done some work on quadratures. He had tried, unsuccessfully, to square the cycloid [2] and some correspondence with Cavalieri and Torricelli proves that he had already been working on the problem of quadratures at the end of the 1630's and early 1640's, see [Mahoney 1994, p. 244].

A description of the contents of the *Treatise* can be found in two important works: Zeuthen's [1895] and Mahoney's [1994]. Mahoney's is a book published originally in 1973 with a second printing in 1994 and can be considered as the current obliged reference on Fermat's mathematical work.

In 1644, according to [Zeuthen 1895, pp. 41–45], Fermat was already in possession of the proof of the computation of the quadrature on $[0, b]$ of the parabolas with equation $b^m y^n = b^n x^m$, with m, n positive integers, and b a given constant. [3] This was precisely the year in which Fermat sent his results to Cavalieri via father Mersenne. The complete transcription of his work on quadrature into the *Treatise* must have taken place after 1657, most likely in 1659 [Mahoney 1994, pp. 244–245, 421], [Zeuthen 1895, p. 45]. In that same year he included the quadrature on $[b, \infty)$ of the higher hyperbolas $x^m y^n = b^{n+m}, \quad m > n,$ [4] using an appropriate partition of the coordinate axes with the help of geometrical progressions. The details can be found in [Mahoney 1994, pp. 245–254], [Bos et al. 1980; Boyer 1945; Katz 1993]. Respect this part of the *Treatise* we have nothing new to say. It has been thoroughly studied for its great importance within the history of integration since it goes apace with the research of other mathematicians of the 17$^{\text{th}}$ century as Pascal, Cavalieri, Torricelli, Wallis, Barrow, etc. who were working on the problem of the integration of x^n .

2 This is a problem that Wallis solves in his *Tractatus duo. Prior de cycloide [...]* (1659) using his method of "interpolation by analogy", see [Whiteside 1961, pp. 242-243]. The first to square the cycloid was Roberval in 1634 in his *Traité des indivisibles*, (first published in Paris 1693), see [Walker 1932].

3 Fermat multiplies each side of the equation by the constant b raised to the necessary power in order to maintain the homogeneity of dimensions. See later note 15.

4 With the exception $xy = b^2$.

Fermat, after having squared the higher parabolas and hyperbolas, tackles the second part of the *Treatise* and says: [5]

> It is remarkable how the theory just presented [the quadrature of the higher parabolas and hyperbolas] can help to advance the work on quadratures since it allows for the easy quadrature of an infinity of curves which no geometer, neither ancient nor modern, has thought of; this will be summarized in some brief rules. [6]

It is precisely about this second part that we think our article adds something to the existing literature. The excellent contributions of Zeuthen [1895] and Mahoney [1994], despite highlighting the importance of the *Treatise*, cannot devote much space to it since their aim is much broader. Therefore their treatment of the second part of the *Treatise* is rather descriptive and does not unravel the logical thread that conducts all the examples presented by Fermat. Zeuthen describes quite accurately the method and each of the individual examples but he neither delves into the method's more delicate aspects nor considers the examples as a whole. Moreover, he does not pay any attention to the question of the limits of integration, which are almost completely disregarded by Fermat. As far as the method and its details are concerned, Mahoney is more thorough but he does not look into all the examples.

In section 2 we have a look at the mathematical context in which Fermat's method was immersed. As we will see, the method, for different reasons, was completely unnoticed. In section 3 we undertake the revision of the basis of Fermat's method which consists of his proof of the linear character of the squaring of sums of parabolas and hyperbolas. Section 4 is devoted to the two instruments of Fermat's method: a particular instance of what we call today the formula of integration by parts (we call this result the *General Theorem*) and the *change of variables*. Sections 5 to 8 are devoted to the core of Fermat's paper: the first quadratures beyond the higher parabolas and hyperbolas (section 5); the quadrature of the folium

5 All translations into English of Fermat's quotations are the authors'.

6 *Ex supradictis mirum quantam opus tetragonismicum consequator accessionem: infinitæ enim exinde figuræ, curvis contentæ de quibus nihil adhuc nec veteribus nec novis geometris in mentem venit, facillimam sortiuntur quadraturam; quod in quasdam regulas breviter contrahemus* [Fermat c. 1659, pp. 266-267].

of Descartes as the first important example of the power of his method (section 6); the quadrature of the witch of Agnesi and the cissoid of Diocles (section 7). In appendix A we reconstruct this last quadrature since Fermat only mentions in passing that it can be carried out in a similar way to the quadrature of Agnesi's curve. This reconstruction is important because the quadrature of the cissoid reduces to the quadrature of an odd power of the ordinates of a circle, that is to say, the area under the graph of a function like $y = \sqrt[m]{b^2 - x^2}$, (odd m). This is precisely what is done in section 8. Section 9, finally, studies the last example presented by Fermat, a rather involved quadrature that requires several iterations of his method. Lastly, some concluding remarks are offered in section 10 followed by four mathematical appendixes.

2. MATHEMATICAL CONTEXT

Henk Bos, [1989] or [1993, p. 1], talking about *recognition* and *wonder* as the "ingredients" that make history an interesting subject of study said:

> The unexpected, the essentially different nature of occurrences in the past excites the interest and raises the expectation that something can be discovered and learned.

For us, the second part of Fermat's *Treatise* has a much greater interest than the first since, using Bos' description, it awakes in us a feeling of *wonder*. In Mahoney's words:

> In this second part, Fermat grouped together all his mathematical forces—his analytic geometry, his method of maxima and minima, his method of tangents, and his direct quadrature of the higher parabolas and hyperbolas—to construct a brilliant "reduction analysis" for the quadrature of curves [Mahoney 1994, p. 254].

These words reflect the importance of this second part where Fermat develops a true "method" in order to reduce the quadrature of a wide class of algebraic curves to known quadratures among the higher parabolas and hyperbolas as well as the reduction of the quadrature of other curves to the quadrature of a circle. These procedures constitute one of the most interesting lines of research of Fermat's and their success can be attested by the quadrature of some well-known curves: the folium of Descartes, the witch of Agnesi (the *versiera* or *versaria*), and the cissoid of Diocles. [7] Both Zeuthen and Mahoney coincide in pointing out that the words *transmutation* and *alteration* in the title of the *Treatise* clearly show two things:

1) that the *Treatise*'s goal was much more ambitious than the simple quadrature of parabolas and hyperbolas of its first part, and

2) that Fermat was imitating Viète who in his *De Æquationum recognitione et emendatione* (published posthumously in 1615) had studied the solubility of algebraic equations with the help of their "transmutation" and "alteration". Fermat wanted to do the same with the algebraic equations of curves to determine their "quadrability".

Notwithstanding these two points in mind, a first reading of the second part of the *Treatise* leaves the reader with the impression that Fermat treats the quadrature of a few particular curves in a disconnected and confused way. Hence, the hasty reader tends to disregard this part of the paper as a simple speculation without great actual importance. This is probably what happened to Huygens when he read the *Treatise* the year it was published, 1679. In a letter to Leibniz he says talking of the *Treatise*:

> [...] J'ai recherché la dessus ce que me souvenois d'avoir vu dans les œuvres posthumes de Mr. Fermat [*Varia Opera*], mais ce Traité est imprimé avec tant de fautes, et de plus si obscur, et avec des demonstrations suspectes d'erreur que je n'en ai pas scu profiter. [Letter of Huygens to Leibniz, 1 September 1691, Huygens 1905, p. 132].

Whether the difficulties of Huygens had to do with the first or the second part of the paper, is not clear as he does not mention it. [8]

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7 The quadrature of some of these curves is by no means trivial even with our modern integration techniques, see [Paradís et al. 2004].

8 As one of the referees pointed out, the Latin original of the *Treatise* contains also quite a number of mistakes and errors. Thus, Huygens, after reading a few pages could already have been deterred from continuing. However, it is worth noting that the quoted letter addresses the problem of the squaring of the catenaria. In the *Acta eruditorum* of May and June 1691, both Leibniz and Johann Bernoulli had published the quadrature of the catenaria alongside with other results concerning this curve. Huygens confesses that, by his own means, he had been unable to reach these results and he concludes that it must be *"votre nouvelle façon de calculer, qui vous offre, à ce qu'il semble, des veritez, que vous n'avez pas même cherchées"*, [Huygens 1905, p. 129]. Huygens says he had been able to reduce the construction of the catenaria to the quadrature of

In our opinion the *Treatise*, and specifically its second part, is not the confused piece of work Huygens suggests. We hope to show that the whole of the *Treatise* possesses great depth of thought and the presentation, once understood, shows the great internal coherence of the ideas considered (see the quotation on page 8). It is worth mentioning that the *Treatise* passed unnoticed by Fermat's contemporaries,[9] possibly because it was not divulged before its publication. Fermat probably wrote it in response to Wallis's *Arithmetica infinitorum* of 1656 but there is no mention of it in any of Wallis's later papers or correspondence, see [Mahoney 1994, p. 244].

Later, in the 1680's, when the *Treatise* was published, the interest aroused in the scientific community was also very little. Without a deeper analysis it is difficult to account for this lack of interest but we dare point out a few—rather obvious—reasons why this was so. First, the scientific focus was placed on the calculus of Newton and Leibniz, which was flowering with great force at the time[10] and whose emphasis was on the relation between area and derivative.[11] Fermat's methods, algebraic and geometric, based on the comparison of the quadrature of two algebraic curves related by a change of variable, were far from the trend of thought of this

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the curve of equation $x^2y^2+a^2y^2 = a^4$ and he was now able to see, thanks to the papers of Leibniz and Bernoulli, that the quadrature of this last curve reduced to that of the common hyperbola but he could not work out how that could be done. This makes us think that Huygens had read enough of the *Treatise* to have reached the second part and have been baffled by it.

9 At the end of his note [Aubry 1912], Aubry remarks that Fermat's "ingenious procedure of variable substitution as well as his concern to avoid radicals, both in the tracing of tangents and in the quadratures, have had a certain influence on Leibniz and on the updating of his *Nova methodus*" [our translation], but he does not substantiate his assertion and we have not been able to find any evidence of it. Despite the existence of the *Treatise* and the extant correspondence Fermat–Cavalieri on integration, Andersen [1985] says: "Fermat never disclosed his ideas about the foundation of arithmetical integration". She does not mention any influence between both mathematicians' ideas on integration. We are of the opinion that the question deserves a little more attention but this is not the place to study it.

10 It is interesting to keep in mind the date of the first—unpublished—paper of Newton, "The October 1666 Tract on Fluxions", see [Edwards 1979, pp. 191 and ff.].

