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DETECTING TRIVIAL ELEMENTS OF PERIODIC QUOTIENT OF HYPERBOLIC GROUPS

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Abstract. — In this article we give a sufficient and necessary condition to determine whether an element of the free group induces a nontrivial element of the free Burnside group of sufficiently large odd exponents. Although this result is "well known" among specialists, it has never been stated with such a level of simplicity. Moreover, our proof highlights some important differences between the Delzant-Gromov approach to the Burnside problems and others that exist. This criterion can be stated without any knowledge regarding Burnside groups, in particular about the proof of its infiniteness. Therefore, it also provides a useful tool to study outer automorphisms of Burnside groups. In addition, we state an analogue result for periodic quotients of torsion-free hyperbolic groups.

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Résumé (*Un critère pour détecter les éléments triviaux dans un quotient périodique d'un groupe hyperbolique*). — Dans cet article, on propose une condition nécessaire et suffisante pour déterminer si un élément du groupe libre induit ou non un élément trivial dans les groupes de Burnside libre d'exposants impairs suffisamment grands. Bien que ce résultat soit « bien connu » des spécialistes, il n'a jamais été énoncé avec un tel niveau de simplicité. En outre la preuve met en lumière les principales différences entres l'approche de Delzant-Gromov du problème de Burnside et les autres existants dans la littérature. Ce critère peut être énoncé sans aucun pré-requis sur les groupes de Burnside. En particulier il n'est pas nécessaire de comprendre pourquoi les groupes de Burnside sont infinis pour l'appliquer. Pour cette raison il fournit un outil effectif qui nous permettra plus tard d'étudier les automorphismes du groupe de Burnside. Nous donnons aussi un résultat analogue pour les quotients périodiques d'un groupe hyperbolique sans torsion.

Introduction

Let n be an integer. A group G is said to have exponent n if for every $g \in G$, $g^n = 1$. In 1902, W. Burnside asked whether a finitely generated group with a finite exponent is necessarily finite [3]. In order to study this question, the natural object to look at is the free Burnside group of rank r and exponent n denoted by $\mathbf{B}_r(n)$. It is the quotient of the free group $\mathbf{F}_r$ of rank r by the (normal) subgroup $\mathbf{F}_r^n$ generated by the n-th power of all elements of $\mathbf{F}_r$. Therefore, it is the largest group of rank r and exponent n. For a long time, one only knew that for some small exponents, $\mathbf{B}_r(n)$ was finite ($n = 2$ Burnside [3], $n = 3$ Burnside [3] Levi and van der Waerden [16], $n = 4$ Sanov [22], $n = 6$ Hall [14]). In 1968, P.S. Novikov and S.I. Adian achieved a breakthrough. In a series of three papers, they proved that if $r \geq 2$ and $n \geq 4381$ is odd then $\mathbf{B}_r(n)$ is infinite [19]. This result has been improved in many directions. A.Y. Ol'shanskiĭ proposed a different proof of the Novikov-Adian theorem using graded diagrams [20]. S.V. Ivanov [15] and I.G. Lysenok [17] solved the case of even exponents.

The different works cited here develop a similar general strategy. They construct by induction a sequence of groups (G_k) whose direct limit is $\mathbf{B}_r(n)$. Each group is built from the previous one by adjoining new relations. Through a deep study of the properties of these relations, they prove the following key fact. Let p be an integer. Let w be a reduced word of $\mathbf{F}_r$. If w does not contain a subword of the form u^p, then w induces a nontrivial element of $\mathbf{B}_r(n)$ for exponents n very large compared to p (the precise values of p and n depend on the techniques used during the construction). In particular, two distinct reduced words not containing "large power" induce distinct elements in the Burnside group of sufficiently large exponents. The infiniteness of $\mathbf{B}_r(n)$ follows then from the existence of an infinite word without third-power (for instance the Thue-Morse word [23, 24] or other examples [1, Paragraph 3.5]).

More recently, T. Delzant and M. Gromov provided an alternative proof for the infiniteness of Burnside groups [12]. Their work relies on a purely geometrical point of view of small cancellation theory. Using this tool, they also construct a sequence of groups (G_k) whose limit is $\mathbf{B}_r(n)$. However, contrary to the previous approaches, they *do not need* a criterion to distinguish elements in $\mathbf{B}_r(n)$. The infiniteness of Burnside groups indeed follows from the hyperbolic structure of the approximation groups G_k.

One goal of this paper is to highlight the differences and similarities between the Delzant-Gromov approach and other methods. We explain how hyperbolic geometry can be used to recover a sufficient and necessary condition to determine whether an element of a free group induces a trivial element of the free Burnside groups of sufficiently large exponents. Let ξ and n be two integers. An (n,ξ)-*elementary move* consists in replacing a reduced word of the form $pu^ms \in \mathbf{F}_r$ by the reduced representative of $pu^{m-n}s$, provided m is an integer larger than $n/2-\xi$. Note that an elementary move may increase the length of the word.

Theorem. — *There exist numbers ξ and n_0 such that for all odd integers $n \geq n_0$ we have the following property. Let w be a reduced word of $\mathbf{F}_r$. The element of $\mathbf{B}_r(n)$ defined by w is trivial if and only if there exists a finite sequence of (n,ξ)-elementary moves that sends w to the empty word.*

This statement is "well known" from the specialists of Burnside groups. To our knowledge, however, it has never been formulated at such a level of simplicity. The reader can, for instance, compare our definition of elementary moves with that of *simple r-reversal of rank α* used by S.I. Adian [1, Paragraph 4.18 and Pages 8-16 for the prerequisites]. Adian's [1] or Ol'shanskiĭ's work [20, Lemma 5.5], as they appear in the literature, would lead to a similar result, but with a *weaker* requirement to perform elementary moves ($m \geq 90$ and $m \geq n/3$, respectively). It may be possible to adapt their techniques to obtain the same theorem. Nevertheless, the purpose of our proof is to emphasize how all this combinatorial machinery can be reinterpreted in terms of stability of quasi-geodesics in an appropriate hyperbolic space.

Before describing our method, we would like to discuss other motivations for this work. We wish to investigate the outer automorphisms of Burnside groups. Since $\mathbf{F}_r^n$ is a characteristic subgroup of $\mathbf{F}_r$, the projection $\mathbf{F}_r \twoheadrightarrow \mathbf{B}_r(n)$ induces a map $\operatorname{Out}(\mathbf{F}_r) \to \operatorname{Out}(\mathbf{B}_r(n))$. This map is neither one-to-one nor onto. Nevertheless, it provides numerous examples of automorphisms of free Burnside groups. For instance, if n is an odd exponent large enough, the image of $\operatorname{Out}(\mathbf{F}_r)$ in $\operatorname{Out}(\mathbf{B}_r(n))$ contains free groups of arbitrary rank [7]. One natural question is: which automorphisms of $\mathbf{F}_r$ induce automorphisms of infinite order of $\mathbf{B}_r(n)$? In [7], the author provided a large class of automorphisms of $\mathbf{F}_r$ with this property. However, we are looking for a sufficient and necessary condition to characterize them. To understand the difficulties that may appear, let us have a look at a simple example that was studied by E.A. Cherepanov [4]. Let $\{a,b\}$ be a free basis of $\mathbf{F}_2$. Let φ be the automorphism of $\mathbf{F}_2$ defined by $\varphi(a)=ab$ and $\varphi(b)=a$. Let us compute the orbit of b under φ.

$$\begin{array}{ll} \varphi^1(b)=a & \varphi^5(b)=abaababa \\ \varphi^2(b)=ab & \varphi^6(b)=abaababaabaab \\ \varphi^3(b)=aba & \varphi^7(b)=abaababaabaababaababa \\ \varphi^4(b)=abaab & \dots \end{array}$$

