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HOLOMORPHIC RIEMANNIAN METRIC AND THE FUNDAMENTAL GROUP

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ABSTRACT. — We prove that compact complex manifolds bearing a holomorphic Riemannian metric have an infinite fundamental group.

RÉSUMÉ (*Métrique riemannienne holomorphe et groupe fondamental*). — Nous démontrons que les variétés complexes compactes admettant une métrique riemannienne holomorphe ont un groupe fondamental infini.

1. Introduction

The complex analogue of a (pseudo)-Riemannian metric is a *holomorphic Riemannian metric*. Recall that a holomorphic Riemannian metric g on a complex manifold X is a holomorphic section of the vector bundle $\mathrm{S}^2(T^*X)$ of complex quadratic forms on the holomorphic tangent bundle TX which is nondegenerate at every point of X (see Definition 2.1).

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Given a holomorphic Riemannian metric g on X, there is a unique torsion-free holomorphic affine connection ∇ on the holomorphic tangent bundle TX such that g is parallel with respect to ∇, or in other words,

$$\xi \cdot (g(s,t)) = g(\nabla_\xi s, t) + g(s, \nabla_\xi t)$$

for all locally defined holomorphic vector fields ξ, s and t; this unique connection ∇ is known as the *Levi-Civita connection* for g. The curvature tensor of ∇ vanishes identically if and only if g is locally isomorphic to the standard flat model $dz_1^2 + \ldots + dz_n^2$ on $\mathbb{C}^n$, where $n = \dim_{\mathbb{C}} X$. More details on the geometry of holomorphic Riemannian metrics can be found in [18, 10, 11].

Compact complex manifolds X bearing holomorphic Riemannian metrics are rather special. First notice that a holomorphic Riemannian metric g on X produces a holomorphic isomorphism between TX and its dual T^*X. In particular, the canonical line bundle and the anticanonical line bundle of X are holomorphically isomorphic, which implies that the canonical bundle is of order two (this implies that the canonical line bundle of a certain unramified double cover of X is trivial). Moreover, if X is Kähler, the classical Chern–Weil theory shows that the Chern classes with rational coefficients $c_i(X, \mathbb{Q})$ must vanish [2, pp. 192–193, Theorem 4]. It now follows, using Yau's theorem proving Calabi's conjecture [25] (see also [3] and [16]), that X admits a flat Kähler metric, and consequently, X admits a finite unramified cover which is a complex torus. Note that any holomorphic Riemannian metric on a complex torus is necessarily translation invariant and, consequently, flat.

An interesting family of compact complex non-Kähler manifolds which generalizes complex tori consists of those complex manifolds whose holomorphic tangent bundle is holomorphically trivial. They are known as *parallelizable manifolds*. Any parallelizable manifold is biholomorphic to the quotient of a complex Lie group G by a co-compact lattice Γ in G [24]. A parallelizable manifold G/Γ is Kähler if and only if G is abelian [24]. Any nondegenerate complex quadratic form on the Lie algebra of G uniquely defines a right invariant holomorphic Riemannian metric on G which descends to a holomorphic Riemannian metric on the quotient G/Γ of G by a lattice Γ. In particular, the Killing quadratic form on the Lie algebra of a complex semi-simple Lie group G, being nondegenerate and invariant under the adjoint representation, furnishes a bi-invariant holomorphic Riemannian metric on G and a G-invariant holomorphic Riemannian metric on any quotient G/Γ by a lattice Γ.

When G is $\mathrm{SL}(2,\mathbb{C})$, exotic deformations of parallelizable manifolds $\mathrm{SL}(2,\mathbb{C})/\Gamma$ bearing holomorphic Riemannian metrics were constructed by Ghys in [14]. Let us briefly recall Ghys' construction. Choose a uniform lattice Γ in $\mathrm{SL}(2,\mathbb{C})$ as well as a group homomorphism $u : \Gamma \longrightarrow \mathrm{SL}(2,\mathbb{C})$, and consider the embedding

$$\Gamma \longrightarrow \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}), \quad \gamma \longmapsto (u(\gamma), \gamma).$$

