Bull. Soc. Math. France
147 (2), 2019, p. 289-302

NON-COMPACT FORM OF THE ELEMENTARY DISCRETE INVARIANT

by Raphaël Fino

Abstract. — We determine the non-compact form of Vishik's elementary discrete invariant for quadrics. As an application, we obtain new restrictions on the possible values of the elementary discrete invariant by studying the action of Steenrod operations on the algebraic cycles defining the non-compact form.

Résumé (*Forme non-compacte de l'invariant discret élémentaire*). — On détermine la forme non-compacte de l'invariant discret élémentaire de Vishik pour quadriques. Comme application, on obtient de nouvelles restrictions sur les valeurs possibles de l'invariant discret élémentaire en étudiant l'action des opérations de Steenrod sur les cycles algébriques définissant la forme non-compacte.

1. Introduction

Let X be a smooth projective quadric of dimension n over a field F associated with a non-degenerate F-quadratic form q. The *splitting pattern* of X is a discrete invariant that measures the possible Witt indices of q_E over all field extensions E/F (see [4] and [5]). The *motivic decomposition type* of X is a discrete invariant which measures in what pieces the Chow motive of X can be

Texte reçu le 19 décembre 2017, modifié le 21 août 2018, accepté le 4 septembre 2018.

Raphaël Fino, Instituto de Matemáticas, Ciudad Universitaria, UNAM, DF 04510, México
• *E-mail :* `fino@im.unam.mx` • *Url :* `http://www.matem.unam.mx/fino`

Mathematical subject classification (2010). — 14C25,11E39.

Key words and phrases. — Chow groups, quadratic forms, grassmannians, Steenrod operations.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2019/289/$ 5.00

doi:10.24033/bsmf2782

decomposed. Moreover, Alexander Vishik noticed in [6] that the study of the interaction between these two invariants provides further information about both of them.

For this reason, he introduced the *generic discrete invariant* of quadrics, a bigger discrete invariant containing the splitting pattern and the motivic decomposition type invariants as faces, see [7] and [9]. The Generic Discrete Invariant $GDI(X)$ is defined as follows. Let K/F be a splitting field extension of q. Let us denote $[n/2]$ as d. For any $i \in \{0,\dots,d\}$, we write G_i for the grassmannian of i-dimensional totally q-isotropic subspaces (in particular G_0 is the quadric X). Then $GDI(X)$ is the collection of the subalgebras of rational elements

$$\overline{\mathrm{Ch}}^*(G_i) := \mathrm{Image}\left(\mathrm{Ch}^*(G_i) \to \mathrm{Ch}^*(G_{iK})\right)$$

for $i \in \{0,\dots,d\}$, where Ch stands for the Chow ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients (an algebraic cycle already defined at the level of the base field F is called *rational*).

In his paper [10] dedicated to the Kaplansky's conjecture on the u-invariant of a field, A. Vishik used the *elementary discrete invariant* of quadrics, a handier invariant than the GDI as it only deals with some particular cycles in $\mathrm{Ch}^*(G_{iK})$. More precisely, for any $i \in \{0,\dots,d\}$, we denote by $\mathcal{F}(0,i)$ the partial orthogonal flag variety of q-isotropic lines contained in i-dimensional totally q-isotropic subspaces. One can consider the diagram

$$X \xleftarrow[\pi_{(\underline{0},i)}]{} \mathcal{F}(0,i) \xrightarrow[\pi_{(0,\underline{i})}]{} G_i,$$

given by the natural projections and, for $0 \le j \le d$, we set

$$Z^i_{n-i-j} := \pi_{(0,\underline{i})_*} \circ \pi^*_{(\underline{0},i)}(l_j) \in \mathrm{CH}^{n-i-j}\left(G_{iK}\right),$$

where CH stands for the Chow ring with $\mathbb{Z}$-coefficients and l_j is the class in $\mathrm{CH}_j(X_K)$ of a j-dimensional totally isotropic subspace of $\mathbb{P}\left((V_q)_K\right)$ (with V_q the F-vector space associated with q). We set $z^i_{n-i-j} := Z^i_{n-i-j} \pmod 2 \in \mathrm{Ch}^{n-i-j}(G_{iK})$, with Ch being the Chow ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients. The cycles z^i_{n-i-j} are the elementary classes defining the elementary discrete invariant $EDI(X)$:

Definition 1.1. — The *elementary discrete invariant* $EDI(X)$ is the collection of subsets $EDI(X,i)$ consisting of those integers m such that z^i_m is rational.

Furthermore, for any $r \geq 1$, the Chow motive of X^r with $\mathbb{Z}/2\mathbb{Z}$-coefficients decomposes into a direct sum of shifts of the motive of some G_i, see [2, Corollary 91.8]. Therefore, knowing $GDI(X)$ is the equivalent to knowing

$$\overline{\mathrm{Ch}}^*(X^r) := \mathrm{Image}\left(\mathrm{Ch}^*(X^r) \to \mathrm{Ch}^*(X^r_K)\right)$$

for all $r \geq 1$. Hence, the collection of the latter subalgebras constitutes a *non-compact* (in the sense that one has to consider infinitely many objects) form of $GDI(X)$. For the same reason, there exists a non-compact form of $EDI(X)$ (with defining cycles living in $\mathrm{Ch}^*(X_K^r)$), which we determine in the this work: for any $i \in \{0, \ldots, d\}$, let us denote by sym : $\mathrm{CH}^*(X^{i+1}) \to \mathrm{CH}^*(X^{i+1})$ the homomorphism $\Sigma_{s \in S_{i+1}} s_*$, where $s : X^{i+1} \to X^{i+1}$ is the isomorphism associated with a permutation s. For $0 \leq j \leq d$, we set

$$\rho_{i,j} := \mathrm{sym}\left(\left(\times_{k=0}^{i-1} h^k\right) \times l_j\right) \in \mathrm{CH}^{n-j+i(i-1)/2}\left(X_K^{i+1}\right),$$

where $\times$ is the external product and h^k is the k-th power of the hyperplane section class $h \in \mathrm{CH}^1(X)$ (always rational). Note that $\rho_{0,j} = Z_{n-j}^0 = l_j$. The symmetric cycles $\rho_{i,j}$ (mod 2) are the classes defining the non-compact form of $EDI(X)$:

