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ON THE COHOMOLOGICAL ACTION OF AUTOMORPHISMS OF COMPACT KÄHLER THREEFOLDS

by F. Lo Bianco

Abstract. — Extending well-known results on surfaces, we give bounds on the cohomological action of automorphisms of compact Kähler threefolds. More precisely, if the action is virtually unipotent we prove that the norm of $(f^n)^*$ grows at most as cn^4; in the general case, we give a description of the spectrum of f^*, and bounds on the possible conjugates over $\mathbb{Q}$ of the dynamical degrees $\lambda_1(f), \lambda_2(f)$. Examples on compact complex tori show the optimality of the results.

Résumé (*Sur l'action cohomologique des automorphismes des variétés compactes kähleriennes de dimension 3*). — Nous étendons des résultats bien connus sur l'action en cohomologie des automorphismes des surfaces aux variétés compactes Kähler de dimension 3. Plus précisément, si l'action est virtuellement unipotente nous montrons que la norme de $(f^n)^*$ croît au plus comme cn^4; dans le cas général, nous donnons une description du spectre de f^* et des bornes sur les possibles conjugués sur $\mathbb{Q}$ des degrés dynamiques $\lambda_1(f), \lambda_2(f)$. Des exemples sur les tores complexes compacts montrent l'optimalité de ces résultats.

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An automorphism $f\colon X \to X$ of a compact Kähler manifold induces by pull-back of forms, a linear automorphism

$$f^*\colon H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$$

which preserves the cohomology graduation, the Hodge decomposition, complex conjugation, wedge product and Poincaré duality.

Question 0.1. — *What else can one say on f^*? More precisely, can one give constraints on f^* which depend only on the dimension of X (and not on the dimension of $H^*(X)$)?*

This is an interesting question since the cohomology of a manifold is a powerful tool to describe its geometry; furthermore, the cohomological action of an automorphism is relevant when studying its dynamics: one can deduce its topological entropy from its spectrum (see Theorem 1.6), and in the surface case knowing f^* allows establishing of the existence of f-equivariant fibrations (see Theorem 2.4). It turns out that the restriction of f^* to the even cohomology encodes most of the interesting informations (see Section 1.3), therefore we focus on this part of the action; furthermore, in dimension 3 the action on $H^0(X)$ and on $H^6(X)$ is trivial, and the action on $H^4(X)$ can be deduced from the action on $H^2(X)$ (see Proposition 1.1.(3)), so we only describe the latter.

The situation of automorphisms (and, more generally, of birational transformations) of curves and surfaces is well understood (see Section 2). We address the three-dimensional case here.

The first result describes the situation where f^* does not have any eigenvalue of modulus > 1, i.e. the dynamical degrees $\lambda_i(f)$ are equal to 1 (see Definition 1.4).

Theorem A. — *Let X be a compact Kähler threefold and let $f\colon X \to X$ be an automorphism such that $\lambda_1(f) = 1$ and whose action on $H^*(X)$ has infinite order. Then the induced linear automorphism $f_2^*\colon H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C})$ is virtually unipotent and has a unique Jordan block of maximal dimension $m = 3$ or 5. In particular, the norm of $(f^n)^*$ grows either as cn^2 or as cn^4 as n goes to infinity.*

For the proof of slightly more general results, see Theorem 3.1 and Proposition 3.4.

Next we give a description of the spectrum of f^* in terms of the dynamical degrees:

Theorem B. — *Let X be a compact Kähler threefold and let $f\colon X \to X$ be an automorphism having dynamical degrees $\lambda_1 = \lambda_1(f)$ and $\lambda_2 = \lambda_2(f)$ (see Definition 1.4). Let λ be an eigenvalue of $f_2^*\colon H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C})$; then there exists a positive integer N such that*

$$|\lambda|^{(-2)^N} \in \left\{1, \lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2\right\}.$$

Furthermore, if $N \geq 1$ then $|\lambda|^{(-2)^k}$ is an eigenvalue of f_2^* for all $k = 1, \ldots, N$.

For a proof see Proposition 4.10 (for the case where λ_1 and λ_2 are multiplicatively independent) and 4.12 (for the case where λ_1 and λ_2 have a multiplicative dependency).

Remark 0.2. — While proving Theorem B, we will also obtain that if $\lambda_2 \notin \{\lambda_1^2, \sqrt{\lambda_1}\}$, then λ_1 and λ_2^{-1} are the only eigenvalues of f_2^* having modulus λ_1 or λ_2^{-1}.

This had already been proved in much wider generality by Truong in [24].

Finally, we describe the (moduli of) Galois conjugates of $\lambda_1(f)$ over $\mathbb{Q}$:

Theorem C. — *Let X be a compact Kähler threefold and let $f\colon X \to X$ be an automorphism having dynamical degrees $\lambda_1 = \lambda_1(f)$ and $\lambda_2 = \lambda_2(f)$. Then λ_1 is an algebraic integer, all of whose conjugates over $\mathbb{Q}$ have a modulus belonging to the following set:*

$$\left\{\lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2, \sqrt{\lambda_1^{-1}}, \sqrt{\lambda_2}, \sqrt{\lambda_1\lambda_2^{-1}}\right\}.$$

See Proposition 5.11 and 5.13 for a proof and for a more detailed description of all possible subcases.

In Section 1 we introduce the problem and the tools which will be used in the proofs, namely the generalized Hodge index theorem, an application of Poincaré's duality and some elements of the theory of algebraic groups; in Section 2 we present the known results in dimension two. In the rest of the paper we treat the case of dimension three: in Section 3 we give a proof of Theorem A and describe examples on complex tori which show the optimality of the result; similarly, in Section 4 and Section 5 we prove Theorem B and C respectively, and describe further examples on tori which show the optimality of the claims; finally, in Section 6 we address the problem to determine whether f^* can be neither (virtually) unipotent nor semisimple (see Proposition 6.2).

1. Introduction and main tools

Throughout this section, we denote by $f\colon X \to X$ an automorphism of a compact Kähler manifold X of complex dimension d and by

$$f^*\colon H^*(X, \mathbb{R}) \to H^*(X, \mathbb{R})$$

the induced linear automorphism on cohomology. We still denote by $f^*\colon H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$ the complexification of f^*, and by f_k^* (resp. $f_{p,q}^*$) the restriction of f^* to $H^k(X, \mathbb{R})$ (resp. to $H^{p,q}(X)$).

1.1. First constraints. — The induced linear automorphism f^* preserves some additional structure on the cohomology space $H^*(X,\mathbb{R})$:

1. the graduation and Hodge decomposition: in other words $f^*(H^i(X,\mathbb{R})) = H^i(X,\mathbb{R})$ for all $i = 0,\dots,2d$ and $f^*(H^{p,q}(X)) = H^{p,q}(X)$ for $p,q = 0,\dots d$;
2. the wedge (or cup) product: in other words $f^*(u\wedge v) = f^*(u)\wedge f^*(v)$ for all $u,v\in H^*(X,\mathbb{R})$;
3. Poincaré's duality: in other words the isomorphism $H^i(X,\mathbb{R}) \cong H^{2d-i}(X,\mathbb{R})^\vee$ induces an identification
$$f_i^* = ((f_{2d-i}^*)^{-1})^\vee.$$

Furthermore

(4) f^* and $(f^*)^{-1}$ are defined over $\mathbb{Z}$; in other words, the coefficients of f^* and $(f^*)^{-1}$ with respect to an integral basis of $H^*(X,\mathbb{R})$ are integers;

(5) f^* preserves the convex salient cones $\mathcal{K}_p\subset H^{p,p}(M,\mathbb{R})$ generated by the classes of positive currents (see [6]).

Note that properties (1)–(4) are algebraic, while property (5) is not.

The rough strategy for the proofs will be to consider the action of f^* on elements of the form $u\wedge v$, $u,v\in H^2(X,\mathbb{C})$, and relate f_4^* with f_2^* by using property (3).

1.2. The unipotent and semisimple parts. — Let V be a finite dimensional real vector space and let $g\colon V\to V$ be a linear endomorphism. It is well-known that there exists a unique decomposition
$$g = g_u\circ g_s = g_s\circ g_u,$$
where g_u is unipotent (i.e. $(g_u-\mathrm{id}_V)^{\dim V} = 0$) and g_s is semisimple (i.e. diagonalizable over $\mathbb{C}$). This is a special case of the following more general statement:

THEOREM 1.1 (Jordan decomposition). — *Let V be a finite dimensional real vector space and let $G\subset \mathrm{GL}(V)$ be a commutative algebraic group. Then, denoting by $G_u\subset G$ (respectively by $G_s\subset G$) the subset of unipotent (respectively semisimple) elements of G, G_u and G_s are closed subgroups of G and the product morphism induces an isomorphism of real algebraic groups*
$$G\cong G_u\times G_s.$$

Let us go back to the context of automorphisms of a compact Kähler manifold. Let $f\colon X\to X$ be an automorphism of a compact Kähler manifold, $V = H^*(X,\mathbb{R})$, $f^*\colon V\to V$ and
$$G = \overline{\bigcup_{n\in\mathbb{Z}}(f^n)^*}^{Zar}\subset \mathrm{GL}(V);$$

here $\overline{A}^{Zar}$ denotes the Zariski-closure of a set $A \subset \mathrm{GL}(V)$, i.e. the smallest Zariski-closed subset of $\mathrm{GL}(V)$ containing A. Then, since $\langle g \rangle$ is a commutative group, G is a commutative real algebraic group, and by Theorem 1.1 we have an isomorphism of real algebraic groups

$$G \cong G_u \times G_s;$$

this means in particular that, writing the Jordan decomposition $f^* = f_u^* \circ f_s^*$, we have $f_u^*, f_s^* \in G$, i.e. if f^* satisfies some algebraic constraint, then so do f_u^* and f_s^*. We have therefore:

Lemma 1.2. — *f_u^* and f_s^* preserve the cohomology graduation, Hodge decomposition, wedge product and Poincaré duality, and they are defined over $\mathbb{Z}$ (properties (1)–(4) in Section 1.1).*

Note however that, since preserving a cone is not an algebraic property, f_u^* and f_s^* may not preserve the positive cones $\mathcal{K}_p$.

Remark 1.3. — Let $g \in \mathrm{GL}(H^*(X, \mathbb{R}))$ be a linear automorphism preserving the cohomology graduation, Hodge decomposition, wedge product and Poincaré duality. Then we may put (the complexification of) g in Jordan form so that each Jordan block corresponds to a subspace of $H^{p,q}(X)$ for some p, q.

Furthermore, if all eigenvalues of g are real, we can put g itself in Jordan form so that each Jordan block corresponds to a subspace of $(H^{p,q}(X) \oplus H^{q,p}(X))_{\mathbb{R}}$ for some p, q.

1.3. Dynamical degrees. — In this paragraph only, we allow $f\colon M \dashrightarrow M$ to be a dominant meromorphic self-map of a compact Kähler manifold M.

Definition 1.4. — The p-th dynamical degree of f is defined as

$$\lambda_p(f) = \limsup_{n \to +\infty} \|(f^n)^*_{p,p}\|^{\frac{1}{n}},$$

where $\|\cdot\|$ is any matrix norm on the space $\mathcal{L}(H^{p,p}(X, \mathbb{R}))$ of linear maps of $H^{p,p}(X, \mathbb{R})$ into itself.

In the meromorphic case the pull-backs $f^*_{p,p}$ are defined in the sense of currents (see [6]).

One can prove that

$$\lambda_p(f) = \lim_{n \to +\infty} \left(\int_M (f^n)^* \omega^p \wedge \omega^{d-p} \right)^{\frac{1}{n}} \tag{1}$$

for any Kähler form ω; see [11, 10] for details.

In the case of holomorphic maps, we have $(f^n)^* = (f^*)^n$, so that $\lambda_p(f)$ is the spectral radius (i.e. the maximal modulus of eigenvalues) of the linear map $f^*_{p,p}$; since f^* also preserves the positive cone $\mathcal{K}_p \subset H^{p,p}(M, \mathbb{R})$, a theorem of Birkhoff [2] implies that $\lambda_p(f)$ is a positive real eigenvalue of f_p^*. In particular, $\lambda_p(f)$ is an algebraic integer. However it should be noted that in the meromorphic setting we have in general $(f^n)^* \neq (f^*)^n$.

At least in the projective case, the p-th dynamical degree measures the exponential growth of the volume of $f^{-n}(V)$ for subvarieties $V \subset M$ of codimension p, see [16].

Remark 1.5. — By definition $\lambda_0(f) = 1$; $\lambda_d(f)$ is called the topological degree of f: it is equal to the number of points in a generic fibre of f.

The *topological entropy* of a continuous map of a topological space is a non-negative number, possibly infinite, which gives a measure of the chaos created by the map and its iterates; for a precise definition see [17]. The computation of the topological entropy of a map is usually complicated, and requires ad hoc arguments; however, in the case of dominant holomorphic self-maps of compact Kähler manifolds one can apply the following result due to Yomdin [25] and Gromov [15]:

Theorem 1.6 (Yomdin-Gromov). — *Let $f\colon M \to M$ be a dominant holomorphic self-map of a compact Kähler manifold of dimension d; then the topological entropy of f is given by*

$$h_{top}(f) = \max_{p=0,\dots,d} \log \lambda_p(f).$$

In the meromorphic setting one can only prove that the topological entropy is bounded from above by the maximum of the logarithms of the dynamical degrees (see [11, 10]).

1.3.1. *Concavity properties.* —

Theorem 1.7 (Teissier-Khovanskii, see [15]). — *Let X be a compact Kähler manifold of dimension d, and $\Omega := (\omega_1, \dots, \omega_k)$ be k-tuple of Kähler forms on X. For any multi-index $I = (i_1, \dots, i_k)$ let $\Omega^I = \omega_1^{i_1} \wedge \dots \wedge \omega_k^{i_k}$.*
Fix $i_3, \dots, i_k$ so that $i := \sum_{h\geq 3} i_h \leq d$, and let $I_p = (p, d-i-p, i_3, \dots, i_k)$; then the function

$$p \mapsto \log\left(\int_X \Omega^{I_p}\right)$$

is concave on the set $\{0, 1, \dots, d-i\}$.

One can use Theorem 1.7 to prove the following log-concavity result:

Proposition 1.8. — *Let $f\colon X \dashrightarrow X$ be a dominant meromorphic self-map of a compact Kähler manifold X of dimension d. Then the function*

$$p \mapsto \log \lambda_p(f)$$

is concave on the set $\{0, 1, \dots, d\}$. In particular, if $\lambda_1(f) = 1$ then $\lambda_p(f) = 1$ for all $p = 0, \dots, d$.

The following result was proved in [8] for automorphisms, but the same proof adapts to the case of dominant meromorphic self-maps.

Proposition 1.9. — *Let $f\colon X \dashrightarrow X$ be a dominant meromorphic self-map of a compact Kähler manifold X, and let*

$$r_{p,q}(f) = \lim_{n\to+\infty} \|(f^n)^*_{p,q}\|^{\frac{1}{n}};$$

here we use the convention that the norm of the identity on the null vector space is equal to 1.

Then

$$r_{p,q} \leq \sqrt{\lambda_p(f)\lambda_q(f)}.$$

In particular

$$\lim_{n+\infty} \|(f^n)^*\|^{\frac{1}{n}} = \max_p \lambda_p(f).$$

In other words, the exponential growth of the matrix norm of $(f^n)^*$ can be detected on the restriction to the intermediate Hodge spaces $H^{p,p}(X)$.