Using this homomorphism, the group Γ acts on $\mathrm{SL}(2,\mathbb{C})$ via the left and right translations of $\mathrm{SL}(2,\mathbb{C})$. More precisely, the action is given by:

$$(\gamma, x) \longmapsto u(\gamma^{-1})x\gamma \in \mathrm{SL}(2,\mathbb{C})$$

for all $(\gamma, x) \in \Gamma \times \mathrm{SL}(2,\mathbb{C})$. It is proved in [14] that for u close enough to the trivial homomorphism, the group Γ acts properly and freely on $\mathrm{SL}(2,\mathbb{C})$ such that the corresponding quotient $M(u,\Gamma)$ is a compact complex manifold (covered by $\mathrm{SL}(2,\mathbb{C})$). For a generic homomorphism u, these examples do not admit a parallelizable manifold as a finite cover (see Corollary 5.4 in [14] and its proof which shows that these generic examples and their finite covers admit no nontrivial holomorphic vector fields). Since the Killing quadratic form on the Lie algebra $\mathrm{Lie}(\mathrm{SL}(2,\mathbb{C}))$ is invariant under the adjoint representation, the induced holomorphic Riemannian metric is bi-invariant on $\mathrm{SL}(2,\mathbb{C})$ and hence it descends to the quotient $M(u,\Gamma)$. Notice that a holomorphic Riemannian metric g on $M(u,\Gamma)$ constructed this way is locally isomorphic to the complexification of the spherical metric on S^3 and it has constant non-zero sectional curvature.

The general case of a compact complex threefold X bearing a holomorphic Riemannian metric g shares many features of the previous construction of Ghys. In this direction, it was proved in [10, 11] that g is necessarily *locally homogeneous* (see Section 2), and X admits a finite unramified cover bearing a holomorphic Riemannian metric of constant sectional curvature. In view of this we make the following conjecture:

Conjecture 1.1. — *Any holomorphic Riemannian metric on a compact complex manifold X is locally homogeneous.*

Conjecture 1.1 implies that X must have an infinite fundamental group (see Section 4).

The main result proved here is the following (see Theorem 4.2):

Every compact complex manifold bearing a holomorphic Riemannian metric has infinite fundamental group.

This generalizes Corollary 4.5 and Theorem 4.6 in [5], where the same result was proved under the extra hypothesis that the algebraic dimension of X is either zero or one.

Some parts of the method of proof of Theorem 4.2 generalize to the broader framework of *rigid geometric structures* in the sense of Gromov [7, 15] and give the following (see Theorem 4.4):

Let X be a compact complex manifold with a trivial canonical bundle and an algebraic dimension equal to one. If X admits a holomorphic rigid geometric structure, then the fundamental group of X is infinite.

Theorem 4.4 was proved in [5] under the hypothesis that X is of algebraic dimension zero. It may be mentioned that for general rigid geometric structures, some hypothesis on the algebraic dimension is needed. Indeed, projective embeddings of a compact projective Calabi–Yau manifold in some complex projective space are holomorphic rigid geometric structures that are not locally homogeneous [7, 15].

Nevertheless, we make the following general conjecture which encapsulates the case of holomorphic Riemannian metrics:

Conjecture 1.2. — *Any holomorphic geometric structure of affine type ϕ on a compact complex manifold with trivial canonical bundle X is locally homogeneous. This implies that if ϕ is rigid, then the fundamental group of X is infinite.*

Conjecture 1.2 was proved to be true in the following contexts:

- the complex manifold is Kähler, so it is a Calabi–Yau manifold [9];
- the holomorphic tangent bundle of the manifold is polystable with respect to some Gauduchon metric on it [5];
- the manifold is Moishezon [4];
- the manifold is a complex torus principal bundle over a compact Kähler (Calabi–Yau) manifold [4].

2. Geometric structures and Killing fields

Let X be a complex manifold of complex dimension n.

Definition 2.1. — A *holomorphic Riemannian metric* on X is a holomorphic section

$$g \in H^0(X, \mathrm{S}^2((TX)^*))$$

such that for every point $x \in X$ the quadratic form $g(x)$ on T_xX is nondegenerate.