This sequence converges for the prefix topology to a right-infinite word

$$\varphi^\infty(b)=abaababaabaababaababaabaababaabaab\dots$$

which does not contain a subword that is a fourth-power [18]. Using the criterion of P.S. Novikov and S.I. Adian, the $\varphi^k(b)$'s define pairwise distinct elements of $\mathbf{B}_r(n)$ for some large n. In particular, φ induces an automorphism of infinite order of free Burnside groups of large exponents. For an arbitrary automorphism the situation becomes considerably more complicated. Let $\{a,b,c,d\}$ be a free basis of $\mathbf{F}_4$. Consider the automorphism ψ of $\mathbf{F}_4$ defined by $\psi(a)=a$, $\psi(b)=ba$, $\psi(c)=c^{-1}bcd$ and $\psi(d)=c$. As previously, we compute the orbit of d under ψ.

$$\begin{aligned} \psi^1(d) &= c \\ \psi^2(d) &= c^{-1}\mathbf{b}cd \\ \psi^3(d) &= d^{-1}c^{-1}b^{-1}c\mathbf{ba}c^{-1}bcdc \\ \psi^4(d) &= c^{-1}d^{-1}c^{-1}b^{-1}ca^{-1}b^{-1}c^{-1}bcd\mathbf{ba}^2d^{-1}c^{-1}b^{-1}cbac^{-1}bcdbcd \\ \psi^5(d) &= d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}b^{-1}ca^{-1}b^{-1}c^{-1}bcda^{-2}b^{-1}d^{-1}c^{-1}b^{-1}c\dots \\ &\quad bac^{-1}bcdc\mathbf{ba}^3c^{-1}d^{-1}c^{-1}b^{-1}ca^{-1}b^{-1}c^{-1}bcdba^2d^{-1}c^{-1}b^{-1}c\dots \\ &\quad bac^{-1}bcdcbac^{-1}bcdc \end{aligned}$$

Note that each time $\psi^k(d)$ contains a subword ba^m, then $\psi^{k+1}(d)$ contains ba^{m+1}. In other words, the $\psi^k(d)$'s contain arbitrary large powers of a; hence the previous strategy does not apply. This pathology cannot be avoided by choosing the orbit of an another element. Therefore, we need a more accurate criterion to distinguish two different elements of $\mathbf{B}_r(n)$, which is provided by our main theorem. This question about automorphisms of $\mathbf{B}_r(n)$ is solved in a joint work with Hilion [10].

Another motivation for this work is to understand the exponential growth rate of free Burnside groups. Let G be a group generated by a finite set S. In order to measure its "size" one defines the *(exponential) growth rate* with respect to S by

$$\lambda_G=\lim_{\ell\to+\infty}\sqrt[\ell]{|B(\ell)|},$$

where $|B(\ell)|$ stands for the cardinality of the ball of radius ℓ for the word metric induced by S. The group G is said to have *exponential growth* if $\lambda_G>1$ (this notion does not depend on the generating set S). For instance, the growth rate of $\mathbf{F}_r$ with respect to a free basis is $2r-1$. Free Burnside groups of large exponents are known to have exponential growth [1, Theorem 2.15]. The author proved that the exponential growth rate of $\mathbf{B}_r(n)$ (with respect to the image of a free basis of $\mathbf{F}_r$) can actually be made arbitrarily close to $2r-1$ [6]. To that end, he bounds from above the difference between the respective growth rate of $\mathbf{F}_r$ and $\mathbf{B}_r(n)$ as a function of r and n. The sharpness of these estimates directly depends on the accuracy of the criterion used to characterize the trivial elements of $\mathbf{B}_r(n)$. Our criterion provides the best known estimate of the growth rate of $\mathbf{B}_r(n)$.

Let us now say a few words about the proof of the main theorem. In [12], T. Delzant and M. Gromov construct a sequence of groups $\mathbf{F}_r \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \dots$ whose direct limit is $\mathbf{B}_r(n)$. At each step the groups G_k have – among others – the following properties.

- G_{k+1} is a small cancellation quotient of G_k
- The relations that define the quotient $G_k \twoheadrightarrow G_{k+1}$ are n-th powers of elements of G_k.

The particularity of the Delzant-Gromov approach is to make *explicit* use of hyperbolic geometry. As a small cancellation quotient, each G_k is indeed a hyperbolic group. Actually, they act on a hyperbolic space X_k whose geometry is finer than the one of the Cayley graph. The geometry of X_k reflects the combinatorial properties of G_k. Let w be a reduced word of $\mathbf{F}_r$. The first space X_0 is just the Cayley graph of $\mathbf{F}_r$. Hence, w can be thought as a path of X_0. If w does not contain large powers (i.e. no large subword of a relation), then it induces a local quasi-geodesic in each X_k. As a consequence of the stability of quasi-geodesics, it cannot loop back to its starting point, thus does not represent a trivial element of $\mathbf{B}_r(n)$. If the word w contains large powers the argument is slightly more complicated. However, the negative curvature of the spaces X_k allows us to understand in a systematic way what prevents w to induce a local quasi-geodesic. To detail this idea let us have a look at the first step $G_0 \to G_1$ of the process.

Given a small cancellation group, one knows an algorithm solving the word problem. Consider for instance w a reduced word of $\mathbf{F}_r$ which is trivial in the first quotient G_1. According to the Greendlinger Lemma, w contains a subword that equals $3/4$ of a relation. In our situation, this means that w can be written $w = pu^m s$ where $m \geq 3n/4$. Applying an elementary move, we obtain a new word w', which represents $pu^{m-n}s$ and is shorter than the previous one. Moreover, w' is still trivial in G_1. By iterating the process we get a sequence of elementary moves that sends w to the empty word.

For the Burnside groups the process is more tricky. Let w be a reduced word of $\mathbf{F}_r$, which is trivial in $\mathbf{B}_r(n)$. Since $\mathbf{B}_r(n)$ is the direct limit of the G_ks, there exists a step k such that w is trivial in G_{k+1} but not in G_k. Roughly speaking, the Greendlinger Lemma tells us that a geodesic word of G_k representing the same element as w contains $3/4$ of a relation, i.e. a subword of the form u^m with $m \geq 3n/4$. We would like to apply an elementary move. However, there is no reason that u^m should be a subword of w in $\mathbf{F}_r$. Consider the following example. Let u and v be two reduced words of $\mathbf{F}_r$. Assume that u^n is trivial in G_1. Let $w = (u^\ell v)^q (u^{\ell-n} v)^{n-q}$, where $\ell \approx n/2$. As an element of G_1, w represents $(u^\ell v)^n = (u^{\ell-n} v)^n$, which contains an n-th power. Nevertheless, this does not hold in $\mathbf{F}_r$. The fact is that the previous relations (here u^n) mess up the powers. However, even though w does not contain an n-th power of $u^\ell v$, it contains a large power of u. Thus, q elementary moves send w to $w' = (u^{\ell-n} v)^n$. We can now "read" the power of $u^{\ell-n} v$ directly on w' and apply an elementary move to reduce the length of this last word. This example actually describes the general situation. To be rigorous, we have to formulate the ideas presented above in a hyperbolic framework, taking care of many parameters (hyperbolicity constants, small cancellation parameters, etc). However, the proof of the main theorem, done by induction on k, does not use any argument other than this simple observation.

Our study works, in fact, in a more general situation. Let (X, x_0) be a pointed δ-hyperbolic geodesic proper space and G a non-elementary torsion-free group acting properly co-compactly by isometries on it. A.Y. Ol'shanskiĭ proved that for every sufficiently large odd exponent n, the quotient G/G^n is infinite [21]. We provide a sufficient and necessary condition to detect elements of G, which are trivial in such quotients. For this purpose we need to extend the definition of elementary moves to this context. Let v be a nontrivial isometry of G. Since G is torsion-free, it fixes two points v^- and v^+ of ∂X, the boundary at infinity of X. We denote by Y_v the set of points of X which are 20δ-close to some bi-infinite (quasi-)geodesic joining v^- and v^+ (see Section 1.3). This subset is quasi-isometric to a line. Moreover, v roughly acts on it by translation of length $[v]$. An (n, ξ)-*elementary move* consists in replacing a point $y \in X$ by $v^{-n} y$, provided that we have in X

$$\mathfrak{D}([x_0, y], Y_v) \geq [v^m], \text{ where } m \geq n/2 - \xi.$$

Here, $\mathfrak{D}([x_0, y], Y_v)$ is a quantity that measures for how long $[x_0, y]$ and Y_v fellow travel together.

