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A PROOF OF THE ℓ-ADIC VERSION OF THE INTEGRAL IDENTITY CONJECTURE FOR POLYNOMIALS

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Abstract. — It is well known that the integral identity conjecture is of prime importance in Kontsevich-Soibelman's theory of motivic Donaldson-Thomas invariants for non-commutative Calabi-Yau threefolds. In this article we consider its numerical version and give a complete demonstration of the case where the potential is a polynomial and the ground field is algebraically closed. The fundamental tool is the Berkovich spaces whose crucial point is how to use the comparison theorem for nearby cycles as well as the Künneth isomorphism for cohomology with compact support.

Résumé (*Une preuve de la version ℓ-adique de la conjecture de l'identité intégrale pour des polynômes*). — Il est bien connu que la conjecture de l'identité intégrale est de première importance de la théorie de Kontsevich-Soibelman des invariants de Donaldson-Thomas motiviques pour des variétés de Calabi-Yau de dimension 3 non commutatives. Dans cet article nous considérons sa version numérique et donnons une complète démonstration dans le cas où le potentiel est un polynôme et le corps de base est algébriquement clos. L'outil fondamental pour la preuve est les espaces de Berkovich dont le point crucial est l'utilisation du théorème de comparaison entre des cycles proches ainsi que l'isomorphisme de Künneth en cohomologie à support compact.

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1. Introduction

Let us start by outlining the concept of motivic Donaldson-Thomas invariants that concern the integral identity conjecture [11]. These invariants are introduced in [10] in the framework for Calabi-Yau threefolds and the motivic Hall algebra. The latter generalizes the derived Hall algebra of Toën [19].

Let $\mathcal{C}$ be an ind-constructible triangulated A_∞-category over a field κ. By giving a constructible stability condition on $\mathcal{C}$ one considers a collection of full subcategories $\mathcal{C}_V \subseteq \mathcal{C}$, with V strict sectors in $\mathbb{R}^2$. The stability condition depends on homomorphisms $cl : K_0(\mathcal{C}) \to \Gamma$ and $Z : \Gamma \to \mathcal{C}$, where Γ is a free abelian group endowed with a skew-symmetric integer-valued bilinear form $\langle \bullet, \bullet \rangle$. A choice of V gives rise to a cone $C(V,Z)$ contained in $\Gamma \otimes \mathbb{R}$ to which one associates a complete motivic Hall algebra $\hat{H}(\mathcal{C}_V)$. Define A_V^{Hall} as invertible in $\hat{H}(\mathcal{C}_V)$ as characteristic functions of the stacks of objects of $\mathcal{C}_V$. The *generic* elements satisfy the factorization property

$$A_V^{\mathrm{Hall}} = A_{V_1}^{\mathrm{Hall}} \cdot A_{V_2}^{\mathrm{Hall}}$$

with $V = V_1 \sqcup V_2$ and with the decomposition taken clockwise.

If the field κ has characteristic zero, the motivic quantum torus $\mathcal{R}_\mathcal{C}$ is defined to be an associative algebra generated by symbols $\hat{e}_\gamma$, for γ in Γ, with the usual relations

$$\hat{e}_{\gamma_1}\hat{e}_{\gamma_2} = [\mathbb{A}_\kappa^1]^{\frac{1}{2}\langle \gamma_1,\gamma_2\rangle}\hat{e}_{\gamma_1+\gamma_2},\ \hat{e}_0 = 1,$$

where $[\mathbb{A}_\kappa^1]^{\frac{1}{2}}$ is a square root of $[\mathbb{A}_\kappa^1]$. The coefficient ring C_0 for the quantum torus $\mathcal{R}_\mathcal{C}$ can be any commutative ring, where the two most important candidates should be a certain localization of the Grothendieck ring of algebraic κ-varieties and its ℓ-adic version.

By choosing in addition the so-called orientation data (its existence depends on another conjecture) and using Denef-Loeser's theory of motivic Milnor fiber (e.g. the motivic Thom-Sebastiani theorem) of the potential of an object of the category $\mathcal{C}$, from [10, Section 6], there is a map $\Phi_V : \hat{H}(\mathcal{C}_V) \to \mathcal{R}_{\mathcal{C}_V}$ for each V, which is nice enough in the sense that if it were a homomorphism the factorization property would be preserved. This is in fact obstructed because of the lack of an assertion of the integral identity. In the case where the above C_0 is a certain localization of the ring $\mathcal{M}_\kappa^{\hat{\mu}}$, one faces the full version of the integral identity conjecture. If well passed, $A_V^{\mathrm{mot}} := \Phi_V(A_V^{\mathrm{Hall}})$ would be invariants in the category of non-commutative Calabi-Yau threefolds, namely *motivic* Donaldson-Thomas invariants. Also, if C_0 is a variant of the Grothendieck ring $K_0(D^b_{\mathrm{constr,aut}}(\mathrm{Spec}(\kappa), \mathbb{Q}_\ell))$, one meets the ℓ-adic version of the conjecture, and in this case, the corresponding invariants are *numerical* Donaldson-Thomas invariants.

In the context of non-Archimedean complete discretely valued fields K of equal characteristic zero, with valuation ring R and residue field κ, the motivic nearby cycles $\mathcal{S}_\mathfrak{f}$ of a formal function $\mathfrak{f} : \mathfrak{X} \to \mathrm{Spf}(R)$ were defined (see [10, 13]).

To do this, it needs to use Denef-Loeser's formula on the motivic nearby cycles of a regular function (cf. [7, 8]) as well as the fact that resolution of singularities of $(\mathfrak{X}, \mathfrak{X}_0)$ exists (see Temkin [17]). Let $\int_{\mathcal{U}}$ be the forgetful morphism for $\mathcal{U}$ a subvariety of $\mathfrak{X}_0$.

Conjecture 1.1 (Integral identity [10]). — *Let f be in $\kappa[[x, y, z]]$ invariant by the natural $\kappa^\times$-action of weight $(1, -1, 0)$, with $f(0, 0, 0) = 0$. Denote by $\mathfrak{X}$ the formal completion of $\mathbb{A}^d_\kappa$ along $\mathbb{A}^{d_1}_\kappa$ with structural morphism $\widehat{f}$ induced by f, and by $\mathfrak{Z}$ the formal completion of $\mathbb{A}^{d_3}_\kappa$ at the origin with structural morphism $\widehat{f}_{\mathfrak{Z}}$ induced by $f(0, 0, z)$. Then the identity $\int_{\mathbb{A}^{d_1}_\kappa} \mathcal{S}_{\widehat{f}} = [\mathbb{A}^1_\kappa]^{d_1} \mathcal{S}_{\widehat{f}_{\mathfrak{Z}}}$ holds in $\mathcal{M}^{\hat{\mu}}_\kappa$.*

In [12] and [13], the author proves the conjecture in some important cases under certain conditions on f or on the base field κ. In [16], Nicaise and Payne obtain a crucial improvement in proving the strong form of the conjecture for polynomials when the base field κ contains all roots of unity. More recently, Nguyen and the author [14] refine the result of Nicaise-Payne with the absence of additional conditions on κ. The purpose of the present article is to show that the ℓ-adic version of the integral identity conjecture holds if f is a polynomial and κ is an algebraically closed field. We denote by $R\psi$ the nearby cycles functor, which was already defined in [3, 4] and will be briefly recalled here in Subsection 2.5.

Theorem 1.2. — *With the assumptions as in Conjecture 1.1, and, moreover, κ algebraically closed, f a polynomial, there is a canonical quasi-isomorphism of complexes $R\Gamma_c(\mathbb{A}^{d_1}_\kappa, R\psi_{\widehat{f}}\mathbb{Q}_\ell) \to R\Gamma_c(\mathbb{A}^{d_1}_\kappa, \mathbb{Q}_\ell) \otimes (R\psi_{\widehat{f}_{\mathfrak{Z}}}\mathbb{Q}_\ell)$.*

Our approach uses Kontsevich-Soibelman's idea in [10, Proposition 9], concerning Berkovich spaces. The fundamental tools are the comparison theorem for nearby cycles and the Künneth isomorphism for étale cohomology with compact support.

2. Preliminaries on the Berkovich spaces

2.1. Notation. — Let $\mathcal{A}$ be a commutative Banach ring, i.e., a commutative ring with unity endowed with a *Banach norm* $\|\cdot\|$ such that every Cauchy sequence has a limit. Here, by a Banach norm on a commutative ring with unity we mean a non-negative real-valued function $\|\cdot\|$ with properties: $\|f\| = 0$ iff $f = 0$, $\|fg\| \leq \|f\| \cdot \|g\|$ and $\|f + g\| \leq \|f\| + \|g\|$. A nonzero bounded multiplicative seminorm on such an $\mathcal{A}$ is a function $|\cdot| : \mathcal{A} \to \mathbb{R}_+$ such that $|f| \neq 0$ for some $f \in \mathcal{A}$, $|gf| = |f| \cdot |g|$, $|f + g| \leq |f| + |g|$ and that there exists a real number $C > 0$ with $|f| \leq C\|f\|$ for all $f \in \mathcal{A}$. The *spectrum* of the commutative Banach ring $\mathcal{A}$, $\mathcal{M}(\mathcal{A})$, is the set of all nonzero bounded multiplicative seminorms on $\mathcal{A}$. For a point x of $\mathcal{M}(\mathcal{A})$ we denote by $|.|_x$ the corresponding seminorm and get the prime ideal $\ker(|\cdot|_x)$ and the domain $\mathcal{A}/\ker(|\cdot|_x)$. Then

$|\cdot|_x$ induces a Banach norm on this domain and a multiplicative Banach norm on its fraction field, whose completion will be denoted by $\mathcal{H}(x)$. Thus $\mathcal{H}(x)$ is a complete field with respect to a valuation $|\cdot|$. By writing $f(x)$ for the image of $f \in \mathcal{A}$ under the bounded character $\mathcal{A} \to \mathcal{H}(x)$ we get $|f|_x = |f(x)|$. The spectrum $\mathcal{M}(\mathcal{A})$ is endowed with the weakest topology such that every function $x \mapsto |f(x)|$ for some $f \in \mathcal{A}$ is continuous, for which it is a locally compact topological space ($\mathcal{A} \neq \{0\}$).

Let K be a non-Archimedean complete discretely valued field of equal characteristics zero, with valuation ring R, maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. Denote by $\mathbb{A}^n_{K,\mathrm{Ber}}$ the n-dimensional K-analytic affine space, which is by definition the set of all multiplicative seminorms on the ring of polynomials $K[T_1, \ldots, T_n]$ whose restriction to K is bounded with respect to the Banach norm on K (see [1]). As previously stated, $\mathbb{A}^n_{K,\mathrm{Ber}}$ is endowed with the weakest topology so that every function $x \mapsto |f(x)|$ for some $f \in \mathcal{A}$ is continuous and $\mathbb{A}^n_{K,\mathrm{Ber}}$ is a locally compact topological space. For any $x \in \mathbb{A}^n_{K,\mathrm{Ber}}$, we define

$$|x| := \max_{1 \leq i \leq n} |T_i(x)|.$$

The subspace of $\mathbb{A}^n_{K,\mathrm{Ber}}$ defined by $|x| \leq 1$ (resp. $|x| < 1$) is called *the n-dimensional unit closed disc* (resp. *the n-dimensional unit open disc*). One denotes these spaces by $E^n(0;1)$ and $D^n(0;1)$, respectively.