Holomorphic Riemannian metrics and holomorphic affine connections are *rigid geometric structures* in the sense of Gromov [15] (see also [7]). Let us briefly recall the definition of rigidity in the holomorphic category.

For any integer $k \geq 1$, we associate the principal bundle of k-frames $R^k(X) \longrightarrow X$, which is the bundle of k-jets of local holomorphic coordinates on X. The corresponding structural group D^k is the group of k-jets of local biholomorphisms of $\mathbb{C}^n$ fixing the origin. This D^k is known to be a complex algebraic group.

Definition 2.2. — A holomorphic geometric structure ϕ of order k on X is a holomorphic D^k-equivariant map from $R^k(X)$ to a complex algebraic manifold Z endowed with an algebraic action of D^k. The geometric structure ϕ is said to be of affine type if Z is a complex affine variety.

Holomorphic tensors are holomorphic geometric structures of affine type of order one, and holomorphic affine connections are holomorphic geometric structures of affine type of order two. Holomorphic embeddings in projective spaces, holomorphic foliations and holomorphic projective connections are holomorphic geometric structure of non-affine type [7, 15].

A (local) biholomorphism f between two open subsets of X is called a (local) *isometry (automorphism)* for a geometric structure ϕ if the canonical lift of f to $R^k(X)$ preserves the fibers of ϕ.

The associated notion of a (local) infinitesimal symmetry is the following:

Definition 2.3. — A (local) holomorphic vector field Y is a (local) *Killing field* of a holomorphic geometric structure $\phi : R^k(X) \longrightarrow Z$ if its canonical lift to $R^k(X)$ preserves the fibers of ϕ.

In other words, Y is a Killing field of ϕ if and only if its (local) flow preserves ϕ. The Killing vector fields form a Lie algebra with respect to the Lie bracket operation of vector fields.

A classical result in Riemannian geometry shows that Y is a Killing field of a holomorphic Riemannian metric g on X if and only if

$$TX \longrightarrow TX, \quad v \longmapsto \nabla_v Y$$

is a skew-symmetric section of $\mathrm{End}(TX)$ with respect to g, where ∇ is the Levi-Civita connection of g [17].

A holomorphic geometric structure ϕ is *rigid* of order l in the sense of Gromov if any local automorphism of ϕ is completely determined by its l-jet at any given point (see [7, 15]).

Holomorphic affine connections are rigid of order one in the sense of Gromov. The rigidity arises from the fact that the local biholomorphisms fixing a point and preserving a connection actually linearize in exponential coordinates, so they are completely determined by their differential at the fixed point. Holomorphic Riemannian metrics, holomorphic projective connections and holomorphic conformal structures for dimensions of at least three are rigid holomorphic geometric structures. On the other hand, holomorphic symplectic structures and holomorphic foliations are not rigid [7].

Local Killing fields of a holomorphic rigid geometric structure ϕ form a locally constant sheaf of Lie algebras [7, 15]. The typical fiber in that case is a finite dimensional Lie algebra called the Killing algebra of ϕ. The geometric structure ϕ is called *locally homogeneous* if its Killing algebra acts transitively on X.

The standard facts about smooth actions of Lie groups preserving an analytic rigid geometric structure and a finite volume are adapted to our holomorphic set-up (compare with [15, Section 3.5] and also with [13, Section 5]).

Lemma 2.4. — *Let X be a compact complex manifold endowed with a holomorphic rigid geometric structure g. Assume that the automorphism group G of (X, g) preserves a smooth volume on X and that G is noncompact. Then at any point $x \in X$, there exists at least one nontrivial local Killing field Y of g such that $Y(x) = 0$.*

Proof. — Let $\phi : R^k(X) \longrightarrow Z$ be a holomorphic rigid geometric structure of order k. Then there exists $l \in \mathbb{N}$ large enough such that the l-jet

$$\phi^{(l)} : R^{k+l}(X) \longrightarrow Z^{(l)}$$

of ϕ satisfies the condition that the orbits of the local automorphisms of ϕ are the projections on X of the inverse images, through $\phi^{(l)}$, of the D^{k+l}-orbits in $Z^{(l)}$ [7, 15]. Recall that the map $\phi^{(l)}$ is D^{k+l}-equivariant.