Let us compare this definition with the previous one. Let X be the Cayley graph of $\mathbf{F}_r$ and x_0 the vertex representing 1. Let $g \in \mathbf{F}_r$. Assume that g can be written as a reduced word $g = pu^m s$. Then the geodesic $[x_0, gx_0]$, labeled by $pu^m s$, intersects the axis of $v = pup^{-1}$ along a path of length $[v^m]$. Moreover, $v^{-n} g$ can be represented by the word $pu^{m-n} s$. Hence, the later definition coincides with the one for free groups. The next theorem is a generalization for hyperbolic groups of the previous one.

Theorem. — *Let (X, x_0) be a pointed geodesic proper hyperbolic space. Let G be a non-elementary torsion-free group acting properly co-compactly by isometries on X. There exist numbers n_2 and ξ such that for every odd integer $n \geq n_2$, the following holds. Let y be a point of X. An element $g \in G$ belongs to G^n if and only if there exist two sequences of elementary moves which send y and gy, respectively, to the same point.*

This statement gives a criterion to detect trivial elements of G/G^n. For people familiar with graded diagrams, such a criterion can certainly be recovered by adapting Ol'shanskiĭ's work [21]. Our main theorem tells us a bit more. Choose two elements h and h' of G. Then, h and h' have the same image in G/G^n if and only if there exist two sequences of elementary moves which send hx_0 and $h'x_0$, respectively, to the same point. This is the form that is used to study automorphisms of free Burnside groups in [10].

In [9], the author constructs infinite partial periodic quotients of mapping class groups, free products, relatively hyperbolic groups, and so on. The same approach would provide a criterion to detect trivial elements in these quotients. However, this would increase the technical level considerably. Therefore, we prefer to stick with quotients of hyperbolic groups.

Outline of the article. In Section 1, we review some of the standard facts on hyperbolic geometry. Since the proofs in the rest of the article are already quite technical, we have also tried to compile in this section all the results that only require hyperbolic geometry. Section 2 investigates the cone-off construction used by T. Delzant and M. Gromov in [12]. In particular we compare at a large scale the relation between the geometry of the cone-off over a metric space and the one of its base. Section 3 is devoted to the study of small cancellation theory. Our goal is to understand how to lift figures from a small cancellation quotient $\bar{G} = G/K$ in the group G. For instance, let g be an element of G. Assume that a geodesic of $\bar{G}$ representing the image of g contains a large power. Under which conditions does g already contain a large power? If not, what kind of transformations could send g to an element containing a large power? In the last section we summarize all these results in an induction that will prove our main theorem.

1. Hyperbolic spaces

Let X be a metric length space. Given two points $x, y \in X$, we denote by $|x - y|_X$ (or simply $|x - y|$) the distance between them. When it exists, we write $[x, y]$ for a geodesic between x and y. Note that this geodesic, however, may not be unique. The Gromov product of three points $x, y, z \in X$ is defined by

$$\langle x, y \rangle_z = \frac{1}{2}\left(|x - z| + |y - z| - |x - y|\right).$$

From now on, we assume that X is δ-hyperbolic, which means that for all $x, y, z, t \in X$,

$$(1) \qquad \langle x, z\rangle_t \geq \min\left\{\langle x, y\rangle_t, \langle y, z\rangle_t\right\} - \delta,$$

or equivalently, for all $x, y, z, t \in X$,

$$(2) \qquad |x-y| + |z-t| \leq \left\{|x-z| + |y-t|, |x-t| + |y-z|\right\} + 2\delta.$$

Although most of the results hold for $\delta = 0$, we will always assume that the hyperbolicity constant δ is positive. The hyperbolicity constant of the hyperbolic plane $\mathbf{H}$ will play a particular role. We denote it by $\boldsymbol{\delta}$. We write ∂X for the boundary at infinity of X. See [5, Chapitre 2] for the definition and the main properties.

In this article, our approach of hyperbolic geometry essentially relies on the four-point inequality. For instance, whenever we write an equality such as $\langle x, y\rangle_z \leq c$, we have in mind that the point z roughly lies on a geodesic $[x, y]$ joining x to y; see [5, Chapitre 3, Lemme 2.7]. The quantity c quantifies this approximation. This point of view has the advantage of working in spaces that are eventually not geodesic and in providing compact proofs. Nevertheless, it can be confusing at first reading. To help the reader navigate through the article, we add comments here and there to explain the meaning of the major statements.

It is known that geodesic triangles in a hyperbolic spaces are uniformly thin. This can be stated through the following metric inequalities.

Lemma 1.1 ([8, Lemma 2.2]). — *Let x, y, z, s and t be points of X.*

(i)
$$\langle x, y\rangle_t \leq \max\left\{|x-t| - \langle y, z\rangle_x, \langle x, z\rangle_t\right\} + \delta,$$

(ii)
$$|s-t| \leq \big||x-s| - |x-t|\big| + 2\max\left\{\langle x, y\rangle_s, \langle x, y\rangle_t\right\} + 2\delta,$$

(iii) *The distance $|s-t|$ is bounded above by*

$$\max\left\{\big||x-s| - |x-t|\big| + 2\max\left\{\langle x, y\rangle_s, \langle x, z\rangle_t\right\}, |x-s| + |x-t| - 2\langle y, z\rangle_x\right\} + 4\delta.$$

1.1. Quasi-geodesics. — In this article, every path is continuous and rectifiable by arc length.

Definition 1.2. — Let $k \geq 1$, $l \geq 0$ and $L > 0$. Let I be an interval of $\mathbf{R}$. A path $\gamma : I \to X$ is

- a (k, l)-*quasi-geodesic* if for all $s, t \in I$,
$$|\sigma(s) - \gamma(t)| \leq |s-t| \leq k|\gamma(s) - \gamma(t)| + l.$$
- a L-*local* (k, l)-*quasi-geodesic* if its restriction to every close interval of diameter L is a (k, l)-quasi-geodesic.

One important feature of hyperbolic spaces is the stability of quasi-geodesic paths. See for instance [5, Chapitre 3, Théorèmes 1.2, 1.4 et 3.1]. For our purpose, we are only using L-local $(1,l)$-quasi-geodesics. For this class of paths we have a more accurate statement that we recall next.

Corollary 1.3 ([8, Corollaries 2.6 and 2.7]). — *Let $l \geq 0$. There exists $L = L(l,\delta)$, which only depends on δ and l with the following properties. Let γ be an L-local $(1,l)$-quasi-geodesic.*

(i) *The path γ is a (global) $(2,l)$-quasi-geodesic.*
(ii) *For every $t, t', s \in I$, such that $t \leq s \leq t'$, we have $\langle \gamma(t), \gamma(t')\rangle_{\gamma(s)} \leq l/2 + 5\delta$.*
(iii) *For every $x \in X$, for every y, y' lying on γ, we have $d(x,\gamma) \leq \langle y, y'\rangle_x + l + 8\delta$.*
(iv) *The Hausdorff distance between γ and an other L-local $(1,l)$-quasi-geodesic joining the same endpoints (eventually in ∂X) is at most $2l + 5\delta$.*

The stability of quasi-geodesics has a discrete analogue that can be stated as follows.

Proposition 1.4 (Stability of discrete quasi-geodesics [8, Proposition 2.7]). — *Let $l > 0$. There exists $L = L(l,\delta)$, which only depends on δ and l with the following property. Let $x_0, \dots, x_m$ be a sequence of points of X such that*

(i) *for every $i \in [\![1, m-1]\!]$, $\langle x_{i-1}, x_{i+1}\rangle_{x_i} \leq l$,*
(ii) *for every $i \in [\![1, m-2]\!]$, $|x_{i+1} - x_i| \geq L$.*

Then for all $i \in [\![0, m]\!]$, $\langle x_0, x_m\rangle_{x_i} \leq l + 5\delta$. Moreover, for every $p \in X$ there exists $i \in [\![0, m-1]\!]$ such that $\langle x_{i+1}, x_i\rangle_p \leq \langle x_0, x_m\rangle_p + 2l + 8\delta$.