Corollary 1.10. — *Let $f\colon X \to X$ be a dominant holomorphic endomorphism of a compact Kähler manifold X. Then the following are equivalent:*

1. $\lambda_1(f) = 1$*;*
2. f^* *is virtually unipotent;*
3. $r_{p,q}(f) = 1$ *for all* p, q*.*

Proof. — The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are evident; let us show that (1) $\Rightarrow$ (2).

Since by Proposition 1.8 $\lambda_0(f) = \lambda_1(f) = \ldots = \lambda_d(f) = 1$, Proposition 1.9 implies that $r_{p,q}(f) \leq 1$ for all p, q. Therefore the spectral radii of the linear automorphisms

$$f_k^*\colon H^k(X,\mathbb{Z}) \to H^k(X,\mathbb{Z})$$

are equal to 1. It follows from a lemma of Kronecker that the roots of the characteristic polynomial of f_k^* are roots of unity; therefore, some iterate of f^* has 1 as its only eigenvalue, i.e. it is unipotent. This concludes the proof. □

Note that in the situation of Corollary 1.10 we have $\lambda_d(f) = 1$, so that f is an automorphism.

1.3.2. *Polynomial growth.* — Suppose now that $f\colon X \to X$ is a dominant holomorphic endomorphism, and assume that $\lambda_1(f) = 1$; then by Corollary 1.10 all the eigenvalues of f^* have modulus 1 (and in particular f is an automorphism), and an easy linear algebra argument implies that

$$\|(f^n)^*_{p,p}\| \sim cn^{\mu_p(f)},$$

where $\mu_p(f) + 1$ is the maximal size of Jordan blocks of $f^*_{p,p}$. Then one can define a number measuring the polynomial growth of $(f^n)^*$.

Definition 1.11. — Suppose that $\lambda_p(f)=1$; the p-th polynomial dynamical degree is defined as

$$\mu_p(f)=\lim_{n\to+\infty}\frac{\log\|(f^n)^*_{p,p}\|}{\log n}.$$

The following question is still open even for birational maps of $\mathbb{P}^k(\mathbb{C})$, $k\geq 3$.

Question 1.12. — *Let $f\colon X\dashrightarrow X$ be a meromorphic self-map of a compact Kähler manifold such that $\lambda_1(f)=1$; is it true that $\|(f^n)^*\|$ grows polynomially?*

The inequalities of Tessier-Khovanskii allow proving of an equivalent of Proposition 1.8 and Proposition 1.9:

Proposition 1.13. — *Let $f\colon X\to X$ be an automorphism of a compact Kähler manifold such that $\lambda_p(f)=1$; then*

1. *the function*
$$p\mapsto\mu_p(f)$$
is concave on the set $\{0,\dots,d\}$;
2. $$\lim_{n\to+\infty}\frac{\log\|(f^n)^*\|}{\log n}=\max_p\mu_p(f).$$

1.4. Generalized Hodge index theorem. — The classical Hodge index theorem asserts that if S is a compact Kähler surface, then the intersection product on $H^{1,1}(X,\mathbb{R})$ is hyperbolic, i.e. it has signature $(1,h^{1,1}(S)-1)$; this is a consequence of the Hodge-Riemann bilinear relations, which can be generalized in higher dimensions in order to obtain an analogue of the classical result. We will focus on the second cohomology group, but analogue results exist for cohomology of any order (see [9]).

Let (X,ω) be a compact Kähler manifold of dimension $d\geq 2$; we define a quadratic form q on $H^2(X,\mathbb{R})$ by

$$q(\alpha,\beta):=\int_X(\alpha_{1,1}\wedge\beta_{1,1}-\alpha_{2,0}\wedge\beta_{0,2}-\alpha_{0,2}\wedge\beta_{2,0})\wedge\omega^{d-2}\quad \alpha,\beta\in H^2(X,\mathbb{R}),$$

where $\alpha_{i,j}$ (resp. $\beta_{i,j}$) denotes the (i,j)-part of α (resp. of β).

Note that the decomposition

$$H^2(X,\mathbb{R})=H^{1,1}(X,\mathbb{R})\oplus(H^{2,0}(X)\oplus H^{0,2}(X))_{\mathbb{R}}$$

is q-orthogonal.

Theorem 1.14 (Generalized Hodge index theorem). — *Let (X,ω) be a compact Kähler manifold of dimension $d\geq 2$ and let q be defined as above. Then the restriction of q to $H^{1,1}(X,\mathbb{R})$ has signature $(1,h^{1,1}(X)-1)$; its restriction to $(H^{2,0}(X)\oplus H^{0,2}(X))_{\mathbb{R}}$ is negative definite.*

An immediate consequence, which we will use constantly in the rest of the paper, is the following:

Corollary 1.15. — *If* $V \subset H^2(X,\mathbb{R})$ *is a q-isotropic space, then* $\dim V < 2$*. In particular, if* $u, v \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ *are linearly independent classes, then the classes*

$$u \wedge u, u \wedge v, v \wedge v \in H^4(X,\mathbb{R})$$

cannot all be null. If furthermore $u \in (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$*, then* $u \wedge u \neq 0$*.*

Analogously, if $u, v \in H^{1,1}(X) \cup (H^{2,0}(X) \oplus H^{0,2}(X))$ *are linearly independent classes, then the classes*

$$u \wedge \bar{u}, u \wedge \bar{v}, v \wedge \bar{v} \in H^4(X,\mathbb{C})$$

cannot all be null. If furthermore $u \in H^{2,0}(X) \oplus H^{0,2}(X)$*, then* $u \wedge \bar{u} \neq 0$*.*

2. The case of surfaces

Note first that the case of automorphisms of curves is dynamically not very interesting: indeed, if the genus of the curve is $g \geq 2$, then the group of automorphism is finite; the only non-trivial dynamics arise from automorphisms of $\mathbb{P}^1$ and from automorphisms of elliptic curves (which, up to iteration, are translations), and both are well-understood.

Let us focus then on the surface case: let S be a compact Kähler surface and let $f\colon S \to S$ be an automorphism. By the Hodge index theorem (Theorem 1.14), the generalized intersection form q makes $H^2(X,\mathbb{R})$ into a hyperbolic space; furthermore, q is preserved by f, so that we may consider $g = f_2^*\colon H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ an element of $O(H^2(X,\mathbb{R}), q)$.

2.1. Automorphisms of hyperbolic spaces. — Let (V, q) be a hyperbolic vector space of dimension n and let $\|\cdot\|$ be a norm on the space $\mathcal{L}(V)$ of linear endomorphisms of V.

Definition 2.1. — Let $g \in O(V, q)$. We say that g is

- *loxodromic* (or *hyperbolic*) if it admits an eigenvalue of modulus strictly greater than 1;
- *parabolic* if all its eigenvalues have modulus 1 and $\|g^n\|$ is not bounded as $n \to +\infty$;
- *elliptic* if all its eigenvalues have modulus 1 and $\|g^n\|$ is bounded as $n \to +\infty$.

In each of the cases above, simple linear algebra arguments allow further descriptions of the situation. For the following result see for example [14].

Theorem 2.2. — *Let* $g \in O^+(V, q)$*, and suppose that* g *preserves a lattice* $\Gamma \subset V$*.*

- *If g is loxodromic, then it is semisimple and it has exactly one eigenvalue λ with modulus > 1 and exactly one eigenvalue λ^{-1} with modulus < 1; these eigenvalues are real and simple, so that in particular $\|g^n\| \sim c\lambda^n$. The eigenvalue λ is an algebraic integer whose conjugates over $\mathbb{Q}$ are λ^{-1} and complex numbers of modulus 1, i.e. λ is a quadratic or Salem number.*
- *If g is parabolic, then all the eigenvalues of g are roots of unity, and some iterate of g has the Jordan form*

$$\begin{pmatrix} 1 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 1 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{d-3} \end{pmatrix}.$$

In particular $\|g^n\| \sim cn^2$.
- *If g is elliptic, then it has finite order.*

An automorphism $f\colon S \to S$ of a compact Kähler surface S is called loxodromic, parabolic or elliptic if $g = f_2^*$ is loxodromic, parabolic or elliptic respectively. Note that g preserves the integral lattice $H^2(X, \mathbb{Z})/(\text{torsion})$, so that Theorem 2.2 can be applied to g.

Note that, if f is homotopic to the identity, then its action on cohomology is trivial. Conversely, if f acts trivially on cohomology, then some of its iterates is homotopic to the identity. More precisely:

Theorem 2.3 (Fujiki, Liebermann [12, 20]). — *Let M be a compact Kähler manifold. If $[\kappa]$ is a Kähler class on M, the connected component of the identity* $\mathrm{Aut}(X)^0$ *has a finite index in the group of automorphisms of M fixing $[\kappa]$.*

This implies that a surface automorphism is elliptic if and only if one of its iterates is homotopic to the identity.

2.2. Equivariant fibrations. — In the case of surfaces, it turns out that the cohomological action of an automorphism has consequences on the following property of decomposability of its dynamics. Let $f\colon M \to M$ be an automorphism of a compact Kähler manifold M; we say that a fibration $\pi\colon M \to B$ (i.e. a surjective map with connected fibers) is *f-equivariant* if there exists an automorphism $g\colon B \to B$ such that $\pi \circ f = g \circ \pi$, i.e. the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{g} & B \end{array}.$$

The following theorem was stated and proved in the present form by Cantat [4], and follows from a result of Gizatullin (see [13], or [14] for a survey); see also [7] for the birational case.

THEOREM 2.4. — *Let S be a compact Kähler surface and let f be an automorphism of S.*

1. *If f is parabolic, there exists an f-equivariant elliptic fibration $\pi\colon S \to C$; f doesn't admit other equivariant fibrations.*
2. *Conversely, if a non-elliptic automorphism of a surface $f\colon S \to S$ admits an equivariant fibration $\pi\colon S \to C$ onto a curve, then f is parabolic. In particular, the fibration π is elliptic, and it is the only equivariant fibration.*

In other words, a non-elliptic automorphism of a surface admits an equivariant fibration if and only if its topological entropy is zero.

In higher dimensions, one can ask the following question:

QUESTION 2.5. — *Let $f\colon X \to X$ be an automorphism of a compact Kähler manifold M. Suppose that f_2^* is virtually unipotent of infinite order. Does f admit an equivariant fibration?*

Apart from the case of surfaces, the only situation where the answer is known (and affirmative) is that of irreducible holomorphic symplectic (or hyperkähler) manifolds of deformation type $K3^{[n]}$ or generalized Kummer (see [18]); the proof uses the hyperkähler version of the abundance conjecture, which was proved in this context by Bayer and Macrí [1]. See also [21] for a converse statement, which holds for any irreducible holomorphic symplectic manifold.

3. Automorphisms of threefolds: the unipotent case

Throughout this section let X be a compact Kähler threefold and let

$$g\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$$

be a *unipotent* linear automorphism preserving the cohomology graduation, the Hodge decomposition, the wedge-product and Poincaré's duality (properties (1)–(3) in Section 1.1).

If $f\colon X \to X$ is an automorphism such that $\lambda_1(f) = 1$, then the linear automorphism $f^*\colon H^*(X,\mathbb{R})$ is virtually unipotent, and therefore an iterate $g = (f^N)^*$ satisfies the assumptions above.

More generally, if $f\colon X \to X$ is any automorphism and

$$f^* = g_u g_s = g_s g_u$$

is the Jordan decomposition of f^*, then by Lemma 1.2 the unipotent part $g = g_u$ satisfies the assumptions above.

Theorem A is thus a special case of the following theorem and of Proposition 3.4.

Theorem 3.1. — *Let X be a compact Kähler threefold and let $g\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$ be a non-trivial unipotent linear automorphism preserving the cohomology graduation, the Hodge decomposition, the wedge-product and the Poincaré duality. Then*

1. *the maximal Jordan block of g_2 (for the eigenvalue 1) has dimension ≤ 5;*
2. *if furthermore g_2 preserves the cone $\mathcal{C} = \{v \in H^2(X,\mathbb{R})\,;\, q(v) \geq 0\}$, then its maximal Jordan block has an odd dimension.*

In particular the norm of g_2^n grows as cn^k with $k \leq 4$; and furthermore, if g_2 preserves the positive cone, then k is even.

Remark 3.2. — Let $f \in \mathrm{Aut}(X)$ be an automorphism such that $\lambda_1(f) = 1$, so that, up to iterating f, $g = f^*$ satisfies the assumptions of Theorem 3.1. In this case, by Proposition 1.13, the growth of $\|g^n\|$ is the same as the maximal growths of the $\|g^n_{p,p}\|$, i.e. the growth of $\|g^n_{1,1}\|$ by Poincaré duality.

Furthermore, g preserves the cone $\mathcal{C}$, therefore the maximal Jordan block has an odd dimension by [2].

Proof. — Let $u_1, \ldots, u_k \in H^2(X,\mathbb{R})$ be a basis of a maximal Jordan block satisfying

$$g(u_1) = u_1, \qquad g(u_h) = u_h + u_{h-1} \quad \text{for } h = 2, \ldots, k.$$

By Remark 1.3 we can suppose that $u_i \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ for $i = 1, \ldots, k$.

We show that the norm of g_4^n grows at least as cn^{2k-6}. Let us consider the elements $u_k \wedge u_k, u_{k-1} \wedge u_{k-1} \in H^4(X,\mathbb{R})$.

Using the notation of Lemma 3.3 below, let $P_h = P_h(n)$; then

$$\begin{aligned} &g^n(u_k \wedge u_k) = g^n(u_k) \wedge g^n(u_k) = \\ &= (P_{k-1}u_1 + P_{k-2}u_2 + P_{k-3}u_3 + \ldots) \wedge (P_{k-1}u_1 + P_{k-2}u_2 + P_{k-3}u_3 + \ldots) = \\ &= P_{k-1}^2(u_1 \wedge u_1) + 2P_{k-1}P_{k-2}(u_1 \wedge u_2) + (2P_{k-1}P_{k-3}u_1 \wedge u_3 + P_{k-2}^2 u_2 \wedge u_2) + \ldots \end{aligned}$$

If $u_1 \wedge u_1 \neq 0$ or $u_1 \wedge u_2 \neq 0$, then the norm of g_4^n would grow at least as n^{2k-3}; we can thus assume that $u_1 \wedge u_1 = u_1 \wedge u_2 = 0$. Thus, by Corollary 1.15 we have $u_2 \wedge u_2 \neq 0$.

Let us apply Lemma 3.3 and look at the dominant terms of P_{k-1}, P_{k-2} and P_{k-3}; if we had

$$2\frac{u_1 \wedge u_3}{(k-1)!(k-3)!} + \frac{u_2 \wedge u_2}{((k-2)!)^2} \neq 0, \tag{2}$$

then the norm of g_4^n would grow at least as cn^{2k-4}. We may then assume that (2) is not satisfied.

Now,

$$\begin{aligned} g^n(u_{k-1}\wedge u_{k-1}) &= g^n(u_{k-1})\wedge g^n(u_{k-1}) = \\ = (P_{k-2}u_1 + P_{k-3}u_2 + P_{k-4}u_3 + \ldots) &\wedge (P_{k-2}u_1 + P_{k-3}u_2 + P_{k-4}u_3 + \ldots) = \\ &= 2P_{k-2}P_{k-4}(u_1\wedge u_3) + P_{k-3}^2(u_2\wedge u_2) + \ldots \end{aligned}$$

By the same argument, if

$$2\frac{u_1\wedge u_3}{(k-2)!(k-4)!} + \frac{u_2\wedge u_2}{((k-3)!)^2} \neq 0, \tag{3}$$

then the norm of g_4^n would grow at least as cn^{2k-6}. Since $u_2\wedge u_2 \neq 0$ and the two linear relations 2 and 3 are independent, at least one between (2) and (3) is satisfied.