11 This relationship had been established by Barrow in his *Lectiones Geometricæ*, London 1670. Before that, perhaps in 1645 (see [Itard & Dedron 1959]), Roberval had had an intuition of the same result.

new calculus. Second, Fermat's style is very laconic and, as he was wont to, did not devote much effort or space in making his ideas more comprehensible for the reader.[12] Even in the cases he solves he does not bother in making the calculations explicit and he limits himself to a mere description of how to reach the desired quadrature. Third, Fermat's method of quadrature could only be used on a very particular class of curves with a known algebraic equation and did not apply to the more frequent (and trendy) curves of the time: the quadratrix, cycloid, spiral of Archimedes, etc. Also, Fermat, contrarily to the rest of authors of the time, did not even refer to infinitesimal quantities, the germ of the new calculus. Finally, the technique of the change of variables, with all certainty, was something new, difficult to grasp and, consequently, suspicious of leading to errors.

Before Fermat and Descartes, curves had no equation. They were described by their geometric properties. Consequently the "typical" problems of the tracing of tangents, the obtention of maxima and minima and the quadrature of regions enclosed by curves were carried out using the specific properties of each curve and the language of these calculations was the language of proportions.

After Fermat and Descartes—and we could even include Viète in the same lot—algebraic language is introduced and curves can be studied through their equations.[13] This allows a more methodic treatment of the problems just mentioned and allows a certain classification of curves: algebraic and non-algebraic. Among the algebraic ones, the easiest to treat are, obviously, those expressible as a sum of monomials ax^n. Fermat, though, in the *Treatise*, develops a method of his own to tackle the quadrature of some of the algebraic curves whose equations are more involved, that is to say, with sumands of the form $ax^n y^m$. This is precisely an *implicit* polynomial equation. The treatment of curves with an implicit polynomial equation was carefully avoided by the 17$^{\text{th}}$ century mathematicians.[14] In

12 This is typical of Fermat's writings. See [Mahoney 1994, p. 25].

13 But even after analytic geometry was introduced, it took some time before the equation of a curve was considered enough to "know" it: the construction of the points of the curve required an accurate geometrical procedure to satisfy the feeling of real knowledge of it, see [Bos 1987].

14 There are a few exceptions as in the case of the Fermat-Descartes controversy about the tangents to the folium.

the problem of the tangents, the first geometer that studies the question in the case of a curve with an implicit polynomial equation is Sluse generalizing a method of Hudde for the explicit case, see [Katz 1993, pp. 433–434] and, for more details, [Rosenfeld 1928].

This was in perfect tune of the worries at the 1650's (see [Baron 1987, pp. 122-135; 151-156; 205-213]) but as Fermat did not publish his results in this field, as we have already mentioned, the *Treatise* passed unnoticed.

3. FERMAT'S APPROACH

In the second part of the *Treatise*, after having—in the first—shown how to square higher parabolas and hyperbolas, Fermat begins by stating that the quadrature of a curve whose equation is the addition or subtraction of different expressions can be squared by the addition or subtraction of the quadrature of each separate summand.

Let us consider a curve whose property leads to the following equation [15]

$$b^2 - a^2 = e^2.$$

(It is seen at once that this curve is a circle.)

We can reduce the power of the unknown e^2 to a root through a division (application [16] or parabolism). We can indeed write $e^2 = bu$, as we are free to equate the product of the unknown u and the constant b to the square of the

15 Fermat used, following Viète, vowels for the unknowns. He uses A and E to stand for our usual x and y. In their French translation [Tannery & Henry 1891–1912], Tannery and Henry used a and e instead. They also use other lowercase vowels: $\bar{o}$ instead of Y, i instead of I, and ω instead of O. We will follow their notation in our translations into English but in our comments and appendixes we will use x and y in the present meaning of the abscissa axis and ordinate axis respectively. This is done in order to make the paper more readable. However we will keep the dimensional homogeneity that Fermat maintains in all his equations: all the monomials of an algebraic expression must have the same degree in order to be added or subtracted. This is essentially Viète's Homogeneity Law, [Viète 1646, chap. 3, pp. 2–4]. Thus, we will normally use b—raised to the necessary powers—as a constant that will help us to abide by Viète's law. Fermat follows Viète very closely on this point, not only as a formal requirement but also, as we will see, as a tool that will help him in his calculations. To know more about the Law of Homogeneity see [Freguglia 1999].

16 On the sense of the term *application* see later note 24.

unknown e. We will then have

$$b^2 - a^2 = bu.$$

But the term bu can decompose in as many terms as those present in the other side of the equation, affecting each one of these terms of the same signs as the corresponding terms of the other side. Let us then write

$$bu = bi - b\bar{o},$$

always representing, following Viète, the unknowns by vowels. We will have

$$b^2 - a^2 = bi - b\bar{o}.$$

Let us equate each one of the terms of one side to the corresponding one in the other side. We will obtain

$$b^2 = bi \quad \text{from which} \quad i = b \quad \text{will be given,}$$
$$-a^2 = -b\bar{o} \quad \text{or} \quad a^2 = b\bar{o}.$$

The extremity of the line $\bar{o}$ will be on a primary parabola. Thus, in this case, everything can be reduced to a square. If we order [17] all the e^2 on a given straight line, [18] they become [19] a rectilinear solid, given and known. [20]

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17 Tannery and Henry translate *order* instead of *apply*. We will stick to this translation. Later they use the expression *ordered sum*. See notes 19 and 24.

18 The interval on which we sum.

19 Tannery and Henry use the word *sum* to denote the result of the operation. Later, Fermat uses the word *aggregatum* and further on, the word *sum*.

20 *Sit curva cujus proprietas det æquationem sequentem:*

$$\text{B}q. - \text{A}q. \ \textit{æquale}\ \text{E}q.$$

(apparet autem statim hanc curvam esse circulum); certum est potestatem ignotam, E*q.*, *posse reduci, per applicationem seu parabolismum, ad latus.*
Possumus enim supponere

$$\text{E}q. \ \textit{æquari}\ \text{B}\ \textit{in}\ \text{U},$$

quum sit liberum quantitatem ignotam U, *in notam* B *ductam, æquare quadrato* E *etiam ingnotæ.*
Hoc posito,

$$\text{B}q. - \text{A}q. \ \textit{æquabitur}\ \text{B}\ \textit{in}\ \text{U};$$

homogeneum autem B *in* U *ex tot quantitatibus homogeneis componi potest quot sunt in parte æquationis correlativâ; iisdemque signis hujusmodi homogenea debent notari. Supponatur igitur*

$$\text{B}\ \textit{in}\ \text{U}\ \textit{æquari}\ \text{B}\ \textit{in}\ \text{I} - \text{B}\ \textit{in}\ \text{Y};$$

ex more enim Vietæo, vocales semper pro quantitatibus ignotis sumimus; ergo

$$\text{B}q. - \text{A}q. \ \textit{æquatur}\ \text{B}\ \textit{in}\ \text{I} - \text{B}\ \textit{in}\ \text{Y}.$$

Fermat's next example performs the same decomposition to the curve with equation $x^3 + bx^2 = y^3$. With these two examples, Fermat has just told us that the *sum* of all the powers of an ordinate y^m, when $y^m = \sum a_i x^i$, can be carried out *summing* each one of the parabolas of the right hand side,

> If there are several terms in the equation, each formed with different powers of one or the other unknowns, they will generally be treated with the same method by legitimate reductions.[21]

The same can be made when this right hand side is made of hyperbolas or a sum of parabolas and hyperbolas,

> [...] but we obtain not less quadratures by dyeresis,[22] with the help of hyperbolas, either on their own or in combination with parabolas.[23]

He then presents two examples, one that combines parabolas and hyperbolas,

$$y^2 = \frac{b^6 + b^5 x + x^6}{x^4}, \tag{1}$$

and the other using only hyperbolas,

$$y^3 = \frac{b^5 x - x^6}{x^3}. \tag{2}$$

Æquentur singula membra partis unius singulis membris partis alterius: sit nempe

B*q. æquale* B *in* I;

ergo dabitur

I *æqualis* B.

Æquatur deinde

−A*q.*, −B *in* Y,

hoc est

A*q.*, B *in* Y;

erit extremum punctum rectæ Y *ad parabolem primariam. Omnia igitur in hoc casu ad quadratum reduci possunt, ideoque, si omnia* E quadrata *ad rectam lineam datam applices, fiet solidum rectilineum datum et cognitum* [Fermat c. 1659, p. 267-268].

21 *Si sint plura in æquationibus membra, imo et sub plerisque utriusque quantitatis ignotæ gradibus involuta, ad eamdem ut plurimum methodum, reductionum legitimarum ope, poterunt aptari* [Fermat c. 1659, p. 268].

22 The quadrature by means of parabolas is called *syneresis*.

23 [...] *sed non minus quadrationum ferax est opus per diæresim, quod per hyperbolas, aut solas aut parabolis mixtas, commode parites expeditur* [Fermat c. 1659, p. 269].

The technique is the same in all cases. If we are interested in calculating the *ordered sum*[24] of the y^m, we "linearize" y^m through the term $b^{m-1}u$, that is to say, we effect the change of variable $y^m = b^{m-1}u$, and then introduce as many new variables as necessary. For example in the case of the curve with equation (1), the new variables are $\bar{o}, i$ and ω:

$$y^2 = bu = b\bar{o} + bi + b\omega,$$

which equated term to term with the right-hand side of (1) provide three new curves: two hyperbolas,

$$b^5 = x^4\bar{o}, \quad b^4 = x^3 i,$$

and one parabola

$$x^2 = b\omega.$$

The *ordered sum* of the y^2 of the original curve can be found through the "ordered sums" of the ordinates of the variables $\bar{o}, i$ and ω referred to the quadrable hyperbolas and parabola, i.e. their quadrature. It is important to notice that Fermat makes no comment about the limits of summation of all those expressions. When squaring a parabola, he takes as *base* the interval $[0, b]$ and when squaring a hyperbola, in order to compute the area between the hyperbola and its asymptote (the axis) he takes as *base* the interval $[b, \infty]$. In this sense, the example we have presented is a little confusing as the presence of both curves in the same quadrature will certainly present

24 *Ordered sum* are the words Tannery and Henry use to translate the "application in order" of Fermat. We must recall that Fermat sees the problem of the quadrature of a curve as a purely geometrical problem that can be solved with the help of algebra. Hence the geometrical language he uses and his geometrical way of thinking. The idea is to *sum the ordinates* thinking of the area to calculate as the result of putting together all the ordinates that correspond to the curve in question. In that sense, he reminds us of Cavalieri, with whom he often used to interchange letters, and thus it is not strange that their language resembles. The actual influence of Cavalieri's method of indivisibles on Fermat has not been thoroughly studied. Neither Giusti [1980] nor Andersen [1985, p. 358], mention any evidence of any influence. One should point out, though, that their *summing* methods were entirely different. Fermat, when he speaks of *summing ordinates ordered on a given base,* understands the sum of an infinity of small rectangles—as the first part of the *Treatise* has clearly shown—, whereas Cavalieri's sums of ordinates are a more ambiguous geometrical idea. Mahoney [1994, pp. 255–256] translates *sum of all the ordinates on a given line* as *apply all the ordinates to a given line* and discusses at length the use of the word "application" in this new context. We refer the interested reader to this work.

problems with the limits of summation. Fermat completely ignores this objection but later, when he applies the method, he never mixes parabolas and hyperbolas.[25]

4. THE INSTRUMENTS OF FERMAT'S METHOD

After the result of the previous section, which can be described as the "linearity of the summing operation" (the quadrature of a sum is the sum of the quadratures of the summands), Fermat turns to the first essential element in his method of quadratures. We will call it the *General Theorem*:

> Let $ABDN$ be any curve [see Fig. 1] with base HN and diameter HA. Let CB, FD be the ordinates on the diameter and BG, DE the ordinates on the base. Let

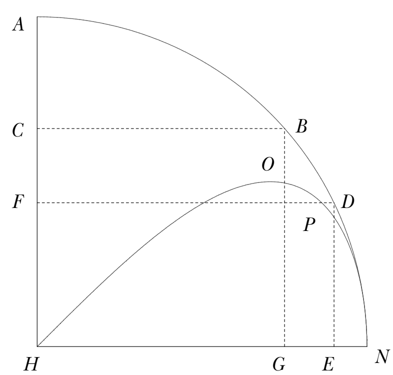

FIGURE 1.