Remark. — Using a rescaling argument, one can see that the optimal value for the parameter $L = L(l,\delta)$ in Corollary 1.3 and Proposition 1.4 satisfies the following property: for all $l, \delta \geq 0$ and $\lambda > 0$, $L(\lambda l, \lambda\delta) = \lambda L(l,\delta)$. In particular, L tends to 0 as l and δ approach 0. For the rest of the article, we denote by L_S the smallest positive number larger than 500 such that $L(10^5\delta, \delta) \leq L_S\delta$.

1.2. Quasi-convex subsets. — Let $\alpha \geq 0$. Let Y be a subset of X. We denote by $Y^{+\alpha}$ the *α-neighborhood of Y*, i.e. the set of points $x \in X$ such that $d(x,Y) \leq \alpha$. A point $x' \in Y$ is called an *η-projection of x on Y* if $|x - x'| \leq d(x,Y) + \eta$. A 0-projection is simply called a *projection.*

Definition 1.5. — Let $\alpha \geq 0$. A subset Y of X is *α-quasi-convex* if for every $x \in X$ and $y_1, y_2 \in Y$, $d(x,Y) \leq \langle y_1, y_2\rangle_x + \alpha$.

Definition 1.6. — Let Y be a subset of X connected by rectifiable paths. We denote by $|\,.\,|_Y$ the length metric on Y induced by the restriction of $|\,.\,|_X$

to Y. We say that Y is *strongly quasi-convex* if Y is 2δ-quasi-convex and for every $y_1, y_2 \in Y$ we have

$$|y_1 - y_2|_X \leq |y_1 - y_2|_Y \leq |y_1 - y_2|_X + 8\delta.$$

Lemma 1.7 (compare [5, Chapitre 10, Proposition 2.1]). — *Let Y be an α-quasi-convex subset of X.*

- *Let $x \in X$ and $y \in Y$. If x' is an η-projection of x on Y, then $\langle x, y\rangle_{x'} \leq \alpha + \eta$.*
- *Let $x_1, x_2 \in X$. If x_1' and x_2' are respectively η_1- and η_2-projections of x_1 and x_2 on Y then,*

  $$|x_1' - x_2'| \leq \max\{\varepsilon, |x_1 - x_2| - |x_1 - x_1'| - |x_2 - x_2'| + 2\varepsilon\},$$

  where $\varepsilon = 2\alpha + \delta + \eta_1 + \eta_2$.

Lemma 1.8. — *Let Y be an α-quasi-convex subset of X. Let x be a point of X and x' an η-projection of x on Y. For every $y \in X$, x' is an ε-projection of y on Y where $\varepsilon = \langle x, x'\rangle_y + 2\alpha + \delta + \eta$.*

Proof. — Let $\eta' > 0$ and y' be an η'-projection of y on Y. The previous lemma combined with the triangle inequality gives $|x' - y'| \leq \varepsilon(\eta')$ where $\varepsilon(\eta') = \langle x, x'\rangle_y + 2\alpha + \delta + \eta + \eta'$. Therefore, x' is an $(\varepsilon(\eta') + \eta')$-projection of y on Y. This property holds for every $\eta' > 0$ which gives the result. □

Lemma 1.9 ([8, Lemma 2.16]). — *Let Y and Z be, respectively, α- and β-quasi-convex subsets of X. For all $A \geq 0$ we have*

$$\operatorname{diam}\left(Y^{+A} \cap Z^{+A}\right) \leq \operatorname{diam}\left(Y^{+\alpha+3\delta} \cap Z^{+\beta+3\delta}\right) + 2A + 4\delta.$$

Remark. — By convention, the diameter of the empty set is zero.

Definition 1.10. — Let Y and Z be two subsets of X; we denote by $\mathfrak{D}(Y, Z)$ the following quantity.

$$\mathfrak{D}(Y, Z) = \frac{1}{2} \sup_{\substack{y_1, y_2 \in Y \\ z_1, z_2 \in Z}} \{0, |y_1 - y_2| + |z_1 - z_2| - |y_1 - z_1| - |y_2 - z_2|\}.$$

Remark. — Let Y and Z be two subsets of X. It follows from the definition that $\mathfrak{D}(Y, Z) \geq \operatorname{diam}(Y \cap Z)$. Actually, if Y and Z are, respectively, α- and β-quasi-convex subsets of X, $\mathfrak{D}(Y, Z)$ roughly measures the diameter of the intersection of Y and Z (see Lemma 1.12). However, this quantity is more compatible with our approach of hyperbolic geometry which relies extensively on the four-point inequality (1). We list below a few easy observations that follow from the triangle inequality.

(i) For all $A, B \geq 0$, $\mathfrak{D}(Y^{+A}, Z^{+B}) \leq \mathfrak{D}(Y, Z) + 2A + 2B$.
(ii) If $[x_1, x_2]$ is a geodesic between $x_1, x_2 \in X$, then $\mathfrak{D}([x_1, x_2], Y) = \mathfrak{D}(\{x_1 x_2\}, Y)$.

(iii) For all $x, y, z \in X$, we have

$$\mathfrak{D}(\{x,z\},Y) \leq \mathfrak{D}(\{x,y\},Y) + \langle x,y\rangle_z. \tag{3}$$

Combining (iii) with the four-point inequality (1) we obtain for all $x, y, z, t \in X$,

$$\mathfrak{D}(\{z,t\},Y) \leq \mathfrak{D}(\{x,y\},Y) + \langle x,y\rangle_z + \langle x,y\rangle_t + \delta. \tag{4}$$

The next proposition quantifies the following fact. Given two points $x_1, x_2 \in X$ and a quasi-convex subset Y of X, a geodesic $[x_1, x_2]$ fellow-travels with Y for a time, which is comparable to the diameter of the projection of $[x_1, x_2]$ on Y.

Proposition 1.11. — *Let Y be an α-quasi-convex subset of X. Let x_1 and x_2 be two points of X. We assume that x_1' and x_2' are respectively η_1- and η_2-projections of x_1 and x_2 on Y. Then*

$$|\mathfrak{D}(\{x_1x_2\},Y) - |x_1' - x_2'|| \leq \varepsilon,$$

where $\varepsilon = 2\alpha + \delta + \eta_1 + \eta_2$.

Proof. — By projection on a quasi-convex, we have

$$\max\{|x_1 - x_2| - |x_1 - x_1'| - |x_2 - x_2'| + 2\varepsilon, \varepsilon\} \geq |x_1' - x_2'|.$$

Therefore,

$$\begin{aligned}\mathfrak{D}(\{x_1x_2\},Y) &\geq \frac{1}{2}\max\{|x_1 - x_2| + |x_1' - x_2'| - |x_1 - x_1'| - |x_2 - x_2'|, 0\} \\ &\geq |x_1' - x_2'| - \varepsilon.\end{aligned}$$

On the other hand, x_1' and x_2', being η_1- and η_2-projections of x_1 and x_2 on Y, respectively, the triangle inequality implies that for every $y_1, y_2 \in Y$

$$\begin{aligned}&\frac{1}{2}\left(|x_1 - x_2| + |y_1 - y_2| - |x_1 - y_1| - |x_2 - y_2|\right) \\ &\qquad \leq |x_1' - x_2'| + \langle x_1, y_1\rangle_{x_1'} + \langle x_2, y_2\rangle_{x_2'} \\ &\qquad \leq |x_1' - x_2'| + 2\alpha + \eta_1 + \eta_2.\end{aligned}$$

This inequality holds for every $y_1, y_2 \in Y$ hence $\mathfrak{D}(\{x_1x_2\},Y) \leq |x_1' - x_2'| + \varepsilon$, which ends the proof. □

As we announced previously, up to a small thickening, the quantity $\mathfrak{D}(Y,Z)$ is roughly the same that the diameter of $Y \cap Z$. More precisely we have the following lemma.