2.2. From special formal schemes to analytic spaces. — A topological R-algebra $\mathcal{A}$ is said to be *special* if $\mathcal{A}$ is a Noetherian adic ring such that, if $\mathcal{J}$ is an ideal of definition of $\mathcal{A}$, the quotient rings $\mathcal{A}/\mathcal{J}^n$, $n \geq 1$, are finitely generated over R. By [4], a topological R-algebra $\mathcal{A}$ is special if and only if it is topologically R-isomorphic to a quotient of the special R-algebra $R\{T_1, \ldots, T_n\}[[S_1, \ldots, S_m]]$. An adic R-algebra $\mathcal{A}$ is *topologically finitely generated over R* if it is topologically R-isomorphic to a quotient algebra of the algebra of restricted power series $R\{T_1, \ldots, T_n\}$. Evidently, any topologically finitely generated R-algebra is a special R-algebra.

A formal R-scheme $\mathfrak{X}$ is *special* if $\mathfrak{X}$ is a separated Noetherian adic formal scheme and if it is a finite union of affine formal schemes of the form $\mathrm{Spf}(\mathcal{A})$ with $\mathcal{A}$ special R-algebras. A formal R-scheme $\mathfrak{X}$ is *topologically of finite type* if it is a finite union of affine formal schemes of the form $\mathrm{Spf}(\mathcal{A})$ with $\mathcal{A}$ topologically finitely generated R-algebras. It is a fact that the category of separated topologically of finite type formal R-schemes is a full subcategory of the category of R-special formal schemes, and both admit fiber products.

A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ of special formal schemes is *locally of finite type* if locally it is isomorphic to a morphism of the form $\mathrm{Spf}(\mathcal{B}) \to \mathrm{Spf}(\mathcal{A})$ with $\mathcal{B}$ topologically finitely generated over $\mathcal{A}$. The morphism φ is of *finite type* if it is a quasi compact morphism of locally finite type.

By [4], there is a canonical functor $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ from the category of special formal R-schemes to that of (Berkovich) K-analytic spaces. In the *standard* affine case, the interpretation of this functor is explicit; namely, if

$$\mathfrak{X} = \operatorname{Spf}\left(R\{T_1, \ldots, T_n\}\,[[S_1, \ldots, S_m]]\right),$$

then one has

$$\mathfrak{X}_\eta = E^n(0;1) \times D^m(0;1).$$

Also, if $\mathfrak{X} = \operatorname{Spf}(\mathcal{A})$, where $\mathcal{A}$ is a quotient of $R\{T_1, \ldots, T_n\}[[S_1, \ldots, S_m]]$ by an ideal $\mathcal{I}$, then $\mathfrak{X}_\eta$ is the closed K-analytic subspace of $X = E^n(0;1) \times D^m(0;1)$ defined by the subsheaf of ideals $\mathcal{I}\mathcal{O}_X$.

In the general case, $\mathfrak{X}_\eta$ is defined by gluing in an appropriate manner of analytic spaces corresponding to affine formal schemes which covers $\mathfrak{X}$ [4].

Remark 2.1. — (i) The functor $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ takes a formal scheme topologically of finite type to a quasi-compact analytic space, and this functor commutes with fiber products.

(ii) The functor $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ takes a morphism of finite type $\varphi : \mathfrak{Y} \to \mathfrak{X}$ to a compact morphism of K-analytic spaces $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$. If φ is finite (resp. flat finite), then so is φ_η.

2.3. The reduction map. — For a special formal R-scheme $\mathfrak{X}$, we denote by $\mathfrak{X}_0$ the closed subscheme of $\mathfrak{X}$ defined by the largest ideal of definition of $\mathfrak{X}$, which is called the *reduction* of $\mathfrak{X}$. Note that $\mathfrak{X}_0$ is a reduced Noetherian scheme, and that the correspondence $\mathfrak{X} \mapsto \mathfrak{X}_0$ is functorial, and that the natural closed immersion $\mathfrak{X}_0 \to \mathfrak{X}$ is a homeomorphism. Moreover, $\mathfrak{X}_0$ is also a separated κ-scheme of finite type.

Let us now recall the construction of the reduction map in the affine case, that is for $\mathfrak{X} = \operatorname{Spf}(\mathcal{A})$ with $\mathcal{A}$ being an adic special R-algebra. Notice that Berkovich did this work in [3, 4] for any special formal R-scheme. The construction of the reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_0$ for $\mathfrak{X} = \operatorname{Spf}(\mathcal{A})$ runs as follows. Note that each point x of $\mathfrak{X}_\eta$ defines a continuous character $\chi_x : \mathcal{A} \to \mathcal{H}(x)$. In its turn, χ_x defines a character $\widetilde{\chi_x} : \mathcal{A}_0 = \mathcal{A}/\mathcal{J} \to \widetilde{\mathcal{H}(x)}$, where $\mathcal{J}$ is the largest ideal of definition of $\mathcal{A}$ and $\widetilde{\mathcal{H}(x)}$ is the residue field of $\mathcal{H}(x)$. Then we assign $\pi(x)$ to the kernel of $\widetilde{\chi}_x$, which is a prime ideal of $\mathcal{A}_0$. This definition guarantees the compatibility of the reduction map with open immersion: if $\mathfrak{Y}$ is an open formal subscheme of $\mathfrak{X}$, then the reduction maps for $\mathfrak{X}$ and $\mathfrak{Y}$ are compatible and $\mathfrak{Y}_\eta \cong \pi^{-1}(\mathfrak{Y}_0)$.

2.4. Étale cohomology of analytic spaces. — A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ of special formal R-schemes is called *étale* if for any ideal of definition $\mathcal{J}$ of $\mathfrak{X}$ the morphism of schemes $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{J}\mathcal{O}_{\mathfrak{Y}}) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is étale. Since the reduction $\mathfrak{X}_0$ is the closed subscheme of $\mathfrak{X}$ defined by the largest ideal of definition of $\mathfrak{X}$, therefore if the morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is étale, the induced morphism $\varphi_0 : \mathfrak{Y}_0 \to \mathfrak{X}_0$ is étale.

By [2], a morphism of K-analytic spaces $\varphi : Y \to X$ is *étale* if for each point $y \in Y$ there exist open neighborhoods V of y and U of $\varphi(y)$ such that φ induces a finite étale morphism $\varphi : V \to U$. By a *finite étale morphism* $\varphi : V \to U$ one means that for each affinoid domain $W = \mathcal{M}(\mathcal{A})$ in U, the preimage $\varphi^{-1}(W) = \mathcal{M}(\mathcal{B})$ is an affinoid domain and $\mathcal{B}$ is a finite étale $\mathcal{A}$-algebra. A morphism of K-analytic spaces $\varphi : Y \to X$ is called *quasi-étale* if for any point $y \in Y$ there exist affinoid domains $V_1, \dots, V_n \subseteq Y$ such that $V_1 \cup \dots \cup V_n$ is a neighborhood of y and each V_i may be identified with an affinoid domain in a K-affinoid space étale over X. By definition, étale morphisms are also quasi-étale.

Lemma 2.2 (Berkovich [4], Proposition 2.1). — *Assume that $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is an étale morphism of special formal R-schemes. Then the following statements hold:*

(i) $\varphi_\eta(\mathfrak{Y}_\eta) = \pi^{-1}(\varphi_0(\mathfrak{Y}_0))$. *As a consequence, $\varphi_\eta(\mathfrak{Y}_\eta)$ is a closed analytic domain in $\mathfrak{X}_\eta$.*

(ii) *The induced morphism $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ of K-analytic spaces is quasi-étale.*

For a K-analytic space X, let $X_{\text{qét}}$ (resp. $X_{\text{ét}}$) be the quasi-étale site (resp. the étale site) of X defined in [3]. Recall that the *quasi-étale topology* on X is the Grothendieck topology on the category of quasi-étale morphisms $U \to X$ generated by the pretopology for which the set of coverings of $(U \to X)$ is formed by the families $\{f_i : U_i \to U\}_{i\in I}$ such that each point of U has a neighborhood of the form $f_{i_1}(V_1) \cup \dots \cup f_{i_n}(V_n)$ for some affinoid domains $V_1 \subseteq U_{i_1}, \dots, V_n \subseteq U_{i_n}$. The *étale topology* on X is the Grothendieck topology on the category of étale morphisms $U \to X$ generated by the pretopology for which the set of coverings of $(U \to X)$ is formed by the families $\{f_i : U_i \to U\}_{i\in I}$ such that $U = \bigcup_{i\in I} f_i(U_i)$. The *quasi-étale topos* (resp. the *étale topos*) of X, i.e., the category of sheaves of sets on $X_{\text{qét}}$ (resp. on $X_{\text{ét}}$), is denoted by $X^{\sim}_{\text{qét}}$ (resp. by $X^{\sim}_{\text{ét}}$). Let $\mathbf{S}(X)$ be the category of abelian étale sheaves on $X_{\text{ét}}$. The cohomology groups of an abelian étale sheaf $F \in \mathbf{S}(X)$ is denoted by $H^i(X, F)$.

Let X be a Hausdorff K-analytic space. As shown in [4], there is a left exact functor $\Gamma_c(X, -)$ from the category of abelian étale sheaves $\mathbf{S}(X)$ to the category of abelian groups given by $\Gamma_c(X, F) = \{f \in F(X) \mid \mathrm{Supp}(f) \text{ is compact}\}$. Then the cohomology groups with compact support of the space X with coefficients in F are defined as $H^i_c(X, F) = R^i\Gamma_c(X, F)$.

We also consider the category of *K-germs* of K-analytic spaces. By definition, cf. [4, Subsection 3.4], such a germ is a pair (X, S) in which X is a K-analytic space and S is a subset of X; a morphism of K-germs $(X, S) \to (Y, T)$

is a morphism of K-analytic spaces $f : X \to Y$ with $f(S) \subseteq T$. The category of *K-Germs* is the localization of the category of K-germs with respect to the system of morphisms $\varphi : (X, S) \to (Y, T)$ such that φ induces an isomorphism of X and an open neighborhood of T in Y. The correspondence $X \mapsto (X, \text{underlying space of } X)$ gives rise to a fully faithful functor from the category of K-analytic spaces to the category of K-Germs. A morphism of K-Germs is called *étale* if it has a representative $\varphi : (X, S) \to (Y, T)$ with $\varphi : X \to Y$ étale and $S = \varphi^{-1}(T)$. In definitions analogous to the previous ones, where $(X, S)_{\text{ét}}$ denotes the étale site of (X, S) and $\mathbf{S}(X, S)$ denotes the category of abelian étale sheaves on $(X, S)_{\text{ét}}$, we obtain the cohomology groups $H^i((X, S), F)$ and, if in addition X is Hausdorff, the cohomology groups with compact support $H^i_c((S, X), F)$ of $F \in \mathbf{S}(X, S)$.