Since the automorphism group G preserves the finite smooth measure (X is compact), by Poincaré recurrence theorem, for any generic point $x \in X$ there exists an unbounded sequence of elements $g_j \in G$, $j \geq 1$, (meaning a sequence leaving every compact subset in G) such that $g_j \cdot x$ converges to x. We lift the G-action on X to the bundle $R^{k+l}(X)$ and we consider the orbit $g_j \cdot \widehat{x}$ of a lift $\widehat{x}$ of x to R^{k+l}. There exists a sequence $\{p_j\}_{j=1}^{\infty}$ in D^{k+l} such that $g_j(\widehat{x}) \cdot p_j^{-1}$ converges to $\widehat{x}$. Notice that $\{p_j\}_{j=1}^{\infty}$ is an unbounded sequence in D^{k+l}, since the lifted G-action on $R^{k+l}(X)$ is proper. Using the equivariance property of $\phi^{(l)}$ we conclude that $p_j \cdot \phi^{(l)}(\widehat{x})$ converges to $\phi^{(l)}(\widehat{x})$.

The action of the algebraic group D^{k+l} on $Z^{(l)}$ is algebraic. This implies that the D^{k+l}-orbits in $Z^{(l)}$ are locally closed [21]. In particular, this also remains valid for the orbit $\mathcal{O}$ of $\phi^{(l)}(\widehat{x})$. Let us denote by I the stabilizer of $\phi^{(l)}(\widehat{x})$ in D^{k+l}. The orbit $\mathcal{O}$ with the induced topology coming from $Z^{(l)}$ is homeomorphic to the quotient D^{k+l}/I. The above observation that $p_j \cdot \phi^{(l)}(\widehat{x})$ converges to $\phi^{(l)}(\widehat{x})$ is equivalent to the existence of a sequence (η_j) in D^{k+l} converging to identity, such that $\eta_j \cdot p_j \in I$. Since I contains an unbounded sequence in D^{k+l} and it is an algebraic group – hence having only finitely many connected components – it follows that its connected component $I^0 \subset I$ containing the identity element is a connected complex algebraic subgroup of D^{k+l} of complex dimension at least one. Any one parameter subgroup in I^0 integrates a local Killing field Y vanishing at x (see Corollary 1.6 C in [15]).

We proved that for any recurrent point $x \in X$, there exists at least one nontrivial local Killing field Y of g such that $Y(x) = 0$. Equivalently, at any recurrent point, the dimension of the local orbit of the Killing algebra is strictly less than the dimension of the Killing algebra (which is same on all of X). This property is thus true on all of X, by the density of recurrent points in X. □

Given a holomorphic Riemannian metric g, there is a holomorphic volume form ω_g associated with it, and hence there is an associated volume form on X given by $(\sqrt{-1})^n \cdot \omega_g \wedge \overline{\omega_g}$. The automorphism group for g preserves the smooth measure on X associated to $(\sqrt{-1})^n \cdot \omega_g \wedge \overline{\omega_g}$.

Recall that the automorphism group of a compact complex simply connected manifold X endowed with a holomorphic rigid geometric structure g is known to admit finitely many connected components [15, Section 3.5]. Therefore, under these assumptions, the automorphism group of (X, g) is compact if and only if its connected component containing the identity element is compact.