This shows that the norm of g_4^n grows at least as cn^{2k-6}.

Now, by Poincaré duality (property (3) in Section 1.1), the norm of $g_4^n = (g_2^{-n})^\vee$ grows exactly as the norm of g_2^n. In particular

$$k-1 \geq 2k-6 \qquad \Rightarrow \qquad k \leq 5,$$

which concludes the proof. □

Lemma 3.3. — *Let*

$$A = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 1 \\ 0 & \ldots & & 0 & 1 \end{pmatrix}$$

be a Jordan block of dimension k. Then

$$A^n = \begin{pmatrix} P_0(n) & P_1(n) & P_2(n) & \ldots & P_{k-1}(n) \\ 0 & P_0(n) & P_1(n) & \ldots & P_{k-2}(n) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & P_0(n) & P_1(n) \\ 0 & \ldots & & 0 & P_0(n) \end{pmatrix},$$

where $P_i(n)$ is a polynomial of degree i in n whose leading term is $n^i/i!$.

Proof. — The functions P_i satisfy the equation

$$P_{i+1}(n) = P_{i+1}(n-1) + P_i(n-1),$$

therefore one obtains recursively

$$P_{i+1}(n) = \sum_{j=0}^{n} P_i(j) \qquad i = 0,\ldots,k-2. \tag{4}$$

The claim is clear for $i = 0$; reasoning inductively, we may assume that P_i is a polynomial of degree i whose leading coefficient is $1/i!$.

In order to conclude, one needs only recall the well-known fact that for any fixed m the function

$$n \mapsto \sum_{k=0}^{n} k^m$$

is a polynomial q_m of degree $m+1$ with leading coefficient $1/(m+1)$. This can be proven by writing

$$n^{m+1} = \sum_{k=0}^{n-1}((k+1)^{m+1} - k^{m+1}) = \sum_{h=0}^{m} \binom{m+1}{h} q_h(n)$$

and by applying induction to m.

Therefore, writing

$$P_{i-1}(n) = \frac{1}{(i-1)!}n^{i-1} + a_{i-1}n^{i-2} + \ldots + a_0,$$

we have

$$P_i(n) = \frac{1}{(i-1)!}q_{i-1}(n) + a_{i-1}q_{i-2}(n) + \ldots + a_0 q_0(n),$$

and the claim follows easily. □

3.1. Bound on the dimension of non-maximal Jordan blocks. —

Proposition 3.4. — *Let X be a compact Kähler threefold and let $g\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$ be a non-trivial unipotent linear automorphism preserving the cohomology graduation, the Hodge decomposition, the wedge-product and the Poincaré duality (properties (1)–(3) in Section 1.1). Then there exists a unique Jordan block of g_2 of maximal dimension $k \leq 5$ (for the eigenvalue 1); more precisely, all other Jordan blocks have dimension $\leq \frac{k+1}{2}$.*

Proof. — Let $v_1, \ldots, v_k \in H^2(X,\mathbb{R})$ form a basis for a maximal Jordan block of g_2, and let $w_1, \ldots, w_l \in H^2(X,\mathbb{R})$ form a Jordan basis for another Jordan block satisfying

$$\begin{aligned} g(v_1) &= v_1, \qquad g(v_i) = v_i + v_{i-1} \quad \text{for } i = 2, \ldots, k,\\ g(w_1) &= w_1, \qquad g(w_j) = w_j + w_{j-1} \quad \text{for } j = 2, \ldots, l. \end{aligned}$$

By Remark 1.3 we can suppose that $v_i, w_j \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$.

We will suppose that $l > 1$ (otherwise the claim is evident), and consider the action of g_4 on the classes $v_k \wedge v_k, v_k \wedge w_l, w_l \wedge w_l \in H^4(X,\mathbb{R})$.

By Corollary 1.15, the classes $v_1 \wedge v_1, v_1 \wedge w_1, w_1 \wedge w_1 \in H^4(X,\mathbb{R})$ cannot be all null; since $g^n v_k \sim c n^{k-1} v_1$ and $g^n w_l \sim c' n^{h-1} w_1$, this implies that $\|g_4^n\|$

grows at least as $c''n^{2(l-1)}$. Since by Poincaré duality $\|g_2^n\|$ and $\|g_4^n\|$ have the same growth, we get

$$2(l-1) \leq k-1 \qquad \Rightarrow \qquad l \leq \frac{k+1}{2},$$

which concludes the proof. □

3.2. Unipotent examples on complex tori. — Examples on complex tori of dimension 3 show the optimality of Theorem 3.1 and Proposition 3.4. Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, where Λ is a lattice of $\mathbb{C}$, and let

$$X := E \times E \times E = {}^{\mathbb{C}^3}\!\!\diagup_{\Lambda \times \Lambda \times \Lambda}.$$

Every matrix $M \in \mathrm{SL}_3(\mathbb{Z})$ acts linearly on $\mathbb{C}^3$ preserving the lattice $\Lambda \times \Lambda \times \Lambda$, and therefore induces an automorphism $f\colon X \to X$. One can easily show that, if dx, dy, dz are holomorphic linear coordinates on the three factors respectively, then the matrix of $f^*_{1,0}$ with respect to the basis dx, dy, dz of $H^{1,0}(X)$ is exactly the transposed M^T. Since the wedge product of forms induces an isomorphism

$$H^{1,1}(X) \cong H^{1,0}(X) \otimes H^{0,1}(X),$$

the matrix of $f^*_{1,1}$ with respect to the basis $dx \wedge d\bar{x}, dx \wedge d\bar{y}, \dots, dz \wedge d\bar{z}$ is

$$M_{1,1} = M^t \otimes \overline{M^t} := (m_{j,i}\overline{m_{l,k}})_{i,j,k,l=1,2}.$$

Example 3.5. — Let

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then

$$M_{1,1} = \begin{pmatrix} 1&1&0&1&1&0&0&0&0 \\ 0&1&1&0&1&1&0&0&0 \\ 0&0&1&0&0&1&0&0&0 \\ 0&0&0&1&1&0&1&1&0 \\ 0&0&0&0&1&1&0&1&1 \\ 0&0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

$M_{1,1}$ is unipotent and its Jordan blocks have dimension 1, 3 and 5.

Example 3.6. — Let

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$M_{1,1} = \begin{pmatrix} 1&1&0&1&1&0&0&0&0\\ 0&1&0&0&1&0&0&0&0\\ 0&0&1&0&0&1&0&0&0\\ 0&0&0&1&1&0&1&1&0\\ 0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

$M_{1,1}$ is unipotent and its Jordan blocks have dimension 2, 2, 2 and 3.

4. Automorphisms of threefolds: the semisimple case, proof of Theorem B

Throughout this section let X be a compact Kähler threefold and let

$$g\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$$

be a *semisimple* linear automorphism preserving the cohomology graduation, the Hodge decomposition, the wedge-product and Poincaré's duality (properties (1)–(3) in Section 1.1).

If $f\colon X \to X$ is an automorphism and

$$f^* = g_u g_s = g_s g_u$$

is the Jordan decomposition of f^*, then by Lemma 1.2 the semisimple part $g = g_s$ satisfies the assumptions above.

Let $\lambda_1 = \lambda_1(g)$ and $\lambda_2 = \lambda_2(g)$ be the dynamical degrees of g, i.e. the spectral radii of g_2 and g_4 respectively, and let Λ be the spectrum of g_2, i.e. the set of complex eigenvalues of g_2 with multiplicities; we will say that two elements $\lambda, \lambda' \in \Lambda$ are distinct if either $\lambda \neq \lambda'$ or $\lambda = \lambda'$ is an eigenvalue with multiplicity ≥ 2.

The main ingredient of the proofs in the rest of this section is the following lemma.

Lemma 4.1. — *Let $g \in \mathrm{GL}(H^*(X,\mathbb{R}))$ be a semisimple linear automorphism preserving the cohomology graduation, the Hodge decomposition, the wedge-product and Poincaré's duality and let Λ be its spectrum (with multiplicities taken into account).*

If $\lambda, \lambda' \in \Lambda$ are distinct elements, then

$$\left\{ \frac{1}{|\lambda|^2}, \frac{1}{\lambda\bar{\lambda}'}, \frac{1}{|\lambda'|^2} \right\} \cap \Lambda \neq \emptyset.$$

Proof. — By Remark 1.3, we may pick eigenvectors $v, v' \in H^{2,0}(X) \cup H^{1,1}(X) \cup H^{0,2}(X)$ for the eigenvalues λ, λ' respectively. By Corollary 1.15, the wedge products

$$v \wedge \bar{v}, v \wedge \bar{v}', v' \wedge \bar{v}' \in H^4(X, \mathbb{C})$$

cannot all be null. A non-null wedge product gives rise to an eigenvector for g_4; in particular, denoting by Λ_4 the spectrum of g_4, we have

$$\left\{|\lambda|^2, \lambda\bar{\lambda}', |\lambda'|^2\right\} \cap \Lambda_4 \neq \emptyset.$$

Now, by assumption (3) g_4 can be identified with $(g_2^{-1})^\vee$, and in particular

$$\Lambda = \Lambda_4^{-1} = \{\lambda^{-1}\, ,\, \lambda \in \Lambda_4\}.$$

This concludes the proof. □

Before passing to the actual proof of Theorem B let us outline the strategy we will adopt.

1. In Section 4.1 we will study the number $r(g)$ of multiplicative parameters which describe the moduli of eigenvalues of g_2; this can be formally defined as the split-rank of the real algebraic group G generated by g_2.
2. In Section 4.2 we define the *weights* of g (or rather of g_2) as the real characters

   $$w_\lambda \colon G \to \mathbb{R}^*$$

   such that $g \mapsto |\lambda|$ for any eigenvalue λ; for the sake of clarity we will adopt an additive notation on weights.

   Using an immediate consequence of Lemma 4.1 (Lemma 4.7), in Lemma 4.9 we prove that

   $$r(g) \leq 2.$$

3. We conclude by considering the cases $r(g) = 2$ and $r(g) = 1$ separately (Section 4.3 and Section 4.4 respectively). In both cases the proof relies essentially on Lemma 4.7.

4.1. Structure of the algebraic group generated by g_2. — For the content of this Section we refer to [3, Section 8]. Let g be as above and let

$$G = \overline{\langle g_2 \rangle}^{Zar} \leq \mathrm{GL}(H^2(X, \mathbb{R}))$$

be the Zariski-closure of the group generated by g; it is a real algebraic group by [3, Proposition I.1.3]; furthermore, since $\langle g \rangle$ is diagonalizable over $\mathbb{C}$ and commutative, so is G.

The Zariski-connected component of the identity G_0 of G is thus a real algebraic torus; we define $G_d \leq G_0$ as the subgroup generated by real one-parameter subgroups of G_0, and $G_a \leq G_0$ as the intersection of the kernels of real characters of G_0. Then we have the following classical result:

Proposition 4.2. — *Let G be as above; then*

1. $G_d \cong (\mathbb{R}^*)^r$ *is the maximal split subtorus;*
2. $G_a \cong (S^1)^s$ *is the maximal anisotropic subtorus;*
3. *the product morphism $G_d \times G_a \to G_0$ is an isogeny (i.e. it is surjective and with finite kernel).*

The number $r \geq 0$ is the (real) split-rank of G; we will denote it by $r(g)$ and call it the *rank of* g; informally, $r(g)$ (respectively $s(g)$) is the number of multiplicative parameters which are necessary to describe the moduli (respectively, the arguments) of the complex eigenvalues of g_2 (see Lemma 4.5).

4.2. Weights of *g*. — Let $\lambda \in \Lambda$ be a complex eigenvalue of g_2; then the group homomorphism

$$\begin{aligned} \langle g \rangle &\to \mathbb{R}^* \\ g^n &\mapsto |\lambda|^n \end{aligned}$$

is algebraic, and therefore can be extended to a non-trivial real character of G. Upon restriction to G_0 and pull-back to $G_d \times G_a \cong (\mathbb{R}^*)^r \times (S^1)^s$, this yields a non-trivial morphism of real algebraic groups

$$\rho_\lambda \colon (\mathbb{R}^*)^r \times (S^1)^s \to \mathbb{R}^*.$$

Since all morphisms of real algebraic groups $S^1 \to \mathbb{R}^*$ are trivial, we have

$$\rho_\lambda(x_1 \dots, x_r, \theta_1, \dots, \theta_s) = x_1^{m_1(\lambda)} \cdots x_r^{m_r(\lambda)}, \qquad m_i \in \mathbb{Z}.$$

For the sake of simplicity, we will adopt an additive notation, so that the character ρ_λ is identified with the vector $w_\lambda = (m_1(\lambda), \dots, m_r(\lambda)) \in \mathbb{R}^r$.

Definition 4.3. — The *weight* of the eigenvalue $\lambda \in \Lambda$ of g_2 is the vector $w_\lambda = (m_1(\lambda), \dots, m_r(\lambda)) \in \mathbb{R}^r$. We denote by W the set of all weights of eigenvalues of g_2 with multiplicities; as for the elements of Λ, we say that two elements w, w' of W are distinct if either $w \neq w'$ or $w = w'$ has multiplicity > 1.

Remark 4.4. — Note that $w_\lambda = w_{\bar{\lambda}}$. Therefore, if λ is a non-real eigenvalue of g_2, the weight w_λ will be counted twice, once for λ and once for $\bar{\lambda}$.

We say that a weight $w_0 \in W$ is *maximal* for a linear functional $\alpha \in (\mathbb{R}^r)^\vee$ if $|\alpha(w_0)| = \max_{w \in W} |\alpha(w)|$; we simply say that w is maximal if it is maximal for some linear functional. The maximal weights are exactly those belonging to the boundary of the convex hull of W in $\mathbb{R}^r$.

Lemma 4.5. — *The weights of g_2 span a real vector space of dimension $r(g)$.*

Proof. — Note first that $r(g)$ is equal to the rank over $\mathbb{Z}$ of the group of real characters

$$G_0 \to \mathbb{R}^*.$$

Therefore, in order to prove the claim one only needs to show that any real character of G_0 can be written as a product of the ρ_λ. Of course, one can check such a property on g and its iterates.

Since g is semisimple, one can find a real basis of $V = H^*(X, \mathbb{R})$ in which g is written as

$$g = \begin{pmatrix} \lambda_1 R_{\theta_1} & & & & & & \\ & \lambda_2 R_{\theta_2} & & & & & \\ & & \ddots & & & & \\ & & & \lambda_k R_{\theta_k} & & & \\ & & & & \lambda_{k+1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n \end{pmatrix},$$

where R_θ denotes the rotation matrix for the angle θ and $\lambda_i \in \mathbb{R}^*$; actually, after possibly replacing g by g^2 (which doesn't change G_0) we may assume that $\lambda_i \in \mathbb{R}^+$.