> us assume that the ordinates decrease constantly from the base to the summit, as shown in Fig. 1; that is to say $HN > FD$; $FD > CB$ and so on.
>
> The figure formed by the squares of HN, FD, CB, ordered on the line AH,[26] that is to say, the solid
>
> $$CB^2 \times CA + \cdots + FD^2 \times FC + \cdots + NH^2 \times HF + \cdots$$

25 Tannery and Henry in their Latin version of the *Treatise* insert a footnote in this sense, see [Fermat c. 1659, p. 268].

26 When Fermat says, "on the line AH" he means the summation on the interval $[A, H]$.

is always equal to the figure formed by the rectangles $BG \times GH$, $DE \times EH$, doubled and ordered on the base HN,[27] that is to say, the solid

$$2BG \cdot GH \cdot GH + \cdots + 2DE \cdot EH \cdot EG + \cdots$$

assuming both series of terms unlimited. As for the other powers of the ordinates, the reduction of the terms on the diameter to the terms on the base is carried out with the same ease; and this observation leads to the quadrature of an infinity of curves unknown till today.[28]

If $AH = b$ and $HN = d$, Fermat's result, in modern notation would amount to

$$\int_0^b x^2\,dy = 2\int_0^d xy\,dx.$$

He also states the result for the case of the sum of the cubes, x^3, and the bi-squares, x^4

$$\int_0^b x^3\,dy = 3\int_0^d x^2 y\,dx; \qquad \int_0^b x^4\,dy = 4\int_0^d x^3 y\,dx.$$

As the reader can see at once, the *General Theorem* consists of a geometrical result equivalent in modern language to the following equation, nothing but an application of the formula of integration by parts (we regain the usual role of x and y):

$$\int_0^d y^n\,dx = n\int_0^b y^{n-1} x\,dy$$

---

27 Fermat uses the expression "on the base" or "on the diameter" to indicate, first, the axis on which the infinite partition has to be considered, that is, the modern dx and dy; second, it is his reference to the interval on which to carry the summation.

28 *Sit in quarta figura curva quælibet* ABDN*, cujus basis* HN*, diameter* HA*, applicatæ ad diametrum* CB*,* FD*, et applicatæ ad basim* BG, DE*; et decrescant semper applicatæ a base ad verticem, ut hic* HN *est major* FD *et* FD *major est* CB *et sic semper.*
Figura composita ex quadratis HN, FD, CB*, ad rectam* AH *applicatis (hoc est solidum sub* CB *quadrato in* CA *et sub* FD *quadrato in* FC *et sub* NH *quadrato in* HF*) æqualis est semper figuræ sub rectangulis* BG *in* GH*,* DE *in* EH*, bis sumptis et ad basim* HN *applicatis (hoc est solido sub* BG *in* GH *bis in* GH *et sub* DE *in* EH *bis in* EG*) etc. utrimque in infinitum.*
In reliquis autem in infinitum potestatibus, eadem facilitate fit reductio homogeneorum ad diametrum ad homogenea ad basim. Quæ observatio curvarum infinitarum hactenus ignotarum detegit quadrationem [Fermat c. 1659, pp. 271-272].

where $y(x)$ represents any curve decreasing from the value b to the value 0 as shown in Fig. 2.[29] Fermat, without stating it, will use the theorem even if the value 0 is reached at infinity, as is the case of Fig. 3 for which

$$\int_0^\infty x^n\,dy = n\int_0^d x^{n-1}y\,dx.$$

Fermat's *General Theorem* is stated without proof. In the case $n = 2$, as can

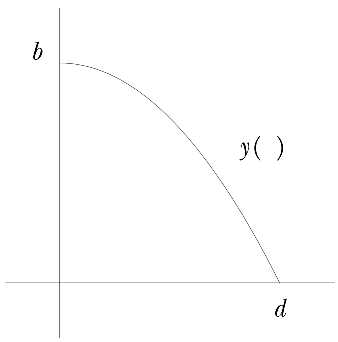

FIGURE 2.

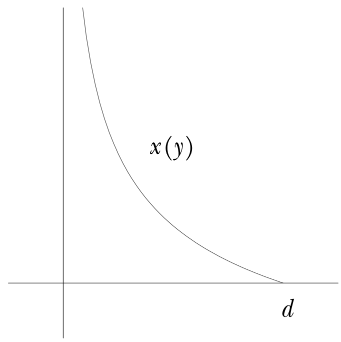

FIGURE 3.

be judged from the quotation above, a sort of three-dimensional argument is used which gives a hint for a possible proof, see [Zeuthen 1895, p. 51]. The cases $n = 3$ and $n = 4$ are merely worded without more ado.

A geometrical proof of the theorem can be found in a work by Pascal published in 1659, *Traité des trilignes rectangles et de leurs onglets*, see [Flad 1963, pp. 142–143] or [Struik 1986, pp. 241–244]. Pascal's result is more general but the proof he offers is consistent with Fermat's geometrical arguments. It is likely, then, that Fermat was familiar with Pascal's theorem and its proof through some correspondence exchanged in 1659 [Tannery & Henry 1891–1912, Fermat to Carcavi, 16 February 1659].

With the *General Theorem*, Fermat is already in possession of one the keys to his method. In his own words:

29 We will use the modern integral notation to indicate Fermat's *ordered sums*. Thus, the sum of y^2 on the base x will be denoted by $\int y^2 dx$. We are conscious of the dangers of misinterpretation that this notation has, but the advantages it offers for the modern reader surpass in our eyes this inconvenience.

From here, as we will see, there will derive an infinite number of quadratures.[30]

The second instrument he needs is the transformation of equations[31] with the help from the technique of the change of variables.

The first example he offers begins with the equation of a circle.

> [...] let, for instance, $b^2 - a^2 = e^2$ be the equation that constitutes the curve (which will be a circle). According to the general theorem above, the sum of the e^2 ordered on the line b [the diameter] equals the sum of the products $HG \cdot GB$ [see Fig. 1] doubled and ordered on the line HN or d [the base]; but the sum of the e^2, ordered on b equals, as has been proven above, a given rectilinear area. Consequently, the sum of the products $HG \cdot GB$, doubled and ordered on the base d constitute a given rectilinear area. If we half it, the sum of the products $HG \cdot GB$, ordered on the base d will also constitute a given rectilinear area.[32]

Fermat applies his *General Theorem* (to $x(y)$) and obtains the result (in the case of his example, $b^2 - y^2 = x^2$)

$$\int_0^d xy\,dx = \frac{1}{2}\int_0^b x^2\,dy.$$

In this particular case $d = b$.[33] But none of these two "summations" corresponds, from Fermat's point of view, to a proper *ordered sum of ordinates applied to a segment.* For this reason he needs to "linearize" the product xy in order to have a properly quadrable (and new) curve.

---

30 *Inde emanant infinitæ, ut statim patebit, quadraturæ* [Fermat c. 1659, p. 272].

31 Notice that the main emphasis in the title of the *Treatise* is on the *transformation and alteration* of equations. See the introduction, page 10.

32 [...] *et sit, verbi gratia, æquatio curvæ constitutiva*

B*q*. – A*q* *æquale* E*q*.,

quod in circulo ita se habet.

Quum ergo, ex prædicto theoremate universali, omnia E quadrata *ad rectam* B *applicata sint æqualia omnibus productis ex* HG *in* GB ⟨ *bis sumptis et* ⟩ *ad basim* HN *sive ad* D *applicatis; sint autem omnia* E quadrata, *ad* B *applicata, æqualia spatio rectilineo dato, ut superius probatum est: ergo omnia producta ex* HG *in* GB, *bis sumpta et ad basim* D *applicata, continent spatium rectilineum datum. Ergo, sumendo dimidium, omnia producta ex* HG *in* GB *ad basim* D *applicata erunt æqualia spatio rectilineo dato* [Fermat c. 1659, pp. 272–273].

33 We must keep in mind that Fermat always uses examples to present theoretical results. Thus, while the example he offers refers to the circle, he speaks as if the curve were the curve *ABDN* of the *General Theorem.*

This is the purpose of the second essential element of his method: the change of variables.

In order to pass easily and without the burden of radicals [34] from the first curve to the new one, we have to employ an artifice which is always the same and which is the essence of our method.

Let $HE \cdot ED$ [see Fig. 1] be any of the products we have to order on the base. In the same way that we call analytically e the ordinate FD or its parallel HE and we call a the coordinate FH or its parallel DE, we will call ea the product $HE \cdot ED$. Let us equate this product ea, formed by two lines unknown and undetermined, to bu, that is to say, the product of the given b by an unknown u and let us suppose that u equals EP taken on the same line than DE. We will have

$$\frac{bu}{e} = a.$$

But according to the specific property of the first curve, [35] $b^2 - a^2 = e^2$. Replacing a by its new value bu/e we will have $b^2e^2 - b^2u^2 = e^4$ or, transposing,

$$b^2e^2 - e^4 = b^2u^2,$$ [36]

equation that constitutes the new curve $HOPN$ [Fig. 1], derived from the first. For this curve it is proven that the sum of the bu ordered on b is given. Dividing by b, the sum [37] of the u ordered on the base, that is to say, the surface $HOPN$ will be given as a rectilinear area and we will consequently obtain its quadrature. [38]

34 This is one of the important consequences of Fermat's method: the sum of radical powers of ordinates avoiding the use of radicals. In this first example, Fermat wants to calculate

$$\int_0^b x\sqrt{b^2 - x^2}\,dx.$$

The change of variable will "linearize" xy and convert it to bu, where u will be the ordinate of a new curve. See [Zeuthen 1895, p. 55].