3. Algebraic reduction and orbits of Killing fields

Recall that the *algebraic dimension* of a compact complex manifold X is the transcendence degree of the field of meromorphic functions $\mathcal{M}(X)$ on X over the field of complex numbers. The algebraic dimension of a projective manifold coincides with its complex dimension. In general, the algebraic dimension of compact complex manifolds of complex dimension n may be less than n and in fact takes all integral values between 0 and n. Compact complex manifolds of maximal algebraic dimension n are called Moishezon manifolds. Moishezon manifolds are known to be bi-meromorphic to projective manifolds [19]. More generally there is the following classical result called the *algebraic reduction* theorem (see [23]):

Theorem 3.1 ([23]). — *Let X be a compact connected complex manifold of algebraic dimension $a(X) = d$. Then there exists a bi-meromorphic modification*

$$\Psi : \widetilde{X} \longrightarrow X,$$

and a holomorphic map

$$t : \widetilde{X} \longrightarrow V$$

with connected fibers onto a d-dimensional algebraic manifold V, such that $t^*(\mathcal{M}(V)) = \Psi^*(\mathcal{M}(X))$.

In the statement of Theorem 3.1 and in the sequel, algebraic manifold means a smooth complex projective manifold.

Let $\pi : X \longrightarrow V$ be the meromorphic map given by $t \circ \Psi^{-1}$; it is called the algebraic reduction of X.

Theorem 3.2. — *Let X be a compact, connected and simply connected complex manifold of complex dimension n and of algebraic dimension d. Suppose that X admits a holomorphic rigid geometric structure g. Then $H^0(X, TX)$ admits an abelian subalgebra A acting on X preserving g and satisfying the condition that each generic fiber of the algebraic reduction of X lies in some orbit of A (hence the dimension of A is at least $n - d$). Moreover, A is the Lie algebra of the connected component, containing the identity element, of the automorphism* *group of the rigid geometric structure g' which is a juxtaposition of g with a maximal family of commuting Killing fields of g.*

Notice that in Theorem 3.2 the generic fibers form an analytic open dense subset of X.

Proof. — By the main theorem in [8] (see also [9]) the Lie algebra of local holomorphic vector fields on X preserving g acts on X with generic orbits containing the fibers of the algebraic reduction π of X.

Since X is simply connected, by a result due to Nomizu, [20], generalized first by Amores, [1], and then by D'Ambra–Gromov, [7, p. 73, 5.15], local vector fields preserving g extend to all of X. Thus we get a finite dimensional complex Lie algebra formed by holomorphic vector fields X_i preserving g, which acts on X with orbits containing the generic fibers of the algebraic reduction π. This finite dimensional complex Lie algebra of holomorphic vector fields will be denoted by $\mathcal{G}$.

Now put together g and a family of global holomorphic vector fields X_i spanning $\mathcal{G}$, to form another rigid holomorphic geometric structure $g' = (g, X_i)$; see [7] (Section 3.5.2 A) for details about the fact that the juxtaposition of a rigid geometric structure with another geometric structure is still a rigid geometric structure in the sense of Gromov. Considering g' instead of g and repeating the same argument as before, the complex Lie algebra A of those holomorphic vector fields preserving g' acts on X with generic orbits containing the fibers of the algebraic reduction π. But preserving g' means preserving g and commuting with the vector fields X_i. Hence A coincides with the center of $\mathcal{G}$. In particular, A is a complex abelian Lie algebra acting on X preserving g and with orbits containing the generic fibers of the algebraic reduction π. □

3.1. Maximal algebraic dimension.

— Assume that X is a Moishezon manifold, so the algebraic dimension of X is $n = \dim_{\mathbb{C}} X$.

PROPOSITION 3.3. — *If TX admits a holomorphic connection, then X admits a finite unramified covering by a compact complex torus.*

The first step of the proof of Proposition 3.3 is the following:

LEMMA 3.4. — *Let X be a complex manifold endowed with an affine holomorphic connection. Then there is no nonconstant holomorphic map from $\mathbb{CP}^1$ to X.*

Proof. — Let ∇ be a holomorphic connection on X. Let

$$f : \mathbb{CP}^1 \longrightarrow X$$

be a holomorphic map. Consider the pulled back connection $f^*\nabla$ on f^*TX. Since $\dim_{\mathbb{C}} \mathbb{CP}^1 = 1$, the connection $f^*\nabla$ is flat. Moreover, $\mathbb{CP}^1$ being simply connected, $f^*\nabla$ has trivial monodromy, implying that the holomorphic vector bundle f^*TX is trivial.