Now let $G_\mathbb{C}$ be the complexification of G_0; then, upon diagonalizing g in $\mathrm{GL}(H^*(X, \mathbb{C}))$, we can naturally see $G_\mathbb{C}$ as a subgroup of a complex torus

$$(\mathbb{C}^*)^{n+k},$$

where the first n coordinates correspond to the λ_i's and the last k correspond to the rotations R_{θ_j}. More explicitly, considering the above base of $H^*(X, \mathbb{R})$ and the corresponding matrix coordinates $a_{i,j}$ of $\mathrm{GL}(H^*(X, \mathbb{R}))$, define

$$\begin{aligned} x_i &= a_{2i-1,2i-1}a_{2i,2i} - a_{2i,2i-1}a_{2i-1,2i} \qquad i = 1, \ldots, k \\ x_i &= a^2_{i+k,i+k} \qquad i = k+1, \ldots, n \\ y_j &= (a_{2j-1,2j-1} + i a_{2j,2j-1})/x_j \qquad j = 1, \ldots, k. \end{aligned}$$

Then it is not hard to see that the x_i's and the y_j's are multiplicative coordinates of a complex torus D_{n+k} containing $G_\mathbb{C}$.

The (x, y) coordinates of the element g are

$$x_i = \lambda_i^2 \quad i = 1, \ldots, n, \qquad y_j = e^{i\theta_j} \quad j = 1, \ldots, k.$$

Now, every complex character

$$G_\mathbb{C} \to \mathbb{C}^*$$

can be written as the restriction to $G_\mathbb{C}$ of a product

$$\chi\colon (x_1, \ldots, x_n, y_1, \ldots, y_k) \mapsto x_1^{A_1} \cdots x_n^{A_n} y_1^{B_1} \cdots y_k^{B_k}.$$

In order for χ to be real in restriction to G_0, we need to have

$$\chi(\lambda_1^2, \ldots, \lambda_n^2, e^{i\theta_1}, \ldots, e^{i\theta_k}) \in \mathbb{R}.$$

This implies that

$$\sum_{j=1}^{k} B_j \theta_j \in \pi\mathbb{Z},$$

meaning that, in restriction to g and its iterates, χ coincides, maybe up to a sign, with

$$\chi' \colon (x_1, \ldots, x_n, y_1, \ldots, y_k) \mapsto x_1^{A_1} \cdots x_n^{A_n}.$$

However, two real characters on a connected group which can at most differ by a sign are equal, thus $\chi_{|G_0} = \chi'_{|G_0}$. By the initial remark, this proves the claim. □

Lemma 4.6. — *there exists a basis $w_1, \ldots, w_r$ of $\mathbb{R}^r$ and a basis $\alpha_1, \ldots, \alpha_r$ of $(\mathbb{R}^r)^\vee$ such that*

1. *the w_i belong to W;*
2. *w_i is α_i-maximal for all i;*
3. *if $i > j$, then $\alpha_i(w_j) = 0$.*

Proof. — First, by Lemma 4.5 the elements of W span the vector space $\mathbb{R}^r$.

We construct the adapted basis inductively. Since W is finite, there exists a maximal weight, say w_1, for a functional α_1.

Now, suppose that $w_1, \ldots, w_k \in W \subset \mathbb{R}^r$ and $\alpha_1, \ldots, \alpha_k \in (\mathbb{R}^r)^\vee$ are linearly independent and satisfy properties (1)–(3). Pick any

$$\alpha_{k+1} \in \{\alpha \in (\mathbb{R}^r)^\vee \mid \alpha(w_1) = \ldots = \alpha(w_k) = 0\} \setminus \{0\} \subset (\mathbb{R}^r)^\vee,$$

and let $w_{k+1} \in W$ be α_{k+1}-maximal. By the condition on α_{k+1}, w_{k+1} does not belong to the span of $w_1, \ldots, w_k$. This completes the proof by induction. □

In the language of weights, Lemma 4.1 becomes the following:

Lemma 4.7. — *Let $w, w' \in W$ be distinct elements; then*

$$\{-2w, -w - w', -2w'\} \cap W \neq \emptyset.$$

If furthermore $w = w_\lambda$ is the weight of an eigenvalue λ of $f_{2,0}^*$ or $f_{0,2}^*$, then $-2w \in W$.

Remark 4.8. — If $\lambda \in \Lambda$ has maximal weight, then $|\lambda|^{-2} \notin \Lambda$ (and in particular the weight $w_\lambda \in W$ is simple by Lemma 4.1); therefore, if w, w' are maximal weights, by Lemma 4.7, $-w - w' \in W$.

As a preliminary result, we bound the rank of g:

Lemma 4.9. — *The rank of g satisfies $r(g) \leq 2$.*

Proof. — Let us fix bases $w_1, \ldots, w_r$ and $\alpha_1, \ldots, \alpha_r$ of $\mathbb{R}^r$ and $(\mathbb{R}^r)^\vee$ respectively as in Lemma 4.6, and suppose by contradiction that $r \geq 3$.

Since w_1, w_2 and w_3 are maximal, we have $-2w_1, -2w_2, -2w_3 \notin W$, and therefore by Lemma 4.7

$$-w_2 - w_3, -w_3 - w_1 \in W.$$

Since $-w_2 - w_3$ and $-w_3 - w_1$ are both maximal for α_3, by Remark 4.8 we have

$$w_1 + w_2 + 2w_3 \in W.$$

However this contradicts the α_3-maximality of w_3. □

We will show later that the rank of g is < 2 if and only if its dynamical degrees $\lambda_1(g)$ and $\lambda_2(g)$ satisfy a multiplicative relation:

$$\lambda_1(g)^m = \lambda_2(g)^n, \qquad (m, n) \in \mathbb{N}^2 \setminus \{(0,0)\};$$

see Corollary 4.11.

4.3. The case $r(g) = 2$. — Throughout this section, we assume that the rank of g (i.e. the split-rank of $G = \overline{\langle g \rangle}^{Zar}$, see Section 4.1) is equal to 2; in other words, the elements of W span a real vector space of dimension 2.

Proposition 4.10. — *Let $g \in \mathrm{GL}(H^*(X, \mathbb{R}))$ be a semisimple linear automorphism preserving the cohomology graduation, Hodge decomposition, wedge product and Poincaré duality. Denote by $\lambda_1 = \lambda_1(g)$ and $\lambda_2 = \lambda_2(g)$ the spectral radii of g_2 and g_4 respectively, by Λ the spectrum of g_2 (with multiplicities) and by W the set of weights of eigenvalues $\lambda \in \Lambda$ (with multiplicities).*

Let w_1 (respectively w_2) be the weight of W associated to the eigenvalue $\lambda_1 \in \Lambda$ (respectively $\lambda_2^{-1} \in \Lambda$).

Assume that the rank of g is equal to 2. Then

1. *$\lambda_1^{-1}\lambda_2$ is an element of Λ, whose weight is $w_3 := -w_1 - w_2$;*
2. *w_1, w_2 and w_3 are maximal weights of W, and in particular they have multiplicity 1 in W; in other words, $\lambda_1, \lambda_2^{-1}$ and $\lambda_1^{-1}\lambda_2$ are simple eigenvalues g_2, and no other eigenvalue of g_2 has modulus $\lambda_1, \lambda_2^{-1}$ or $\lambda_1^{-1}\lambda_2$;*
3. *for any eigenvalue $\lambda \in \Lambda$ such that $|\lambda| \notin \{\lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2, 1\}$, we have $|\lambda|^{-2} \in \Lambda$;*
4. *there exists $n_1, n_2, n_3 \geq 0$ such that, up to multiplicities,*

$$W \setminus \{0\} = \bigcup_{i=1,2,3} \left\{ \frac{w_i}{(-2)^n} \,;\, n = 0, \ldots, n_i \right\}.$$

Proof. — Let us fix an adapted basis w_1, w_2 of $\mathbb{R}^2$ as in Lemma 4.6, and let $w_3 := -w_1 - w_2$. We show first properties $(2) - (4)$ for these w_1, w_2, w_3, and then that, after possibly permuting indices, w_1, w_2 and w_3 are the weights of $\lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2 \in \Lambda$ respectively.

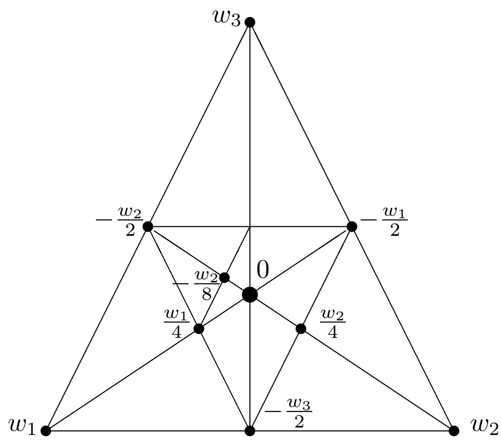

FIGURE 4.1. An example of the structure of $W \subset \mathbb{R}^2$ (without taking multiplicities into account) in the case $r(g) = 2$; here, with the notation of Proposition 4.10, $n_1 = 2, n_2 = 1, n_3 = 3$.

The maximality of w_1, w_2 is part of Lemma 4.6; since $\alpha_2(w_1) = 0$, w_3 is also α_2-maximal. Property (2) then follows from Remark 4.8.

Now let $\lambda \in \Lambda$ be an eigenvalue of g_2 whose weight w is different from w_1, w_2, w_3; we want to show that $|\lambda|^{-2} \in \Lambda$.

- Suppose first that $w = 0$ (i.e. $|\lambda| = 1$), and let $\lambda' \in \Lambda$ be an eigenvalue whose weight is w_1; recall that by maximality $-2w_1 \notin W$ and w_1 is a simple weight. If $|\lambda|^{-2} = 1 \notin \Lambda$, then by Lemma 4.1 we would have $\lambda\lambda' \in \Lambda$, which contradicts the simplicity of w_1.
- Now suppose that $\alpha_2(w) \neq 0$; since $\alpha_2(w_2)$ and $\alpha_2(w_3)$ have different signs, we have either $|\alpha_2(-w - w_2)| > |\alpha_2(w_2)|$ or $|\alpha_2(-w - w_3)| > |\alpha_2(w_3)| = |\alpha_2(w_2)|$, so that by maximality $-w - w_2$ and $-w - w_3$ cannot be both weights of W. Since, again by maximality, $-2w_2, -2w_3 \notin W$, by Lemma 4.1 $|\lambda|^{-2} \in \Lambda$.
- Finally suppose that $w \neq 0$ and $\alpha_2(w) = 0$, so that $w \in \mathbb{R}w_1$. We repeat the inductive construction of an adapted basis as in the proof of Lemma 4.6 starting with $w_1' := w_2$, which is maximal for $\alpha_1' := \alpha_2$; pick any non-trivial $\alpha_2' \in w_2^\perp \subset (\mathbb{R}^2)^\vee$ and $w_2' \in W$ maximal for α_2'. If we had again $\alpha_2'(w) = 0$, then $w \in \mathbb{R}w_1 \cap \mathbb{R}w_2 = \{0\}$, a contradiction; thus we can conclude as above.

This shows property (3).

Property (4) follows from property (3): indeed, if a w didn't satisfy property (4), by (3) we would have $(-2)^n w \in W \setminus \{0\}$ for all $n \in \mathbb{N}$, contradicting the finiteness of W.

Now let us show that, after permuting indices, w_1 and w_2 are the weights of λ_1 and λ_2^{-1} respectively. Since $\lambda_2(g) = \lambda_1(g^{-1})$, it is enough to show that the weight of λ_1 is one of the w_i.

Suppose by contradiction that the weight w of λ_1 is not one of the w_i; then by property (4) there exists $k > 0$ and $i \in \{1, 2, 3\}$ such that

$$w = \frac{w_i}{(-2)^k}.$$

Since λ_1 is the spectral radius of g_2, we have $\lambda_1^4 \notin \Lambda$; since by property (3) we have $(-2)^h w \in W$ for all $h = 0, \dots k$, we must have $k = 1$. Up to permuting the indices, we may suppose that $i = 1$, so that

$$2w = -w_1 = w_2 + w_3.$$

Denoting by $\lambda, \lambda' \in \Lambda$ the eigenvalues associated to w_2 and w_3, this means that

$$|\lambda\lambda'| = \lambda_1^2;$$

since λ_1 is the spectral radius of g_2, this implies that $|\lambda| = |\lambda'| = \lambda_1$, contradicting the assumption that $r(g) = 2$.

This shows that we may assume that w_1 and w_2 are the weights of the eigenvalues $\lambda_1, \lambda_2^{-1} \in \Lambda$. Since $w_3 = -w_1 - w_2$ has multiplicity 1 in W, it is associated to a real simple eigenvalue, which is $\lambda_1^{-1}\lambda_2$ by Lemma 4.1. This concludes the proof. □

Proposition 4.10 shows in particular that, if $r(g) = 2$, then λ_1 and λ_2 are multiplicatively independent:

$$\lambda_1(g)^m = \lambda_2(g)^n,\ m, n \in \mathbb{Z} \quad \Leftrightarrow \quad m = n = 0.$$

Indeed, the weights w_1 and w_2 form a base of $\mathbb{R}^2$.

Conversely, if $r = 1$, since all the weights of g_2 can be interpreted as integers, λ_1 and λ_2 satisfy a non-trivial equation $\lambda_1(g)^m = \lambda_2(g)^n$. Thus we have the following:

Corollary 4.11. — *The rank of g is equal to 2 if and only if the dynamical degrees of g are multiplicatively independent.*

4.4. The case $r(g) = 1$. — Recall that we denote by $\lambda_1 = \lambda_1(g)$ and $\lambda_2 = \lambda_2(g)$ the dynamical degrees of g, by Λ the spectrum of g_2 (with multiplicities) and by W the set of weights of eigenvalues $\lambda \in \Lambda$ (with multiplicities).

Throughout all this section, we assume that the rank of g (i.e. the split-rank of $G = \overline{\langle g \rangle}^{Zar}$, see Section 4.1) is equal to 1; in other words, the elements of W span a real vector space of dimension 1. In this case the weights are equipped with a natural order: $w_\lambda > w_{\lambda'}$ if and only if $|\lambda| > |\lambda'|$; for $w \in W$ we set $|w| := \max\{w, -w\}$.

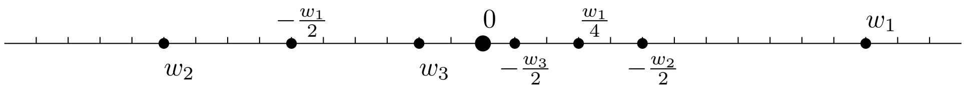

FIGURE 4.2. Example of weights of $W \subset \mathbb{R}$ (without taking multiplicities into account) in the case $r(g) = 1$; here, with the notation of Proposition 4.12, $n_1 = 2, n_2 = 1, n_3 = 1$.

PROPOSITION 4.12. — *Let $g \in \mathrm{GL}(H^*(X, \mathbb{R}))$ be a semisimple linear automorphism preserving the cohomology graduation, Hodge decomposition, wedge product and Poincaré duality. Denote by $\lambda_1 = \lambda_1(g)$ and $\lambda_2 = \lambda_2(g)$ the spectral radii of g_2 and g_4 respectively, by Λ the spectrum of g_2 (with multiplicities) and by W the set of weights of eigenvalues $\lambda \in \Lambda$ (with multiplicities).*

Let w_1 (respectively w_2) be the weight of W associated to the eigenvalue $\lambda_1 \in \Lambda$ (respectively $\lambda_2^{-1} \in \Lambda$).