35 That is to say, its analytic equation.

36 Notice that $bu = x\sqrt{b^2 - x^2}$.

37 Fermat uses the word *sum* for the first time.

38 *Ut autem facillima et nullis asymmetriis involuta fiat translatio prioris curvæ ad novam, ita constanti artificio, quæ est nostra methodus, operari debemus.*
Sit quodlibet ex productis ad basim applicandis, HE *in* ED. *Quum igitur* FD *sive* HE, *ipsi parallela, vocetur in analysi* E, *et* FH *sive* DE, *ipsi parallela, vocetur* A, *ergo productum sub* HE *in* ED *vocabitur* E *in* A. *Ponatur illud productum* E *in* A, *quod sub duabus ignotis et indefinitis rectis comprehenditur, æquari* B *in* U, *sive producto ex* B *data in* U *ignotam, et intelligatur* EP, *in directum ipsi* DE *posita, æquari* U. *Ergo*

$$\frac{\text{B } \textit{in} \text{ U}}{\text{E}} \textit{ æquabitur } \text{A}.$$

Fermat effects the change of variable $y = bu/x$ [39] and the circle transforms into a new curve in the xu-plane (see Fig. 4). As the change amounts

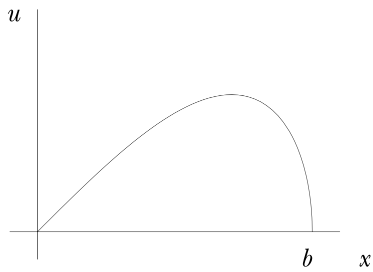

FIGURE 4. The new curve, $b^2x^2 - x^4 = b^2u^2$.

to $xy = bu$, the new curve is quadrable since the new ordinates u can be summed when ordered on the line b:

$$\int_0^b u\,dx = \frac{1}{b}\int_0^b xy\,dx = \frac{1}{2b}\int_0^b x^2\,dy,$$

and as the sum of the x^2 can be obtained squaring two parabolas,

$$\int_0^b x^2\,dy = \int_0^b (b^2 - y^2)\,dy = \frac{2b^3}{3},$$

we have

$$\int_0^b u\,dx = \frac{b^2}{3}.$$

Quum autem B*q.*—A*q. æquatur, ex proprietate specifica prioris curvæ, ipsi* E*q.*, *ergo subrogando, in locum* A, *ipsius novum valorem*

$$\frac{\mathrm{B}\ in\ \mathrm{U}}{\mathrm{E}},$$

fiet

$$\mathrm{B}q.\ in\ \mathrm{E}q. - \mathrm{B}q.\ in\ \mathrm{U}q.\ æquale\ \mathrm{E}qq.,$$

sive, per antithesim,

$$\mathrm{B}q.\ in\ \mathrm{E}q. - \mathrm{E}qq.\ æquale\ \mathrm{B}q.\ in\ \mathrm{U}q.,$$

quæ est æquatio novæ HOPN *curvæ ex priore oriundæ constitutiva, in qua, quum omnia producta ex* B *in* U *dentur, ut jam probatum est, si omnia ad* B *applicentur, dabitur summa omnium* U *ad basim applicatarum, hoc est, dabitur planum* HOPN ⟨ *in* ⟩ *rectilineis, ideoque ipsius quadratura* [Fermat c. 1659, p. 273].

39 Notice the homogeneity of the dimensions. The constant b is introduced not only to keep the dimensions right but also to keep the “limit” of summation under control. Notice that the point $(b, 0)$ in the xy-plane becomes $(b, 0)$ in the xu-plane.

Fermat's method in this first example consists essentially of the following. We start from an algebraic equation $y^n = \sum a_i x^i + \sum b_j / x^j$ and, consequently we know how to calculate $\int_0^b y^n\,dx$. We then proceed in two steps:

1) Apply the *General Theorem*: $\int_0^b y^n\,dx = n\int_0^d y^{n-1}x\,dy$;

2) Linearize the integrand through an appropriate change of variable: $y^{n-1}x = b^{n-1}u$, where u is the ordinate of a new—and *a fortiori*, quadrable—curve.

In Fermat's account it is worth mentioning the absolute lack of references to the region which is actually squared in each curve. It goes without saying that if the curve draws a closed region this is precisely the area to be squared. If the curve has an asymptote, the region to be squared is the one enclosed by the curve, the asymptote (always an axis) and an appropriate ordinate which is almost self-evident. [40]

5. A MORE DIFFICULT EXAMPLE

The next example is a bit more elaborate as it involves a curve with an implicit algebraic equation. The starting curve is the cubic

$$y^3 = bx^2 - x^3. \tag{3}$$

The new curve, however, ends up being an algebraic curve of degree 9 in the variable y and degree 3 in the new variable u. The squaring of this last curve is a challenge even if one is equipped with all the artillery our calculus provides us with.

Fermat offers this example not only as a second instance of his method but also to exemplify a situation which needs an improved version of the *General Theorem*.

Fermat reminds us that the *sum of all* y^3 on the interval $[0, b]$ is immediately obtained as a sum of two quadrable parabolas (see Fig. 5),

$$\int_0^b y^3\,dx = \int_0^b (bx^2 - x^3)\,dx = \frac{b^4}{12}.$$

On the other hand, the *General Theorem* says that

40 Fermat only considers positive values of the variables and consequently, his quadratures limit themselves to the first quadrant.

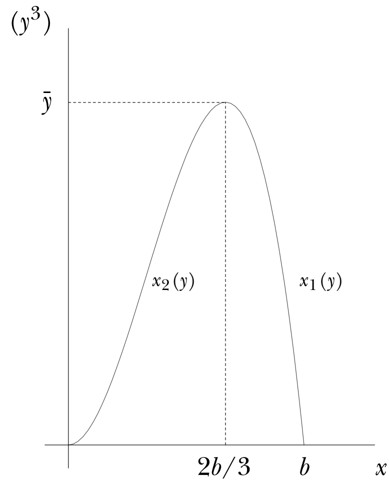

FIGURE 5. $f(x) = bx^2 - x^3$.

$$\int_0^b y^3\,dx = 3\int_0^{\bar{y}} y^2 x\,dy. \tag{4}$$

Fermat now makes the change of variable that provides the linearization part of the method,

$$x = \frac{b^2 u}{y^2}$$

which takes curve (3) into the curve with equation (see Fig. 6):

$$b^5 u^2 y^2 - y^9 = b^6 u^3. \tag{5}$$

The change effected on the integral of the right-hand side of (4) is

$$\int_0^{\bar{y}} y^2 x\,dy = b^2 \int_0^{\bar{y}} u\,dy.$$

Notice the upper limit of integration in the new integral: $\bar{y}$. If you look at Fig. 6, you will clearly see that the "sum of all the u" has to be "ordered on the line $\bar{y}$". But actually, the value of $\bar{y}$ is irrelevant for Fermat's purposes as the quadrature he is interested in is represented exactly by that last integral which will be calculated going backwards in the chain of integrals obtained so far:

$$\int_0^{\bar{y}} u\,dy = \frac{1}{b^2}\int_0^{\bar{y}} y^2 x\,dy = \frac{1}{3b^2}\int_0^b y^3\,dx = \frac{b^2}{36}.$$

Thus, the quadrature of the new curve (5) is $b^2/36$.

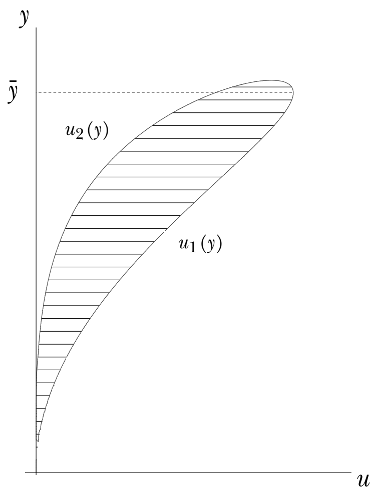

FIGURE 6. The curve $b^5u^2y^2 - y^9 = b^6u^3$.

As we mentioned before, in this example, some comments are really necessary to fully understand Fermat's technique.

The initial curve, $y^3 = bx^2 - x^3$, is not decreasing, a necessary condition for the *General Theorem* to hold. In fact, seen as a function $y(x)$ it increases on the interval $[0, 2b/3]$, and decreases from there until reaching the value 0 for $x = b$. The highest value it attains is $\bar{y} = \sqrt[3]{4b^2}/3$. In terms of y^3, this maximum is, obviously, $\bar{y}^3 = 4b^2/27$.

This means that when y varies between 0 and $\bar{y}$, for each value of the variable y, two values are obtained for the variable x. Let us denote each of these values by x_1 and x_2, as shown in Fig. 5. We can think of $x_1(y)$ and $x_2(y)$ as two different functions. The same considerations have to be made about the new curve, see Fig. 6. Again, for a given y, two values of u have to be considered, u_1 and u_2.

Thus, to be rigorous, Fermat's procedure should be rewritten as follows:

$$\int_0^b y^3\,dx = 3\int_0^{\bar{y}} y^2(x_1 - x_2)\,dy = 3b^2\int_0^{\bar{y}} (u_1 - u_2)\,dy,$$

this last integral representing the lined area in Fig. 6.

Fermat is conscious that this example is not exactly covered in his *General Theorem* and proceeds to offer a new version when the curve is not

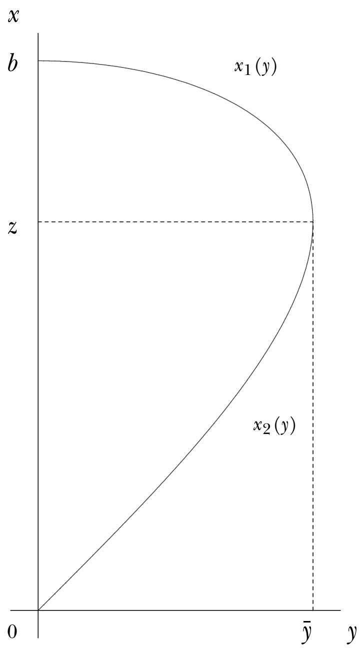

FIGURE 7a.