Now consider the differential of f

$$df : T\mathbb{CP}^1 \longrightarrow f^*TX.$$

There is no nonzero holomorphic homomorphism from $T\mathbb{CP}^1$ to the trivial holomorphic line bundle, because $2 = \text{degree}(T\mathbb{CP}^1) > \text{degree}(\mathcal{O}_{\mathbb{CP}^1}) = 0$. This implies that $df = 0$. Therefore, f is a constant map. □

Proof of Proposition 3.3. — Since the Moishezon manifold X does not admit any nonconstant holomorphic map from $\mathbb{CP}^1$, it is a complex projective manifold [6, p. 307, Theorem 3.1].

As TX admits a holomorphic connection, and X is a complex projective manifold, we have $c_i(X, \mathbb{Q}) = 0$ for all $i > 0$ [2, p. 192–193, Theorem 4], where $c_i(X, \mathbb{Q})$ denotes the i-th Chern class of TX with rational coefficients. Therefore, X being complex projective, from Yau's theorem proving Calabi's conjecture, [25], it follows that X admits a finite unramified covering by a compact complex torus (see also [3, p. 759, Theorem 1] and [16]). □

4. Rigid geometric structures and fundamental group

In this section we prove the two main results mentioned in the introduction.

Let us first address the easy case where the geometric structure is taken to be locally homogeneous.

Proposition 4.1. — *Let X be a compact complex manifold with trivial canonical bundle. If X is endowed with a locally homogeneous holomorphic rigid geometric structure g, then the fundamental group of X is infinite.*

Proof. — Assume, by contradiction, that the fundamental group of X is finite. So replacing X by its universal cover we may assume that X is simply connected. Since g is locally homogeneous, and local Killing fields extend to all of X by Nomizu's theorem [20, 1, 15], it follows that TX is generically spanned by globally defined holomorphic Killing vector fields. Let $\{X_1, \cdots, X_n\}$ be a family of linearly independent holomorphic vector fields on X which span TX at the generic point. Consider a nontrivial holomorphic section vol of the canonical line bundle, and evaluate it on $X_1 \wedge \cdots \wedge X_n$ to get a holomorphic function $vol(X_1 \wedge \cdots \wedge X_n)$. This holomorphic function on X is constant and nonzero at the generic point, hence it is nowhere zero. This immediately implies that $\{X_1, \cdots, X_n\}$ span TX at every point in X. Consequently, TX admits a holomorphic trivialization and hence, by Wang's theorem [24], this X is a quotient of a connected complex Lie group by a lattice in it. In particular, the fundamental group of X is infinite: a contradiction. □

Theorem 4.2. — *Let X be a compact complex manifold admitting a holomorphic Riemannian metric g. Then the fundamental group of X is infinite.*

Proof. — Assume, by contradiction, that X is endowed with a holomorphic Riemannian metric g and has a finite fundamental group. Replacing X by its universal cover we can assume that X is simply connected. Denote also by g the symmetric bilinear form associated to the quadratic form g. The holomorphic tangent bundle TX is endowed with the (holomorphic) Levi-Civita connection associated to g. If X is a Moishezon manifold, Proposition 3.3 shows that X admits a finite unramified cover which is a complex torus: a contradiction (under the assumption that X is Moishezon).

Now consider the situation where the algebraic dimension d of M is strictly less than the complex dimension n of M. Consequently the algebraic reduction of M admits fibers of positive dimension $n-d$ (see Theorem 3.1).

By Theorem 3.2, there exists a finite dimensional abelian Lie algebra A lying inside the Lie algebra of global holomorphic vector fields on X such that

- A preserves g, and
- the generic fibers of the algebraic reduction are contained in the leaves of the foliation generated by A.

Let $X_1, X_2, \cdots, X_k \in H^0(X, TX)$ be a basis of A; so we have $k \geq n-d > 0$.

For all $i, j \in \{1, \cdots, k\}$, the functions $g(X_i, X_j)$ on X are holomorphic and hence constant.