Assume that the rank of g is equal to 1 (i.e. $\lambda_1 > 1$ and λ_1 and λ_2 are not multiplicatively independent). Then

1. *$\lambda_1^{-1}\lambda_2$ is an element of Λ, whose weight is $w_3 := -w_1 - w_2$;*
2. *for any eigenvalue $\lambda \in \Lambda$ such that $|\lambda| \notin \{\lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2, 1\}$, we have $|\lambda|^{-2} \in \Lambda$;*
3. *there exists $n_1, n_2, n_3 \geq 0$ such that, up to multiplicities,*

$$W \setminus \{0\} = \bigcup_{i=1,2,3} \left\{ \frac{w_i}{(-2)^n} \,;\, n = 0, \dots, n_i \right\}.$$

REMARK 4.13. — Let $g = f_s^*$, where $f\colon X \to X$ is an automorphism and f_s^* denotes the semisimple part of the induced linear automorphism $f^* \in \mathrm{GL}(H^*(X, \mathbb{R}))$. Then $w_2 \notin \{-2w_1, -w_1/2\}$ if and only if the log-concavity inequalities

$$\sqrt{\lambda_1(f)} \leq \lambda_2(f) \leq \lambda_1(f)^2$$

are strict (see Proposition 1.8). If this is the case, then by Proposition 4.10 and 4.12 λ_1 and λ_2^{-1} are the only eigenvalues of f_2^* having such modulus and they are simple. This was already proven in greater generality by Truong in [24].

Proof. — After possibly replacing g by g^{-1}, we may suppose that $w_1 = |w_1| \geq |w_2| = -w_2$, so that w_1 is maximal; let $v_1, v_2 \in H^2(X, \mathbb{R})$ denote eigenvectors for the eigenvalues $\lambda_1, \lambda_2^{-1} \in \Lambda$.

Note that, if $v \in H^2(X, \mathbb{C})$ is an eigenvector whose eigenvalue $\lambda \in \Lambda$ has weight $w \in]0, w_1[$, then by Lemma 4.1 applied to v and v_1 and by maximality of w_1 we have $v \wedge \bar{v} \neq 0$.

Let us prove first that $\lambda_1^{-1}\lambda_2 \in \Lambda$. If $w_1 = -2w_2$, then $\lambda_1^{-1}\lambda_2 = \lambda_2^{-1}$ and the claim is evident; therefore we may suppose that $w_1 \neq -2w_2$. We observe first that, since λ_2^{-1} is the minimal modulus of eigenvalues of g_2, w_2 is the minimal weight for the natural order introduced above. Now, if we had $\lambda_1^{-1}\lambda_2 \notin \Lambda$, then by Lemma 4.1 we would have $\lambda_2^2 \in \Lambda$ and in particular $-2w_2 \in W$; since we supposed that $-2w_2 \neq w_1$, by the above remark this implies that $4w_2 \in W$, contradicting the minimality of w_2. This shows (1).

Now let us show that for any $\lambda \in \Lambda$ whose weight is $w \in W \setminus \{0, w_1, w_2, w_3\}$ and for any eigenvector $v \in H^2(X, \mathbb{C})$ with eigenvalue λ we have $v \wedge \bar{v} \neq 0$. The case $w > 0$ (i.e. $|\lambda| \geq 1$) has been treated above; let $w < 0$, and suppose by contradiction that $v \wedge \bar{v} = 0$. Then by Lemma 4.1 we get $-w - w_2 \in W$, and since $-w - w_2 > 0$ and $-w - w_2 \neq w_1$ by assumption, we also have $2w + 2w_2 \in W$ (see the remark at the beginning of the proof); this contradicts the minimality of w_2 for the natural order, and concludes the proof of (2).

Property (3) follows from (2) by induction: indeed, if a w didn't satisfy property (3), by (2) we would have $(-2)^n w \in W \setminus \{0\}$ for all $n \in \mathbb{N}$, contradicting the finiteness of W.

Now assume that $w_1 \neq -2w_2$ and suppose by contradiction that w_2 has multiplicity > 1 in W. Then by Lemma 4.7 $-2w_2 \in W$, and since $-2w_2 > 0$ we also have $4w_2 \in W$. This contradicts the minimality of λ_2 for the natural order and proves (4). □

5. Automorphisms of threefolds: the semisimple case, proof of Theorem C

As in Section 4, let X be a compact Kähler threefold and let $g \in \mathrm{GL}(H^*(X, \mathbb{R}))$ be a *semisimple* linear automorphism preserving the Hodge decomposition, the wedge product and Poincaré duality; furthermore, suppose now that g and g^{-1} are defined over $\mathbb{Z}$. These are properties (1)–(4) in Section 1.1.

We denote as usual by λ_1 and λ_2 the dynamical degrees of g (i.e. the spectral radii of g_2 and g_4 respectively), by Λ the spectrum of g_2 (with multiplicities) and by W the set of weights of g_2 (with multiplicities).

Recall that we pick all eigenvectors of g_2 inside the union of subspaces $H^{1,1}(X) \cup (H^{2,0}(X) \oplus H^{0,2}(X))$ (see Remark 1.3).

Let $P(T)$ be the minimal polynomial of g_2; since g_2 is defined over $\mathbb{Z}$, we have $P(T) \in \mathbb{Z}[T]$. Since g is semisimple, we can write

$$P(T) = P_1(T) \cdots P_n(T),$$

where the $P_i \in \mathbb{Z}[T]$ are distinct and irreducible over $\mathbb{Q}$. Let P_1 be the factor having $\lambda_1(g)$ as a root, and denote by $\Lambda_i \subset \Lambda$ (respectively $W_i \subset W$) the set of roots of P_i (respectively the set of weights of roots of P_i).

For $i = 1, \ldots, n$ let

$$V_i = \ker P_i(g_2) \subset V := H^2(X, \mathbb{C}).$$

Since g is semisimple, we have

$$V = \bigoplus_{i=1}^{n} V_i.$$

For a polynomial $Q \in \mathbb{C}[T]$ define

$$Q^\vee(T) = T^{\deg Q} \cdot Q(T^{-1}).$$

Poincaré's duality allows identifying of $H^4(X,\mathbb{C})$ with $H^2(X,\mathbb{C})^\vee = V^\vee$; under this identification, we have $g_4 = (g_2^{-1})^\vee$, so that the minimal polynomial of g_4 is

$$P^\vee(T) = P_1^\vee(T) \cdots P_n^\vee(T).$$

Since g_4 is semisimple, we have

$$V^\vee = H^4(X,\mathbb{C}) = \bigoplus_{i=1}^{n} V_i^\vee.$$

Finally, let us define the bilinear map

$$\begin{aligned} \theta\colon H^2(X,\mathbb{C}) \times H^2(X,\mathbb{C}) &\to H^4(X,\mathbb{C}) \\ (u,v) &\mapsto u \wedge v. \end{aligned}$$

Remark 5.1. — The V_i and the $V_i^\vee$ are g-invariant subspaces defined over $\mathbb{Q}$; furthermore, if the roots of some P_i are simple eigenvalues of g_2 (or, equivalently, if P_i is a simple factor of the characteristic polynomial of g_2), then V_i is minimal for such a property: $\{0\}$ is the only proper subspace of V_i which is g_2-invariant and defined over $\mathbb{Q}$. The same holds for the action of g_4 on $V_i^\vee$.

The goal of this section is to describe the possible (moduli of) roots of a given P_i, most importantly for the factor having λ_1 as a root.

In what follows, we say for short that $\lambda, \lambda' \in \Lambda$ are conjugate if they are conjugate over $\mathbb{Q}$.

Remark 5.2. — Since $P_i(0) = \pm 1$, we have

$$\prod_{\lambda \in \Lambda_i} \lambda = \pm 1, \qquad \sum_{w \in W_i} w = 0.$$

Definition 5.3. — Let $\lambda \in \Lambda$; we say that a weight $w \in W$ is conjugate to λ if one of the conjugates of λ has weight w.

The main technical tool for the proofs in this section is the following basic result in Galois theory (see for example [19]).

Lemma 5.4. — *Let $\alpha, \beta \in \overline{\mathbb{Q}}$ be two algebraic numbers. If α and β are conjugate, then there exists $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \{\rho \in \mathrm{Aut}(\overline{\mathbb{Q}}) \mid \rho_{|\mathbb{Q}} = \mathrm{id}_{\mathbb{Q}}\}$ such that $\rho(\alpha) = \beta$.*

Note that, since elements of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act as the identity on $\mathbb{Q}$, the polynomials P_i are fixed; in particular $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by permutations on each Λ_i and on each W_i.

Recall that the rank $r(g)$ of g (i.e. the split-rank of $G = \overline{\langle g\rangle}^{Zar}$, see Section 4.1), which is the number of multiplicative parameters necessary to describe the moduli of the eigenvalues of g_2, is no greater than 2 by Lemma 4.9.

Furthermore, we saw in Corollary 4.11 that $r(g) = 2$ if and only if λ_1 and λ_2 are multiplicatively independent. We will distinguish the two cases $r(g) = 2$ and $r(g) = 1$.

5.1. The case $r(g) = 2$. — Let us treat first the case where the rank of g (see Section 4.1) is equal to 2.

We denote as usual by $w_1, w_2, w_3 \in W$ the weights of the eigenvalues of g_2

$$\alpha_1 := \lambda_1, \quad \alpha_2 := \lambda_2^{-1}, \quad \alpha_3 := \lambda_1^{-1}\lambda_2,$$

and fix non-null eigenvectors $v_1, v_2, v_3 \in H^2(X, \mathbb{R})$ for these eigenvalues.

5.1.1. *Algebraic properties of the eigenvalues.* —

Lemma 5.5. — *Let $r = 2$ and $\lambda \in \Lambda$. If one of the conjugates of λ has modulus 1, then λ is a root of unity.*

Proof. — By a lemma of Kronecker, if all the conjugates of an algebraic integer λ have modulus 1, then λ is a root of unity. Therefore, we only need to show that, if a conjugate of λ has modulus 1, then λ has also modulus 1. Suppose by contradiction that this is not the case, and let μ be a conjugate of λ such that $|\mu| = 1$.

Let $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ be such that $\rho(\mu) = \lambda$; since $\mu \cdot \bar{\mu} = 1$, we have

$$\lambda \cdot \rho(\bar{\mu}) = 1,$$

so that $\rho(\bar{\mu}) = \lambda^{-1}$. In terms of weights, this means that w_λ and $w_{\lambda^{-1}} = -w_\lambda$ are both non-trivial weights of W. This contradicts Proposition 4.10 and concludes the proof. □

Proposition 5.6. — *Let $r = 2$. Then for all $1 \leq k \leq n$ there exists $n_i = n_i(k)$, $i = 1, 2, 3$, such that, without taking multiplicities into account,*

$$W_k \subset \left\{ \frac{w_1}{(-2)^{n_1}}, \frac{w_1}{(-2)^{n_1+1}}, \frac{w_2}{(-2)^{n_2}}, \frac{w_2}{(-2)^{n_2+1}}, \frac{w_3}{(-2)^{n_3}}, \frac{w_3}{(-2)^{n_3+1}} \right\}.$$

Proof. — Let $\lambda \in \Lambda_i$, and let $w = w_\lambda$ be its weight. We will prove that if a weight w' collinear to w is conjugate to λ, then

$$w' \in \left\{ -\frac{w}{2}, w, -2w \right\}.$$

The claim then follows easily.

Suppose by contradiction that $w' = w_{\lambda'} \notin \{-w/2, w, -2w\}$; remark first that by Lemma 5.5 w and w' are both non-trivial. By Proposition 4.10, after maybe swapping λ and λ', we have

$$w' = (-2)^k w, \quad k \geq 2,$$

which means that

$$\lambda\bar{\lambda} = (\lambda'\bar{\lambda}')^{(-2)^k}.$$

Now, let $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be an automorphism such that $\rho(\lambda)$ is a conjugate of λ whose weight can be written as $w_a/(-2)^{n_a}$, $a \in \{1,2,3\}$, with n_a maximal. Let $\alpha, \beta, \gamma, \delta$ denote the images of $\lambda, \bar{\lambda}, \lambda', \bar{\lambda}$ under ρ, and let

$$w_\alpha = \frac{w_a}{(-2)^{n_a}}, \quad w_\beta = \frac{w_b}{(-2)^{n_b}}, \quad w_\gamma = \frac{w_c}{(-2)^{n_c}}, \quad w_\delta = \frac{w_d}{(-2)^{n_d}}$$

denote their weights; here $a, b, c, d \in \{1,2,3\}, n_a, n_b, n_c, n_d \geq 0$ and n_a is maximal. Since

$$\alpha\beta = (\gamma\delta)^{(-2)^k},$$

in terms of weights we get

$$\frac{w_a}{(-2)^{n_a}} + \frac{w_b}{(-2)^{n_b}} = (-2)^k \frac{w_c}{(-2)^{n_c}} + (-2)^k \frac{w_d}{(-2)^{n_d}},$$

so that

$$w_a + (-2)^{n_a - n_b} w_b = (-2)^{k + n_a - n_c} w_c + (-2)^{k - n_a - n_d} w_d.$$

Let $\Gamma = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \subset \mathbb{R}^2$ be the lattice generated by w_1, w_2. Since $k \geq 2$ we have

$$w_a + (-2)^{n_a - n_b} w_b \equiv 0 \qquad \mod 4\Gamma,$$

which is impossible. This leads to a contradiction and concludes the proof. □

Corollary 5.7. — *Let $r = 2$ and $\lambda \in \Lambda$. If λ is not a root of unity, then its degree over $\mathbb{Q}$ is a multiple of 3.*

Proof. — Fix $\lambda \in \Lambda$ which is not a root of unity, and let P_j be the unique factor of P having λ as a root; according to Proposition 5.6, there exists $n_1, n_2, n_3 \geq 0$ such that the weights of conjugates of λ are elements of the set

$$\left\{ \frac{w_1}{(-2)^{n_1}}, \frac{w_1}{(-2)^{n_1+1}}, \frac{w_2}{(-2)^{n_2}}, \frac{w_2}{(-2)^{n_2+1}}, \frac{w_3}{(-2)^{n_3}}, \frac{w_3}{(-2)^{n_3+1}} \right\}.$$

Since by Remark 5.2 we have

$$\sum_{w \in W_i} w = 0,$$

we get

$$\left(\frac{k_1}{(-2)^{n_1}}+\frac{h_1}{(-2)^{n_1+1}}\right)w_1+\left(\frac{k_2}{(-2)^{n_2}}+\frac{h_2}{(-2)^{n_2+1}}\right)w_2 + \left(\frac{k_3}{(-2)^{n_3}}+\frac{h_3}{(-2)^{n_3+1}}\right)w_3=0,$$

where the k_i and the h_i are the multiplicities of the weights in W_j. Since the only linear dependency among the w_i is $w_1+w_2+w_3=0$, this implies that there exists a constant $c\in\mathbb{Z}[1/2]$ such that

$$\frac{k_i}{(-2)^{n_i}}+\frac{h_i}{(-2)^{n_i+1}}=c, \qquad i=1,2,3.$$

This implies that

$$k_i+h_i\equiv c \qquad \mod 3,$$

so that in particular $\sum_i(k_i+h_i)\equiv 0$ modulo 3. □

5.1.2. *Algebraic properties of* λ_1. — Now let us focus on the factor P_1 having λ_1 as a root. Recall that we denoted by θ the map on $H^2(X,\mathbb{C})\times H^2(X,\mathbb{C})$ defined by

$$\theta\colon (u,v)\mapsto u\wedge v\in H^4(X,\mathbb{C}).$$

Lemma 5.8. — *Let $r=2$, and let P_1 be the factor of P having λ_1 as a root. If either $v_2,v_3\in V_1$ or $v_2,v_3\notin V_1$, then*

$$\theta(V_1\times V_1)=V_1^\vee.$$

If either $v_2\in V_1, v_3\in V_i$ or $v_2\in V_i, v_3\in V_1$ for some $i\neq 1$, then

$$\theta(V_1\times V_1)=V_1^\vee\oplus V_i^\vee.$$

Proof. — Without loss of generality, in the second case we may assume that $i=2$.