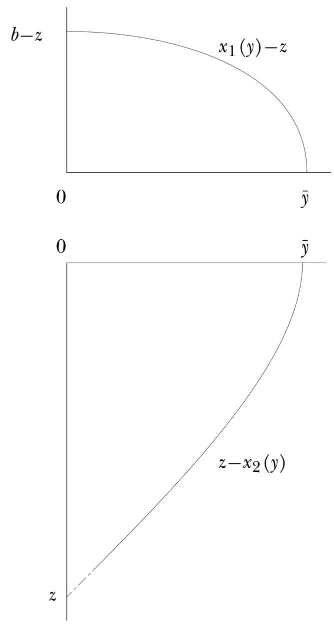

FIGURE 7b AND 7c.

decreasing. His explanation amounts to saying that if a curve as the one shown in Fig. 7a is given, the *General Theorem* can be applied first to the decreasing portion of the curve, $x_2(y)$ from $x = 0$ to $x = z$ where the maximum is reached. A different procedure, though, has to be used for the increasing portion, the one we have called $x_1(y)$ that increases from $x = b$ to $x = z$. Essentially what Fermat does is change the axes in such way as to have the increasing portion as a decreasing curve. Consider $x = z$ as the new "base". On the one hand we have the curve $z - x_2$ which is decreasing from the new base to $x = z$ (we have to think of positive x downwards), see Fig. 7b; and on the other hand we have the curve $x_1 - z$ which decreases from the new base to $x = b - z$, see Fig. 7c. The sum of the y^n ordered on $[0, b]$ can obviously be decomposed into the two portions. His arguments, translated into our notation would lead to:

$$\begin{aligned}\int_0^b y^n\,dx &= \int_0^z y^n\,dx + \int_z^b y^n\,dx\\ &= n\int_0^{\bar{y}} y^{n-1}(x_1 - z)\,dy + n\int_0^{\bar{y}} y^{n-1}(z - x_2)\,dy\\ &= n\int_0^{\bar{y}} y^{n-1}(x_1 - x_2)\,dy.\end{aligned}$$

Obviously, the value z of y where the maximum $\bar{y}$ for x is attained can be obtained by his *method of maxima and minima*, developed twenty years before. Paradoxically, these values are of no importance as they only are intermediate values that are not explicitly needed to carry out the quadrature.[41] This is probably one of the reasons why Fermat pays no attention at all to the limits of summation in the intermediate curves he uses.

6. THE QUADRATURE OF THE FOLIUM OF DESCARTES

The next example Fermat offers has the clear intention of creating an impression on the reader.

> Just to show clearly that our method provides new quadratures which had never even been suspected before among the moderns, let the curve considered before be proposed[42] with equation
>
> $$\frac{b^5x - b^6}{x^3} = y^3.$$
>
> It has been proven that the sum of the y^3 is given as a rectilinear area. Transforming them on the base[43] we will have, according to the preceding method, $b^2u/y^2 = x$. Replacing the new value of x and finishing the calculations according to the rules,[44] we will arrive at the new equation $y^3 + u^3 = byu$, which provides a curve from the side of the base. It is the one from Schooten, who gave its construction in his *Miscellanea*, section 25, page 493.[45] The curvilinear figure

41 Mahoney [1994, p. 264] says on this point that "Fermat employs his method of maxima and minima to determine the value of x for which y attains a maximum and the value of that maximum". This is not really so as the actual values of both, the maximum and the value of x where it is attained, are irrelevant in Fermat's method.

42 It is example (2) of page 16.

43 That is, using the *General Theorem*.

44 The rules of algebra, of course.

45 The curve is the folium of Descartes.

> *AKOGDCH* [the loop in Fig. 8] of this author is, consequently, easily quadrable according to the preceding rules. [46]

The folium of Descartes appeared during the controversy that confronted Fermat and Descartes around 1637 on the methods for the tracing of tangents to curves. After quite an acrid exchange of letters with examples and counterexamples to prove the superiority of each other's method, Descartes ended by challenging Fermat to find the tangents to the curve of his invention with equation (see Fig. 8).

$$x^3 + y^3 = bxy. \tag{6}$$

Fermat not only solved the problem but also offered a general solution that allowed him to find the two tangents of a given slope (see [Duhamel 1864] or [Mahoney 1994, p. 181 & ff] for more details about the controversy). Descartes, after this *tour de force* of his opponent, had to admit Fermat's superiority and the merit of being one of the greatest geometers of the moment. It is not strange, then, that twenty years later, Fermat used the same curve to test his method. [47]

If you follow the exasperatingly short description of Fermat in the quotation above, one realizes that Fermat starts with an apparently innocent curve that, as if by chance, gets transformed into the equation of the folium. It is obvious that Fermat proceeded just in the contrary direction.

---

46 *Ut autem pateat novas ex nostra hac methodo emergere quadraturas, de quibus nondum recentiorum quisquam est aliquid subodoratus, proponatur præcedens curva, cujus æquatio*

$$\frac{\mathrm{B}qc.\ in\,\mathrm{A} - \mathrm{B}cc.}{\mathrm{A}c.}\ \textit{æqualis}\ \mathrm{E}c.$$

Dantur omnes E *cubi in rectilineis, ut jam probatum est. Quibus ad basim translatis, fiet, ex superiori methodo,*

$$\frac{\mathrm{B}q.\ in\,\mathrm{U}}{\mathrm{E}q.}\ \textit{æquale}\ \mathrm{A},$$

et, omnibus secundum artem novo ipsius A *valori accommodatis, evadet tandem nova æquatio quæ dabit curvam ex parte basis; cujus æquatio dabit*

$$\mathrm{E}c. + \mathrm{U}c.\ \textit{æqualis}\ \mathrm{B}\ in\,\mathrm{E}\ in\,\mathrm{U},$$

quæ est curva Schotenii, cujus constructionem tradit in Sectione 25 Miscellanearum*, pag. 493. Figura itaque curvæ* AKOGDLA *quæ apud illum autorem delineatur, ex superioribus præceptis quadrationem suam commode nanciscetur* [Fermat c. 1659, pp. 275-276].

47 We coincide with [Mahoney 1994, p. 265, n. 67] when he insinuates that Fermat deliberately slights Descartes as the author of the curve and attributes it to van Schooten.

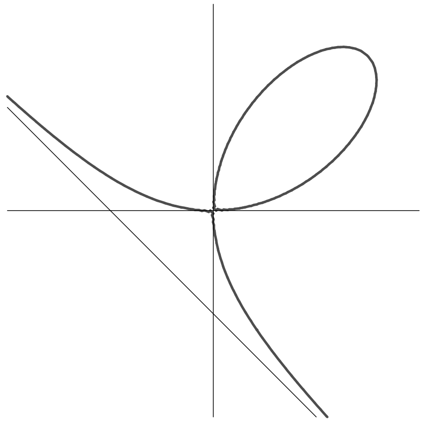

FIGURE 8. The folium of Descartes, $x^3 + y^3 = bxy$.

From the equation of the folium, he derived an equation which had the necessary features for his method to be applied, that is, an equation of the form

$$y^m = \sum a_i x^i + \sum \frac{b_j}{x^j}. \tag{7}$$

We can now present the chain of integrals of Fermat's method (see Fig. 9a and 9b):

$$\int_0^{\bar{y}} (x_1 - x_2)\, dy = \frac{1}{b^2} \int_0^{\bar{y}} (u_1 - u_2) y^2\, dy$$
$$= \frac{1}{3b^2} \int_b^{\infty} y^3\, du = \frac{b^3}{3} \int_b^{\infty} \frac{u-b}{u^3}\, du = \frac{b^2}{6}.$$

Needless to say that Fermat does not bother to calculate the actual area $b^2/6$. See Appendix A for an alternative way of solving the problem and some interesting generalizations.

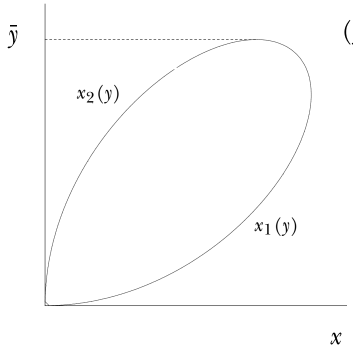

FIGURE 9a. $x^3 + y^3 = bxy$.

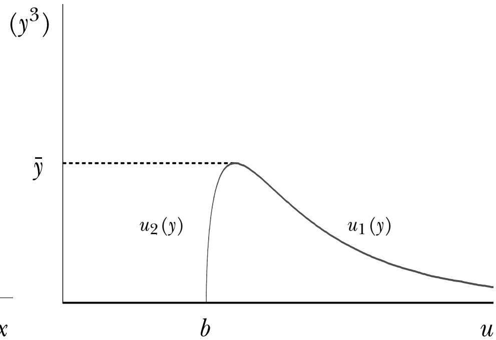

FIGURE 9b. $f(u) = b^5(u - b)/u^3$.

7. THE QUADRATURE OF THE WITCH OF AGNESI AND THE CISSOID OF DIOCLES

Fermat now opens a new front. He proceeds to reduce the quadrature of some curves to that of the circle. In this sense, he first tackles the quadrature of the curve known today as the witch of Agnesi. [48] This curve seems to have been brought to Fermat's attention by the geometer Lalouvère who might have asked Fermat about its quadrature. [49] Immediately after doing that, Fermat adds:

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48 This curve was studied in 1748 by Maria Gaetana Agnesi (1718-1799) and had already been object of attention by Guido Grandi (1703) who gave it the curious name of *versiera* or *versaria* (see [Gray & Malakyan 1999; Mulcrone 1957; Truesdell 1989] for the history of the name and other details about the curve itself). In English it is known as the *witch of Agnesi* or the *curve of Agnesi.*

49 Fermat in the *Treatise*, after constructing geometrically the versiera and after giving us the value of its quadrature, comments: "It is so that we have solved at once that question proposed to us by a learned geometer" [*Hanc vero quæstionem, ab erudito geometra nobis propositam, ita statim expedivimus* [Fermat c. 1659, p. 281]]. Aubry [1909, p. 85] is of the opinion that the "learned geometer" mentioned by Fermat is none other than Antoine de Lalouvère, a Jesuit from Toulouse and frequent correspondent of Fermat. Anyway, we have not been able to find a previous mention of a curve like the versiera in the literature, and this leads us to think that Fermat might be the real author of the curve or, at least of its algebraic equation.

> [...] with the same method I have squared the cissoid of Diocles or, I had rather say that I have reduced its quadrature to that of the circle.[50]

Fermat in the *Treatise* does not give any more indications about how he reached the quadrature of the cissoid.[51] However, the two curves, the versiera and the cissoid, have similar cartesian equations, a fact that makes a common treatment with Fermat's method possible. In fact, going clearly beyond Fermat's work, in Appendix B we treat a more general family of curves that include both the versiera and the cissoid and can be tackled in the same way.

The versiera (Fig. 10) is the curve of equation:

$$b^3 = xy^2 + b^2x$$

which can be written

$$xy^2 = b^2(b - x). \tag{8}$$

The quadrature of the versiera (8) corresponds to the area between the curve and the two axes—the vertical axis being the asymptote of the curve. Fermat obtains it with the help, in this case, of two changes of variables. In the first place,

$$x = \frac{z^2}{b}$$

which leads to the new curve

$$z^2y^2 = b^2(b^2 - z^2).$$

50 [...] *eadem methodo spatium a Dioclea comprehensum quadravimus, vel ad circuli quadraturam reduximus* [Fermat c. 1659, p. 281].