Assume first that there exists $X_i \in A$ as above such that the dual one-form ω_i defined by

$$\omega_i(v) := g(X_i, v)$$

vanishes on all tangent vectors tangent to the orbits of A (equivalently, $g(X_i, X_j) = 0$, for all $j \in \{1, \cdots, k\}$). This implies that ω_i vanishes on the generic fibers of the algebraic reduction

$$\pi : X \longrightarrow V.$$

We observe that $\Psi^*(\omega_i)$ (see Theorem 3.1 for the map Ψ) defines a holomorphic one-form on $\widetilde{X}$. To see that $\Psi^*(\omega_i)$ is a holomorphic one-form on entire $\widetilde{X}$, first note that the meromorphic map Ψ is holomorphic away from the indeterminacy set S which is of complex codimension of at least two in $\widetilde{X}$. Consequently, $\Psi^*(\omega_i)$ is a holomorphic one-form on the complement $\widetilde{X} \setminus S$, and hence by Hartog's theorem can be uniquely extended to a holomorphic one-form on entire $\widetilde{X}$ (for more details the reader is referred to Proposition 1.2 in pages 282–283 of [22]).

Since $\Psi^*(\omega_i)$ vanishes on the generic fibers of t (and hence on all fibers), which are compact and connected, it can be shown that there is a holomorphic one-form $\widetilde{\omega_i}$ defined on the compact complex projective manifold V such that $\Psi^*(\omega_i)$ is the pull-back $t^*(\widetilde{\omega_i})$. To prove this, let us first define the one-form $\widetilde{\omega_i}$

on the complement $V \setminus S_0$, where S_0 is the set of critical values of t. Take any $v \in V \setminus S_0$ and $w \in T_v V$, and define

$$\widetilde{\omega_i}(v) \cdot w := \Psi^*(\omega_i)(\widetilde{v}) \cdot \widetilde{w},$$

where $\widetilde{v} \in t^{-1}(v)$ is any element of the (regular) fiber above v and $\widetilde{w} \in T_{\widetilde{v}} \widetilde{X}$ is such that $dt(\widetilde{v}) \cdot \widetilde{w} = w$. It should be clarified that this definition does not depend on the choice of $\widetilde{w}$ because $\Psi^*(\omega_i)$ vanishes on the fiber $t^{-1}(v)$. Moreover this definition does not depend either on the choice of $\widetilde{v} \in t^{-1}(v)$, since $t^{-1}(v)$ being a compact and connected manifold, any holomorphic map from $t^{-1}(v)$ to $(T_v V)^*$ must be a constant one. This construction furnishes a holomorphic one-form $\widetilde{\omega_i}$ on the complement $V \setminus S_0$ such that $\Psi^*(\omega_i) = t^*(\widetilde{\omega_i})$ on $t^{-1}(V \setminus S_0)$.

Finally, in order to extend $\widetilde{\omega_i}$ to all of V we make use of the following Lemma 3.3 in [12] (page 57).

Lemma 4.3 ([12, Lemma 3,3]). — *Let $t : \widetilde{X} \longrightarrow V$ be a holomorphic mapping with connected fibers between compact complex manifolds $\widetilde{X}$ and V. Let $S_0 \subset V$ be the set of all critical values of t. Assume that V is Kähler. Then*

$$H^0\left(\widetilde{X}, \Omega^1_{\widetilde{X}}\right) \cap H^0\left(\widetilde{X} \setminus t^{-1}(S_0), t^*\Omega^1_V\right) \subset t^* H^0\left(V, \Omega^1_V\right).$$

Notice that Lemma 4.3 directly applies to our situation since we already proved that

$$\Psi^*(\omega_i) \in H^0\left(\widetilde{X}, \Omega^1_{\widetilde{X}}\right) \cap H^0\left(\widetilde{X} \setminus t^{-1}(S_0), t^*\Omega^1_V\right).$$

Hence there exists a holomorphic one-form $\widetilde{\omega_i}$ on V such that $\Psi^*(\omega_i) = t^*(\widetilde{\omega_i})$.