Lemma 1.12. — *Let Y and Z be α- and β-quasi-convex subsets of X, respectively. We have the following.*

$$|\mathfrak{D}(Y,Z) - \operatorname{diam}\left(Y^{+\alpha+3\delta} \cap Z^{+\beta+3\delta}\right)| \leq 2\alpha + 2\beta + 25\delta.$$

Proof. — It follows from the triangle inequality that

$$\operatorname{diam}\left(Y^{+\alpha+3\delta} \cap Z^{+\beta+3\delta}\right) \leq \mathfrak{D}(Y^{+\alpha+3\delta}, Z^{+\beta+3\delta}) \leq \mathfrak{D}(Y, Z) + 2\alpha + 2\beta + 12\delta,$$

which provides one half of the statement. Let us focus on the other inequality. Without loss of generality we may assume that $\mathfrak{D}(Y, Z) > 25\delta$. Let $\eta > 0$ such that $\mathfrak{D}(Y, Z) > 25\delta + 9\eta$. There exist $y_1, y_2 \in Y$ and $z_1, z_2 \in Y$ such that $\mathfrak{D}(\{y_1, y_2\}, \{z_1, z_2\}) > \mathfrak{D}(Y, Z) - \eta$. Let $\gamma \colon I \to X$ be a $(1, \eta)$-quasi-geodesic joining z_1 to z_2 Let y_1' and y_2' be η-projections of y_1 and y_2, respectively, on γ. Proposition 1.11 yields

$$|y_1' - y_2'| \geq \mathfrak{D}(\{y_1, y_2\}, \{z_1, z_2\}) - 7\delta - 4\eta \geq \mathfrak{D}(Y, Z) - 7\delta - 5\eta > 18\delta + 4\eta$$

Thus, by projection on a quasi-convex we get $\langle y_1, y_2 \rangle_{y_1'} \leq 7\delta + 4\eta$. However, Y is α-convex thus y_1' lies in the $(\alpha + 7\delta + 4\eta)$-neighborhood of Y. On the other hand, y_1' being a point on γ, $\langle z_1, z_2 \rangle_{y_1'} \leq \eta/2$; thus y_1' belongs to the $(\beta + \eta/2)$-neighborhood of Z. The same facts hold for y_2'. Consequently, by Lemma 1.9

$$\begin{aligned} |y_1' - y_2'| &\leq \operatorname{diam}\left(Y^{+\alpha+7\delta+4\eta} \cap Z^{+\beta+\eta/2}\right) \\ &\leq \operatorname{diam}\left(Y^{+\alpha+3\delta} \cap Z^{+\beta+3\delta}\right) + 2\max\{\alpha, \beta\} + 18\delta + 8\eta \end{aligned}$$

which leads to

$$\mathfrak{D}(Y, Z) \leq \operatorname{diam}\left(Y^{+\alpha+3\delta} \cap Z^{+\beta+3\delta}\right) + 2\max\{\alpha, \beta\} + 25\delta + 13\eta.$$

This inequality holds for every sufficiently small η, which completes the proof. □

1.3. Isometries of a hyperbolic space. — In this section we assume that X is a proper geodesic δ-hyperbolic space. Let x be a point of X. An isometry g of X is either

- *elliptic*, i.e. the orbit $\langle g \rangle \cdot x$ is bounded,
- *loxodromic*, i.e. the map from $\mathbf{Z}$ to X that sends m to $g^m x$ is a quasi-isometry,
- or *parabolic*, i.e. it is neither loxodromic or elliptic.

Note that these definitions do not depend on the point x. The *axis* of g, denoted by A_g, is the set of points $x \in X$ such that $|gx - x| < [g] + 8\delta$. It is 10δ-quasi-convex [8, Proposition 2.28]. Note that this definition makes sense for any isometry of X, not just a loxodromic one. In order to measure the action of g on X, we define two translation lengths. By the *translation length* $[g]_X$ (or simply $[g]$), we mean

$$[g]_X = \inf_{x \in X} |gx - x|.$$

The *asymptotic translation length* $[g]_X^\infty$ (or simply $[g]^\infty$) is

$$[g]_X^\infty = \lim_{n \to +\infty} \frac{1}{n} |g^n x - x|.$$

These two lengths are related as follows.

Proposition 1.13 ([5, Chapitre 10, Proposition 6.4]). — *Let g be an isometry of X. Its translation lengths satisfy*

$$[g]^\infty \leq [g] \leq [g]^\infty + 32\delta.$$

The isometry g is loxodromic if and only if its asymptotic translation length is positive. In this case, g fixes exactly two points of ∂X, namely

$$g^- = \lim_{n\to-\infty} g^n x \quad \text{and} \quad g^+ = \lim_{n\to+\infty} g^n x$$

Let Γ_g be the union of all $(1,\delta)$-quasi-geodesic joining g^- to g^+. The *cylinder* of g, denoted by Y_g, is the 20δ-neighborhood or Γ_g. It is a g-invariant, strongly quasi-convex subset of X [8, Lemma 2.31].

Proposition 1.14 ([8, Lemma 2.32]). — *Let g be a loxodromic isometry of X. Let Y be a g-invariant α-quasi-convex subset of X. Then Y_g is contained in the $(\alpha+42\delta)$-neighborhood of Y. In particular, Y_g lies in the 52δ-neighborhood of A_g.*

Let g be an isometry of X such that $[g] > L_S\delta$ (in particular, g is loxodromic). For every $l \in (0,\delta]$ there exists $T \in \mathbf{R}$ with $[g] \leq T \leq T + l$ and a T-local $(1,l)$-quasi-geodesic $\gamma\colon \mathbf{R} \to X$ such that for every $t \in \mathbf{R}$, $\gamma(t+T) = g\gamma(t)$. We call such a path an *l-nerve* of g and T its *fundamental length* (see [8, Definition 2.29]). It is a very convenient tool for the proofs. Indeed, γ is quasi-isometric to a line on which g acts by translation of length almost $[g]$. Moreover, the Hausdorff distance between γ and Y_g is at most 27δ. Therefore, up to a small error, one can advantageously replace Y_g by γ. We summarize here some basic properties of nerves. They directly follow from the stability of the local geodesics and the projection on quasi-convex subsets. In order to lighten the proofs, we will later use these facts without any reference.

- The l-nerve γ is $(l+8\delta)$-quasi-convex.
- Let x be a point of X and $x' = \gamma(s)$ a projection of x on γ. Let $y = \gamma(t)$ be another point of γ. We have $\langle x,y\rangle_{x'} \leq l + 8\delta$. Moreover, for every $u \in [s,t]$, $\langle x,y\rangle_{\gamma(u)} \leq 3l/2 + 13\delta$.
- Let x_1, x_2 be two points of X and $x'_1 = \gamma(s_1)$, $x'_2 = \gamma(s_2)$ respective projections of x_1 and x_2 on γ. If $|x'_1 - x'_2| > 2l + 17\delta$, then $\langle x_1,x_2\rangle_{x'_1} \leq 2l + 17\delta$. Moreover, for all $t \in [s_1,s_2]$, $\langle x_1,x_2\rangle_{\gamma(t)} \leq 5l/2 + 22\delta$.
- For all $x,y \in X$, we have

$$|d(x,\gamma) - d(x,Y_g)| \leq 27\delta \quad \text{and} \quad |\mathfrak{D}(\{x,y\},\gamma) - \mathfrak{D}(\{x,y\},Y_g)| \leq 54\delta.$$

Lemma 1.15. — *Let g be an isometry of X such that $[g] > L_S\delta$. For all $x \in X$ we have*

$$|\langle gx, g^{-1}x\rangle_x - d(x,Y_g)| \leq 52\delta.$$

Proof. — Let γ be a δ-nerve of g and x' a projection of x on γ. It follows from the hyperbolicity condition (1) that

$$\begin{aligned}\langle gx, g^{-1}x\rangle_x - \langle gx, g^{-1}x\rangle_{x'} &\leq |x-x'| \\ &\leq \langle gx, g^{-1}x\rangle_x + \max\left\{\langle x, gx\rangle_{x'}, \langle x, g^{-1}x\rangle_{x'}\right\} + \delta.\end{aligned}$$

However, $[g] > L_S\delta$ hence $|gx' - g^{-1}x'| > 19\delta$. Consequently, $\langle x, gx\rangle_{x'} \leq 19\delta$, $\langle x, g^{-1}x\rangle_{x'} \leq 19\delta$ and $\langle g^{-1}x, gx\rangle_{x'} \leq 25\delta$. It follows that $|\langle gx, g^{-1}x\rangle_x - |x-x'|| \leq 25\delta$. However, $|x-x'|$ is exactly $d(x,\gamma)$. Hence, $|\langle gx, g^{-1}x\rangle_x - d(x, Y_g)| \leq 52\delta$. □

The next lemma has the following meaning. Let $x, y \in X$ and g be a loxodromic isometry of X. Assume that the geodesic $[x, y]$ fellow-travels with the axis of g for a "sufficiently long" time – by sufficiently long, here we mean longer than half of $[g]$. Then g can be used to move x closer to y.