51 The same Fermat in a brief note titled *De cissoide fragmentum* [Fermat 1662] squares the cissoid by purely geometrical methods without using any of the methods of the *Treatise*. The result he obtains in that short "fragment" comes to say that the area trapped between the cissoid and its asymptote is the triple of the area of the semicircle used in its geometrical construction. Details of this construction can be found in [Truesdell 1989]. Aubry [1912] offers a reconstruction of the quadrature of the cissoid of doubtful likelihood. He freely uses differentials and the full formula of integration by parts, poles apart from Fermat's method. More than from Fermat, Aubry seems to borrow from Johann Bernoulli, who in [Bernoulli 1692, pp. 399–407] had carried out the quadrature of the folium, the versiera and some other curves treated by Fermat. Bernoulli's procedure, though vaguely reminiscent of Fermat by the changes of variables used, is definitely far from the method of the French mathematician.

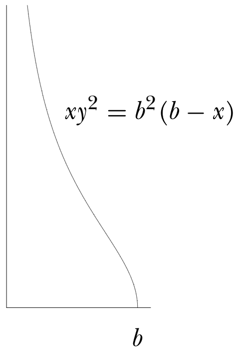

FIGURE 10. Versiera.

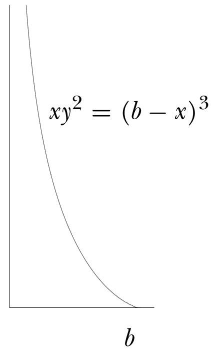

FIGURE 11. Cissoid.

Thus,

$$\text{Area} = \int_0^\infty x\,dy = \frac{1}{b}\int_0^\infty z^2\,dy.$$

He then applies the *General Theorem*:

$$\frac{1}{b}\int_0^\infty z^2\,dy = \frac{2}{b}\int_0^b yz\,dz.$$

And the second change,

$$y = \frac{bu}{z}$$

that leads to the new curve

$$u^2 = b^2 - z^2,$$

a circle, to whose quadrature the quadrature of the versiera reduces:

$$\frac{2}{b}\int_0^b yz\,dz = 2\int_0^b u\,dz.$$

Summing up,

$$\text{Area versiera} = \int_0^\infty x\,dy = 2\int_0^b u\,dz.$$ [52]

As we see, the quadrature of our first curve depends on the sum of all the u on the interval $[0, b]$,

$$\int_0^b u\,dz,$$

52 Fermat does not mention it, but the final value is $\pi b^2/2$.

where u is the ordinate of a circle of radius b.

Fermat, as we mentioned before, says that the cissoid of Diocles (Fig. 11) may be squared similarly, which is true, but he does not mention that, in this last case, the quadrature reduces to that of an *odd power* of the ordinate of a circle. In fact, the same method applied to the cissoid of equation

$$xy^2 = (b - x)^3$$

leads to the quadrature of u^3 where, as before, $u^2 = b^2 - z^2$: [53]

$$\text{Area cissoid} = \int_0^\infty x\,dy = \frac{2}{b^2}\int_0^b u^3\,dz.$$

(See Appendix B for the details). As the case of the cissoid demands, Fermat will now turn to the problem of summing different powers of the ordinates of a circle.

8. THE SUM OF THE POWERS OF THE ORDINATES OF A CIRCLE

We now come across one of the reasons why the reading of the *Treatise* is so puzzling. Fermat, apparently, stops analyzing the quadrature of curves and turns to solve the problem of finding the sum of the powers of the ordinate of a circle. This is only clear if the reader has taken the trouble of reducing the quadrature of the cissoid to that of an odd power of the ordinates of a circle which is not obvious at all.

He begins by considering the equation of the circle $y^2 = b^2 - x^2$. He has already remarked that the sum of even powers of y poses no problem. The odd powers, he asserts, can be reduced through his method to the quadrature of the circle.

Fermat considers only the case y^3 and informs us that the generalization to all odd powers is very easy. [54]

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53 Here Fermat again faces the problem of the sum of a radical power of the ordinates. His method circumvents the difficulty.

54 As [Zeuthen 1895, pp. 57–58] says, Fermat's method reduces the sum of y^{2n+1} to the sum of z^n where z is the ordinate of a circle of radius $b/2$ and not centered on the origin. This reduction is faster than the one we would undertake today if we had to calculate

$$\int_0^b (b^2 - x^2)^{(2n+1)/2}\,dx$$

An application of the *General Theorem* gives

$$\int_0^b y^3\,dx = 3\int_0^b y^2 x\,dy.$$

There are two changes of variable. The first,

$$x = \frac{bu}{y}$$

leads to the new curve

$$b^2u^2 = y^2(b^2 - y^2),$$

and the corresponding quadrature,

$$3\int_0^b y^2 x\,dy = 3b\int_0^b yu\,dy.$$

If the *General Theorem* is applied again,

$$3b\int_0^b yu\,dy = \frac{3}{2}\int_0^{b/2} y^2\,du.$$

The second change is

$$y^2 = bv$$

which gives the curve

$$u^2 = bv - v^2.$$

and the last quadrature is

$$\frac{3}{2}\int_0^{b/2} y^2\,du = \frac{3}{2}b\int_0^{b/2} v\,du.$$

Since Fermat presents only the case of the quadrature of y^3, he finds no difficulties as the sum of the v is simply half the area of a circle of radius $b/2$. But for the general case, y^{2n+1} for $n > 1$ one must still reduce the new circle to another circle, this time, centered on the origin in order to be able to iterate the procedure. See Appendix C for the details.

---

integrating directly by parts. This would imply the differentiation of $(b^2 - x^2)^{(2n+1)/2}$ and a reduction formula that reduces the degree in 2 units at a time. If we bother to do the necessary calculations, we get the reduction formula

$$\int_0^b (b^2 - x^2)^{(2n+1)/2}\,dx = \frac{(2n+1)b^2}{2n+2}\int_0^b (b^2 - x^2)^{(2n-1)/2}\,dx.$$

Fermat's reduction halves the degree each time. In Appendix C we develop the general case with all the necessary details.

9. THE LAST TURN OF THE SCREW

Fermat, to close his paper yields to the temptation of presenting the quadrature of a curve that needs up to eight changes of variable to be reduced.[55]

> As for the rest, it often occurs that, strangely enough, in order to reach the simple measure for a proposed equation of locus we need to carry our analysis through a great number of curves.[56]

This last example is, obviously enough, a *tour de force* to present an almost impossible quadrature. But after careful analysis we can see that it is not only that. It can be placed along the class of curves that lead to the quadrature of the folium of Descartes. The difference lies in the fact that now Fermat wants to find the quadrature of the first curve instead of starting with the known sum of the power of the ordinates of a curve in order to derive the quadrature of a new curve. In this, the example differs from the previous ones.

Fermat's initial equation is

$$y^2 = \frac{b^7(x-b)}{x^6}.$$ [57]

The aim of Fermat is to square this curve (see Fig. 12), that is, to compute

$$\int_b^\infty y\, dx$$

or, what amounts to the same,

$$\int_0^{\bar{y}} x\, dy.$$

55 Tannery and Henry remark in a footnote that in this part, a series of mistakes in the names of the successive curves (for instance, *quarta* instead of *tertia*), seem to indicate that the original text may have been edited by someone who wanted to clarify it.

56 *Sæpius autem contingit et miraculi instar est per plurimas numero curvas incedendum et exspatiandum esse analystae, ut ad simplicem æquationis localis propositæ dimensionem perveniatur* [Fermat c. 1659, p. 282].

57 It is worth mentioning that Fermat uses numbers to denote high powers: 7, 8, etc. Tannery and Henry remark that this is rather suspicious as he has not done it before and makes no note of the change of notation. Again, some edition of the original manuscript may have occurred.

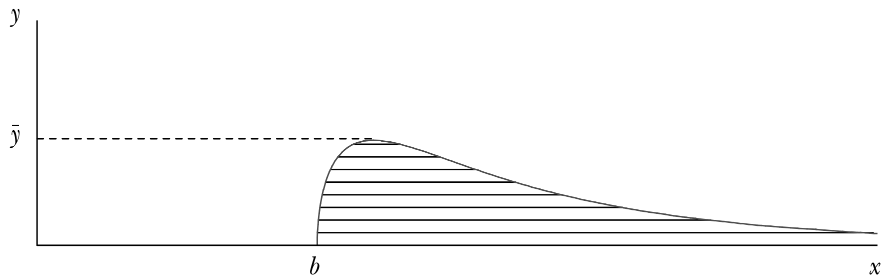

FIGURE 12. $y^2 = b^7(x-b)/x^6$.

The different quadratures and the corresponding changes of variable with the resulting curves are the following (we denote by T the aplications of the *General Theorem* and by CV a change of variables, all listed below together with the different curves, C, obtained through them), see [Fermat c. 1659, pp. 283–285]:

$$\begin{aligned}
\int_0^{\bar{y}} x\,dy &\overset{\text{CV1}}{=} \frac{1}{b}\int_0^{\bar{y}} z^2\,dy \overset{\text{T}}{=} \frac{2}{b}\int_b^{\infty} zy\,dz \overset{\text{CV2}}{=} \frac{2}{b}\int_b^{\infty} u^2\,dz \\
&\overset{\text{T}}{=} \frac{4}{b}\int_0^{\bar{u}} uz\,du \overset{\text{CV3}}{=} 4\int_o^{\bar{u}} v\,du \\
&\overset{*}{=} 4\int_0^{\bar{v}} u\,dv \overset{\text{CV4}}{=} \frac{4}{b}\int_0^{\bar{v}} vw\,dv \\
&\overset{\text{T}}{=} \frac{2}{b}\int_0^b v^2 dw \overset{\text{CV5}}{=} 2\int_0^b s dw \overset{\text{CV6}}{=} \frac{2}{b^2}\int_0^b w^2 t dw \\
&\overset{\text{T}}{=} \frac{2}{(3)b^2}\int_0^b w^3 dt;
\end{aligned}$$

CV1 :	$x = z^2/b$	$C_1 : y^2 z^{12} = b^{12}(z^2 - b^2)$
CV2 :	$y = u^2/z$	$C_2 : u^4 z^{10} = b^{12}(z^2 - b^2)$
CV3 :	$z = bv/u$	$C_3 : v^{10} = b^4)(v^2 - u^2)u^4$
CV4 :	$u = vw/b$	$C_4 : b^2 v^4 = (b^2 - w^2)w^4$
CV5 :	$v^2 = bs$	$C_5 : b^4 s^2 = (b^2 - w^2)w^4$
CV6 :	$s = w^2 t/b^2$	$C_6 : w^2 = b^2 - t^2.$

A few remarks are in order. First, notice that the quadrature of the initial curve ends by depending directly on the sum of the powers of the ordinates of the circle, already studied by Fermat. Second, the example chosen allows him to exhibit eight changes of variable—the last three, though, are only needed for summing the third power of the ordinates of a circle. Third, in this example, he uses for the first time a quite obvious result which can be seen as the *General Theorem* for the case $n = 1$. The area of a figure is the same whether the sum of the ordinates is taken on the base or the sum of the abscissas is taken on the diameter. That is to say,

$$\int x\,dy = \int y\,dx.$$

It is the step marked above with the symbol $\stackrel{*}{=}$.