Theorem 1.2. — *Let $1 \leq i \leq d$ and $0 \leq j \leq d$. The cycle z_{n-i-j}^i is rational if and only if the cycle $\rho_{i,j}$ (mod 2) is rational.*

Because of the stability of rational cycles under pull-backs of diagonal morphisms and the possibility of a refined use of Steenrod operations of cohomological type, studying the non-compact form provides new restrictions on the possible values of $EDI(X)$, as illustrated by Sections 4 and 5.

Moreover, Theorem 1.2 reduces certain questions about the rationality of algebraic cycles on orthogonal grassmannians to the sole level of quadrics. For example, it allows one to reformulate both Vishik's conjecture [8, Conjecture 3.11] and the conjecture [9, Conjecture 0.13] on the *dimensions of Bruno Kahn.*

In Section 2, we introduce some basic tools which are required in Section 3, where we prove Theorem 1.2, using mainly compositions of correspondences and Chern classes of vector bundles over orthogonal grassmannians.

2. Preliminaries

In this section, we continue to use the notation introduced in Section 1.

2.1. Rational cycles on powers of quadrics. — We refer to [2, § 68] for an introduction to cycles on powers of quadrics. For any $1 \leq i \leq d$ and $0 \leq j \leq i-1$, we set

$$\Delta_{i,j} := \mathrm{sym}\left(\left(\times_{k=0}^{i-1} h^k\right) \times l_j\right) + \sum_{m=i}^{d} \mathrm{sym}\left(\left(\times_{\substack{k=0 \\ k\neq j}}^{i-1} h^k\right) \times h^m \times l_m\right)$$

in $\mathrm{Ch}^{n-j+i(i-1)/2}(X_K^{i+1})$. If $n = 2d$, we choose an orientation l_d of the quadric.

Lemma 2.1. — *For any $1 \leq i \leq d$ and $0 \leq j \leq i-1$, the cycle $\Delta_{i,j}$ is rational.*

Proof. — We proceed by induction on i. In $\mathrm{Ch}^n(X_K^2)$, the cycle $\Delta_{1,0}$ or $\Delta_{1,0}+h^d\times h^d$, depending on whether $l_d^2=0$ or not, is the class of the diagonal. Therefore, the cycle $\Delta_{1,0}$ is rational. Let $\sigma\in S_{i+1}$ be a cyclic permutation (with $i\geq 2$). For $0\leq j\leq i-2$, the induction hypothesis step is provided by the identity

$$\Delta_{i,j}=\sum_{l=0}^{i}\sigma_*^l\left(\Delta_{i-1,j}\times h^{i-1}\right)\quad \text{in}\quad \mathrm{Ch}\left(X_K^{i+1}\right).$$

It just remains to show that the cycle $\Delta_{i,i-1}$ is rational to complete the proof. In $\mathrm{Ch}(X_K^{i+1})$, one has

$$\begin{aligned}\Delta_{i,i-1}&=\sum_{m=i-1}^{d}\mathrm{sym}\left(\left(\times_{k=0}^{i-2}h^k\right)\times l_m\times h^m\right)\\&=\sum_{m=0}^{d}\mathrm{sym}\left(\left(\times_{k=0}^{i-2}h^k\right)\times l_m\times h^m\right)\end{aligned}$$

and the latter sum can be rewritten as

$$\sum_{s\in A_{i+1}}s_*\left(\left(\times_{k=0}^{i-2}h^k\right)\times\Delta_{1,0}\right).$$

Thus, the cycle $\Delta_{i,i-1}$ is rational. □

2.2. Correspondences. — We refer to [2, § 62] for an introduction to Chow-correspondences.

For any $1\leq i\leq d$, we denote by θ_i the class of the subvariety

$$\{(y,x_1,\dots,x_{i+1})\mid x_1,\dots x_{i+1}\in y\}\subset G_i\times X^{i+1}$$

in $\mathrm{CH}(G_i\times X^{i+1})$ and we view the cycle θ_i as a correspondence $G_i\rightsquigarrow X^{i+1}$.

We set

$$\eta_i:=\prod_{k=1}^{i}\left(\mathrm{Id}_{G_i}\times p_{X_k^i}\right)^*([\mathcal{F}(i,0)])\in\mathrm{CH}(G_i\times X^i),\tag{1}$$

with $p_{X_k^i}$ the projection from X^i to the k-th coordinate. For any integer $i\leq s\leq d$, we write

$$W_{s-i}^i:=\pi_{(0,\underline{i})_*}\circ\pi_{(\underline{0},i)}^*(h^s)\in\mathrm{CH}^{s-i}(G_i),$$

and $w_{s-i}^i:=W_{s-i}^i \pmod 2\in\mathrm{Ch}^{s-i}(G_i)$. Since the variety X_K is cellular, the cycle $[\mathcal{F}(i,0)]$ decomposes as

$$[\mathcal{F}(i,0)]=\sum_{s=0}^{d}z_{n-i-s}^i\times h^s+\sum_{s=i}^{d}w_{s-i}^i\times l_s\ \text{in}\ \mathrm{Ch}(G_{iK}\times X_K),\tag{2}$$

where l_d has to be replaced by the other class l'_d of maximal totally isotropic subspaces if $n = 2d$ and l_d^2 is not zero, i.e., if four divides n (see [2, Theorem 66.2]).