If X is a Hausdorff K-analytic space and if S is either an analytic domain of X, or a closed K-analytic subset of X (i.e., $S \to X$ is a closed immersion), there are isomorphisms of abelian groups

$$H^i_c(S, F) \cong H^i_c((S, S), F) \cong H^i_c((X, S), F) \tag{2.1}$$

for any abelian constructible sheaf F (cf. [4] or [15]). More generally, it is proved in [4] that if $\varphi : (X, S) \to (Y, T)$ is a *quasi-immersion* then the induced functor $\mathbf{S}(X, S) \to \mathbf{S}(Y, \varphi(S))$ is an equivalence of categories.

Let $\mathcal{A}$ be an affinoid K-algebra over K, $\mathcal{B}$ an $\mathcal{A}$-algebra of finite type, and $\mathfrak{X} = \operatorname{Spec}(\mathcal{B})$. A *semi-algebraic* subset of $\mathfrak{X}^{\text{an}}$ is a finite Boolean combination of subsets of the form $\{x \in \mathfrak{X}^{\text{an}} \mid |f(x)| \leq \lambda |g(x)|\}$ with $f, g \in \mathcal{B}$ and $\lambda > 0$.

Proposition 2.3 (Martin [15], Proposition 4.2). — *Let $\mathfrak{X}$ be as previously stated, with in addition $\mathfrak{X}$ being separated and the ground valued field K being algebraically closed. Let S be a locally closed semi-algebraic subset of $\mathfrak{X}^{\text{an}}$. Then, for any torsion abelian constant sheaf F in $\mathbf{S}(\mathfrak{X}^{\text{an}}, S)$, the groups $H^i_c((\mathfrak{X}^{\text{an}}, S), F)$ are finite.*

By Proposition 2.3, one can introduce in a natural way the ℓ-adic cohomology of a locally closed semi-algebraic subset of the analytification of a variety. Namely, with $\mathfrak{X}$ and S as in that proposition, one defines

$$H^*_c((\mathfrak{X}^{\text{an}}, S), \mathbb{Z}_\ell) := \varprojlim_n H^*_c((\mathfrak{X}^{\text{an}}, S), \mathbb{Z}/\ell^n\mathbb{Z})$$

and

$$H^*_c((\mathfrak{X}^{\text{an}}, S), \mathbb{Q}_\ell) := H^*_c((\mathfrak{X}^{\text{an}}, S), \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Notice that, in the rest of this article, if (2.1) happens or if $\mathfrak{X}^{\text{an}}$ is fixed, we shall write for simplicity $H^*_c(S, \mathbb{Q}_\ell)$ instead of $H^*_c((\mathfrak{X}^{\text{an}}, S), \mathbb{Q}_\ell)$.

Let $\widehat{K^s}$ be the completion of a separable closure of K. For a K-analytic space X, there is a canonical morphism $b : \overline{X} := X \widehat{\otimes}_K \widehat{K^s} \to X$. If Y is an analytic subspace of the X, denote by $\overline{Y}$ or by $Y \widehat{\otimes}_K \widehat{K^s}$ the preimage of Y in $\overline{X}$ under b.

Proposition 2.4 (Martin [15]). — *Let $\mathcal{Y}$ and $\mathcal{Y}'$ be affine algebraic K-varieties, let Y and Y' be locally closed semi-algebraic subsets of $\mathcal{Y}^{\mathrm{an}}$ and $\mathcal{Y}'^{\mathrm{an}}$, respectively.*

(i) *If U is an open semi-algebraic subset of Y, and $V := Y \setminus U$, there is an exact sequence*

$$\cdots \to H_c^m(\overline{V}, \mathbb{Q}_\ell) \to H_c^{m+1}(\overline{U}, \mathbb{Q}_\ell) \to H_c^{m+1}(\overline{Y}, \mathbb{Q}_\ell) \to H_c^{m+1}(\overline{V}, \mathbb{Q}_\ell) \to \cdots .$$

(ii) *There is a canonical Künneth isomorphism of complexes*

$$R\Gamma_c(\overline{Y}, \mathbb{Q}_\ell) \otimes R\Gamma_c(\overline{Y'}, \mathbb{Q}_\ell) \cong R\Gamma_c(\overline{Y \times Y'}, \mathbb{Q}_\ell).$$

2.5. The nearby cycles functor. — For a K-analytic space X, there is a morphism of sites $\mu : X_{\mathrm{q\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$, and the functor $\mu^* : X^{\sim}_{\mathrm{\acute{e}t}} \to X^{\sim}_{\mathrm{q\acute{e}t}}$ is a fully faithful functor, according to [3].

Let $\mathfrak{X}$ be a special formal R-scheme. By [3], the correspondence $\mathfrak{Y} \mapsto \mathfrak{Y}_0$ induces an equivalence between the category of formal schemes étale over $\mathfrak{X}$ and the category of schemes étale over $\mathfrak{X}_0$. We fix the functor $\mathfrak{Y}_0 \mapsto \mathfrak{Y}$ which is inverse to the previous correspondence $\mathfrak{Y} \mapsto \mathfrak{Y}_0$. The composition of the functor $\mathfrak{Y}_0 \mapsto \mathfrak{Y}$ with the functor $\mathfrak{Y} \mapsto \mathfrak{Y}_\eta$ induces a morphism of sites $\nu : \mathfrak{X}_{\eta\mathrm{q\acute{e}t}} \to \mathfrak{X}_{0\mathrm{\acute{e}t}}$. By [4], this construction also holds over a separable closure K^s of K, therefore we shall also denote by ν the corresponding morphism of sites $\mathfrak{X}_{\overline{\eta}\mathrm{q\acute{e}t}} \to \mathfrak{X}_{\overline{0}\mathrm{\acute{e}t}}$, where $\mathfrak{X}_{\overline{\eta}} := \mathfrak{X}_\eta \widehat{\otimes}_K \widehat{K^s}$ and $\mathfrak{X}_{\overline{0}} := \mathfrak{X}_0 \otimes_\kappa \kappa^s$.

Now let us consider the composition functor $\nu_* \circ \mu^* : \mathfrak{X}^{\sim}_{\overline{\eta}\mathrm{\acute{e}t}} \to \mathfrak{X}^{\sim}_{\overline{0}\mathrm{\acute{e}t}}$ of the functors $\mu^* : \mathfrak{X}^{\sim}_{\overline{\eta}\mathrm{\acute{e}t}} \to \mathfrak{X}^{\sim}_{\overline{\eta}\mathrm{q\acute{e}t}}$ and $\nu_* : \mathfrak{X}^{\sim}_{\overline{\eta}\mathrm{q\acute{e}t}} \to \mathfrak{X}^{\sim}_{\overline{0}\mathrm{\acute{e}t}}$. Composing with the pullback functor of the canonical morphism $\mathfrak{X}_{\overline{\eta}} \to \mathfrak{X}_\eta$ it yields a functor $\psi : \mathfrak{X}^{\sim}_{\eta\mathrm{\acute{e}t}} \to \mathfrak{X}^{\sim}_{\overline{0}\mathrm{\acute{e}t}}$, which is called the *nearby cycles functor* (see [3, 4]). It is a left exact functor, thus we can involve right derived functors $R^i\psi : \mathbf{S}(\mathfrak{X}_\eta) \to \mathbf{S}(\mathfrak{X}_{\overline{0}})$ and $R\psi : D^+(\mathfrak{X}_\eta) \to D^+(\mathfrak{X}_{\overline{0}})$, the latter is exact while the others are right exact functors. If necessary, we can write $R^i\psi_{\mathfrak{f}}$ and $R\psi_{\mathfrak{f}}$ labeling $\mathfrak{f}$ the structural morphism of $\mathfrak{X}$.

Lemma 2.5 (Berkovich [4], Corollary 2.3). — *Let $\varphi : \mathfrak{Y} \to \mathfrak{X}$ be an étale morphism of special formal R-schemes and F in $\mathbf{S}(\mathfrak{X}_\eta)$. Then, for any $m \geq 0$, we have $(R^m\psi F)|_{\mathfrak{Y}_{\overline{0}}} \cong R^m\psi(F|_{\mathfrak{Y}_\eta})$.*

2.6. The comparison theorem for nearby cycles. — From [4, Theorem 3.1], we get the comparison theorem for nearby cycles functor working on a Henselian ring R. Let $\mathcal{E}$ be a scheme locally of finite type over R with the structural morphism f; and let $\mathcal{E}_0$ be the zero locus of f, which is a κ-scheme. Then we have $\mathcal{E}_0 = \widehat{\mathcal{E}}_0$, where the scheme on the right is the reduction of the completion $\widehat{\mathcal{E}}$ of the scheme $\mathcal{E}$. For a subscheme $\mathcal{Y} \subseteq \mathcal{E}_0$, let $\widehat{\mathcal{E}}_{/\mathcal{Y}}$ denote the formal $\mathfrak{m}$-adic completion of $\widehat{\mathcal{E}}$ along $\mathcal{Y}$. A result of [4], this shows that there is a canonical isomorphism of K-analytic spaces $(\widehat{\mathcal{E}}_{/\mathcal{Y}})_\eta \cong \pi^{-1}(\mathcal{Y})$, where π is the reduction map $\widehat{\mathcal{E}}_\eta \to \mathcal{E}_0$. For a sheaf $\mathcal{F} \in \mathcal{E}^{\sim}_{\eta\text{ét}}$, with $\mathcal{E}_\eta := \mathcal{E} \otimes_R K$, let $\widehat{\mathcal{F}}_{/\mathcal{Y}}$ denote the pullback of $\mathcal{F}$ on $(\widehat{\mathcal{E}}_{/\mathcal{Y}})_\eta$. The nearby cycles functor for $\mathcal{E}$, for $\widehat{\mathcal{E}}$ and for $(\widehat{\mathcal{E}}_{/\mathcal{Y}})_\eta$ will be denoted by the same symbol ψ. If $\mathcal{Y}$ is an (ordinary) κ-scheme, we define $\overline{\mathcal{Y}} := \mathcal{Y} \otimes_\kappa \kappa^s$.