Holomorphic forms on algebraic manifolds being closed, it follows that

$$d\Psi^*(\omega_i) = d\widetilde{\omega_i} = 0.$$

Consequently, we have $d\omega_i = 0$.

Since X is simply connected, the closed form ω_i must be exact. This implies that ω_i vanishes identically, and hence the vector field X_i vanishes identically: a contradiction.

Thus we are left with the case where g restricted to A is nondegenerate. So assume that g restricted to A is nondegenerate. This immediately implies that the vector fields in A do not vanish at any point of X. Consequently, the foliation generated by A is nonsingular and is of complex dimension k. Now Lemma 2.4 implies that the corresponding connected Lie group G, meaning the connected component, containing the identity element, of the automorphism group of the holomorphic rigid geometric structure $g' = (g, X_1, \cdots, X_k)$, is compact. Consequently, G must be isomorphic to a compact complex torus T of dimension $n - d$.

The action of $G = T$ on X is locally free. We will show that this action must be free. For this, assume that an element $f \in G$ fixes $x_0 \in X$. Since G is abelian, the differential $df(x_0)$ at x_0 acts trivially on Ax_0, where $Ax_0 \subset T_{x_0}X$ denotes the infinitesimal orbit of A (meaning the evaluation of the Lie algebra of vector fields A at x_0). It must preserve its g-orthogonal part $(Ax_0)^\perp$. Moreover, since f preserves each orbit of G, the action of $df(x_0)$ must also be trivial on $(Ax_0)^\perp$. Recall that g restricted to A is nondegenerate, which implies that

$$Ax_0 \oplus (Ax_0)^\perp = T_{x_0}X.$$

It now follows that $df(x_0)$ is the identity map, and so f must be trivial: it is the identity element in G.

The action of G being free, X is a holomorphic principal G–bundle over the compact complex manifold $N := X/G$. Since X is simply connected and G is connected, it follows that N is simply connected. The action of G on X preserves A and g, hence the restriction of g to $A^\perp$ defines a transverse holomorphic Riemannian metric transverse to the foliation defined by the G-action. This transverse holomorphic Riemannian metric descends to a holomorphic Riemannian metric on N. But the complex dimension of N is strictly less then the complex dimension of X. Now an induction on the complex dimension of X finishes the proof; note that the only compact Riemann surfaces admitting holomorphic Riemannian metrics are elliptic curves and they have an infinite fundamental group. □

Recall that it was proved in [5, Proposition 4.4] that *compact complex manifolds with trivial canonical bundle and algebraic dimension zero admitting holomorphic rigid geometric structures have an infinite fundamental group.*

We prove the following here:

Theorem 4.4. — *Let X be a compact complex manifold with trivial canonical bundle and assume its algebraic dimension is one. If X admits a holomorphic rigid geometric structure, then the fundamental group of X is infinite.*

Proof. — Let X be a compact complex manifold bearing a holomorphic rigid geometric structure g. Assume, by contradiction, that the fundamental group of X is finite. Replacing X by its universal cover we assume that X is simply connected.

We now use Theorem 3.2 to get an abelian subalgebra A of $H^0(X, TX)$ which acts on X preserving g and satisfying the condition that each generic fiber of the algebraic reduction of X lies in some A-orbit (hence the dimension of A is at least $n-1$). Choose elements $X_1, \cdots, X_{n-1}$ of A which span, at the generic point $x \in X$, the tangent space of the fiber $\pi^{-1}(\pi(x))$ of the algebraic reduction π of X.

Consider a nontrivial holomorphic section vol of the canonical bundle of X. Then the holomorphic one-form ω on X defined by $v \longmapsto vol(X_1 \wedge \ldots \wedge X_{n-1} \wedge v)$ vanishes on the fibers of the algebraic reduction π. As in the proof of Theorem 4.2, the form $\Psi^*(\omega)$ (see Theorem 3.1 for Ψ) descends to the complex projective manifold V, the base manifold of the algebraic reduction (once again, extension across singular fibers is not an issue in view of Lemma 4.3). In particular, the form ω is closed. This implies that the fundamental group of X is infinite. □

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