Lastly, to emphasize the great internal coherence of the *Treatise*, it is worth noting that this final example is the quadrature of a curve of the same class as the first he had used to obtain the quadrature of the folium (see footnote 60). Thus, this last example closes the paper with a spectacular display of his method and, at the same time closes a circle returning to the starting point. See Appendix D for a more general treatment of the example and some more interesting comments.

Fermat's last words clearly show the pride of the author for his creation:

> We have thus used up to nine [actually eight, see footnote 55] different curves to reach the knowledge of the first.[58]

10. CONCLUSIONS

One of the more momentous conquests of the first third of the seventeenth century was the expression of a curve by the means of a mathematical equation expressed by a polynomial.

In fact, if a general method for determining properties of curves from their algebraic equations could be found, a giant step would have been taken, since in this case, important parts of mathematics would achieve their independence from pure geometry.

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58 *Beneficio igitur novem curvarum inter se diversarum ad notitiam prioris pervenimus* [Fermat c. 1659, p. 285].

In this direction, Descartes' finding in *La Géométrie* is crucial: as a curve can be expressed by the use of a polynomial equation, $P(x, y) = 0$, the normal at a given point (x_0, y_0) of the curve can be found (and consequently also the tangent). The method consists of cutting the curve with a circle of unknown center $O = (r, s)$ and imposing that the resulting polynomial $Q(x) = 0$ have $x = x_0$ as a double root. A great success for a good method. It always depends, of course, on the degree of the polynomial equation of the curve.

More or less at the same time, the geometers of the seventeenth century came to realize the importance of squaring the curves of the form $y^m = b^{m \pm n} x^{\mp n}$. They devoted a great deal of energy to achieve these quadratures and they strived to find

$$\int_0^b x^n\, dx, \quad \int_b^\infty x^{-n}\, dx, \quad \int_0^b x^{+m/n}\, dx, \quad \int_b^\infty x^{-m/n}\, dx.$$

So, from Cavalieri to Newton and Leibniz, with different techniques and different epistemological frameworks, they carried out their calculations and arrived at

$$\int_0^b x^{\pm m/n}\, dx = \frac{b^{\pm m/n+1}}{\pm m/n + 1},$$

except in the case in which the exponent is -1. The success was so spectacular that Newton considered as the explicit analytical expression of a function its power series expansion and thus developed a sort of algebra of infinite series (see [Stillwell 1989, p. 107]).

It is precisely in this context where Fermat's contributions to algebraic geometry, tangents to curves, lengths of curves and quadratures have to be analyzed. In this last subject, the quadrature of curves, Fermat finds a method similar to the ones he has found in the other areas mentioned. This is what our reading of the *Treatise* tries to show.

In a first part, Fermat establishes a general method to find the quadrature of all higher parabolas and hyperbolas. Next, he sets himself the problem of determining the quadrature of an algebraic curve given by an implicit equation $P(x, y) = 0$ using the quadratures he has just calculated. This is the difficult part of his paper and the one analyzed in the present article.

To achieve his aim, he seeks a new curve, quadrable, whose quadrature is expressible through the known quadratures of the curves at his disposal, the higher parabolas and hyperbolas.

Thus, given an equation of the form

$$y^n = \sum a_i x^i + \sum b_j / x^j \tag{9}$$

Fermat is able to obtain the sum of the y^n through the squaring of the parabolas and hyperbolas of the right-hand side. He then applies the *General Theorem* to reduce the degree and proceeds to determine a new curve by a change of variable that either linearizes ($y^n = b^{n-1}u$) or reduces even more the degree.

In order to enlarge the class of reducible quadratures, he has to add the circle to his stock of known quadratures. He then realizes that the squaring of curves like

$$(b^2 - x^2)^{n/2} \tag{10}$$

will lead to the possibility of squaring more curves. The case in which n is even presents no problem as $y^2 = b^2 - x^2$, and for odd n he manages to circumvent the difficulty of the radicals by a masterful use of his method applied to y^{2m+1}, where $y^2 = b^2 - x^2$.

We could describe in a few words the essence of Fermat's method (leaving apart the last example of the *Treatise* in which he deviates from the previous ones while maintaining the spirit) as follows. Fermat knows how to compute

$$\int_0^b y^n \, dx, \tag{11}$$

either by squaring directly higher parabolas or hyperbolas, (9), or as the sum of the ordinates of a circle, (10). Now, by the *General Theorem*,

$$\int_0^b y^n \, dx = n \int_0^{\bar{y}} x y^{n-1} \, dy.$$

A change of variable of the style

$$x y^{n-1} = b^{n-q} u^q$$

can be carried out with a suitable q. So, in (9) or (10), x can be replaced by $b^{n-q}u^q/y^{n-1}$ in order to obtain a new curve

$$P(y,u) = 0.$$

for which

$$\int_0^{\bar{y}} u^q\, dy$$

is computable in terms of (11). The process can be iterated until reaching

$$\int_\alpha^\beta z dw$$

which is the actual quadrature of a curve $F(w,z) = 0$.

Strictly following the previous process, it seems that the "new" quadrable curve, $F(w,z) = 0$, appears at the end of the process as a sort of surprise. Fermat—and we hope our new reading of the *Treatise* will have made this clear—is conscious that the process can be reversed at least for certain families of algebraic curves with a "standard" equation.

Fermat's method of quadratures is, as has been shown, highly original and powerful, but only applicable to a certain class of algebraic curves. It could be argued that he sought a general method to square curves with an *implicit* polynomial equation. He did not succeed but he managed to find a workable method for a limited amount of curves. In fact this limitation partly explains the sparse attention the method received in its time.

In our opinion, the history of mathematics consists of understanding the writings of great mathematicians, their internal coherence, the methodology that has been used, the extension of the methods deployed. All this independently of the measure of success of those writings. A paradigmatic text in this sense is the *Lettres de Dettonville* by Blaise Pascal. Fermat's *Treatise on quadratures* is another one which we hope we have contributed to vindicate at least for its great intellectual value. Our work is neither a historiographic analysis of Fermat's text nor a study of its ulterior influence—which has been almost non-existent—but offers a complete detailed analysis of all of its examples showing its inter-dependence and the logical thread that conducts them all. In some occasions we dare reconstruct in an appendix obscure parts of Fermat's exposition but we do so in the hope that these reconstructions shed some light on the method Fermat is trying to develop.

APPENDIX A
THE FOLIUM OF DESCARTES

In the case of the folium, Fermat's most likely train of thought would have been to essay a change of variable that replaced x in (6) by an expression involving the new variable u and the old y in such a way that after making the change the new equation would look like (7). In order to achieve this it is enough to make the change of variable[59]

$$x = \frac{uy^2}{b^2},$$

which alters (6) into

$$\frac{u^3y^6}{b^6} + y^3 = \frac{uy^3}{b}$$

or, after simplifying y^3 from each side and rearranging,

$$y^3 = \frac{b^5(u-b)}{u^3}. \qquad (12)$$

The graph of y^3 as a function of u can be seen in Fig. 13b.[60] We can also ask

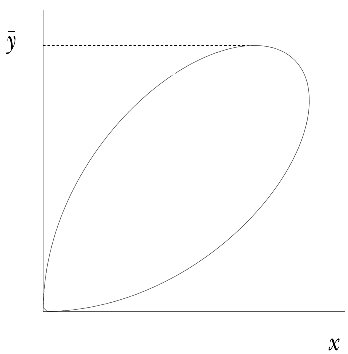

FIGURE 13a. The loop of the folium.

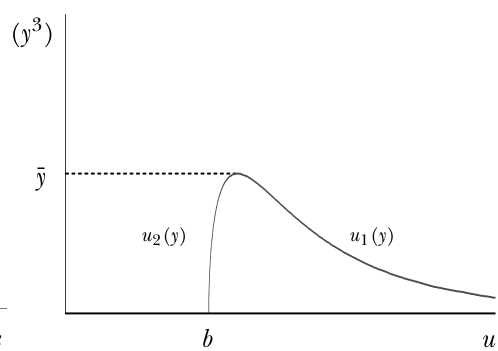

FIGURE 13b. $f(u) = b^5(u-b)/u^3$.

59 The change $x = b^2u/y^2$ is an alternative that also solves the problem. Johann Bernoulli [1692, p. 403] uses this last change of variable in order to square the folium, but here ends all similitude with Fermat's method, despite what Aubry [1912] says. See also footnote 51.

60 It must be noticed that equation (12) or, if you prefer, equation (13), has a very special structure, which, as it happens, occurs almost in the same form in many of Fermat's examples.

ourselves (see [Paradís et al. 2004]) about the possibility that Fermat carried out a few trials on curves composed by higher hyperbolas with equations of the form:

$$y^m = \frac{b^{m+k-1}(x-b)}{x^k}; \quad m \geq 2, \quad k > 2. \tag{13}$$

The graph corresponding to y in (13) and of y^m, for $m > 0$, are essentially the same and very similar to the one depicted in Fig. 13b.

Proceeding *à la Fermat* we make the change of variable

$$x = \frac{b^{m-1}z}{y^{m-1}},$$

and we undertake the chain of integrals:

$$\int_b^\infty y^m \, dx = m \int_0^{\bar{y}} y^{m-1} x \, dy = mb^{m-1} \int_0^{\bar{y}} z \, dy.$$

The new curve's equation will be

$$b^{(m-2)k-m} z^k + y^{(m-1)k-m} = b^{m-2} z y^{(m-1)(k-1)-m}.$$

This family of curves, in the first quadrant have a loop similar to the loop of the folium—which is the curve given by $m = 3$ and $k = 3$ (see Fig. 13a). The areas of these loops, that is to say

$$\int_0^{\bar{y}} z \, dy$$

are

$$A(m,k,b) = \frac{b^2}{m(k-1)(k-2)}.$$

It is seen at once that Fermat's method also solves in a quite straightforward way the quadrature of the generalized folia of [Bullard 1916] with equation

$$x^{2q+1} + y^{2q+1} = (2q+1) b x^q y^q,$$

where q is a positive integer.