Let us first prove the $\subseteq$ inclusions. Denote by θ_1 the restriction of θ to $V_1\times V_1$; let

$$\pi_1\colon V^\vee=\bigoplus_{i=1}^{n}V_i^\vee\to\bigoplus_{i=2}^{n}V_i^\vee$$

be the projection onto the last $n-1$ factors and

$$\pi_{1,2}\colon V^\vee=\bigoplus_{i=1}^{n}V_i^\vee\to\bigoplus_{i=3}^{n}V_i^\vee$$

be the projection onto the last $n-2$ factors.

For $\pi\in\{\pi_1,\pi_{1,2}\}$, the subspace

$$\ker(\pi\circ\theta_1):=\{u\in V_1\,;\,\pi\circ\theta(u,v)=0\quad\text{for all } v\in V_1\}\subset V_1$$

is g_2-invariant and defined over $\mathbb{Q}$. By minimality of V_1 (see Remark 5.1), we then have either $\ker(\pi\circ\theta_1)=0$ or $\ker(\pi\circ\theta_1)=V_1$. Therefore, in order to show the inclusions, we only need to prove that $v_1 \in \ker(\pi \circ \theta_1)$ ($\pi = \pi_1$ in the first case and $\pi = \pi_{1,2}$ in the second case); since g_2 is semisimple, it is enough to check that $\pi \circ \theta(v_1, v) = 0$ for all eigenvectors $v \in V_1$.

Let β be the eigenvalue associated to an eigenvector $v \in V_1$, and let $w = w_\beta$ be its weight. We distinguish the following subcases:

- $w = 0$ is excluded by Lemma 5.5;
- if $w \notin \{w_2, w_3, -w_1/2\}$, then $-w_1 - w \notin W$, so that $v_1 \wedge v = 0$ and in particular $\pi \circ \theta(v_1, v) = 0$;
- if $w = -w_1/2$ and $v_1 \wedge v \neq 0$, then $v_1 \wedge v$ is an eigenvector with eigenvalue $\beta\lambda_1 = \bar{\beta}^{-1}$. Since β is conjugated to λ_1, so is $\bar{\beta}$, and thus $v \wedge v_1 \in V_1^\vee$ and $\pi \circ \theta(v_1, v) = 0$;
- if $w = w_2$, by simplicity of the weight w_2 we have $\beta = \alpha_2$; in particular $v_2 \in V_1$. Then $v_1 \wedge v = v_1 \wedge v_2$ is an eigenvector for g_4 with eigenvalue $\alpha_1\alpha_2 = \alpha_3^{-1}$. If also $v_3 \in V_1$, then $v_1 \wedge v \in V_1^\vee$; if $v_3 \notin V_1$, say $v_3 \in V_2$, then $v_1 \wedge v \in V_2^\vee$. In both cases, choosing the right $\pi \in \{\pi_1, \pi_{1,2}\}$ we get $\pi \circ \theta(v_1, v) = 0$;
- the case $w = w_3$ is analogous to the case $w = w_2$.

This concludes the proof of the $\subseteq$ inclusions.

Let us now prove the other inclusions $\supseteq$.

Suppose first that either $v_2, v_3 \in V_1$ or $v_2, v_3 \notin V_1$, so that $\theta(V_1 \times V_1) \subseteq V_1^\vee$. Since $\theta(V_1 \times V_1)$ is g-invariant and defined over $\mathbb{Q}$, by minimality of $V_1^\vee$ we only need to show that $\theta(V_1 \times V_1) \neq \{0\}$. Since $\dim V_1 \geq 2$, this follows from Lemma 4.1.

Now suppose that $v_2 \in V_1, v_3 \in V_i$ for some $i \neq 1$ (the proof of the other case being analogous), so that $\theta(V_1 \times V_1) \subseteq V_1^\vee \oplus V_i^\vee$. Then $v_1 \wedge v_2 \in V^\vee$ is an eigenvector with eigenvalue α_3^{-1}, so that $v_1 \wedge v_2 \in V_i^\vee$. Since $\theta(V_1 \times V_1)$ is g-invariant and defined over $\mathbb{Q}$, by minimality of $V_1^\vee$ and $V_i^\vee$ we have either

$$\theta(V_1 \times V_1) = V_i^\vee$$

or

$$\theta(V_1 \times V_1) = V_1^\vee \oplus V_i^\vee.$$

The first case contradicts Lemma 5.9 below, so equality must hold and the proof is complete. □

Lemma 5.9. — *Let $r = 2$, and let P_1 be the factor of P having λ_1 as a factor. Suppose that*

$$\theta(V_1 \times V_1) \subseteq V_i^\vee$$

for some $1 \leq i \leq n$. Then $i = 1$.

Proof. — Assume by contradiction that $i \neq 1$, say $i = 2$. By the $\subseteq$ inclusions in Lemma 5.8 (whose proof is independent on the result we want to prove here), we may then assume that $v_2 \in V_1, v_3 \in V_2$.

Let us prove first that $-w_1/2, -w_2/2 \notin W_1$. Indeed, suppose for example that $-w_1/2 \in W_1$, and let $v \in V_1$ be an eigenvector whose eigenvalue λ has weight $-w_1/2$. Then by Lemma 4.1 we have $v \wedge \bar{v} \neq 0$, so that $|\lambda|^2 = \lambda_1^{-1}$ is an eigenvalue of the restriction of g_4 to $\theta(V_1 \times V_1) = V_2^\vee$. This contradicts the fact that λ_1^{-1} is an eigenvalue of g_4 restricted to $V_1^\vee$, and proves that $-w_1/2 \notin W_1$; the proof for $-w_2/2$ is analogous.

Now let us prove that $w_3/(-2)^n \notin W_1$ for $n \geq 2$. Suppose by contradiction that $v \in V_1$ is an eigenvector whose eigenvalue λ has weight $-w_3/(-2)^n$, $n \geq 2$. Then by Lemma 4.1 $v \wedge \bar{v} \neq 0$ is a non-trivial eigenvector with eigenvalue $\mu = |\lambda|^2$; since $\theta(V_1 \times V_1) = V_2^\vee$, μ is conjugated to $\mu' = \alpha_3^{-1}$, and these two algebraic integers satisfy an algebraic equation

$$\mu^{(-2)^N} = \mu' \qquad N \geq 1.$$

Using Lemma 5.4 it is not hard to see that for this to happen we need to have $\mu = \mu' = 1$, a contradiction. This proves that $w_3/(-2)^n \notin W_1$ for $n \geq 2$.

Now, by Proposition 5.6 this implies that, up to multiplicities,

$$W_1 \subset \left\{ w_1, w_2, -\frac{w_3}{2} \right\}.$$

This however contradicts the equation

$$\sum_{w \in W_1} w = 0.$$

The claim is then proved. □

Remark 5.10. — Lemma 5.8 and 5.9 still hold if one permutes α_1, α_2 and α_3; the proofs are completely analogous.

We are ready to state and prove Theorem C in the case where λ_1 and λ_2 are multiplicatively independent (i.e. the rank of g is equal to 2).

Proposition 5.11. — *Let $g \in \mathrm{GL}(H^2(X, \mathbb{R}))$ be a semisimple linear automorphism preserving the cohomology graduation, Hodge decomposition, wedge product and Poincaré duality, and such that g and g^{-1} are defined over $\mathbb{Z}$. Let λ_1 and λ_2 be the spectral radii of g_2 and g_4 respectively, and suppose that λ_1 and λ_2 are multiplicatively independent.*

Then the conjugates of $\lambda_1(g)$ have all modulus belonging to the following set:

$$\left\{ \lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2, \sqrt{\lambda_1^{-1}}, \sqrt{\lambda_2}, \sqrt{\lambda_1 \lambda_2^{-1}} \right\}.$$

More accurately, there exists a permutation $\alpha_1, \alpha_2, \alpha_3$ of the eigenvalues $\lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2$ of g_2, such that one of the following is true:

1. *α_1, α_2 and α_3 are all cubic algebraic integers without real conjugates;*
2. *α_1 is a cubic algebraic integer without real conjugates; α_2 and α_3 are conjugate to one another, and their other conjugates are pairs of conjugate complex numbers with modulus $\alpha_1^{-1/2}, \alpha_3^{-1/2}, \alpha_3^{-1/2}$ ($k, k+1$ and $k+1$ pairs respectively, $k \geq 0$);*
3. *α_1, α_2 and α_3 are conjugate, and their other conjugates are pairs of conjugate complex numbers with modulus $\alpha_1^{-1/2}, \alpha_3^{-1/2}, \alpha_3^{-1/2}$ (k pairs for each module, $k \geq 0$).*

Proof. — Thanks to Lemma 5.8 and Remark 5.10, up to permutation of indices only three situations are possible:

1. α_1, α_2 and α_3 are not mutually conjugate. In this case, denoting by P_i the factor of P having α_i as a factor, Lemma 5.8 implies that
$$\theta(V_i \times V_i) = V_i^\vee.$$
Then $W_i = \{w_i, -w_i/2\}$; indeed, suppose by contradiction that $w = w_j/(-2)^n \in W_i$ for some $j \neq i$, and let $v \in V_i$ be an eigenvector whose eigenvalue has weight w. Then by Lemma 4.1, either $n = 0$ or $v \wedge \bar{v} \neq 0$, so that $-2w \in W_i$. By a recursive argument, this proves that $w_j \in W_i$, which contradicts the assumption that α_j and α_i are not conjugate. Therefore, by Proposition 5.6,
$$W_i \subseteq \left\{ w_i, -\frac{w_i}{2} \right\}.$$
Since $\sum_{w \in W_i} w = 0$, the multiplicity of $-w_i/2$ must be 2, which implies that the conjugates of α_i are two conjugate complex numbers. This concludes the proof of case (1).
2. α_2 and α_3 are conjugate, while α_1 is not. The above proof shows that α_1 is cubic without real conjugates. Let P_1 (respectively P_2) be the factor of P having α_1 (respectively α_2 and α_3) as a root; by Lemma 5.8 and Remark 5.10 we have
$$\theta(V_2 \times V_2) = V_1^\vee \oplus V_2^\vee.$$
Suppose by contradiction that $w = w_1/(-2)^n \in W_2$ for some $n \geq 2$, and let $v \in V_2$ be an eigenvector whose eigenvalue λ has weight w. Then by Lemma 4.1 $v \wedge v \neq 0$, so that $|\lambda|^{-2} \in \Lambda_1 \cup \Lambda_2$; since α_1 is a non-trivial power of $|\lambda|^{-2}$, these two numbers cannot be conjugate, therefore $|\lambda|^{-2} \in \Lambda_2$. Inductively, this shows that $\beta = \alpha_1^{-1/2} \in \Lambda_1$ is conjugate to α_2 and α_3. We can write
$$\beta^2 = \alpha_2 \alpha_3.$$
Since
$$\{\alpha_2, \alpha_3\} \cap \{\lambda_1, \lambda_2^{-1}\} \neq \emptyset,$$

we can find $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho(\beta) = \lambda_1$ or $\rho(\beta) = \lambda_2^{-1}$; both cases lead to a contradiction by maximality of λ_1 (respectively, by minimality of λ_2^{-1}) among the moduli of eigenvalues of g_2. Therefore $w_1/(-2)^n \notin W_2$ for $n \geq 2$ and $n = 0$.

By Proposition 5.6 this implies that, up to multiplicities,

$$W_2 \subset \left\{ w_2, w_3, -\frac{w_1}{2}, -\frac{w_2}{2}, -\frac{w_3}{2} \right\}.$$

The equation

$$\sum_{w \in W_2} w = 0$$

implies that the multiplicities of $-w_1/2, -w_2/2, -w_3/2$ are $h, h+2, h+2$ respectively for some $h \geq 0$. Since α_2 cannot be conjugate to $\alpha_2^{-1/2}$, $h = 2k$ is even, which concludes the proof of case (2).

3. α_1, α_2 and α_3 are conjugate. Then, by Proposition 5.6, up to multiplicities,

$$W_2 \subset \left\{ w_1, w_2, w_3, -\frac{w_1}{2}, -\frac{w_2}{2}, -\frac{w_3}{2} \right\}.$$

The equation

$$\sum_{w \in W_2} w = 0$$

implies that the multiplicities of $-w_1/2, -w_2/2, -w_3/2$ are all equal to h for some $h \geq 0$. Since α_1 cannot be conjugate to $\alpha_1^{-1/2}$, $h = 2k$ is even, which concludes the proof of case (3).

□

5.2. The case $r(g) = 1$. — Let us now suppose that the rank of g is equal to 1 (see Section 4.1). Recall that in this case the weights are equipped with a natural order: $w_\lambda > w_{\lambda'}$ if and only if $|\lambda| > |\lambda'|$; for $w \in W$ we set $|w| := \max\{w, -w\}$.

Denote as usual by $w_1, w_2, w_3 \in W$ the weights of the eigenvalues $\lambda_1, \lambda_2^{-1}$, $\lambda_1^{-1}\lambda_2 \in \Lambda$ respectively.

Lemma 5.12. — *Suppose that $r = 1$ and let $\lambda \neq \lambda_1$ be a real conjugate of λ_1; then $\lambda = \lambda_1^{-1}$. In this case $\lambda_1 = \lambda_2$.*

Proof. — Since $r = 1$, there exists integers m, n, not both equal to 0, such that

$$\lambda^m = \lambda_1^n.$$

Suppose that $|m| \geq |n|$ (the case $|n| \geq |m|$ is proved in the same way) and let $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be such that $\rho(\lambda_1) = \mu$, where μ is a conjugate of λ_1 whose weight has maximal modulus. Denoting by w and w' the weights of μ and $\rho(\lambda_1)$ respectively, the above equation implies that

$$mw = nw';$$

by maximality of $|w|$ we get $|m| = |n|$, so that $\lambda = \lambda_1^{-1}$ as claimed.

In order to prove that in this case $\lambda_1 = \lambda_2$, it suffices to apply Proposition 4.12: if this were not the case, since $\lambda_1^{-1} \in \Lambda$, then either $\lambda_1^{-1} = \lambda_1^{-1}\lambda_2$, a contradiction, or $\lambda_1^2 \in \Lambda$, contradicting the maximality of λ_1. □

The following proposition implies Theorem C in the case where λ_1 and λ_2 are not multiplicatively independent.

Proposition 5.13. — *Let $g \in \mathrm{GL}(H^2(X, \mathbb{R}))$ be a semisimple linear automorphism preserving the cohomology graduation, Hodge decomposition, wedge product and Poincaré duality, and such that g and g^{-1} are defined over $\mathbb{Z}$. Let λ_1 and λ_2 be the spectral radii of g_2 and g_4 respectively, and suppose that the rank of g is equal to 1 (i.e. $\lambda_1 > 1$ and λ_1 and λ_2 are not multiplicatively independent). Then*

- *either λ_1 and λ_2 are both cubic without real conjugates;*
- *or $\lambda_2 = \lambda_1 = \lambda$, λ and λ^{-1} are conjugate, and all of their other conjugates are pairs of conjugate complex numbers of modulus $\sqrt{\lambda}, 1$ or $\sqrt{\lambda^{-1}}$ (k, k' and k pairs respectively, $k, k' \geq 0$).*

Proof. — Denote by P_1 the factor of P having λ_1 as a root.