LEMMA 1.16. — *Let $A \geq 0$. Let g be an isometry of X such that $[g] > L_S\delta$. Let x and y be two points of X. We assume that*

$$\mathfrak{D}(\{x, y\}, Y_g) > [g]/2 + A > 92\delta$$

Then the following holds.

- (i) *There exists $k \in \mathbf{Z}$, such that $|g^k x - y| < |x-y| - A + 112\delta$.*
- (ii) *There exists $\varepsilon \in \{\pm 1\}$ such that $|g^\varepsilon x - y| < |x-y| - \min\{[g] - 49\delta, 2A - 186\delta\}$*

Proof. — Let γ be a δ-nerve of g and T its fundamental length. Its 27δ-neighborhood contains Y_g; therefore, $\mathfrak{D}(\{x, y\}, \gamma) > [g]/2 + A - 54\delta$. We denote by $x' = \gamma(s)$ and $y' = \gamma(t)$ respective projections of x and y on γ. Proposition 1.11 gives

$$|s-t| \geq |x'-y'| > [g]/2 + A - 73\delta > 19\delta.$$

Combined with the projection on γ we obtain

$$|x-y| \geq |x-x'| + |x'-y'| + |y'-y| - 38\delta > |x-x'| + [g]/2 + A + |y'-y| - 111\delta.$$

On the other hand, g acts on γ by translating the parameter by T. Hence, there exists $k \in \mathbf{Z}$ such that $|g^k x' - y'| \leq [g]/2 + \delta$. The triangle inequality yields

$$|g^k x - y| \leq |x-x'| + [g]/2 + |y'-y| + \delta < |x-y| - A + 112\delta,$$

which provides the first point. The second point allows us to move x only by g or its inverse. We proceed as follows. Up to replacing g by its inverse, we may assume that $s \leq t$. Hence, $|(s+T) - t| = \max\{|s-t| - T, T - |s-t|\}$. If $|s-t| \leq T$, using the fact that γ is a T-local $(1,\delta)$-quasi-geodesic we get $|gx' - y'| \leq |x'-y'| - 2A + 148\delta$. In the other case, we get $\langle x', y'\rangle_{gx'} \leq 11\delta/2$. Consequently, $|gx' - y'| \leq |x'-y'| - [g] + 11\delta$. We conclude with the triangle inequality as noted previously. □

The next lemma is a variation of the previous one. We prove that if g and h are two loxodromic isometries of X whose respective axes fellow-travel for a "sufficiently long" time, then g can be used to shorten the translation length of h.

Lemma 1.17. — *Let $A \geq 0$. Let g and h be two isometries of X such that $[g] > L_S\delta$. We assume that*

$$\min\{[h], \mathfrak{D}(Y_h, Y_g)\} > [g]/2 + A > 196\delta.$$

Then, there exists $k \in \mathbf{Z}$ such that $[g^k h] < [h] - A + 328\delta$.

Proof. — Let γ be a δ-nerve of h and T its fundamental length. Since Y_h lies in the 27δ-neighborhood of γ, we have $\mathfrak{D}(Y_g, \gamma) > [g]/2 + A - 54\delta$. Hence, there exist x and y in Y_g such that $\mathfrak{D}(\{x, y\}, \gamma) > [g]/2 + A - 54\delta$. We denote by $x' = \gamma(s)$ and $y' = \gamma(t)$ the respective projections of x and y on γ. Up to changing the role of x and y, we can assume that $s \leq t$. Recall that γ is parametrized by arc length. Hence, Proposition 1.11 gives

$$|s - t| \geq |x' - y'| > [g]/2 + A - 73\delta > 19\delta.$$

On the other hand, it follows from our assumption that $T > [g]/2 + A$. Consequently, there exists $u \in \mathbf{R}$ with $s \leq s + u \leq \min\{t, s + T\}$ such that $|x' - z| = [g]/2 + A - 73\delta$, where $z = \gamma(s + u)$. The isometry h acts on γ by translation of length almost $[h]$. More precisely, $hx' = \gamma(s + T)$. Consequently, $\langle x', hx' \rangle_z \leq 6\delta$ and

$$|x - y| \geq |x - x'| + |x' - z| + |z - y| - 50\delta.$$

In particular, $\mathfrak{D}(Y_g, \{x', z\}) \geq |x' - z| - 25\delta$. It follows from (3) that

$$\begin{aligned}\mathfrak{D}(Y_g, \{x', hx'\}) \geq \mathfrak{D}(Y_g, \{x', z\}) - \langle x', hx' \rangle_z &\geq |x' - z| - 31\delta \\ &\geq [g]/2 + A - 104\delta > 92\delta.\end{aligned}$$

According to Lemma 1.16, there exists $k \in \mathbf{Z}$ such that $|g^k hx' - x'| < |hx' - x'| - A + 216\delta$. However, x' is a point of a δ-nerve of h and thus of the cylinder of h. It follows that x' lies in the 52δ-neighborhood of A_h (Proposition 1.14). Consequently, $[g^k h] \leq [h] - A + 328\delta$. □

The goal of the next two results is to describe a figure that will naturally arise in Section 3. Since the proof only requires some basic properties of hyperbolicity, we give it here. It will significantly lighten the proofs involving foldable configurations (see Sections 3.3 and 3.5). The constants A, B and C which appear in the following statements will be made precise in Section 3. They represent distances which are large in comparison to δ but small compared to $[g]$.

Proposition 1.18. — *Let $A, B, C \geq 0$. Let g be an isometry of X such that $[g] > \max\{L_S\delta, 2A + 2B + 2C + 350\delta\}$. Let x, y and z be three points of X. We assume that there exists a point $s \in X$ such that $\mathfrak{D}(\{s, y\}, Y_g) \leq [g]/2 + A$ and $|x - s| \leq \langle y, z\rangle_x + B$. Let γ be a δ-nerve of g. We denote by $y' = \gamma(a)$ and $z' = \gamma(b)$ the respective projections of y and z on γ. Let x' be a projection of x on $\gamma(I)$, where I is the closed interval whose endpoints are a and b. If $\mathfrak{D}(\{y, z\}, Y_g) \geq [g] - C$, then we have*

(i) $|y' - z'| \geq [g] - C - 73\delta$,
$|x' - y'| \leq [g]/2 + A + B + 83\delta$,
$|x' - z'| \geq [g]/2 - A - B - C - 156\delta$,

(ii) $\langle x, y\rangle_z \geq \langle x, y\rangle_{x'} + |x' - z'| + |z' - z| - 62\delta$.

Remark. — The conditions on s have the following significance. Consider a geodesic triangle whose apices are x, y and z. By hyperbolicity, this triangle is thin. The part of side $[x, y]$ that is close to $[y, z]$ cannot fellow-travel to a great extent with Y_g (see Figure 1.1). We could have chosen for s the point of $[x, y]$ such that $|x - s| = \langle y, z\rangle_x$ and asked that $\mathfrak{D}(\{s, y\}, Y_g) \leq [g]/2 + A$. However, in Section 3, we will need this more general assumption.