The two following lemmas, where we write p with underlined target for projections, can be proven the same way [3, Lemmas 3.2 and 3.10] have been proven but with Z^i_{n-i-j} (resp. z^i_{n-i-j}) instead of Z^i_{n-i} (resp. z^i_{n-i}).

Lemma 2.2. — *For any $1 \leq i \leq d$, $0 \leq j \leq d$ and $x \in CH(X_K)$, one has*

$$\left((\theta_i)_*(Z^i_{n-i-j})\right)_*(x) = p_{G_i \times \underline{X^i}\,*}\left(p^*_{\underline{G_i} \times X^i}\left(\pi_{(0,\underline{i})\,*} \circ \pi^*_{(\underline{0},i)}(x) \cdot Z^i_{n-i-j}\right) \cdot \eta_i\right),$$

where the cycle $(\theta_i)_*(Z^i_{n-i-j})$ is viewed as a correspondence $X_K \rightsquigarrow X^i_K$.

For any $1 \leq i \leq d$, we write $\mathcal{F}(i-1,i)$ for the partial orthogonal flag variety of $(i-1)$-dimensional totally isotropic subspaces contained in i-dimensional totally isotropic subspaces and we consider the diagram

$$G_{i-1} \xleftarrow[\pi_{(\underline{i-1},i)}]{} \mathcal{F}(i-1,i) \xrightarrow[\pi_{(i-1,\underline{i})}]{} G_i,$$

given by the natural projections.

Lemma 2.3. — *For any $2 \leq i \leq d$, $0 \leq j \leq d$ and $i \leq m \leq d$, the cycle*

$$p_{G_i \times \underline{X^i}\,*}\left(w^i_{m-i} \cdot z^i_{n-i-j} \cdot \eta_i\right) \in Ch(X^i_K),$$

where we write η_i for $\eta_i \pmod 2$, and this can be rewritten as

$$\sum_{s=0}^{m} \sum_{k=max(i-s,0)}^{min(m-s,i)} p_{G_{i-1} \times \underline{X^{i-1}}\,*}\left(w^{i-1}_{m-s-k} \cdot \sigma^k_{i-1} \cdot z^{i-1}_{n-i+1-j} \cdot \eta_{i-1}\right) \times h^s,$$

with $\sigma^k_{i-1} = \pi_{(\underline{i-1},i)\,*} \circ \pi^*_{(i-1,\underline{i})}(z^i_{n-2i+k}) \in Ch^j(G_{i-1\,K})$.

3. Equivalence

In this section, we continue to use notation and material introduced in the previous sections and we prove Theorem 1.2.

For $1 \leq i \leq d$ and $0 \leq j \leq i-1$, we set

$$\alpha_{i,j} := (\theta_i)_*(Z^i_{n-i-j}) + \rho_{i,j} \in \mathrm{CH}\left(X^{i+1}_K\right),$$

and we view the cycle $\alpha_{i,j}$ as a correspondence $X_K \rightsquigarrow X^i_K$.

Proposition 3.1. — *One has*

$$(\alpha_{i,j}\ (mod\,2))_*\,(h^m) = \begin{cases} sym\left(\times_{k=0}^{i-1} h^k\right) & if\ m=j;\\ sym\left(\left(\times_{\substack{k=0\\k\neq j}}^{i-1} h^k\right)\times h^m\right) & if\ i\le m\le d;\\ 0 & otherwise.\end{cases}$$

Proof. — For any $x \in \mathrm{CH}^m(X_K)$ with $m \le i-1$, the cycle $\pi_{(0,\underline{i})_*}\circ\pi^*_{(\underline{0},i)}(x)$ is trivial for dimensional reasons. Thus, by Lemma 2.2, the cycle $\left((\theta_i)_*(Z^i_{n-i-j})\right)_*(x)$ is also trivial. Therefore, since $(\rho_{i,j})_*(h^m) = \mathrm{sym}\left(\times_{k=0}^{i-1} h^k\right)$ if $m=j$ and is trivial otherwise, one gets the conclusion of Proposition 3.1 for the cases $m \le i-1$.

Moreover, for $i \le m \le d$, Lemma 2.2 provides the identity

$$(3)\qquad (\alpha_{i,j}(\mathrm{mod}\,2))_*\,(h^m) = p_{G_i\times \underline{X^i}}{}_*\left(w^i_{m-i}\cdot z^i_{n-i-j}\cdot \eta_i\right)\ \text{ in } \mathrm{Ch}(X^i_K).$$

We prove the cases $i \le m \le d$ of Proposition 3.1 by descending induction on i. The base of the descending induction $i=d$ (so $i=m=d$) is obtained by combining the identities (1), (2) and (3) for $i=d$ (recall also that, by [10, Proposition 2.1], one has $W^i_0 = 1$ for any $0 \le i \le d$) with the fact that, for any integers $0 \le a_0 \le a_1 \le \cdots \le a_e \le d$, with $e \le d$, one has

$$\deg\left(\prod_{k=0}^{e} z^d_{n-d-a_k}\right) = \begin{cases} 1 & \text{if } \{a_0,a_1,\ldots,a_e\} = \{0,1,\ldots,d\};\\ 0 & \text{otherwise,}\end{cases}$$

where $\deg : \mathrm{Ch}(G_{d_K}) \to \mathrm{Ch}(\mathrm{Spec}(K)) = \mathbb{Z}/2\mathbb{Z}$ is the homomorphism associated with the push-forward of the structure morphism, see [2, Lemma 87.6].