Case 1: $\lambda_2 \notin \{\sqrt{\lambda_1}, \lambda_1, \lambda_1^2\}$. We show first that

$$\theta(V_1 \times V_1) = V_1^\vee.$$

Since λ_1 is a simple eigenvalue of g_2 by Proposition 4.12, V_1 is minimal among the g-invariant subspaces defined over $\mathbb{Q}$ (see Remark 5.1), and so is $V_1^\vee$; therefore, as in the proof of Lemma 5.8, in order to prove that $\theta(V_1 \times V_1) \subseteq V_1^\vee$ we only need to show that

$$v_1 \wedge v \in V_1^\vee$$

for all eigenvectors $v \in V_1$. Let $v \in V_1$ be an eigenvector with eigenvalue λ and let $w = w_\lambda$, and let, as usual, $w_3 := -w_1 - w_2$.

- If $w \notin \{-w_1/2, w_2, w_3\}$, then $v_1 \wedge v = 0$. Indeed, if this were not the case, then $-w_1 - w \in W$; by the assumption and Proposition 4.12,the smallest weights of W (with respect to the natural order) are

  $$w_2 < -\frac{w_1}{2}.$$

  Therefore

  $$w > -w_1/2 \qquad \Rightarrow \qquad -w_1 - w < -\frac{w_1}{2},$$

  which implies that $-w_1 - w = w_2$, i.e. $w = w_3$, contradicting the assumption.

- If $w = -w_1/2$, i.e. $\lambda\bar{\lambda} = \lambda_1^{-2}$, then $v_1 \wedge \bar{v}$ is an eigenvector with eigenvalue $\bar{\lambda}^{-1}$; since λ and $\bar{\lambda}$ are conjugate, this implies that $v_1 \wedge v \in V_1^\vee$.
- If $w = w_2$ or $w = w_3$, then $\lambda = \lambda_2^{-1}$ or $\lambda = \lambda_1^{-1}\lambda_2$ is a real conjugate of λ_1; but then by Lemma 5.12 we have $\lambda = \lambda_1^{-1}$, which contradicts the assumptions on λ_2. Thus this case cannot occur.

We have showed that $\theta(V_1 \times V_1) \subseteq V_1^\vee$.

The vector space $\theta(V_1 \times V_1)$ is non-empty by Lemma 4.1, g-invariant and defined over $\mathbb{Q}$. Therefore, equality follows from minimality.

Now let us show that λ_1 is cubic without real conjugates. Since

$$\sum_{w \in W_1} w = 0$$

and since w_1 has multiplicity 1 in W, we only need to show that the conjugates of λ_1 have weight $-w_1/2$. Let λ be a conjugate of λ_1 with weight w and let v be an eigenvector for λ.

- If we had $w = w_2$, then by simplicity of such weight we have $\lambda = \lambda_2^{-1}$, contradicting Lemma 5.12 since $\lambda_2 \neq \lambda_1$.
- If we had $w = 0$, a conjugate λ of λ_1 would satisfy

  $$\lambda\bar{\lambda} = 1.$$

  Applying $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho(\lambda) = \lambda_1$, we would have that λ_1^{-1} is a conjugate of λ_1, so that $\lambda_1 = \lambda_2$, a contradiction.
- If $w = w_3$, since

  $$\lambda_1^{-1}\lambda_2 \neq \lambda_1, \lambda_1^{-1},$$

  Lemma 5.12 implies that $\lambda \notin \mathbb{R}$. Lemma 4.1 applied to v and v_1 implies that $v \wedge \bar{v} \neq 0$: indeed otherwise we would have $v_1 \wedge \bar{v}_1 \neq 0$ (contradicting the minimality of $\lambda_2^{-1} > \lambda_1^{-2}$) or $v_1 \wedge \bar{v} \neq 0$ (contradicting the simplicity of the weight w_2).

  Therefore, since $\theta(V_1, V_1) = V_1^\vee$, we have

  $$|\lambda|^{-2} = \lambda_1^2\lambda_2^{-2} \in \Lambda_1$$

  and by Lemma 5.12 either $\lambda_1^2\lambda_2^{-2} = \lambda_1$, i.e. $\lambda_1 = \lambda_2^2$, a contradiction or $|\lambda|^{-2} = \lambda_1^{-1} \in \Lambda$ and $\lambda_1 = \lambda_2$, again a contradiction.
- Finally, if $w \notin \{0, w_1, -w_1/2, w_2, w_3\}$, then by Proposition 4.12 $v \wedge \bar{v} \neq 0$ would be an eigenvector with (real) eigenvalue $|\lambda|^2$. Since $\theta(V_1 \times V_1) = V_1^\vee$, this implies that $|\lambda|^{-2} \in \Lambda_1$; by Lemma 5.12 we would have $|\lambda|^2 = \lambda_1^{\pm 1}$, a contradiction.

This shows that λ_1 is cubic without real conjugates; the proof for λ_2 is completely analogous.

Case 2: $\lambda_2 \in \{\sqrt{\lambda_1}, \lambda_1^2\}$. Up to replacing g by g^{-1}, we may assume that $\lambda_2 = \sqrt{\lambda_1}$; let us show that λ_1 and λ_2 are both cubic without real conjugates.

Note that by Proposition 4.12, up to multiplicities

$$W \setminus \{0\} = \left\{ \frac{w_1}{(-2)^n}, n = 0, \dots, N \right\}.$$

Let us show first that λ_1 is cubic without real conjugates. Since

$$\sum_{w \in W_1} w = 0$$

and since w_1 has multiplicity 1 in W, we only need to show that $W_1 \subset \{w_1, -w_1/2\}$. Let $w \in W_1$ be the weight of an eigenvalue $\lambda \in \Lambda_1$.

- If $w = 0$, we show as in Case 1 that $\lambda_1 = \lambda_2$, a contradiction.
- If $w = w_1/(-2)^n$ and $n \geq 2$, then we argue as in the proof of Proposition 5.6 to obtain a contradiction: indeed in this case

  $$(\lambda\bar{\lambda})^k = \lambda_1, \qquad 2|k.$$

  Applying $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho(\lambda_1)$ has weight $w_1/(-2)^n$ with n maximal and letting w' and w'' be the weights of $\rho(\lambda)$ and $\rho(\bar{\lambda})$ respectively, we would have

  $$k(w' + w'') = \frac{w_1}{(-2)^n},$$

  a contradiction modulo $\mathbb{Z}w_1/(-2)^{n-1}$.

Therefore $W_1 \subset \{w_1, -w_1/2\}$ and thus λ_1 is cubic without real conjugates.

Now let us prove that λ_2 is also cubic without real conjugates; this is equivalent to λ_2^{-1} being cubic without real conjugates. Let P_2 be the factor of P having λ_2^{-1} as a root. Since $\lambda_2 = \sqrt{\lambda_1}$, λ_2 has degree 3 or 6 over $\mathbb{Q}$; the same proof as above and the simplicity of the weight w_1 show that

$$W_2 \subset \left\{ w_2, -\frac{w_2}{2} \right\}.$$

Since

$$\sum_{w \in W_2} w = 0,$$

if λ_2 had degree 6 then the multiplicity of the weight w_2 in W_2 would be equal to 2, contradicting the fact that λ_2^{-1} is a real eigenvalue with weight w_2. Therefore λ_2 is cubic, and by Lemma 5.12 it doesn't have any real conjugate.

Case 3: $\lambda_1 = \lambda_2$. Suppose that λ_1 is not a cubic algebraic integer without real conjugates. Denote by P_1 the factor of P having λ_1 as a root; since

$$\sum_{w \in W_1} w = 0,$$

and since the weight $w_1 \in W$ has multiplicity 1, λ_1 is *not* cubic without real conjugates if and only if some conjugate λ of λ_1 has weight $w \notin \{w_1, -w_1/2\}$. Let us prove first that in this case λ_1 and λ_1^{-1} are conjugate. We distinguish the following sub-cases:

- $w = -w_1$. Then, since λ_1^{-1} is the only eigenvalue with weight $-w_1$, λ_1 and λ_1^{-1} are conjugate.
- $0 \leq |w| < w_1/2$. Since the rank r is equal to 1, λ and λ_1 satisfy an equation

$$(\lambda\bar{\lambda})^m = \lambda_1^n,$$

and since $|w| < w_1/2$ we have $|n| < |m|$. By Lemma 5.4 there exists $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho(\lambda) = \lambda_1$; let $\lambda' = \rho(\bar{\lambda}), \lambda'' = \rho(\lambda_1)$, and let w', w'' be their weights respectively. Then the above equation implies that

$$mw_1 + mw' = nw'' \quad \Leftrightarrow \quad w_1 = -w' + \frac{n}{m}w''. \tag{5}$$

This implies that either $w' = w_2$ or $w'' = w_2$; indeed, if this were not the case, by Proposition 4.12 we would have

$$|w'|, |w''| \leq \max\left\{\frac{|w_1|}{2}, \frac{|w_2|}{2}, |w_1 + w_2|\right\} = \frac{|w_1|}{2};$$

this would contradict equation 5 because $|n/m| < 1$.

We have shown that w_2 is a conjugate weight of λ_1; since λ_1^{-1} is the only eigenvalue with weight w_2, this means that λ_1 and λ_1^{-1} are conjugate as claimed.
- $w = w_1/2$. We may assume that we don't fall in one of the cases above, i.e. that

$$W_1 \subset \left\{w_1, \frac{w_1}{2}, -\frac{w_1}{2}\right\}.$$

We show that $\theta(V_1 \times V_1) = V_1^\vee$; in order to do that it suffices to show that $v_1 \wedge v \in V_1^\vee$ for every eigenvector $v \in V_1$ (see Case 1 above and the proof of Lemma 5.8). If the eigenvalue μ of the eigenvector v has weight w_1 or $w_1/2$, then $v_1 \wedge v = 0$; if the weight is $-w_1/2$, then $v_1 \wedge v$ is an eigenvector with eigenvalue $\bar{\mu}^{-1}$, hence $v_1 \wedge v \in V_1^\vee$. Therefore $\theta(V_1 \times V_1) = V_1^\vee$ as claimed.

Now let $v \in V_1$ be an eigenvector with eigenvalue of weight $w_1/2$; by Proposition 4.12, $v \wedge \bar{v} \neq 0$, so that $-w_1 \in W_1$. This shows that we fall in one of the above cases, and in particular λ_1 and λ_1^{-1} are conjugate.

We have shown that λ_1 and λ_1^{-1} are conjugate. By Lemma 5.12 there are no other real conjugates, therefore in order to complete the proof we only need to show that

$$\pm\frac{w_1}{2^n} \notin W_1 \qquad \text{for } n \geq 2.$$

This can be proved exactly as in Case 2. □

5.3. Examples on tori. — In this section we provide examples of automorphisms of compact complex tori of dimension 3 which show that (almost) all of the sub-cases of Proposition 5.11 and 5.13 can actually occur. For more examples see [23, 22].

LEMMA 5.14. — *Let $P \in \mathbb{Z}[T]$ be a Monica polynomial of degree $2n$ all of whose roots are distinct and non-real and such that $P(0) = 1$. Then there exists a compact complex torus X of dimension n and an automorphism*

$$f\colon X \to X$$

such that the characteristic polynomial of the linear automorphism $f_1^*\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C})$ is equal to P.

Proof. — Let

$$P(T) = T^{2n} + a_{2n-1}T^{2n-1} + \ldots + a_1T + 1 \in \mathbb{Z}[T]$$

be any polynomial.

We will prove first that there exists a linear diffeomorphism f of the real torus $M = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ such that the induced linear automorphism $f_1^* \in \mathrm{GL}(H^1(M,\mathbb{R}))$ has characteristic polynomial P. Indeed, the companion matrix

$$A = A(P) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & 0 & 1 & 0 & -a_{2n-2} \\ 0 & \cdots & 0 & 0 & 1 & -a_{2n-1} \end{pmatrix}$$

has characteristic polynomial P; since $A \in \mathrm{SL}_{2n}(\mathbb{Z})$, the induced linear automorphism f of $\mathbb{R}^{2n}$ preserves the lattice $\mathbb{Z}^{2n}$ and so does its inverse. Hence, A induces a linear automorphism, which we denote again by f:

$$f\colon \mathbb{R}^{2n}/\mathbb{Z}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n}.$$

Let dx_i be a coordinate on the i-th factor of $M = \mathbb{R}^{2n}/\mathbb{Z}^{2n} = (\mathbb{R}/\mathbb{Z})^{2n}$. In the basis $dx_1, \ldots, dx_{2n}$ of $H^1(X,\mathbb{R})$, the matrix of f_1^* is exactly the transposed A^T; in particular, the characteristic polynomial of f_1^* is equal to P.

In order to conclude the proof, we will show that, if the roots of P are all distinct and non-real, then M can be endowed with a complex structure J such that f is holomorphic with respect with the structure J. Let

$$\beta_1, \bar{\beta}_1, \ldots, \beta_n, \bar{\beta}_n \in \mathbb{C} \setminus \mathbb{R}$$

be the roots of P, and let

$$V_i = \ker(f - \beta_i I)(f - \bar{\beta}_i I) \subset \mathbb{R}^{2n},$$

where we have identified f with the linear automorphism of $\mathbb{R}^{2n}$ induced by the matrix A.

The V_i are planes such that

$$\mathbb{R}^{2n} = \bigoplus_{i=1}^{n} V_i.$$

The restriction of f to V_i is diagonalizable with eigenvalues β_i and $\bar{\beta}_i$; therefore there exists a unique complex structure J_i on V_i such that, with respect to a holomorphic coordinate z_i on $V_i \cong \mathbb{C}$, the action of f is the multiplication by β_i:

$$f|_{V_i}(z_i) = \beta_i z_i.$$

The complex structures J_i induce a complex structure on $\mathbb{R}^{2n}$; by canonically identifying $\mathbb{R}^{2n}$ with the tangent space at any point of M, we get an almost-complex structure on M. It is not hard to see that J is integrable, and that f is holomorphic with respect to J. This concludes the proof. □

Let us apply Lemma 5.14 to the three-dimensional case: fix a monic polynomial $P \in \mathbb{Z}[T]$ of degree 6 such that $P(0) = 1$, and suppose that its roots

$$\beta_1, \beta_2, \beta_3, \beta_4 = \bar{\beta}_1, \beta_5 = \bar{\beta}_2, \beta_6 = \bar{\beta}_3$$

are all distinct and non-real.

By Lemma 5.14, there exists a 3-dimensional complex torus $X = \mathbb{C}^3/\Lambda$ and an automorphism $f\colon X \to X$ such that the induced linear automorphism $f_1^*\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C})$ has characteristic polynomial P. Note that the proof of the Lemma shows something more precise: the restriction of f_1^* to $H^{1,0}(X)$ (respectively to $H^{0,1}(X)$) is diagonalizable with eigenvalues $\beta_1, \beta_2, \beta_3$ (respectively $\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3$).