Theorem 2.1 (Berkovich [4], Theorem 3.1). — *Let $\mathcal{F}$ be an étale abelian constructible sheaf on $\mathcal{E}_\eta$. For $i \geq 0$, there is a canonical isomorphism $(R^i\psi\mathcal{F})|_{\overline{\mathcal{Y}}} \cong R^i\psi(\widehat{\mathcal{F}}_{/\mathcal{Y}})$.*

The previous theorem is widely known as the Berkovich's comparison theorem for nearby cycles, while the full version is in fact stated for both nearby cycles functor and vanishing cycles functor and it is motivated by a conjecture of Deligne. Part of the conjecture claims that the restrictions of the vanishing cycles sheaves of a scheme $\mathcal{E}$ of finite type over a Henselian discrete valuation ring to the subscheme $\mathcal{Y} \subseteq \widehat{\mathcal{E}}_0$ depend only on the formal $\mathfrak{m}$-adic completion $\widehat{\mathcal{E}}_{/\mathcal{Y}}$ of $\mathcal{E}$ along $\mathcal{Y}$, and that the automorphism group of $\widehat{\mathcal{E}}_{/\mathcal{Y}}$ acts on them. By proving this comparison theorem, Berkovich [4] provided the positive answer to Deligne's conjecture.

The following corollary runs over any complete discretely valued field.

Corollary 2.6 (Berkovich [4], Corollary 3.6). — *Let $\mathcal{S}$ be an R-scheme of locally finite type, $\mathfrak{X}$ a special formal $\widehat{\mathcal{S}}$-scheme which is locally isomorphic to the formal $\mathfrak{m}$-adic completion of a $\mathcal{S}$-scheme of finite type along a subscheme of its reduction, F an étale sheaf on $\mathfrak{X}_\eta$ locally in the étale topology of $\mathfrak{X}$ isomorphic to the pullback of a constructible sheaf on $\widehat{\mathcal{S}}_\eta$. Then $R\psi(F)$ is constructible and, for any subscheme $\mathcal{Y} \subseteq \mathfrak{X}_0$, there is a canonical isomorphism of complexes*

$$R\Gamma(\overline{\mathcal{Y}}, (R\psi F)|_{\overline{\mathcal{Y}}}) \xrightarrow{\sim} R\Gamma(\overline{\pi^{-1}(\mathcal{Y})}, F).$$

If, in addition, the closure of $\mathcal{Y}$ in $\mathfrak{X}_0$ is proper, there is a canonical isomorphism

$$R\Gamma_c(\overline{\mathcal{Y}}, (R\psi F)|_{\overline{\mathcal{Y}}}) \xrightarrow{\sim} R\Gamma_{\overline{\pi^{-1}(\mathcal{Y})}}(\mathfrak{X}_{\overline{\eta}}, F).$$

3. The polynomial f and comparisons

From this section, the condition that κ is an algebraically closed field is used because we apply the comparison theorem for nearby cycles of Berkovich. We write R and K for $\kappa[[t]]$ and $\kappa((t))$, respectively.

3.1. Resetting the data. — Let $f(x,y,z)$ be in $\kappa[x,y,z]$ such that $f(0,0,0)=0$ and $f(\tau x, \tau^{-1}y, z) = f(x,y,z)$ for $\tau \in \kappa^\times$. Let us consider the following R-schemes with the structural morphisms

$$\begin{aligned} \mathcal{E} &:= \mathrm{Spec}(R[x,y,z]/(f(x,y,z)-t)) \to \mathrm{Spec}(R), \\ \mathcal{W} &:= \mathrm{Spec}(R[z]/(f(0,0,z)-t)) \to \mathrm{Spec}(R) \end{aligned} \tag{3.1}$$

given by $t = f(x,y,z)$, $t = f(0,0,z)$, respectively. Note that $\mathbb{A}_{\kappa}^{d_1}$ is a closed subvariety of κ-variety $\mathcal{E}_0 = f^{-1}(0)$, and that $\mathfrak{X} = \widehat{\mathcal{E}}_{/\mathbb{A}_{\kappa}^{d_1}}$ and $\mathfrak{Z} = \widehat{\mathcal{W}}_{/0}$, where the formal schemes on the left hand sides were already defined in the first section. We also consider the reduction maps $\pi : \mathfrak{X}_{\eta} \to \mathfrak{X}_0$ and $\pi_{\mathcal{W}} : \mathfrak{Z}_{\eta} \to \mathfrak{Z}_0$.

3.2. Applying the comparison theorem. — Let $\mathbf{f}$ be the homogenization of f, i.e. $\mathbf{f}(x,y,z,\xi)$ is homogeneous in $d+1$ variables with $\mathbf{f}(x,y,z,1) = f(x,y,z)$ and $\deg(\mathbf{f}) = \deg(f) = n$. Note that the R-scheme

$$\mathbf{E} := \mathrm{Proj}\left(R[x,y,z,\xi]/(\mathbf{f}(x,y,z,\xi) - t\xi^n)\right)$$

is locally of finite type. Let us consider the t-adic completion $\widehat{\mathbf{E}}$, which is a formal R-scheme canonically glued from the following affine formal R-schemes

(3.2)

$$\mathrm{Spf}\left(R\left\{\frac{x}{x_i},\frac{y}{x_i},\frac{z}{x_i},\frac{\xi}{x_i}\right\}\Big/\left(\mathbf{f}\left(\frac{x}{x_i},\frac{y}{x_i},\frac{z}{x_i},\frac{\xi}{x_i}\right) - t\left(\frac{\xi}{x_i}\right)^n\right)\right), \quad i = 1,\ldots,d_1,$$

$$\mathrm{Spf}\left(R\left\{\frac{x}{y_j},\frac{y}{y_j},\frac{z}{y_j},\frac{\xi}{y_j}\right\}\Big/\left(\mathbf{f}\left(\frac{x}{y_j},\frac{y}{y_j},\frac{z}{y_j},\frac{\xi}{y_j}\right) - t\left(\frac{\xi}{y_j}\right)^n\right)\right), \quad j = 1,\ldots,d_2,$$

$$\mathrm{Spf}\left(R\left\{\frac{x}{z_l},\frac{y}{z_l},\frac{z}{z_l},\frac{\xi}{z_l}\right\}\Big/\left(\mathbf{f}\left(\frac{x}{z_l},\frac{y}{z_l},\frac{z}{z_l},\frac{\xi}{z_l}\right) - t\left(\frac{\xi}{z_l}\right)^n\right)\right), \quad l = 1,\ldots,d_3,$$

$$\mathrm{Spf}\left(R\left\{\frac{x}{\xi},\frac{y}{\xi},\frac{z}{\xi}\right\}\Big/\left(f\left(\frac{x}{\xi},\frac{y}{\xi},\frac{z}{\xi}\right) - t\right)\right) \cong \widehat{\mathcal{E}}.$$

The reduction $\widehat{\mathbf{E}}_0 = \mathbf{E}_0$ is the hypersurface $\{\mathbf{f} = 0\}$ in the projective space $\mathbb{P}_{\kappa}^d$, it admits the inclusions $\mathbb{A}_{\kappa}^{d_1} \subseteq \mathcal{E}_0 \subseteq \mathbf{E}_0$.

Let $\tilde{\mathbb{A}}_{\kappa}^{d_1}$ be the closure of $\mathbb{A}_{\kappa}^{d_1}$ in $\mathbf{E}_0$. By construction, the embedding of $\widehat{\mathcal{E}}$ in $\widehat{\mathbf{E}}$ is an open immersion of formal R-schemes (hence it is an étale morphism). By [9, Corollary 10.9.9], the formal R-scheme $\mathfrak{X} = \widehat{\mathcal{E}}_{/\mathbb{A}_{\kappa}^{d_1}}$ can be identified to the fiber product of $\widehat{\mathcal{E}} \to \widehat{\mathbf{E}}$ and $\mathbf{X} := \widehat{\mathbf{E}}_{/\tilde{\mathbb{A}}_{\kappa}^{d_1}} \to \widehat{\mathbf{E}}$. Since étale morphisms are preserved under a base change, the induced morphism $\mathfrak{X} \to \mathbf{X}$ is also étale (it is even an open immersion). We denote by $\widehat{\mathbf{f}}$ the structural morphism of $\mathbf{X}$, which is induced by $\mathbf{f}$. We shall use the following notation

- ⋆ $i : \mathfrak{X}_{\overline{\eta}} \to \mathbf{X}_{\overline{\eta}}$ is the embedding of analytic spaces,
- ⋆ $j : \mathfrak{X}_0 \to \mathbf{X}_0$, $k : \mathbf{X}_0 \setminus \mathfrak{X}_0 \to \mathbf{X}_0$, $u : \mathbb{A}_{\kappa}^{d_1} \to \mathfrak{X}_0$ and $v : \mathbb{A}_{\kappa}^{d_1} \to \mathbf{X}_0$ are the embeddings of κ-schemes (note that $v = j \circ u$).

Let F denote the constant sheaf $(\mathbb{Z}/\ell^n\mathbb{Z})_{\mathfrak{X}_{\overline{\eta}}}$ in $\mathbf{S}(\mathfrak{X}_{\overline{\eta}})$, $n \geq 1$. By Lemma 2.5, for any $m \geq 0$, we have $j^*R^m\psi_{\widehat{\mathbf{f}}}(i_!F) \cong R^m\psi_{\widehat{f}}F$, hence $j_!j^*R^m\psi_{\widehat{\mathbf{f}}}(i_!F) \cong j_!R^m\psi_{\widehat{f}}F$. In the latter, the complex on the right hand side fits into the exact triangle

$$\to j_!R^m\psi_{\widehat{f}}F \to R^m\psi_{\widehat{\mathbf{f}}}(i_!F) \to k_*k^*R^m\psi_{\widehat{\mathbf{f}}}(i_!F) \to .$$

The functor v^* being exact, we have the following exact triangle

$$\to u^* R^m \psi_{\widehat{f}} F \to v^* R^m \psi_{\widehat{\mathbf{f}}}(i_! F) \to v^* k_* k^* R^m \psi_{\widehat{\mathbf{f}}}(i_! F) \to . \tag{3.3}$$

We observe that the source of the map v is $\mathbb{A}_\kappa^{d_1}$, which is a subset of $\mathfrak{X}_0$, while that of k is $\mathbf{X}_0 \backslash \mathfrak{X}_0$, and the two subsets $\mathbb{A}_\kappa^{d_1}$ and $\mathbf{X}_0 \backslash \mathfrak{X}_0$ are disjoint in $\mathbf{X}_0$. This fact means that $v^* k_* k^* R^m \psi_{\widehat{\mathbf{f}}}(i_! F) \cong 0$, hence $R^m \psi_{\widehat{f}} F|_{\mathbb{A}_\kappa^{d_1}} \cong R^m \psi_{\widehat{\mathbf{f}}}(i_! F)|_{\mathbb{A}_\kappa^{d_1}}$. One then deduces a quasi-isomorphism of complexes

$$R\Gamma_c(\mathbb{A}_\kappa^{d_1}, R\psi_{\widehat{f}} F|_{\mathbb{A}_\kappa^{d_1}}) \xrightarrow{\text{qis}} R\Gamma_c(\mathbb{A}_\kappa^{d_1}, R\psi_{\widehat{\mathbf{f}}}(i_! F)|_{\mathbb{A}_\kappa^{d_1}}). \tag{3.4}$$