Let $2 \le i \le d$ and $0 \le j \le i-2$. On the one hand, by descending induction hypothesis, for any $i \le m \le d$, one has

$$(4)\qquad p_{G_i\times \underline{X^i}}{}_*\left(w^i_{m-i}\cdot z^i_{n-i-j}\cdot \eta_i\right) = \mathrm{sym}\left(\left(\times_{\substack{k=0\\k\neq j}}^{i-1} h^k\right)\times h^m\right).$$

Therefore, the coordinate of (4) on the top right h^{i-1}, i.e.,

$$p_{\underline{X^{i-1}}\times X}{}_*\left(\left(p_{G_i\times \underline{X^i}}{}_*\left(w^i_{m-i}\cdot z^i_{n-i-j}\cdot \eta_i\right)\right)\cdot [X^{i-1}]\times l_{i-1}\right)$$

is equal to

$$(5)\qquad \mathrm{sym}\left(\left(\times_{\substack{k=0\\k\neq j}}^{i-2} h^k\right)\times h^m\right).$$

On the other hand, by Lemma 2.3, this coordinate is also equal to

$$(6)\qquad \sum_{k=1}^{\min(m-i+1,i)} p_{G_{i-1}\times \underline{X^{i-1}}}{}_*\left(w^{i-1}_{m-i+1-k}\cdot \sigma^k_{i-1}\cdot z^{i-1}_{n-i+1-j}\cdot \eta_{i-1}\right).$$

Let us denote by T_{i-1} the tautological vector bundle on G_{i-1}, i.e., T_{i-1} is given by the closed subvariety of the trivial bundle $V\mathbb{1} = V_q \times G_{i-1}$ consisting of pairs

(u, U) such that $u \in U$. Note that the vector bundle T_{i-1} has rank i. For a vector bundle E over a scheme, we write $c_i(E)$ for the i-th Chern class with value in CH. Since $W^{i-1}_{m-i+1-k} = c_{m-i+1-k}(V\mathbb{1}/T_{i-1})$ (see [10, Proposition 2.1]) and $\sigma^k_{i-1} = c_k(T_{i-1}) (\operatorname{mod} 2)$ (see [3, Lemma 2.6]), by the Whitney Sum Formula (see [2, Proposition 54.7]), one has

$$\text{(7)} \qquad \sum_{k=0}^{\min(m-i+1,i)} w^{i-1}_{m-i+1-k} \cdot \sigma^k_{i-1} = \sum_{k=0}^{\min(m-i+1,i)} c_{m-i+1-k}(V\mathbb{1}/T_{i-1}) \cdot c_k(T_{i-1}) = c_{m-i+1}(V\mathbb{1}).$$

Moreover, one has $c_{m-i+1}(V\mathbb{1}) = 0$ because $m - i + 1 > 0$. Consequently, in view of (5), (6) and (7), one gets

$$\text{(8)} \qquad p_{G_{i-1} \times \underline{X^{i-1}}}{}_* \left(w^{i-1}_{m-i+1} \cdot z^{i-1}_{n-i+1-j} \cdot \eta_{i-1} \right) = \operatorname{sym} \left(\left(\times_{\substack{k=0 \\ k \neq j}}^{i-2} h^k \right) \times h^m \right).$$

By identities (3) and (8), it only remains to prove the case $m = i-1$ to complete the descending induction step. On the one hand, by descending induction hypothesis, the coordinate of $p_{G_i \times \underline{X^i}}{}_* \left(z^i_{n-i-j} \cdot \eta_i \right)$ on the the top right h^i is

$$\operatorname{sym} \left(\times_{\substack{k=0 \\ k \neq j}}^{i-1} h^k \right)$$

(see (4)) and, on the other hand, by Lemma 2.3, it is also equal to

$$p_{G_{i-1} \times \underline{X^{i-1}}}{}_* \left(z^{i-1}_{n-i+1-j} \cdot \eta_{i-1} \right).$$

Proposition 3.1 is proven. □

As a consequence of Proposition 3.1, we partially obtain the first part of Theorem 1.2.

Corollary 3.2. — *Let $1 \leq i \leq d$ and $0 \leq j \leq i-1$. If the cycle z^i_{n-i-j} is rational then the cycle $\rho_{i,j}$ (mod 2) is also rational.*

Proof. — In view of the ring structure of $\mathrm{CH}(X^{i+1}_K)$ (see [2, §68]), and knowing that the cycle $\alpha_{i,j}$ is symmetric, one deduces from Proposition 3.1 that

$$\alpha_{i,j} (\operatorname{mod} 2) = \Delta_{i,j} + \beta$$

with β being a sum of nonessential elements (a nonessential element is an external product of powers of the hyperplane class, it is always rational). Since $\alpha_{i,j} = (\theta_i)_*(Z^i_{n-i-j}) + \rho_{i,j}$ and $\Delta_{i,j}$ is rational (Lemma 2.1), the corollary is proven. □

Remark 3.3. — As a consequence of Proposition 3.1 and its proof, one makes the following observation. Let $1 \leq i \leq d-1$, $0 \leq j \leq i-1$, $i+1 \leq m \leq d$ and $s \in \{0,1,\dots,i\}\backslash\{j\}$. For any integers $0 \leq a_1 \leq a_2 \leq \cdots \leq a_i \leq d$, the integer

$$\deg\left(\left(Z^i_{n-i-j} \cdot \prod_{l=1}^{i} Z^i_{n-i-a_l}\right) \cdot \left(\sum_{k=0}^{i-s} W^i_{m-s-k} \cdot c_k(T_i)\right)\right)$$

is congruent to $1 \,(\mathrm{mod}\, 2)$ if $\{a_1,\dots,a_i\} = \{m\} \cup (\{0,1,\dots,i\}\backslash\{j,s\})$ and to $0 \,(\mathrm{mod}\, 2)$ otherwise.