Since for a complex torus the wedge-product of forms induces isomorphisms

$$\begin{aligned} H^{2,0}(X) &\cong \textstyle\bigwedge^2 H^{1,0}(X), \\ H^{1,1}(X) &\cong H^{1,0}(X) \otimes H^{0,1}(X), \\ H^{0,2}(X) &\cong \textstyle\bigwedge^2 H^{0,1}(X), \end{aligned}$$

the eigenvalues of $f_2^* \in \mathrm{GL}(H^2(X, \mathbb{R}))$ are exactly the 15 numbers $\beta_i\beta_j$, $1 \leq i < j \leq 6$. If $|\beta_1| \geq |\beta_2| \geq |\beta_3|$, then

$$\alpha_1 := \lambda_1 = |\beta_1|^2, \quad \alpha_2 := \lambda_2^{-1} = |\beta_3|^2, \quad \alpha_3 := \lambda_1^{-1}\lambda_2 = |\beta_2|^2.$$

Let

$$Q(T) = \prod_{1 \leq i < j \leq 6} (T - \beta_i\beta_j).$$

Then Q is the characteristic polynomial of f_2^*, and in particular $Q \in \mathbb{Z}[T]$. Let

$$R_P := \{\beta_i\}_{1\le i\le 6} \qquad R_Q = \{\beta_i\beta_j\}_{1\le i<j\le 6}$$
$$K_P := \mathbb{Q}(R_P) \supset K_Q := \mathbb{Q}(R_Q).$$

We are interested in the irreducible factors of Q over $\mathbb{Z}$; assuming that all the roots of Q are distinct, the irreducible factors of Q are in $1:1$ correspondence with the orbits of the action of $\mathrm{Gal}(K_Q/\mathbb{Q})$ on the set of roots $R_Q = \{\beta_i\beta_j\}$. Since each element of $\mathrm{Gal}(K_Q/\mathbb{Q})$ can be extended to an element of $\mathrm{Gal}(K_P/\mathbb{Q})$, we consider instead the orbits under the action of

$$G := \mathrm{Gal}(K_P/\mathbb{Q});$$

G acts by permuting the roots of P, and thus it can be seen as a subgroup of $\mathfrak{S}_6$. Under this identification, the action of G on R_Q is given by the natural action of (subgroups of) $\mathfrak{S}_6$ on the set R_P.

Therefore, as long as we know how $\mathrm{Gal}(K_P/\mathbb{Q})$ permutes the roots of P, we can deduce the number and the degrees of the irreducible factors of Q. This is a classical problem in Galois theory (see [5]), and programs like Magma allow easy computing of this action.

Example 5.15. — Let $P(T) = T^6 - T^5 + T^3 - T^2 + 1$; then $G = \mathfrak{S}_6$ acts transitively on R_P. This means that Q is irreducible, and thus $\alpha_1, \alpha_2, \alpha_3$ are conjugate; their other conjugates are six pairs of complex conjugates, two of modulus $1/\sqrt{\alpha_1}$, two of modulus $1/\sqrt{\alpha_2}$ and two of modulus $1/\sqrt{\alpha_3}$.

This realizes subcase 1 of Proposition 5.11 with $k = 2$.

Example 5.16. — Let $P(T) = T^6 - 3T^5 + 4T^4 - 2T^3 + T^2 - T + 1$; then $G = \langle (1\,3\,4)(2\,6\,5), (1\,4\,3), (1\,6)(2\,3)(4\,5)\rangle$. The action of G on R_P has two orbits, of cardinality 9 and 6 respectively; one can check that the roots of Q are all distinct, so that α_1 and α_2 are not both cubic, and that $\alpha_1 \neq \alpha_2^{-1}$, so that by Proposition 5.13 we have $r = 2$. By Proposition 5.11, the only possibility is that α_1, α_2 and α_3 are conjugate of degree 9; their other conjugates are three pairs of complex conjugates, of modulus $1/\sqrt{\alpha_1}, 1/\sqrt{\alpha_2}$ and $1/\sqrt{\alpha_3}$ respectively.

This realizes subcase 1 of Proposition 5.11 with $k = 1$.

Example 5.17. — Let $P(T) = T^6 + T^5 + 2T^4 - T^3 + 2T^2 - 3T + 1$; then $G = \langle (1\,2\,3)(4\,5\,6), (1\,4\,5)(2\,3\,6), (2\,4)(3\,5), (1\,5\,6\,3)\rangle$. The action of G on R_P has two orbits, of cardinality 12 and 3 respectively. One can check that α_1, α_2 and α_3 are not all conjugate, so that we are in case 2 of Proposition 5.11: after permuting the indices, α_1 is cubic without real conjugates; α_2 and α_3 are conjugate and their other conjugates are 5 pairs of complex conjugates, one of modulus $1/\sqrt{\alpha_1}$, two of modulus $1/\sqrt{\alpha_2}$ and two of modulus $1/\sqrt{\alpha_2}$. Note that, after possibly replacing f by f^{-1} (which replaces P by $P^\vee$, see Section 5), we may assume that λ_1 is not cubic without real conjugates.

This realizes subcase 2 of Proposition 5.11 with $k = 1$.

Example 5.18. — Let $P(T) = T^6 + T^5 + 4T^4 + T^3 + 2T^2 - 2T + 1$; then $G = \langle (1\,2\,5)(3\,6\,4), (1\,3)(2\,4)(5\,6) \rangle$. The action of G on R_P has three orbits of cardinality 3 and one of cardinality 6. Then λ_1 and λ_2 are both cubic without real conjugates:

- if $r = 1$, since it is easy to prove that $\lambda_1 \neq \lambda_2$, this follows from Proposition 5.13;
- if $r = 2$, it can be proven easily that α_1, α_2 and α_3 are not all conjugate, and that all the other eigenvalues of f_2^* are non-real. Therefore, the α_i are all contained in (distinct) orbits of cardinality 3, meaning that they are cubic without real conjugates.

However it is unclear whether $r = 1$ or $r = 2$; if one could prove that $r = 2$, this would realize subcase 3 of Proposition 5.11.

Example 5.19. — Let $P(T) = T^6 + T^4 - 2T^3 + T^2 - T + 1$ and let $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2, \beta_3, \bar{\beta}_3$ be the roots of P, with $|\beta_1| \geq |\beta_2| \geq |\beta_3|$; then one finds out that

$$|\beta_1| = |\beta_2|^{-2} = |\beta_3|^{-2}.$$

In particular $\lambda_1 = \lambda_2^2$, so that $r = 1$. By Proposition 5.13, λ_1 and λ_2 are both cubic without real conjugates.

This realizes subcase 1 of Proposition 5.13.

Remark 5.20. — In order to realize subcase 2 of Proposition 5.13 it would suffice to exhibit an irreducible polynomial $P \in \mathbb{Z}[T]$ of degree 6, without real roots, such that $P(0) = 1$ and having exactly two roots with modulus equal to 1.

6. Automorphisms of threefolds: the mixed case

In this last section, we deal with the case of an automorphism $f\colon X \to X$ of a compact Kähler threefold X such that the action in cohomology $f_2^*\colon H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ is neither (virtually) unipotent nor semisimple. As we saw in Section 2, this situation is not possible in the surface case; in the threefold case, we manage to give some constraints but not to completely exclude this situation. However, due to restriction on the dimension, no examples can be produced on complex tori, and to the best of my knowledge no examples are known at all.

Conjecture 6.1. — *Let $f\colon X \to X$ be an automorphism of a compact Kähler threefold. Then $f_2^*\colon H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ is either semisimple or virtually unipotent.*

Proposition 6.2. — *Let X be a compact Kähler threefold and let $f\colon X \to X$ be an automorphism such that $\lambda_1(f) > 1$ and f_2^* is not semisimple. Then*

1. *$\lambda_2(f) \in \{\sqrt{\lambda_1(f)}, \lambda_1(f)^2\}$; in particular, $r(f) = 1$ and $\lambda_1 = \lambda_1(f)$ and $\lambda_2 = \lambda_2(f)$ are both cubic without real conjugates;*

2. *if* $\lambda_2 = \lambda_1^2$*, then the eigenvalue* λ_1 *has a unique non-trivial Jordan block whose dimension is* $m \leq 3$*; the other eigenvalues having non-trivial Jordan blocks have modulus* $1/\sqrt{\lambda_1}$*, and their non-trivial Jordan blocks have a dimension of at most* $m-1$*;*
3. *analogously, if* $\lambda_2 = \sqrt{\lambda_1}$*, then the eigenvalue* λ_2^{-1} *has a unique non-trivial Jordan block whose dimension is* $m \leq 3$*; the other eigenvalues having non-trivial Jordan blocks have modulus* $\sqrt{\lambda_2}$*, and their non-trivial Jordan blocks have a dimension of at most* $m-1$*.*

In what follows denote by $g = f^* \colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$ the linear automorphism induced by f, and by g_i the restriction of g to $H^i(X,\mathbb{R})$. We will denote by $\lambda_i = \lambda_i(f)$ the dynamical degrees and we will assume that

$$\lambda_2 \geq \lambda_1 > 1;$$

the case $\lambda_1 \geq \lambda_2$ follows from the previous one by replacing f by f^{-1}.

Lemma 6.3. — *If* $\lambda_2 \neq \lambda_1^2$*, then* λ_1 *has no non-trivial Jordan block for* g_2*. If* $\lambda_2 = \lambda_1^2$*, then* g_2 *has at most one non-trivial Jordan block for the eigenvalue* λ_1*, whose dimension is at most 3. In either case,* g_2 *does not have non-trivial Jordan blocks for other eigenvalues of modulus* λ_1*.*

Proof. — Suppose first that $\lambda_2 \neq \lambda_1^2$; then, by Theorem B, w_1 is a simple weight of (the semisimple part of) g. Therefore in particular λ_1 has no non-trivial Jordan block.

Now suppose that $\lambda_2 = \lambda_1^2$. We prove first that the Jordan blocks for the eigenvalue λ_1 have a dimension of at most 3. Suppose by contradiction that there exists a Jordan block of dimension at least 4; then, as in the proof of Theorem 3.1, we may pick $u_1, u_2, u_3, u_4 \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ such that

$$g(u_1) = \lambda_1 u_1, \qquad g(u_i) = u_{i+1} + \lambda_1 u_i \quad i = 1,2,3.$$

Considering

$$g^n(u_4 \wedge u_4), g^n(u_3 \wedge u_3) \in H^4(X,\mathbb{R}),$$

and applying Lemma 4.1 as in the proof of Theorem 3.1, we obtain a class $v \in H^4(X,\mathbb{R})$ such that

$$\|g_4^n v\| \sim cn^k \lambda_1^{2n} = cn^k \lambda_2^n \qquad \text{for some } k \geq 1.$$

This means that the eigenvalue λ_2 has a non-trivial Jordan block for g_4, and since $g_4 = (g_2^{-1})^\vee$, the eigenvalue λ_2^{-1} has a non-trivial Jordan block for g_2. Applying the first part of the claim to g^{-1}, we obtain that $\lambda_1 = \lambda_2^2$, contradicting the assumption that $\lambda_2 = \lambda_1^2$.

This proves that Jordan blocks of g_2 for the eigenvalue λ_1 have a dimension of at most 3.

Now let us prove that there exists a unique non-trivial Jordan block of g_2 for the eigenvalue λ_1. Suppose by contradiction that we can find linearly independent elements $u_1, u_2, v_1, v_2 \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ such that

$$g(u_1) = \lambda_1 u_1, \quad g(u_2) = u_1 + \lambda_1 u_2, \qquad g(v_1) = \lambda_1 v_1, \quad g(v_2) = v_1 + \lambda_1 v_2.$$

Then, considering

$$g^n(u_2 \wedge u_2), \quad g^n(u_2 \wedge v_2), \quad g^n(v_2 \wedge v_2)$$

and applying Lemma 4.1 to the classes u_1 and v_1, we get as before a class $v \in H^4(X,\mathbb{R})$ such that

$$\|g_4^n v\| \sim cn^k \lambda_1^{2n} = cn^k \lambda_2^n \qquad \text{for some } k \geq 1,$$

which yields a contradiction as above. This concludes the proof. □

Proof of Proposition 6.2. — Let $\lambda \in \Lambda$ be an eigenvalue of g_2 with weight w such that g_2 has a non-trivial Jordan block for λ of dimension $k > 1$. As in the proof of Theorem 3.1, we can take a Jordan basis $u_1, \dots, u_k \in H^{1,1}(X) \cup (H^{2,0}(X) \oplus H^{0,2}(X))$ such that

$$g(u_1) = \lambda u_1, \qquad g(u_{i+1}) = u_i + \lambda u_{i+1} \quad i = 1, \dots, k-1.$$

Suppose that $\lambda_2 \geq \lambda_1$, so that by Lemma 6.3 applied to f^{-1} the eigenvalue λ_2 has no non-trivial Jordan block. Let, as usual, w_1, w_2, w_3 be the weights of the eigenvalues $\alpha_1 = \lambda_1, \alpha_2 = \lambda_2^{-1}, \alpha_3 = \lambda_1^{-1}\lambda_2 \in \Lambda$.

We distinguish the following cases:

- $w \notin \{0, w_1, w_2, w_3\}$: by Propositions 4.10.(3) and 4.12.(2) we have $u_1 \wedge \bar{u}_1 \neq 0$. In particular,

  $$g^n(u_k \wedge \bar{u}^k) \sim cn^{2k-2}|\lambda|^{2n}(u_1 \wedge \bar{u}_1),$$

  which means that g_4 has a Jordan block of dimension $\geq 2k-1$ for the eigenvalue $|\lambda|^2$. Since $g_4 = (g_2^{-1})^\vee$, g_2 has a Jordan block of dimension $\geq 2k-1 > k$ for the eigenvalue $|\lambda|^{-2} \in \Lambda$;
- $w = 0$; take $\lambda \in \Lambda$ with weight 0 such that the dimension k of its maximal Jordan block is maximal, and let $v \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ be an eigenvector for the eigenvalue λ_2.

  Since $v \wedge v = 0$, by Lemma 4.1 we have either $u_1 \wedge v \neq 0$ or $u_1 \wedge \bar{u}_1 \neq 0$. In the first case, considering $g^n(u_k \wedge v)$ we obtain a non-trivial Jordan block for an eigenvalue λ' of weight $-w_2$; this implies that $w_1 = -w_2$, and by Proposition 4.12 the weight w_1 is simple, contradicting the existence of a non-trivial Jordan block. In the second case, considering $g^n(u_k \wedge \bar{u}_k)$ we obtain a Jordan block of dimension $\geq 2k-1 > k$ for the eigenvalue $|\lambda|^{-2} = 1$; since 1 has weight 0, this contradicts maximality. Therefore, this case cannot occur;
- $w = w_1$: in this case, by Lemma 6.3 we have $\lambda_2 = \lambda_1^2$, $\lambda = \lambda_1$ and $k \leq 3$;
- $w = w_2$: by Lemma 6.3 applied to f^{-1} this case cannot occur;

- $w = w_3$: let $v \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ be an eigenvector for the eigenvalue λ_2^{-1}. Since $v \wedge v = 0$, by Lemma 4.1 we have either $v \wedge \bar{u}_1 \neq 0$ or $u_1 \wedge \bar{u}_1 \neq 0$.

  In the first case we obtain a non-trivial Jordan block for an eigenvalue of weight w_1; by Lemma 6.3 we have then $\lambda_2 = \lambda_1^2$, thus $w_3 = w_1$ and, again by Lemma 6.3, $\lambda = \lambda_1$.

  In the second case, we get a non-trivial Jordan block for the eigenvalue $|\lambda|^{-2}$ with dimension $> k$.

The above computation shows that, if g_2 has a non-trivial Jordan block of dimension k for the eigenvalue $\lambda \in \Lambda$, then

- either $\lambda = \lambda_1$, in which case $\lambda_2 = \lambda_1^2$;
- or $|\lambda| \neq 1$ and there is a Jordan block of dimension $> k$ for the eigenvalue $|\lambda|^{-2}$.

One proves inductively that g_2 admits a non-trivial Jordan block for the eigenvalue λ_1. By Lemma 6.3, such a block has a dimension of at most 3; the claim follows from the fact that, as we proved above, the dimension of a non-trivial Jordan block of $\lambda \neq \lambda_1$ is strictly smaller than that of a non-trivial Jordan block of $|\lambda|^{-2}$. □

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