Proof. — The 27δ-neighborhood of γ contains Y_g; thus $\mathfrak{D}(\{y, z\}, \gamma) \geq [g] - C - 54\delta$. Since y' and z' are respective projections of y and z on γ, Proposition 1.11 yields $|y' - z'| \geq [g] - C - 73\delta > 19\delta$. This proves the first inequality of Point (i). Moreover, it gives $\langle y, z\rangle_{x'} \leq 25\delta$.

Upper bound of $|x' - y'|$. We may assume that $|x' - y'| > 19\delta$. Using the previous observation, we get $\langle y, z\rangle_x \leq |x - x'| + \langle y, z\rangle_{x'} \leq |x - x'| + 25\delta$. Our second assumption on s yields $|x - s| \leq |x - x'| + B + 25\delta$. Lemma 1.1 (i) gives

$$\langle s, y\rangle_{x'} \leq \max\left\{|y - x'| - \langle x, s\rangle_y, \langle x, y\rangle_{x'}\right\} + \delta$$

which combined with the triangle inequality leads to

$$\langle s, y\rangle_{x'} \leq \max\left\{|x - s| - |x - x'| + 2\langle x, y\rangle_{x'}, \langle x, y\rangle_{x'}\right\} + \delta$$

Since $|y' - x'| > 19\delta$, we have $\langle x, y\rangle_{x'} \leq 19\delta$ thus $\langle s, y\rangle_{x'} \leq B + 64\delta$. The point y' is a projection of y on γ. Hence, by Proposition 1.11 we have

$$\begin{aligned} |x' - y'| \leq \mathfrak{D}(\{x', y\}, \gamma) + 19\delta &\leq \mathfrak{D}(\{s, y\}, \gamma) + \langle s, y\rangle_{x'} + 19\delta \\ &\leq \mathfrak{D}(\{s, y\}, Y_g) + B + 83\delta. \end{aligned}$$

The second inequality of Point (i) now follows from the first assumption on s.

Lower bound of $|x' - z'|$. The third inequality of Point (i) follows by triangle inequality from the two previous ones.

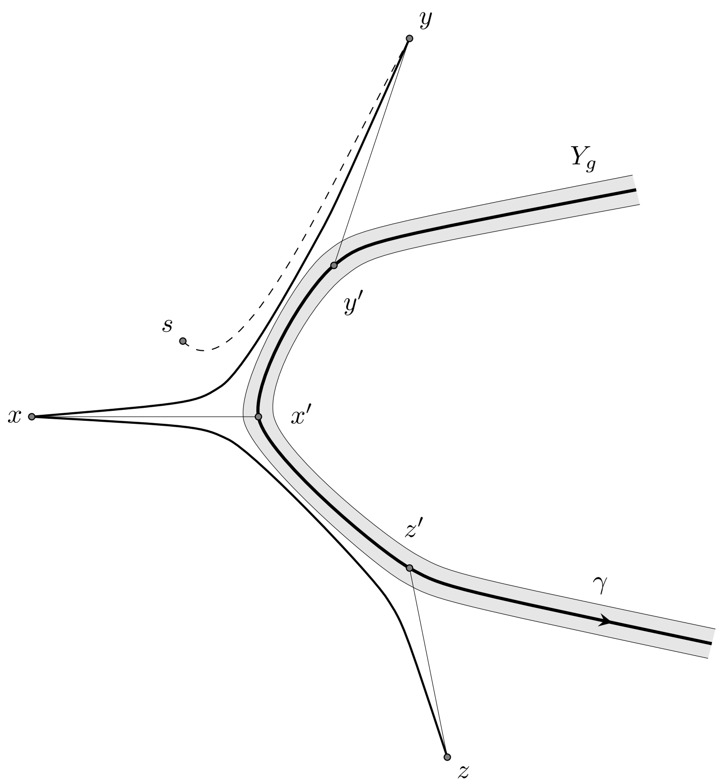


FIGURE 1.1. Significance of the point s

Estimation of $\langle x, y\rangle_z$. As a consequence of Point (i), $|x' - z'| > 19\delta$, thus $\langle x, z\rangle_{x'} \leq 19\delta$. In addition, we proved that $\langle y, z\rangle_{x'} \leq 25\delta$. Consequently,

$$\langle x, y\rangle_z = \langle x, y\rangle_{x'} + |x' - z| - \langle x, z\rangle_{x'} - \langle y, z\rangle_{x'} \geq \langle x, y\rangle_{x'} + |x' - z| - 44\delta.$$

Since z' is a projection of z on γ, we have $|x' - z| \geq |x' - z'| + |z' - z| - 18\delta$, which combined with the previous inequality gives Point (ii). □

PROPOSITION 1.19. — *Let $A, B, C \geq 0$. Let g be an isometry of X such that $[g] > \max\{L_S\delta, 2A + 4B + 2C + 512\delta\}$. Let x, y_1 and y_2 be three points of X. We assume that there exist two points $s_1, s_2 \in X$ such that $\mathfrak{D}(\{s_i, y_i\}, Y_g) \leq [g]/2 + A$ and $|x - s_i| \leq \langle y_1, y_2\rangle_x + B$, for all $i \in \{1, 2\}$. Let γ be a δ-nerve of g. We denote by $x' = \gamma(b)$, $y_1' = \gamma(a_1)$ and $y_2' = \gamma(a_2)$ the respective projections of x, y_1 and y_2 on γ. If $\mathfrak{D}(\{y_1, y_2\}, Y_g) \geq [g] - C$, then the following holds*

(i) *b is between a_1 and a_2,*
(ii) $|y'_1 - y'_2| \geq [g] - C - 73\delta$,

$$[g]/2 - A - B - C - 156\delta \leq |x' - y'_i| \leq [g]/2 + A + B + 83\delta,$$

(iii) $\langle x, y_i\rangle_{x'}, \langle x, y_i\rangle_{y'_i} \leq 19\delta$ *and* $\langle s_i, y_i\rangle_{y'_i} \leq 20\delta$.
(iv) $|\langle y_1, y_2\rangle_x - |x - x'|| \leq 25\delta$.

Remark. — Intuitively, we have Figure 1.2 in mind. The goal of this proposition is to prove that this picture actually corresponds to the reality.

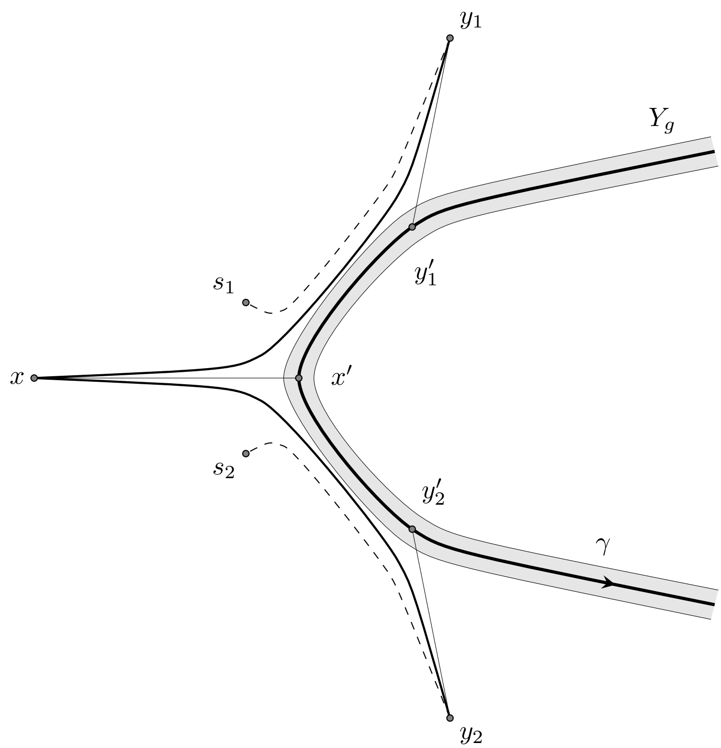


Figure 1.2. Positions of the points y'_1, y'_2 and x'