The following proposition will complete the first part of Theorem 1.2 (see Corollary 3.5).

Proposition 3.4. — *For any $i \leq j \leq d$, one has*

$$\left((\theta_i)_*(z^i_{n-i-j})\right)_*(h^m) = \begin{cases} sym\left(\times_{k=0}^{i-1} h^k\right) & \text{if } m=j; \\ 0 & \text{otherwise.}\end{cases}$$

Proof. — We already know from Lemma 2.2 that $\left((\theta_i)_*(z^i_{n-i-j})\right)_*(h^m) = 0$ for $m \leq i-1$. We prove the cases $i \leq m \leq d$ by descending induction on i. The base of the descending induction (so $i=j=m=d$) is done similarly as the base of the descending induction in the proof of Proposition 3.1.

Let $2 \leq i \leq d$, $i \leq j \leq d$ and $i \leq m \leq d$. On the one hand, by Lemma 2.2 and the descending induction hypothesis, one has

$$\begin{aligned}\left((\theta_i)_*(z^i_{n-i-j})\right)_*(h^m) &= p_{G_i \times \underline{X^i}\,*}\left(w^i_{m-i} \cdot z^i_{n-i-j} \cdot \eta_i\right) \\ &= \begin{cases} \mathrm{sym}\left(\times_{k=0}^{i-1} h^k\right) & \text{if } m=j; \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{9}$$

Therefore, the coordinate of (9) on the top right h^{i-1} is

$$\begin{cases} \mathrm{sym}\left(\times_{k=0}^{i-2} h^k\right) & \text{if } m=j; \\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, by Lemmas 2.2, 2.5 and identity (7), this coordinate is also equal to

$$\left((\theta_{i-1})_*\left(z^{i-1}_{n-i+1-j}\right)\right)_*(h^m).$$

It remains to consider the cases $i \leq j \leq d$ with $m = i-1$ and $j = i-1$ with $i-1 \leq m \leq d$ to complete the descending induction step. Let $i-1 \leq j \leq d$. By Lemma 2.2, one has

$$\left((\theta_{i-1})_*\left(z^{i-1}_{n-i+1-j}\right)\right)_*\left(h^{i-1}\right) = p_{G_{i-1}\times \underline{X^{i-1}}\,*}\left(z^{i-1}_{n-i+1-j} \cdot \eta_{i-1}\right).$$

By Lemma 2.3, the latter cycle is the coordinate on the top right h^i of $\left((\theta_i)_*(z^i_{n-i-j})\right)_*(h^i)$. If $j \geq i$ then this coordinate is trivial by the descending induction hypothesis. Otherwise – if $j = i-1$ – then one has

$$\left((\theta_i)_*\left(z^i_{n-2i+1}\right)\right)_*(h^i) = {\rho_{i,i-1}}_*(h^i) + \left(\alpha_{i,i-1}(\mathrm{mod}\ 2)\right)_*(h^i).$$

By Proposition 3.1, the latter cycle is equal to

$$\mathrm{sym}\left(\left(\times_{k=0}^{i-2} h^k\right) \times h^i\right),$$

whose coordinate on the top right h^i is

$$\mathrm{sym}\left(\times_{k=0}^{i-2} h^k\right).$$

Now suppose that $j = i-1$ and let $i \leq m \leq d$. By Lemmas 2.2, 2.5 and identity (7), the cycle $\left((\theta_{i-1})_*(z^{i-1}_{n-2i+2})\right)_*(h^m)$ is the coordinate of $\left((\theta_i)_*(z^i_{n-2i+1})\right)_*(h^m)$ on the top right h^{i-1}. Since

$$\left((\theta_i)_*\left(z^i_{n-2i+1}\right)\right)_*(h^m) = {\rho_{i,i-1}}_*(h^m) + \left(\alpha_{i,i-1}(\mathrm{mod}\ 2)\right)_*(h^m),$$

${\rho_{i,i-1}}_*(h^m) = 0$ (because $m \neq i-1$) and $\left(\alpha_{i,i-1}\ (\mathrm{mod}\ 2)\right)_*(h^m) = \mathrm{sym}\left(\left(\times_{k=0}^{i-2} h^k\right) \times h^m\right)$ (Proposition 3.1), this coordinate is trivial. This completes the descending induction step. The proposition is proven. □

Corollary 3.5. — *Let $1 \leq i \leq d$ and $i \leq j \leq d$. If the cycle z^i_{n-i-j} is rational then the cycle $\rho_{i,j}$ $(mod\, 2)$ is also rational.*

Proof. — In view of the ring structure of $\mathrm{CH}(X_K^{i+1})$ and knowing that the cycle $(\theta_i)_*(Z^i_{n-i-j})$ is symmetric, one deduces from Proposition 3.4 that

$$(\theta_i)_*(z^i_{n-i-j}) = \rho_{i,j}\ (\mathrm{mod}\ 2) + \beta$$

with β being a sum of nonessential elements. The corollary is proven. □

The next proposition gives the second part of Theorem 1.2.