Now apply Corollary 2.6 to the nearby cycles functor $R\psi_{\widehat{\mathbf{f}}}$. For such an $\mathbf{f}$, the assumptions of that corollary are satisfied: the scheme $\mathbf{E}$ is of finite type over R and the closure of $\mathbb{A}_\kappa^{d_1}$ in $\mathbf{X}_0$ is proper as $\mathbf{X}_0$ is. Let $\tilde{\pi}$ denote the reduction map $\mathbf{X}_\eta \to \mathbf{X}_0$. One then deduces from Corollary 2.6 that

$$R\Gamma_c(\mathbb{A}_\kappa^{d_1}, R\psi_{\widehat{\mathbf{f}}}(i_! F)|_{\mathbb{A}_\kappa^{d_1}}) \xrightarrow{\sim} R\Gamma_{\overline{\tilde{\pi}^{-1}(\mathbb{A}_\kappa^{d_1})}}(\mathbf{X}_{\overline{\eta}}, i_! F). \tag{3.5}$$

3.3. Shrinking analytic domains. — Consider the complex $R\Gamma_{\overline{\tilde{\pi}^{-1}(\mathbb{A}_\kappa^{d_1})}}(\mathbf{X}_{\overline{\eta}}, i_! F)$ in the target of (3.5). Note that the analytic space $\mathbf{X}_{\overline{\eta}}$ is the gluing of $A := \mathfrak{X}_{\overline{\eta}}$ together with other analytic spaces which correspond to the formal schemes in (3.2), each of which is a closed analytic domain in $\mathbf{X}_{\overline{\eta}}$ (Lemma 2.2). Similarly, $\overline{\tilde{\pi}^{-1}(\mathbb{A}_\kappa^{d_1})}$ is the gluing of $X := \overline{\pi^{-1}(\mathbb{A}_\kappa^{d_1})}$ together with others in the same way. We define $P := \mathbf{X}_{\overline{\eta}} \setminus A$ and $T := \mathbf{X}_{\overline{\eta}} \setminus \overline{\tilde{\pi}^{-1}(\mathbb{A}_\kappa^{d_1})}$.

LEMMA 3.1. — *We have a quasi-isomorphism of complexes as follows*

$$R\Gamma_{\overline{\tilde{\pi}^{-1}(\mathbb{A}_\kappa^{d_1})}}(\mathbf{X}_{\overline{\eta}}, i_! F) \xrightarrow{qis} R\Gamma_X(A, F). \tag{3.6}$$

Proof. — Let i_α be the embedding of an $\widehat{K^s}$-analytic space α in $\mathbf{X}_{\overline{\eta}}$, $i_{\alpha,\beta}$ the embedding of α in β (thus $i_A = i$), and $B := A \setminus X$. Now both sides of (3.6) can be rewritten as follows

$$R\Gamma_{\overline{\tilde{\pi}^{-1}(\mathbb{A}_\kappa^{d_1})}}(\mathbf{X}_{\overline{\eta}}, i_! F) \xrightarrow{\text{qis}} R\widehat{\mathbf{f}}_{\overline{\eta}*}\mathrm{Cone}(i_! F \to i_{T*} i_T^* i_! F),$$

$$R\Gamma_X(A, F) \xrightarrow{\text{qis}} R\widehat{f}_{\overline{\eta}*}\mathrm{Cone}(F \to i_{B,A*} i_{B,A}^* F).$$

Note that the embeddings $i_P : P \hookrightarrow \mathbf{X}_{\overline{\eta}}$ and $i : A \hookrightarrow \mathbf{X}_{\overline{\eta}}$ altogether give rise to an exact triangle of complexes on $\mathbf{X}_{\overline{\eta}}$:

$$\to i_{P!} i_P^* \mathrm{Cone}(i_! F \to i_{T*} i_T^* i_! F) \to \mathrm{Cone}(i_! F \to i_{T*} i_T^* i_! F)$$
$$\xrightarrow{h} i_* i^* \mathrm{Cone}(i_! F \to i_{T*} i_T^* i_! F) \to .$$

The sources of i_P and i are disjoint, hence h is a quasi-isomorphism. Rewrite h in the form $h : \mathrm{Cone}(i_! F \to i_{T*} i_T^* i_! F) \to \mathrm{Cone}(i_* F \to i_{B*} i_{B,A}^* F)$. The identity

$i_B = i \circ i_{B,A}$ implies the following isomorphisms of complexes

$$\begin{aligned}\mathrm{Cone}(i_*F \to i_{B*}i_{B,A}^*F) &\cong \mathrm{Cone}(i_*F \to i_*i_{B,A*}i_{B,A}^*F)\\ &\cong i_*\mathrm{Cone}(F \to i_{B,A*}i_{B,A}^*F).\end{aligned}$$

We claim that $R\widehat{\mathbf{f}}_{\overline{\eta}*}i_* = R\widehat{f}_{\overline{\eta}*}$. Indeed, one deduces from [2, Corollary 5.2.4] and $\widehat{\mathbf{f}}_{\overline{\eta}} \circ i = \widehat{f}_{\overline{\eta}}$ that $R\widehat{\mathbf{f}}_{\eta*}Ri_* = R\widehat{f}_{\eta*}$. That $i_* = i_!$ is as A is closed in $\mathbf{X}_{\overline{\eta}}$ (cf. Lemma 2.2), while $i_!$ is exact since the stalk $(i_!F)_{\mathbf{y}}$ is equal to $F_{\mathbf{y}}$ if $\mathbf{y} \in A$, and zero otherwise, thus $Ri_* = i_*$. Finally, taking the exact functor $R\widehat{\mathbf{f}}_{\overline{\eta}*}$ to the quasi-isomorphism h yields a quasi-isomorphism of complexes

$$R\widehat{\mathbf{f}}_{\overline{\eta}*}\mathrm{Cone}(i_!F \to i_{T*}i_T^*i_!F) \xrightarrow{\mathrm{qis}} R\widehat{f}_{\overline{\eta}*}\mathrm{Cone}(F \to i_{B,A*}i_{B,A}^*F),$$

This proves the lemma. □

3.4. Description of *A*, *X* and *D*. — We notice that from now on we shall abuse the notation x, y, z, and others, i.e., we use them in parallel with two different senses. As before (x,y,z) stands for a system of coordinates in $\mathbb{A}_\kappa^d$ $(d = d_1 + d_2 + d_3)$, in what follow it will also denote the corresponding system of coordinates on the analytification $\mathbb{A}_{K^s}^{d,\mathrm{an}}$. Similarly, if τ is an element in the group scheme $\mathbb{G}_{m,\kappa}$, we also write τ for the corresponding element in $\mathbb{G}_{m,K^s}^{\mathrm{an}}$.

Lemma 3.2. — *With f as in Theorem 1.2, the analytic space $A = \mathfrak{X}_{\overline{\eta}}$ is the inductive limit of the compact domains*

$$A_{\gamma,\epsilon} := \{(x,y,z) \in \mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}} : |x| \leq \gamma^{-1}, |y| \leq \gamma\epsilon, |z| \leq \epsilon, f(x,y,z) = t\}$$

with γ,ϵ running over the value group $|(K^s)^*|$ of the absolute value on K^s such that $\gamma,\epsilon \in (0,1)$ and $\gamma,\epsilon \to 1$. In the same way, $X = \overline{\pi^{-1}(\mathbb{A}_\kappa^{d_1})}$ is the inductive limit of

$$X_{\gamma,\epsilon} := \{(x,y,z) \in \mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}} : |x| < \gamma^{-1}, |y| \leq \gamma\epsilon, |z| \leq \epsilon, f(x,y,z) = t\}.$$

Proof. — For each $\gamma \in |(K^s)^*|$, choose an element τ_γ in $\mathbb{G}_{m,\kappa}$ such that its corresponding element τ_γ in $\mathbb{G}_{m,\kappa}^{\mathrm{an}}$ takes the absolute value γ. Since $f(\tau_\gamma x, \tau_\gamma^{-1}y, z)$ $= f(x,y,z)$, the following special R-algebras are isomorphic

$$R\{\tau_\gamma x, \tau_\gamma^{-1}y, z\}/(f(x,y,z) - t) \cong R\{x,y,z\}/(f(x,y,z) - t).$$

Setting

$$A_\gamma := \left(\left(\mathrm{Spf}\frac{R\{\tau_\gamma x, \tau_\gamma^{-1}y, z\}}{(f(x,y,z) - t)}\right)_{/\mathbb{A}_\kappa^{d_1}}\right)_{\overline{\eta}},$$

it is clear that

$$\begin{aligned} A_\gamma &= \left\{(x,y,z)\in \mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}} : |\tau_\gamma x|\leq 1, |\tau_\gamma^{-1}y|<1, |z|<1, f(x,y,z)=t\right\} \\ &= \left\{(x,y,z)\in \mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}} : |x|\leq \gamma^{-1}, |y|<\gamma, |z|<1, f(x,y,z)=t\right\} \end{aligned}$$

and that all the spaces A_γ's, with $\gamma\in|(K^s)^*|$, are analytically isomorphic. The latter implies an analytic isomorphism between any pair $(A_\gamma, A_{\gamma'})$ with γ, γ' in $|(K^s)^*|$, and thus one can establish an inductive system

$$\{\{A_\gamma\},\{A_\gamma\to A_{\gamma'}\}_{\gamma<\gamma'} : \gamma,\gamma'\in|(K^s)^*|\cap(0,1)\}\,.$$

Then A is exactly the inductive limit of this system $\{A_\gamma\}$ when $\gamma\to 1$. On the other hand, the space $\{y : |y|<\gamma\}$ is covered by the compact domains $\{z : |z|\leq\gamma\epsilon\}$ and the space $\{z : |z|<1\}$ is covered by the compact domains $\{z : |z|\leq\epsilon\}$ with $\epsilon\in|(K^s)^*|$ and $0<\epsilon<1$. Therefore A can be viewed as the inductive limit of $A_{\gamma,\epsilon}$'s as above with $\gamma,\epsilon\in|(K^s)^*|\cap(0,1)$ and $\gamma,\epsilon\to 1$.