The change of variable required is $b^{q+1} x^q = u^q y^{q+1}$, and the areas of the loops in the first quadrant are

$$A(q,b) = \frac{2q+1}{2} b^2.$$

More details can be found in [Paradís et al. 2004].

APPENDIX B
THE VERSIERA FAMILY

Let us consider the family of curves[61] with equation

$$b^{N-3}xy^2 = (b-x)^N. \tag{14}$$

For $N = 1$ we have the versiera (Fig. 14) and for $N = 3$ the cissoid (Fig. 15). The quadrature of the family of curves (14) will correspond to

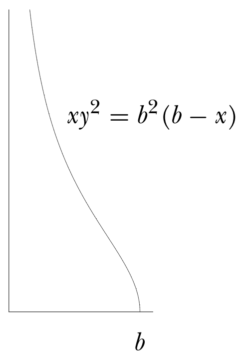

FIGURE 14. Versiera.

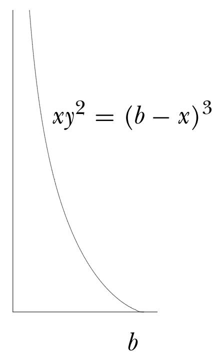

FIGURE 15. Cissoid..

the area trapped between the curve and the two axes—the vertical axis is in fact the asymptote of the curve. It can be obtained with the help, in this case, of two changes of variables. As before, we will use a T upon the equal sign to denote an application of the *General Theorem* and CV to denote a change of variables.

$$\begin{aligned} \text{Area} &= \int_0^\infty x\,dy \\ &\overset{\text{CV1}}{=} \frac{1}{b}\int_0^\infty z^2\,dy \overset{\text{T}}{=} \frac{2}{b}\int_0^b yz\,dz \overset{\text{CV2}}{=} \frac{2}{b^{N-1}}\int_0^b u^N\,dz. \end{aligned} \tag{15}$$

Fermat needed two changes of variable:

$$\text{CV1:} \qquad x = \frac{z^2}{b}$$

61 Notice that these curves are again of the form (13). The only difference is that instead of $x-b$, now we consider $b-x$. See also note 60.

which led to the new curve

$$b^{2N-4}z^2y^2 = (b^2 - z^2)^N,$$

and

$$\text{CV2:} \qquad y = \frac{u^N}{b^{N-2}z}$$

giving the last curve which, independently of N, is the same circle,

$$u^2 = b^2 - z^2.$$

Now, the quadrature of our first curve will depend on the sum of all the u^N on the interval $[0, b]$,

$$\int_0^b u^N \, dz,$$

where u is the ordinate of a circle of radius b. For even values of N, it is clear that the required sum will be very easy to calculate as it will ultimately be a sum of quadratures of parabolas, i.e. the powers $(b^2 - z^2)^{N/2}$. For odd values of N, the required sum will not be so easy to carry out.

The simplest odd case, the case of the versiera ($N = 1$), is easily dealt with. Its quadrature will depend on the quadrature of the circle itself, (formula (15) for $N = 1$)):

$$\text{Area versiera} = 2\int_0^b u \, dz = \frac{\pi b^2}{2}.$$

APPENDIX C

THE QUADRATURE OF $y = (b^2 - x^2)^{m/2}$

Let $A(m, r)$ denote the sum of the y^m where y is the ordinate of a circle of radius r centered on the origin.

$$\begin{aligned} A(2n+1, b) &= \int_0^b y^{2n+1} \, dx \\ &\overset{\text{T}}{=} (2n+1)\int_0^b y^{2n} x \, dy \overset{\text{CV1}}{=} (2n+1)b\int_0^b y^{2n-1} u \, dy \\ &\overset{\text{T}}{=} \frac{2n+1}{2n}\int_0^{b/2} y^{2n} \, du \overset{\text{CV2}}{=} \frac{2n+1}{2n} b^n \int_0^{b/2} v^n \, du. \end{aligned}$$

The changes of variable indicated are, first:

$$\text{CV1:} \qquad x = \frac{bu}{y}$$

which leads to the new curve

$$b^2u^2 = y^2(b^2 - y^2),$$

and then

$$\text{CV2:} \qquad y^2 = bv$$

which produces the curve

$$u^2 = bv - v^2.$$

Let us remark that this last curve is a circle of center $(b/2, 0)$ and radius $b/2$. The sum of the v^n, where v is the ordinate of this circle, has to be taken as the sum of the expressions $v_1^n - v_2^n$, where the v_i are the monotone portions of the circle as shown in Fig. 16. Since Fermat presents only the

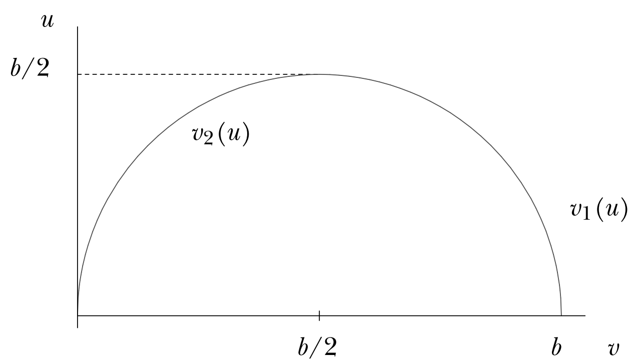

FIGURE 16. $u^2 = bv - v^2$.

case $n = 1$, he finds no difficulties as the sum of the v is simply half the area of a circle of radius $b/2$. But for $n > 1$ one must still reduce the new circle to another circle, this time, centered on the origin in order to be able to iterate the procedure. This can be done with another change of variable:

$$\text{(16)} \qquad \text{CV3:} \qquad v_1 = \frac{b}{2} + t, \quad v_2 = \frac{b}{2} - t,$$

which transforms the sum of the v^n as follows:

$$\int_0^{b/2} (v_1^n - v_2^n)\, du \overset{\text{CV3}}{=} \int_0^{b/2} \left\{ \left(t + \frac{b}{2}\right)^n - \left(t - \frac{b}{2}\right)^n \right\} du$$
$$= 2 \sum_{j=1}^{\lceil n/2 \rceil} \binom{n}{2j-1} \left(\frac{b}{2}\right)^{n-2j+1} \int_0^{b/2} t^{2j-1}\, du.$$

The sum of the odd powers of t corresponds to the circle centered on the origin with equation $t^2 = (b/2)^2 - u^2$. We obtain a recurrence formula for the sum of the odd powers of the ordinates of a circle:

$$A(2n+1, b) = \frac{2n+1}{n} \sum_{j=1}^{\lceil n/2 \rceil} \binom{n}{2j-1} \left(\frac{b}{2}\right)^{n-2j+1} \cdot A\left(2j-1, b/2\right).$$

In this last formula, $\lceil x \rceil$ denotes the ceiling of the number x, i.e. the smallest integer greater than or equal to x.

APPENDIX D

ANALYSIS OF THE LAST EXAMPLE

Instead of studying Fermat's last curve directly, we can deal with his example a little more generally.

Let us consider the curve (see Fig. 17) with equation

$$y^2 = \frac{b^{k+1}(x-b)}{x^k}.$$

The chain of integrals and the corresponding changes of variable with the

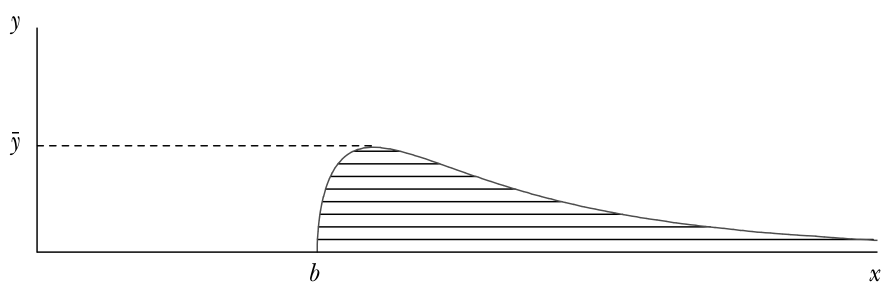

FIGURE 17. $y^2 = b^{k+1}(x-b)/x^k$.

resulting curves is the following:

$$\begin{aligned}\int_0^{\bar{y}} x\,dy &\overset{\text{CV1}}{=} \frac{1}{b}\int_0^{\bar{y}} z^2\,dy \overset{\text{T}}{=} \frac{2}{b}\int_b^{\infty} zy\,dz \overset{\text{CV2}}{=} \frac{2}{b}\int_b^{\infty} u^2\,dz\\
&\overset{\text{T}}{=} \frac{4}{b}\int_0^{\bar{u}} uz\,du \overset{\text{CV3}}{=} 4\int_o^{\bar{u}} v\,du\\
&\overset{*}{=} 4\int_0^{\bar{v}} u\,dv \overset{\text{CV4}}{=} \frac{4}{b}\int_0^{\bar{v}} vw\,dv\\
&\overset{\text{T}}{=} \frac{2}{b}\int_0^{b} v^2 dw \overset{\text{CV5}}{=} 2\int_0^{b} s\,dw \overset{\text{CV6}}{=} \frac{2}{b^{k-4}}\int_0^{b} w^{k-4}t\,dw\\
&\overset{\text{T}}{=} \frac{2}{(k-3)b^{k-4}}\int_0^{b} w^{k-3}dt;\end{aligned}$$

CV1:	$x = z^2/b$	$C_1 : y^2z^{2k} = b^{2k}(z^2 - b^2)$
CV2:	$y = u^2/z$	$C_2 : u^4z^{2k-2} = b^{2k}(z^2 - b^2)$
CV3:	$z = bv/u$	$C_3 : v^{2k-2} = b^4)(v^2 - u^2)u^{2k-8}$
CV4:	$u = vw/b$	$C_4 : b^{2k-10}v^4 = (b^2 - w^2)w^{2k-8}$
CV5:	$v^2 = bs$	$C_5 : b^{2k-8}s^2 = (b^2 - w^2)w^{2k-8}$
CV6:	$s = w^{k-4}t/b^{k-4}$	$C_6 : w^2 = b^2 - t^2.$

Besides the remarks offered in section 9 we see that in order to be able to apply Fermat's method, it is necessary that $k > 3$. This is a condition which, from a modern point of view, makes the improper integral

$$\int_b^{\infty} \frac{b^{k+1}(x-b)}{x^k}\,dx$$

convergent. Assuming that Fermat tried different curves of this nature, we can add that he did not choose $k = 4$ for then the quadrature would have taken only five changes of variable. He also skipped the case $k = 5$ for then the quadrature reduces to that of a higher parabola. Instead he chose $k = 6$ which allowed him to exhibit eight changes of variable—the last three, though, only needed for summing the third power of the ordinates of a circle. For the symbols $T, CV, C, *$ in the above displayed formula, see p. 37.

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