Proposition 3.6. — *Let $1 \leq i \leq d$ and $0 \leq j \leq d$. If the cycle $\rho_{i,j}$ is rational then the cycle Z^i_{n-i-j} is also rational.*

Proof. — Since ${\pi_{(0,\underline{i})}}_* \circ \pi^*_{(\underline{0},i)}(h^i) = [G_i]$ in $\mathrm{CH}^0(G_i)$, for dimensional reasons, one has

$$\left({\pi_{(0,\underline{i})}}_* \circ \pi^*_{(\underline{0},i)}\right)^{\times i+1}\left(h^i \times h^{i-1} \times \cdots \times 1 \cdot \mathrm{sym}\left(\left(\times_{k=0}^{i-1} h^k\right) \times l_j\right)\right) = [G_i]^{\times i} \times Z^i_{n-i-j}. \tag{10}$$

The conclusion follows by taking the image of cycle (10) under the pull-back of the diagonal morphism $X \to X^{i+1}$. □

4. Steenrod operations

In this section, we continue to use notation and material introduced in the previous sections and we assume that the base field F is of characteristic different from 2 because we use *Steenrod operations of cohomological type on Chow groups*. For a smooth scheme Y over a field with a characteristic different from two, Brosnan constructed in [1, § 10] a certain homomorphism

$$S_Y : \mathrm{Ch}(Y) \to \mathrm{Ch}(Y)$$

called the *total Steenrod operation on Y of cohomological type*. For any integer $j \geq 0$, we denote by

$$S_Y^j : \mathrm{Ch}^*(Y) \to \mathrm{Ch}^{*+j}(Y)$$

the *jth Steenrod operation on Y of cohomological type* (or simply by S^j if there is no ambiguity).

In addition to the classical rule

$$m \in EDI(i, X) \Rightarrow m, m-1 \in EDI(i+1, X)$$

(valid in any characteristic, see [10, Proposition 2.5]; note that one easily retrieves that rule by using the cycles $\rho_{i,j} \pmod 2$ defining the non-compact form of the elementary discrete invariant), the action of Steenrod operations on the compact form of the elementary discrete invariant already provides a restriction on the possible values, see [7, Proposition 5.12]. The non-compact form provides new restrictions on the possible values of the elementary discrete invariant as it allows a refined use of Steenrod operations.

4.1. Lower Steenrod operations. — Let δ_i denote the diagonal morphism $X^i \to X^{i+1}$, $(x_1, x_2, \cdots, x_i) \mapsto (x_1, x_1, x_2, \cdots, x_i)$. For any $1 \leq l \leq i-1$ and $l \leq j \leq d$, we set

$$\rho_{i,j,l} := \delta_i^* \circ \left(S^l \times \mathrm{Id}^{\times i}\right)(\rho_{i,j} \pmod 2) \in \mathrm{Ch}^{n-j+l+i(i-1)/2}(X_K^i).$$

Lemma 4.1. — *One has*

$$\rho_{i,j,l} = \sum_{k=l}^{i-1} \sum_{\substack{s=0 \\ s \neq k}}^{i-1} \binom{k}{l} h^{k+l+s} \times sym\left(\left(\times_{\substack{t=0 \\ t\neq s \\ t \neq k}}^{i-1} h^t\right) \times l_j\right)$$
$$+ \sum_{k=l}^{i-1} \binom{k}{l} l_{j-k-l} \times sym\left(\times_{\substack{s=0 \\ s\neq k}}^{i-1} h^s\right)$$
$$+ \sum_{s=0}^{i-1} \binom{n+1-j}{l} l_{j-l-s} \times sym\left(\times_{\substack{t=0 \\ t\neq s}}^{i-1} h^t\right).$$

Proof. — Since $S^l(h^k) = 0$ for $k < l$ (see [2, Theorem 61.13]), $S^l(h^k) = \binom{k}{l}$ for $k \geq l$ and $S^l(l_j) = \binom{n+1-j}{l} l_{j-l}$ for $l \leq j$ (see [2, Corollary 78.5]), one has

$$\left(S^l \times \mathrm{Id}^{\times i}\right)(\rho_{i,j} \pmod 2) = \sum_{k=l}^{i-1} \binom{k}{l} h^{k+l} \times \mathrm{sym}\left(\left(\times_{\substack{s=0\\ s\neq k}}^{i-1} h^s\right) \times l_j\right) + \binom{n+1-j}{l} l_{j-l} \times \mathrm{sym}\left(\times_{s=0}^{i-1} h^s\right).$$

The conclusion follows. □

Corollary 4.2. — *For even i, the cycle $\rho_{i,j}$ (mod 2) is rational if and only if the cycle $\rho_{i,j,1}$ is rational.*

Proof. — If the cycle $\rho_{i,j}$ (mod 2) is rational then the cycle $\rho_{i,j,1}$ is also rational as Steenrod operations of cohomological type commute with pull-backs of morphisms of smooth schemes (see [2, Theorem 61.9]). Moreover, since ${\pi_{(0,\underline{i})}}_* \circ \pi^*_{(\underline{0},i)}(h^i) = [G_i]$ in $\mathrm{CH}^0(G_i)$, one deduces from Lemma 4.1 and dimensional considerations that

$$\left({\pi_{(0,\underline{i})}}_* \circ \pi^*_{(\underline{0},i)}\right)^{\times i}\left(1 \times h^{i-1} \times h^{i-2} \times \cdots \times h^2 \times 1 \cdot \rho_{i,j,1}\right) = \binom{i-1}{1}[G_i]^{\times i-1} \times z^i_{n-i-j}.$$