The inductive system of $X_{\gamma,\epsilon}$'s whose limit describes X is defined by $X_{\gamma,\epsilon} := A_{\gamma,\epsilon}\cap X$, transition morphisms induce from those in the system of $A_{\gamma,\epsilon}$'s. □

We also note that $D := \overline{\pi_{\mathcal{W}}^{-1}(0)}$ is an open and locally compact analytic space, it can be covered by the following compact domains

$$D_\epsilon := \left\{z\in\mathbb{A}^{d_3}_{\widehat{K^s},\mathrm{Ber}} : |z|\leq\epsilon, f(0,0,z)=t\right\},\ \epsilon\in|(K^s)^*|\cap(0,1).$$

Corollary 3.3. — *With the hypothesis as in Theorem 1.2 and with $\gamma\in|(K^s)^*|\cap(0,1)$ fixed, we have*

(i) $R\Gamma_c(\mathbb{A}^{d_1}_\kappa, R\psi_{\widehat{f}}F|_{\mathbb{A}^{d_1}_\kappa})\xrightarrow{qis} R\Gamma_{X_\gamma}(A_\gamma, F^\circ_\gamma)$, *$F^\circ_\gamma$ the pullback of $F\in\mathbf{S}(A)$ via $A_\gamma\cong A$.*

(ii) $(R\psi_{\widehat{f_{\mathfrak{Z}}}}G)_0\xrightarrow{qis} R\Gamma(D, G|_D)$, *for $G\in\mathbf{S}(\mathfrak{Z}_{\overline{\eta}})$.*

Proof. — By the description of A and X, there are isomorphisms of analytic spaces $A_\gamma\cong A$ and $X_\gamma\cong X$ for a fixed γ in $|(K^s)^*|\cap(0,1)$. These together with (3.4), (3.5) and Lemma 3.1 imply (i). Also, (ii) follows from Corollary 2.6. □

Corollary 3.4. — *With the hypothesis as in Theorem 1.2 and with $\gamma\in|(K^s)^*|\cap(0,1)$ fixed, we have*

(i) $R\Gamma_c\left(\mathbb{A}^{d_1}_\kappa, R\psi_{\widehat{f}}\mathbb{Q}_\ell|_{\mathbb{A}^{d_1}_\kappa}\right)\xrightarrow{qis} R\Gamma_{X_\gamma}(A_\gamma,\mathbb{Q}_\ell)$,

(ii) $R\psi_{\widehat{f_{\mathfrak{Z}}}}\mathbb{Q}_\ell\xrightarrow{qis} R\Gamma(D,\mathbb{Q}_\ell)$.

4. Proof of Theorem 1.2

4.1. Using a comparison theorem. — By Corollary 3.4, there is a quasi-isomorphism of complexes

$$R\Gamma_c(\mathbb{A}_\kappa^{d_1}, R\psi_{\widehat{f}}\mathbb{Q}_\ell|_{\mathbb{A}_\kappa^{d_1}}) \stackrel{\text{qis}}{\to} R\Gamma_{X_\gamma}(A_\gamma, \mathbb{Q}_\ell), \tag{4.1}$$

where γ is fixed in $|(K^s)^*|\cap(0,1)$, A_γ is the analytic subspace of $\mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}}$ given by $|x|\le\gamma^{-1}$, $|y|<\gamma$, $|z|<1$ and $f(x,y,z)=t$, and X_γ is defined as A_γ but with $|x|<\gamma^{-1}$ instead of $|x|\le\gamma^{-1}$. The space A_γ is a paracompact $\widehat{K^s}$-analytic space which is a union of the following increasing sequence of compact domains

$$A_{\gamma,\epsilon} := \left\{(x,y,z)\in\mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}} : |x|\le\gamma^{-1}, |y|\le\gamma\epsilon, |z|\le\epsilon, f(x,y,z)=t\right\},$$

for $\epsilon\in|(K^s)^*|\cap(0,1)$. The space X_γ is covered by the corresponding increasing sequence

$$X_{\gamma,\epsilon} = \left\{(x,y,z)\in\mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}} : |x|<\gamma^{-1}, |y|\le\gamma\epsilon, |z|\le\epsilon, f(x,y,z)=t\right\}.$$

Denote $B_\gamma := A_\gamma\setminus X_\gamma$ and $B_{\gamma,\epsilon} := A_{\gamma,\epsilon}\setminus X_{\gamma,\epsilon}$.

Let us now consider the analytic function $f^\gamma := \widehat{f}_{\overline{\eta}} : A_\gamma \cong A \to \mathcal{M}(\widehat{K^s})$, and the function $f^{\gamma,\epsilon}$ which is the restriction of f^γ to $A_{\gamma,\epsilon}$.

Lemma 4.1. — *For any $m\ge1$ and $F\in\mathbf{S}(A_\gamma)$, there is a canonical isomorphism of groups*

$$H^m_{X_\gamma}(A_\gamma,F)\cong \operatorname*{proj\,lim}_{\epsilon\to1} H^m_{X_{\gamma,\epsilon}}(A_{\gamma,\epsilon},F).$$

Proof. — Note that the functors $H^m_{X_\gamma}(A_\gamma,-)$ are the derived functors of the global section functor $H^0_{X_\gamma}(A_\gamma,-)$ defined by

$$H^0_{X_\gamma}(A_\gamma,F)=\ker(F(A_\gamma)\to F(B_\gamma)),$$

which is the kernel of the restriction homomorphism $F(A_\gamma)\to F(B_\gamma)$. If J is an injective abelian sheaf, then the pullback of J on B_γ is acyclic and the homomorphism $J(A_\gamma)\to J(B_\gamma)$ is surjective. Take an injective resolution of F, namely $0\to F\to J^0\to J^1\to\cdots$, and consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \ker(\alpha_0) & \xrightarrow{d^0} & \ker(\alpha_1) & \xrightarrow{d^1} & \ker(\alpha_2) & \xrightarrow{d^2} & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & J^0(A_\gamma) & \xrightarrow{d^0_{A_\gamma}} & J^1(A_\gamma) & \xrightarrow{d^1_{A_\gamma}} & J^2(A_\gamma) & \xrightarrow{d^2_{A_\gamma}} & \cdots \\
& \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \\
0 \longrightarrow & J^0(B_\gamma) & \xrightarrow{d^0_{B_\gamma}} & J^1(B_\gamma) & \xrightarrow{d^1_{B_\gamma}} & J^2(B_\gamma) & \xrightarrow{d^2_{B_\gamma}} & \cdots
\end{array}$$

Then we have

$$H^m_{X_\gamma}(A_\gamma, F) = \ker\left(H^m(A_\gamma, F) \to H^m(B_\gamma, F)\right) \cong \ker(d^m)/\mathrm{im}(d^{m-1}).$$

Analogously, we consider the surjections, say, $\alpha_{m,\epsilon} : J^m(A_{\gamma,\epsilon}) \to J^m(B_{\gamma,\epsilon})$. There is a commutative diagram as follows, where all vertical arrows are surjective,

$$\begin{array}{ccccccccc}
0 \longrightarrow & \ker(\alpha_0) & \xrightarrow{d^0} & \ker(\alpha_1) & \xrightarrow{d^1} & \ker(\alpha_2) & \xrightarrow{d^2} & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \ker(\alpha_{0,\epsilon}) & \xrightarrow{d^0_\epsilon} & \ker(\alpha_{1,\epsilon}) & \xrightarrow{d^1_\epsilon} & \ker(\alpha_{2,\epsilon}) & \xrightarrow{d^2_\epsilon} & \cdots
\end{array}$$

Here, $H^m_{X_{\gamma,\epsilon}}(A_{\gamma,\epsilon}, F) \cong \ker(d^m_\epsilon)/\mathrm{im}(d^{m-1}_\epsilon)$. Then we can use the arguments in the proof of Lemma 6.3.12 of [2] to complete this proof. Note that in this situation the following condition is satisfied: For any $0 < \epsilon < 1$, for any $\epsilon < \epsilon', \epsilon'' < 1$, the image of $H^{m-1}_{X_{\gamma,\epsilon'}}(A_{\gamma,\epsilon'}, F)$ and that of $H^{m-1}_{X_{\gamma,\epsilon''}}(A_{\gamma,\epsilon''}, F)$ coincide in $H^{m-1}_{X_{\gamma,\epsilon}}(A_{\gamma,\epsilon}, F)$ under the restriction homomorphisms (see [5, Lemma 7.4] for a similar argument). □

Here is an important corollary of (4.1) and Lemma 4.1.

Corollary 4.2. — *There is a canonical quasi-isomorphism of complexes*

$$R\Gamma_c(\mathbb{A}^{d_1}_\kappa, R\psi_{\widehat{f}}\mathbb{Q}_\ell|_{\mathbb{A}^{d_1}_\kappa}) \xrightarrow{qis} \operatorname*{proj\,lim}_{\epsilon\to 1} R\Gamma_{X_{\gamma,\epsilon}}(A_{\gamma,\epsilon}, \mathbb{Q}_\ell).$$

Proof. — We deduce from (4.1) and the properties of the mapping cone functor that

$$\begin{aligned}
R\Gamma_c\left(\mathbb{A}^{d_1}_\kappa, R\psi_{\widehat{f}}\mathbb{Q}_\ell|_{\mathbb{A}^{d_1}_\kappa}\right) \xrightarrow{\text{qis}} & R\Gamma_{X_\gamma}(A_\gamma, \mathbb{Q}_\ell) \\
& \cong Rf^\gamma_* \mathrm{Cone}\left(\mathbb{Q}_\ell \to i_{B_\gamma, A_\gamma *} i^*_{B_\gamma, A_\gamma}\mathbb{Q}_\ell\right) \\
& \cong \mathrm{Cone}\left(Rf^\gamma_*\mathbb{Q}_\ell \to R(f^\gamma|_{B_\gamma})_*\mathbb{Q}_\ell\right).
\end{aligned}$$

By the universality of the projective limit, there are canonical morphisms

$$Rf^{\gamma}_{*}\mathbb{Q}_\ell \to \operatorname*{proj\,lim}_{\epsilon\to 1} Rf^{\gamma,\epsilon}_{*}\mathbb{Q}_\ell,$$
$$R(f^{\gamma}|_{B_\gamma})_*\mathbb{Q}_\ell \to \operatorname*{proj\,lim}_{\epsilon\to 1} R(f^{\gamma,\epsilon}|_{B_{\gamma,\epsilon}})_*\mathbb{Q}_\ell.$$

Here, the latter is induced from the former by restriction. Thus there is a canonical morphism of complexes

$$\begin{aligned} R\Gamma_c\left(\mathbb{A}^{d_1}_{\kappa}, R\psi_{\widehat{f}}\mathbb{Q}_\ell|_{\mathbb{A}^{d_1}_{\kappa}}\right) &\to \operatorname{Cone}\left(\operatorname*{proj\,lim}_{\epsilon\to 1} Rf^{\gamma,\epsilon}_{*}\mathbb{Q}_\ell \to \operatorname*{proj\,lim}_{\epsilon\to 1} R(f^{\gamma,\epsilon}|_{B_{\gamma,\epsilon}})_*\mathbb{Q}_\ell\right)\\ &\cong \operatorname*{proj\,lim}_{\epsilon\to 1} \operatorname{Cone}\left(Rf^{\gamma,\epsilon}_{*}\mathbb{Q}_\ell \to R(f^{\gamma,\epsilon}|_{B_{\gamma,\epsilon}})_*\mathbb{Q}_\ell\right)\\ &\cong \operatorname*{proj\,lim}_{\epsilon\to 1} R\Gamma_{X_{\gamma,\epsilon}}(A_{\gamma,\epsilon},\mathbb{Q}_\ell). \end{aligned}$$

This morphism of complexes then induces the isomorphisms in Lemma 4.1. □

The second part of Corollary 3.4 asserts that

$$R\psi_{\widehat{f_3}}\mathbb{Q}_\ell \xrightarrow{\text{qis}} R\Gamma(D,\mathbb{Q}_\ell). \tag{4.2}$$

The space D is open and locally compact, which is covered by the compact domains $D_\epsilon = \{z \in \mathbb{A}^r_{\widehat{K^s},\mathrm{Ber}} : |z| \leq \epsilon, f(0,0,z) = t\}$, for all $\epsilon \in |(K^s)^*| \cap (0,1)$. According to Lemma 6.3.12 of [2], there is a canonical isomorphism of cohomology groups

$$H^m(D,\mathbb{Q}_\ell) \cong \operatorname*{proj\,lim}_{\epsilon\to 1} H^m(D_\epsilon,\mathbb{Q}_\ell)$$

for any $m \geq 0$. Thus by the same arguments as in the proof of Corollary 4.2, one deduces from (4.2) that

$$R\psi_{\widehat{f_3}}\mathbb{Q}_\ell \xrightarrow{\text{qis}} \operatorname*{proj\,lim}_{\epsilon\to 1} R\Gamma(D_\epsilon,\mathbb{Q}_\ell). \tag{4.3}$$

(The readers may compare this with [5, Lemma 7.4].)