Hence, by applying the pull-back of the diagonal morphism $X \to X^i$ to the previous identity, one sees that the rationality of $\rho_{i,j,1}$ implies the rationality of $(i-1)z^i_{n-i-j}$. Theorem 1.2 completes the proof. □

Example 4.3. — Let $i = 2$ and $m = n - 2 - j$ for some $2 \leq j \leq d$. By Theorem 1.2 and Corollary 4.2, the integer m belongs to $EDI(X, 2)$ if and only if the cycle

$$\rho_{2,j,1} = h^2 \times l_j + l_{j-2} \times 1 + (m+3)(l_{j-1} \times h + l_{j-2} \times 1) \in \mathrm{Ch}^{n-j+2}\left(X^2_K\right)$$

is rational. Suppose now that m is odd. Then one has

$$\rho_{2,j-1,1} = h^2 \times l_{j-1} + l_{j-2} \times h = (1 \times h) \cdot \left(h^2 \times l_j + l_{j-2} \times 1\right) = (1 \times h) \cdot \rho_{2,j,1}.$$

Therefore, for odd $n - 2 - d \leq m \leq n - 4$, if $m \in EDI(X, 2)$ then $m + 1 \in EDI(X, 2)$.

Proposition 4.4. — *Let $2 \leq i \leq d$, $1 \leq l \leq i-1$ and $n-i-d \leq m \leq n-i-l$ such that the binomial coefficient $\binom{m+i+1}{l}$ is odd. Assume that there exists an integer $0 \leq a \leq n - i - m - l$ such that, for any $l \leq k \leq i - 1$ satisfying $k + l + a \in \{i, \ldots, d\}$, the binomial coefficient $\binom{k}{l}$ is even. Then one has*

$$m \in EDI(X, i) \Longrightarrow m + l + a \in EDI(X, i).$$

Proof. — Let $l \leq j \leq d$ such that $m = n - i - j$. Since $m \in EDI(X,i)$, the cycle

$$\rho_{i,j,l} \times 1 \in \mathrm{Ch}\left(X_K^{i+1}\right)$$

is rational by Theorem 1.2. Let $\sigma \in S_i$ be a cyclic permutation. It follows from Lemma 4.1 that, because of repetitions, one has

$$\left(\mathrm{Id} \times \left(\sum_{r=0}^{i-1} \sigma_*^r\right)\right)(\rho_{i,j,l} \times 1) = \sum_{k=l}^{i-1} \binom{k}{l} h^{k+l} \times \mathrm{sym}\left(\left(\times_{\substack{s=0\\ s\neq k}}^{i-1} h^s\right) \times l_j\right) + l_{j-l} \times \mathrm{sym}\left(\times_{t=0}^{i-1} h^t\right).$$

Thus, the cycle

$$\sum_{k=l}^{i-1} \binom{k}{l} h^{k+l+a} \times \mathrm{sym}\left(\left(\times_{\substack{s=0\\ s\neq k}}^{i-1} h^s\right) \times l_j\right) + l_{j-l-a} \times \mathrm{sym}\left(\times_{t=0}^{i-1} h^t\right) \tag{11}$$

is rational. Let $\tau \in S_{i+1}$ be a cyclic permutation. For any $l \leq k \leq i-1$ such that $k + l + a \leq i - 1$, one has

$$\sum_{r=0}^{i} \tau_*^r \left(h^{k+l+a} \times \mathrm{sym}\left(\left(\times_{\substack{s=0\\ s\neq k}}^{i-1} h^s\right) \times l_j\right)\right) = 0$$

because of repetitions. Consequently, since $h^t = 0$ in $\mathrm{Ch}^t(X_K)$ for any $t > d$ (see [2, §68]), the image of the rational cycle (11) under $\sum_{r=0}^{i} \tau_*^r$ is equal to $\rho_{i,j-l-a} \pmod 2$. Theorem 1.2 completes the proof. □

Example 4.5. — For odd-dimensional quadratic forms, Proposition 4.4 applies with $i = 3$, $l = 1$, $m = n - 4$ and $a = 0$.

4.2. Higher Steenrod operations. —

Proposition 4.6. — *Let $2 \leq i \leq d$, $i \leq l \leq d$ and $n - i - d \leq m \leq n - i - l$ such that the binomial coefficient $\binom{m+i+1}{l}$ is odd. If $m \in EDI(X,i)$ then, for any $m + l \leq m' \leq n - i$, one has $m' \in EDI(X,i)$.*

Proof. — Let $l \leq j \leq d$ such that $m = n - i - j$. Since $l \geq i$, the operation S^l applied to any hyperplane power class appearing in the decomposition of $\rho_{i,j} \pmod 2$ gives zero. Therefore, one has

$$\left(S^l \times \mathrm{Id}^{\times i}\right)(\rho_{i,j} (\mathrm{mod}\ 2)) = \binom{m+i+1}{l} l_{j-l} \times \mathrm{sym}\left(\times_{k=0}^{i-1} h^k\right) \tag{12}$$

in $\mathrm{Ch}(X_K^{i+1})$. Thus, since the binomial coefficient $\binom{m+i+1}{l}$ is odd, $m \in EDI(X,i)$ and Steenrod operations of cohomological type commute with pull-backs of morphisms of smooth schemes, it follows from Theorem 1.2 and identity (12) that, for any $0 \leq j' \leq j - l$, the cycle

$$l_{j'} \times \mathrm{sym}\left(\times_{k=0}^{i-1} h^k\right) \tag{13}$$

is rational. Let $\sigma \in S_{i+1}$ be a cyclic permutation. Since the image of the cycle (13) under $\sum_{r=0}^{i} \sigma_*^r$ is $\rho_{i,j'} \pmod 2$, the proposition is proven. □

Example 4.7. — Let $J(q)$ be the J-invariant of the quadratic form q, as defined by A. Vishik in [7]. One has $J(q) = EDI(X, d)$. Suppose that $n - 2d \in J(q)$. If the binomial coefficient $\binom{n-d+1}{d}$ is odd then, by Proposition 4.6, one has $n - d \in J(q)$.

Remark 4.8. — For $i = 1$, the proof of Proposition 4.6 provides the following stronger assertion. Let $1 \leq l \leq d$ and $n - 1 - d \leq m \leq n - 1 - l$ such that the binomial coefficient $\binom{m+2}{l}$ is odd. If $m \in EDI(X, 1)$ then $i_0(X) \geq n - m - l$.

5. Witt index

We continue to use notation and material introduced in the previous sections and we do not make any assumption on the characteristic of the base field F. In this section, we assume that the F-quadric X is anisotropic and we study some restrictions on the elementary discrete invariant when the first Witt index i_1 of X is sufficiently large, using the non-compact form (Theorem 1.2).

By [2, Lemmas 73.18 and 73.3], there exists a unique minimal rational cycle in $\mathrm{Ch}^{n-i_1+1}(X_K^2)$ containing $1 \times l_{i_1-1}$. This cycle is symmetric ([2, Lemma 73.17]) and is called the *1-primordial cycle*. We denote it by π.

Proposition 5.1. — *Let $i \in \{1, \ldots, d\}$. Suppose that the quadric X is anisotropic with $i_1 > i$. If $m \in EDI(X, i)$ is such that $n - m \notin \{i_1\} \cup \{2i_1, \ldots, d+1\}$ then $m + 1 \in EDI(X, i - 1)$.*

Proof. — We set $j = n - i - m$. Since $m \in EDI(X, i)$, the cycle $\rho_{i,j} \pmod 2$ is rational by Theorem 1.2. We claim that the hypothesis $i_1 > i$ implies that the cycle $\rho_{i-1,j} \pmod 2$ is also rational, which, by Theorem 1.2, gives the conclusion.

The rational cycle π decomposes as

$$\pi = 1 \times l_{i_1-1} + l_{i_1-1} \times 1 + \sum_{k=i_1}^{d-i_1+1} a_k \left(h^k \times l_{k+i_1-1} + l_{k+i_1-1} \times h^k\right)$$

for some $a_k \in \mathbb{Z}/2\mathbb{Z}$. The fact that one can choose to make the previous sum start from $k = i_1$ is due to [2, Proposition 73.27]. Since $i_1 > i$, and $j \notin \{i_1 - i\} \cup \{2i_1 - i, \ldots, d - i + 1\}$, one has

$$(\rho_{i,j} (\mathrm{mod}\ 2)) \circ \left(\left(1 \times h^{i_1-i}\right) \cdot \pi\right) = 1 \times (\rho_{i-1,j} \ (\mathrm{mod}\ 2)),$$

where $\circ$ stands for the composition of correspondences. Therefore, pulling back the latter identity with respect to the diagonal morphism δ_i, one gets that the cycle $\rho_{i-1,j} \pmod 2$ is rational. □

The following statement is obtained by recursively applying Proposition 5.1.

Corollary 5.2. — *Let $i \in \{1, \ldots, d\}$. Suppose that the quadric X is anisotropic with $i_1 > i$. One has*

(i) *if $m \in EDI(X, i)$ then $n - m \geq i_1$;*

(ii) *if $m \in EDI(X, i)$ and $n - m = i_1 + l$ or $d + 1 + l$ for some $1 \leq l < i$ then $m + l \in EDI(X, i - l)$.*

BIBLIOGRAPHY

[1] P. Brosnan – "Steenrod operations in Chow theory", *Trans. Amer. Math. Soc.* **355** (2003), no. 5, p. 1869–1903.

[2] R. Elman, N. Karpenko & A. Merkurjev – *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, 2008.

[3] R. Fino – "Symmetric correspondences on quadrics", *Manuscripta Math.* **155** (2018), no. 3, p. 523–537.

[4] J. Hurrelbrink & U. Rehmann – "Splitting patterns of excellent quadratic forms", *J. Reine Angew. Math.* **444** (1993), p. 183–192.

[5] M. Knebusch – "Generic splitting of quadratic forms, I.", *Proc. London Math. Soc. (3)* **33** (1976), no. 1, p. 65–93.

[6] A. Vishik – "Motives of quadrics with applications to the theory of quadratic forms", in *Geometric methods in the algebraic theory of quadratic forms*, Lecture Notes in Math., vol. 1835, Springer, 2004, p. 25–101.

[7] ———, "On the Chow groups of quadratic Grassmannians", *Documenta Math.* (2005), no. 10, p. 111–130.

[8] ———, "Generic points of quadrics and Chow groups", *Manuscripta Math* **122** (2007), no. 3, p. 365–374.

[9] ———, "Topics on quadratic forms", in *Some recent developments in algebraic K-theory*, ICTP Lect. Notes, vol. 23, Abdus Salam Int. Cent. Theoret. Phys., 2008, p. 228–275.

[10] ———, "Field of u-invariant 2^r+1", in *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, Progr. Math, vol. 270, Birkhäuser Boston Inc., 2009, p. 661–685.