4.2. Using the Künneth isomorphism. — We now use the Künneth isomorphism for cohomology with compact support mentioned in Proposition 2.4, (iii). To begin, we write $A_{\gamma,\epsilon}$ as a disjoint union $A_{\gamma,\epsilon} = A^0_{\gamma,\epsilon} \sqcup A^1_{\gamma,\epsilon}$ of analytic spaces

$$A^0_{\gamma,\epsilon} := \{(x,y,z) \in A_{\gamma,\epsilon} : |x||y| = 0\},$$
$$A^1_{\gamma,\epsilon} := \{(x,y,z) \in A_{\gamma,\epsilon} : |x||y| \neq 0\}.$$

Similarly, one can write $X_{\gamma,\epsilon}$ as a disjoint union of analytic spaces

$$X^0_{\gamma,\epsilon} := \{(x,y,z) \in X_{\gamma,\epsilon} : |x||y| = 0\},$$
$$X^1_{\gamma,\epsilon} := \{(x,y,z) \in X_{\gamma,\epsilon} : |x||y| \neq 0\}.$$

Observe that we can write $X^0_{\gamma,\epsilon}$ as the product $Y^0_{\gamma,\epsilon}\times D_\epsilon$, where D_ϵ is defined in Subsection 3.4 and $Y^0_{\gamma,\epsilon} := \{(x,y)\in \mathbb{A}^{d_1+d_2}_{\widehat{K^s},\mathrm{Ber}} : |x||y|=0, |x|<\gamma^{-1}, |y|\leq \gamma\epsilon\}$. By the compactness of $A^0_{\gamma,\epsilon}$ and D_ϵ, and by the Künneth isomorphism, we have

$$\begin{aligned} R\Gamma_{X^0_{\gamma,\epsilon}}(A^0_{\gamma,\epsilon},\mathbb{Q}_\ell) \cong R\Gamma_c(X^0_{\gamma,\epsilon},\mathbb{Q}_\ell) &\xrightarrow{\text{qis}} R\Gamma_c(Y^0_{\gamma,\epsilon},\mathbb{Q}_\ell)\otimes R\Gamma_c(D_\epsilon,\mathbb{Q}_\ell)\\ &\xrightarrow{\text{qis}} R\Gamma_c(Y^0_{\gamma,\epsilon},\mathbb{Q}_\ell)\otimes R\Gamma(D_\epsilon,\mathbb{Q}_\ell). \end{aligned} \tag{4.4}$$

Decompose $Y^0_{\gamma,\epsilon}$ into a disjoint union of $Y^{0,1}_{\gamma,\epsilon} := \{(x,0)\in \mathbb{A}^{d_1+d_2}_{\widehat{K^s},\mathrm{Ber}} : |x| < \gamma^{-1}\}$ and $Y^{0,2}_{\gamma,\epsilon} := \{(0,y)\in \mathbb{A}^{d_1+d_2}_{\widehat{K^s},\mathrm{Ber}} : 0<|y|\leq\gamma\epsilon\}$.

Lemma 4.3. — (i) *$R\Gamma_c(\mathbb{A}^{d_1}_\kappa,\mathbb{Q}_\ell) \xrightarrow{qis} R\Gamma_c(Y^{0,1}_{\gamma,\epsilon},\mathbb{Q}_\ell)$;*
(ii) *$R\Gamma_c(Y^{0,2}_{\gamma,\epsilon},\mathbb{Q}_\ell) \xrightarrow{qis} 0$;*
(iii) *$R\Gamma_c(\mathbb{A}^{d_1}_\kappa,\mathbb{Q}_\ell) \xrightarrow{qis} R\Gamma_c(Y^{0}_{\gamma,\epsilon},\mathbb{Q}_\ell)$.*

Proof. — (i) For notational simplicity, let F denote both constant sheaves $\mathbb{Z}/\ell^n\mathbb{Z}$ on $\mathbb{A}^{d_1}_{K^s}$ and on $\mathbb{A}^{d_1,\mathrm{an}}_{K^s} = \mathbb{A}^{d_1}_{\widehat{K^s},\mathrm{Ber}}$. The comparison theorem for cohomology with compact support [2, Theorem 7.1.1] gives an isomorphism of groups

$$H^m_c(\mathbb{A}^{d_1}_{K^s},F)\cong H^m_c(\mathbb{A}^{d_1,\mathrm{an}}_{K^s},F), \tag{4.5}$$

for any $m\geq 0$. Let $V=\mathbb{A}^{d_1,\mathrm{an}}_{K^s}\setminus Y^{0,1}_{\gamma,\epsilon}$. By Proposition 5.2.6 (ii) of [2] (notice that Proposition 2.4 (ii) is the ℓ-adic version of this result), we have an exact sequence

$$\begin{aligned} \cdots\to H^m_c(V,F) \to &H^{m+1}_c(Y^{0,1}_{\gamma,\epsilon},F)\to\\ &\to H^{m+1}_c(\mathbb{A}^{d_1,\mathrm{an}}_{K^s},F)\to H^{m+1}_c(V,F)\to\cdots. \end{aligned} \tag{4.6}$$

We shall prove that $H^m_c(V,F)=0$ for every m.

Let us choose an *open* covering $\{V_i\}_{i\in\mathbb{N}}$ of $V=\mathbb{A}^{d_1,\mathrm{an}}_{K^s}\setminus Y^{0,1}_{\gamma,\epsilon}$ defined as follows:

$$V_i := \{x\in \mathbb{A}^{d_1,\mathrm{an}}_{K^s} : \gamma^{-1}\leq |x| <\gamma_i\},$$

where $\gamma^{-1}<\gamma_i<\gamma_j$ for every $i<j$. Choose an analogous *open* covering $\{V_{ijl}\}_{l\in\mathbb{N}}$ of $V_i\cap V_j$ for each pair i,j. Let α_i and α_{ijl} be the open embeddings $V_i\to V$ and $V_{ijl}\to V$, respectively. Then the following exact sequence

$$\bigoplus_{i,j,l}\alpha_{ijl!}(F_{V_{ijl}})\to\bigoplus_i \alpha_{i!}(F_{V_i})\to F_V\to 0$$

induces an exact sequence

$$\bigoplus_{i,j,l} H^m_c(V_{ijl},F)\to \bigoplus_i H^m_c(V_i,F)\to H^m_c(V,F)\to 0.$$

The étale cohomology groups with compact support $H_c^m(V_{ijl}, F)$ and $H_c^m(V_i, F)$ clearly vanish for $m \geq 0$, thus $H_c^m(V, F) = 0$ for $m \geq 0$. By (4.6), one has $H_c^m(\mathbb{A}_{K^s}^{d_1,\mathrm{an}}, F) \cong H_c^m(Y_{\gamma,\epsilon}^{0,1}, F)$ for $m \geq 0$, which together with (4.5) implies that $H_c^m(\mathbb{A}_{K^s}^{d_1}, F) \cong H_c^m(Y_{\gamma,\epsilon}^{0,1}, F)$ for $m \geq 0$. Now, since κ is algebraically closed and K^s is separably closed (for fields of characteristic zero the concepts "algebraically closed" and "separably closed" coincide), applying a result of SGA4$\frac{1}{2}$ [6, Corollary 3.3], for $m \geq 0$, $H_c^m(\mathbb{A}_{\kappa}^{d_1}, F) \cong H_c^m(\mathbb{A}_{K^s}^{d_1}, F)$. Therefore

$$H_c^m(\mathbb{A}_{\kappa}^{d_1}, F) \cong H_c^m(Y_{\gamma,\epsilon}^{0,1}, F),\ m \geq 0,$$

hence the ℓ-adic version, namely, $H_c^m(\mathbb{A}_{\kappa}^{d_1}, \mathbb{Q}_\ell) \cong H_c^m(Y_{\gamma,\epsilon}^{0,1}, \mathbb{Q}_\ell)$ for $m \geq 0$.

(ii) Let us denote by F the constant sheaf $\mathbb{Z}/\ell^n\mathbb{Z}$, and consider the closed immersion $\mathcal{M}(\widehat{K^s}) \to \mathcal{M}(\widehat{K^s}\{\gamma^{-1}y\})$ of $\widehat{K^s}$-analytic spaces. By [2, Corollary 4.3.2], there is an isomorphism of groups

$$H^m(\mathcal{M}(\widehat{K^s}), F) \cong H^m(\mathcal{M}(\widehat{K^s}\{\gamma^{-1}y\}), F)$$

for each $m \geq 0$. This leads to an isomorphism of groups in the ℓ-adic cohomology. Thus using the exact sequence in Proposition 2.4 (ii), we have $H_c(Y_{\gamma,\epsilon}^{0,2}, \mathbb{Q}_\ell) = 0$, where

(iii) follows from (i) and (ii). □

4.3. The final step of the proof. — The aim of this subsection is to prove the following morphism is quasi-isomorphic:

$$R\Gamma_{X_{\gamma,\epsilon}}(A_{\gamma,\epsilon}, \mathbb{Q}_\ell) \xrightarrow{\text{qis}} R\Gamma_{X_{\gamma,\epsilon}^0}(A_{\gamma,\epsilon}^0, \mathbb{Q}_\ell). \tag{4.7}$$

Assume the quasi-isomorphism (4.7). Then there are quasi-isomorphisms of complexes, due to Corollary 4.2, (4.7), (4.4) and Lemma 4.3,

$$\begin{aligned} R\Gamma_c\left(\mathbb{A}_{\kappa}^{d_1}, R\psi_{\widehat{f}}\mathbb{Q}_\ell|_{\mathbb{A}_{\kappa}^{d_1}}\right) &\xrightarrow{\text{qis}} \operatorname*{proj\,lim}_{\epsilon\to 1}\left(R\Gamma_c(\mathbb{A}_{\kappa}^{d_1}, \mathbb{Q}_\ell)\otimes R\Gamma(D_\epsilon, \mathbb{Q}_\ell)\right) \\ &\xrightarrow{\text{qis}} R\Gamma_c(\mathbb{A}_{\kappa}^{d_1}, \mathbb{Q}_\ell)\otimes \operatorname*{proj\,lim}_{\epsilon\to 1} R\Gamma(D_\epsilon, \mathbb{Q}_\ell). \end{aligned}$$

This together with (4.3) implies Theorem 1.2.

To process a proof for (4.7), we write $R\Gamma_{X_{\gamma,\epsilon}}(A_{\gamma,\epsilon}, \mathbb{Q}_\ell)$ and $R\Gamma_{X_{\gamma,\epsilon}^0}(A_{\gamma,\epsilon}^0, \mathbb{Q}_\ell)$ in the following form:

$$\begin{aligned} &R\Gamma_{X_{\gamma,\epsilon}}(A_{\gamma,\epsilon}, \mathbb{Q}_\ell) \xrightarrow{\text{qis}} Rf_*^{\gamma,\epsilon}\mathrm{Cone}(\mathbb{Q}_{\ell,A_{\gamma,\epsilon}} \to i_{B_{\gamma,\epsilon},A_{\gamma,\epsilon}*}\mathbb{Q}_{\ell,B_{\gamma,\epsilon}}), \\ &R\Gamma_{X_{\gamma,\epsilon}^0}(A_{\gamma,\epsilon}^0, \mathbb{Q}_\ell) \xrightarrow{\text{qis}} R(f^{\gamma,\epsilon}|_{A_{\gamma,\epsilon}^0})_*\mathrm{Cone}(\mathbb{Q}_{\ell,A_{\gamma,\epsilon}^0} \to i_{B_{\gamma,\epsilon}^0,A_{\gamma,\epsilon}^0*}\mathbb{Q}_{\ell,B_{\gamma,\epsilon}^0}), \end{aligned}$$

where $A_{\gamma,\epsilon}^0 := \{(x,y,z) \in A_{\gamma,\epsilon} : |x||y| = 0\}$ and $B_{\gamma,\epsilon}^0 := B_{\gamma,\epsilon} \cap A_{\gamma,\epsilon}^0$. To abuse the notation we shall use from now on $\mathbb{Q}_\ell$ in stead of $\mathbb{Q}_{\ell,A_{\gamma,\epsilon}}$, $\mathbb{Q}_{\ell,B_{\gamma,\epsilon}}$, $\mathbb{Q}_{\ell,A_{\gamma,\epsilon}^0}$ or $\mathbb{Q}_{\ell,B_{\gamma,\epsilon}^0}$.

THEOREM 4.1. — *With the previous notation and hypotheses, there is a canonical quasi-isomorphism of complexes*

$$Rf_*^{\gamma,\epsilon}\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B_{\gamma,\epsilon},A_{\gamma,\epsilon}*}\mathbb{Q}_\ell) \xrightarrow{qis} R(f^{\gamma,\epsilon}|_{A^0_{\gamma,\epsilon}})_*\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B^0_{\gamma,\epsilon},A^0_{\gamma,\epsilon}*}\mathbb{Q}_\ell).$$

Proof. — The space $A^1_{\gamma,\epsilon} := \{(x,y,z)\in A_{\gamma,\epsilon} : |x||y| \neq 0\}$ together with $A^0_{\gamma,\epsilon}$ composing a disjoint union of $A_{\gamma,\epsilon}$, there exists a canonical exact triangle

$$\begin{aligned}&\to R\overline{f}_!^{\gamma,\epsilon}\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B^1_{\gamma,\epsilon},A^1_{\gamma,\epsilon}*}\mathbb{Q}_\ell) \to Rf_*^{\gamma,\epsilon}\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B_{\gamma,\epsilon},A_{\gamma,\epsilon}*}\mathbb{Q}_\ell)\\ &\to \quad R(f^{\gamma,\epsilon}|_{A^0_{\gamma,\epsilon}})_*\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B^0_{\gamma,\epsilon},A^0_{\gamma,\epsilon}*}\mathbb{Q}_\ell) \to,\end{aligned} \tag{4.8}$$

where $\overline{f}^{\gamma,\epsilon} := f^{\gamma,\epsilon}|_{A^1_{\gamma,\epsilon}}$ and $B^1_{\gamma,\epsilon} := B_{\gamma,\epsilon}\cap A^1_{\gamma,\epsilon}$. We are going to verify the following

$$R\overline{f}_!^{\gamma,\epsilon}\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B^1_{\gamma,\epsilon},A^1_{\gamma,\epsilon}*}\mathbb{Q}_\ell) \xrightarrow{\mathrm{qis}} 0. \tag{4.9}$$

Let us consider the action of $\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}$ on $\mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}}$ given by $\tau\cdot(x,y,z) = (\tau x,\tau^{-1}y,z)$ for $\tau\in\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}$ and $(x,y,z)\in\mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}}$. This $\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}$-action is free, since $\tau\cdot(x,y,z) = (x,y,z)$ if and only if $\tau = 1$. Each orbit of the action on $A^1_{\gamma,\epsilon}$ has the following form

$$\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}\cdot(x,y,z)\cap A^1_{\gamma,\epsilon} = \{(\tau x,\tau^{-1}y,z) : \gamma^{-1}\epsilon^{-1}|y| \le |\tau| \le \gamma^{-1}|x|^{-1}\}$$

for $(x,y,z)\in A^1_{\gamma,\epsilon}$. Also, an orbit of $\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}$-action on $B^1_{\gamma,\epsilon}$ is of the form

$$\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}\cdot(x,y,z)\cap B^1_{\gamma,\epsilon} = \{(\tau x,\tau^{-1}y,z) : |\tau| = \gamma^{-1}|x|^{-1}\}$$

for $(x,y,z)\in B^1_{\gamma,\epsilon}$. Furthermore, the $\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}$-action has the following

PROPERTY (*). — *Every orbit on $\mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}}$ intersects with $A^1_{\gamma,\epsilon}$ in a closed annulus C and with $B^1_{\gamma,\epsilon}$ in a thin annulus contained in C.*

Let $\mathcal{P}$ be the space of orbits of $\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}$-action on $\mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}}$. By Lemma 4.4, $\mathcal{P}$ admits an obvious structure of a $\widehat{K^s}$-analytic space. The property (*) deduces that the restriction maps of the natural projection onto $\mathcal{P}$ on $A^1_{\gamma,\epsilon}$ and on $B^1_{\gamma,\epsilon}$, say, $a : A^1_{\gamma,\epsilon}\to\mathcal{P}$ and $b : B^1_{\gamma,\epsilon}\to\mathcal{P}$, are surjective. We note that $\overline{f}^{\gamma,\epsilon}$ and $\overline{f}^{\gamma,\epsilon}|_{B^1_{\gamma,\epsilon}}$ factor through a and b, respectively. Since one has a spectral sequence (the Leray spectral sequence, see Berkovich [2, Theorem 5.2.2])

$$H^n_c(\mathcal{P},R^m a_!\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B^1_{\gamma,\epsilon},A^1_{\gamma,\epsilon}*}\mathbb{Q}_\ell)) \Rightarrow R^{n+m}\overline{f}_!^{\gamma,\epsilon}\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B^1_{\gamma,\epsilon},A^1_{\gamma,\epsilon}*}\mathbb{Q}_\ell),$$

it suffices to verify that $Ra_!\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B^1_{\gamma,\epsilon},A^1_{\gamma,\epsilon}*}\mathbb{Q}_\ell)$ is quasi-isomorphic to 0. Let us consider the following exact triangle of complexes on $\mathcal{P}$:

$$\to Ra_!\mathbb{Q}_\ell \to Rb_!\mathbb{Q}_\ell \to Ra_!\mathrm{Cone}(\mathbb{Q}_\ell \to i_{B^1_{\gamma,\epsilon},A^1_{\gamma,\epsilon}*}\mathbb{Q}_\ell)[+1] \to .$$

Applying the Berkovich's weak base change theorem [2, Theorem 5.3.1], we have

$$(R^m a_! \mathbb{Q}_\ell)_\lambda \cong H_c^m(a^{-1}(\lambda), \mathbb{Q}_\ell), \quad (R^m b_! \mathbb{Q}_\ell)_\lambda \cong H_c^m(b^{-1}(\lambda), \mathbb{Q}_\ell)$$

for $\lambda \in \mathcal{P}$ and $m \geq 0$. The embedding of the thin annulus $b^{-1}(\lambda)$ into the closed annulus $a^{-1}(\lambda)$ inducing an isomorphism on étale cohomology (here since $a^{-1}(\lambda)$ and $b^{-1}(\lambda)$ are compact, their étale cohomology and étale cohomology with compact support are the same), we obtain $(R^m a_! \mathbb{Q}_\ell)_\lambda \cong (R^m b_! \mathbb{Q}_\ell)_\lambda$. In other words, for $\lambda \in \mathcal{P}$ and $m \geq 0$,

$$R^m a_! \mathrm{Cone}(\mathbb{Q}_\ell \to i_{B^1_{\gamma,\epsilon}, A^1_{\gamma,\epsilon} *} \mathbb{Q}_\ell)_\lambda \cong 0.$$

This proves (4.9), which together with (4.8) implies the theorem. ☐

Lemma 4.4. — *There is a natural structure of an analytic space on the quotient*

$$\mathcal{P} = \left(\mathbb{A}^{d_1+d_2}_{\widehat{K^s},\mathrm{Ber}} \setminus \{0\}\right) \times \mathbb{A}^{d_3}_{\widehat{K^s},\mathrm{Ber}} / \mathbb{G}^{\mathrm{an}}_{m,K^s}.$$

Proof. — We endow $\mathcal{P}$ with the quotient topology, then obviously it is a compact Hausdorff space. The construction of an analytic structure on $\mathcal{P}$ is analogous to that of the projective analytic spaces $\mathbb{P}^d_{\widehat{K^s},\mathrm{Ber}}$, where the natural $\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}$-action on $\mathbb{A}^d_{\widehat{K^s},\mathrm{Ber}}$ is replaced by the $\mathbb{G}^{\mathrm{an}}_{m,\widehat{K^s}}$-action given by $\tau \cdot (x, y, z) = (\tau x, \tau^{-1} y, z)$, which is also free. See [18] for the construction in detail of $\mathbb{P}^d_{K,\mathrm{Ber}}$. ☐

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