Proof. — Note that by interchanging, if necessary, y_1 and y_2, we can always assume that $a_1 \leq a_2$. We prove Point (i) by contradiction. Suppose that b does not belong to $[a_1, a_2]$. By symmetry we can assume that a_1 is a point of $[b, a_2]$. Let $a \in [a_1, a_2]$ and put $z = \gamma(a)$. Since x' is a projection of x on γ, $|x - z| \geq |x - x'| + |x' - z| - 18\delta$. However, y'_1 lies on γ between x' and z. Therefore, we obtain $|x-z| \geq |x-y'_1| - 24\delta$. Consequently, y'_1 is a 24δ-projection of x on $\gamma([a_1, a_2])$. By Lemma 1.7, the distance between y_1' and a projection x'' of x on $\gamma([a_1, a_2])$ is at most 86δ. Nevertheless, Proposition 1.18 (i) gives $|y_1' - x''| \geq [g]/2 - A - B - C - 156\delta$. Contradiction. Hence, b belongs to $[a_1, a_2]$. Therefore, Point (ii) follows from Proposition 1.18.

The points x' and y_i' are respective projections of x and y_i on γ. Thus, $\langle x, y_i\rangle_{x'}, \langle x, y_i\rangle_{y_i'} \leq 19\delta$ and $\langle y_1, y_2\rangle_{x'} \leq 25\delta$, which proves in particular the first part of Point (iii). The four-point inequality (1) yields

$$\langle y_1, y_2\rangle_x - \langle y_1, y_2\rangle_{x'} \leq |x - x'| \leq \langle y_1, y_2\rangle_x + \max\left\{\langle x, y_1\rangle_{x'}, \langle x, y_2\rangle_{x'}\right\} + \delta$$

which leads to Point (iv). We are left to show that $\langle s_i, y_i\rangle_{y_i'} \leq 11\delta$. As in the previous proposition, by hyperbolicity we have

$$\langle s_i, y_i\rangle_{y_i'} \leq \max\left\{|x - s_i| - |x - y_i'| + 2\langle x, y_i\rangle_{y_i'}, \langle x, y_i\rangle_{y_i'}\right\} + \delta.$$

However, $\langle x, y_i\rangle_{y_i'} \leq 19\delta$; thus it is sufficient to give an upper bound to $|x - s_i| - |x - y_i'|$. Since x' is a projection of x on γ, one has $|x - y_i'| \geq |x - x'| + |x' - y_i'| - 18\delta$. However, we already proved that $|x - x'| \geq \langle y_1, y_2\rangle_x - 25\delta \geq |x - s_i| - B - 25\delta$. Hence, $|x - y_i'| \geq |x - s_i| + |x' - y_i'| - B - 43\delta$. It follows then from (ii) that

$$|x - s_i| - |x - y_i'| + 2\langle x, y_i\rangle_{y_i'} \leq 2\langle x, y_i\rangle_{y_i'} - |x' - y_i'| + B + 43\delta \leq \langle x, y_i\rangle_{y_i'},$$

which leads to the result. □

1.4. Groups acting on a hyperbolic space. — In this section, we still assume that X is proper geodesic δ-hyperbolic space. We consider a group G acting properly, co-compactly by isometries on X. A subgroup of G is called *elementary* if it is virtually cyclic. Every non-elementary subgroup of G contains a copy of $\mathbf{F}_2$, the free group of rank 2 [13, Chapitre 8, Théorème 37]. Given a loxodromic element g of G, the subgroup of G stabilizing $\{g^-, g^+\} \subset \partial X$ is elementary [5, Chapitre 10, Proposition 7.1].

Definition 1.20. — The *injectivity radius* of G on X is

$$r_{inj}(G, X) = \inf\left\{[g]^\infty \mid g \in G,\ g \text{ loxodromic}\right\}$$

Definition 1.21. — We denote by $\mathcal{A}$ the set of pairs (g, g'), where g and g' are two elements of G whose length is bounded above by $L_S\delta$ and that generate a non-elementary subgroup of G. The parameter $A(G, X)$ is given by

$$A(G, X) = \sup_{(g,g')\in\mathcal{A}} \operatorname{diam}\left(A_g^{+13\delta} \cap A_{g'}^{+13\delta}\right).$$

Proposition 1.22 ([8, Proposition 2.48]). — *Let g and h be two loxodromic elements of G which generate a non-elementary subgroup. If $[g] \leq L_S\delta$, then*

$$\operatorname{diam}\left(A_g^{+13\delta} \cap A_h^{+13\delta}\right) \leq [h] + A(G, X) + 159\delta.$$

Vocabulary. The group G satisfies the *small centralizers hypothesis* if G is non-elementary and every elementary subgroup of G is cyclic.

2. Cone-off over a metric space

In this section we focus on the cone-off over a metric space. Variations on this construction have been used by several authors. A detailed exposition can be found in [8]. Let us fix a positive real number ρ. Its value will be made precise later. It should be thought of as a very large-scale parameter.

2.1. Cone over a metric space. — We review the construction of a cone over a metric space. For additional details, see [2, Chapter I.5]. Let Y be metric space. The *cone of radius ρ over Y*, denoted by $Z_\rho(Y)$ – or simply $Z(Y)$ – is the quotient of $Y \times [0,\rho]$ by the equivalence relation which identifies all the points of the form $(y,0)$. The equivalence class of $(y,0)$ is the *apex* of the cone, denoted by v. We endow Y with a metric characterized as follows. Given any two points $x=(y,r)$ and $x'=(y',r')$ of $Z(Y)$,

$$\cosh|x-x'| = \cosh r\cosh r' - \sinh r\sinh r'\cos\left(\min\left\{\pi, \frac{|y-y'|}{\sinh\rho}\right\}\right).$$

In order to compare the cone $Z(Y)$ and its base Y, we define two maps.

$$\begin{aligned} \iota : Y &\to Z(Y) \\ y &\to (y,\rho) \end{aligned} \qquad\qquad \begin{aligned} p : Z(Y)\setminus\{v\} &\to Y \\ (y,r) &\to y \end{aligned}$$

If y and y' are two points of Y, the distance between $\iota(y)$ and $\iota(y')$ is then given by

$$|\iota(y)-\iota(y')| = \mu\left(|y-y'|\right),$$

where $\mu : \mathbf{R}^+ \to \mathbf{R}^+$ is defined in the following way. For all $t\in\mathbf{R}^+$,

$$\cosh\circ\mu(t) = \cosh^2\rho - \sinh^2\rho\cos\left(\min\left\{\pi, \frac{t}{\sinh\rho}\right\}\right). \tag{5}$$

The next lemma summarizes some properties of μ. Its proof is a straightforward calculus exercise.

Lemma 2.1. — *The map μ is continuous, concave, non-decreasing. Moreover, the following holds.*

(i) *For all $t\geq 0$,*

$$t - \frac{1}{24}\left(1+\frac{1}{\sinh^2\rho}\right)t^3 \leq \mu(t) \leq t.$$

(ii) *For all $t\in[0,\pi\sinh\rho]$, $t\leq \pi\sinh(\mu(t)/2)$.*

(iii) *For every $r,s,t\geq 0$, $\mu(r+s)\leq\mu(r+t)+\mu(t+s)-\mu(t)$.*

Lemma 2.2. — *Let $x=(y,r)$ and $x'=(y',r')$ be two points of $Z(Y)\setminus\{v\}$. If $|x-v|+|v-x'|<|x-x'|$ then*

$$|p(x)-p(x')| \leq \frac{\pi\sinh\rho}{\min\{\sinh r,\sinh r'\}}\sinh\left(\frac{|x-x'|}{2}\right).$$

Proof. — Note that the assumption $|x-v|+|v-x'|<|x-x'|$ exactly means that $|y-y'|<\pi\sinh\rho$. The rest of the proof is a calculus exercise. □

Paths. Let $x=(y,r)$ and $x'=(y',r')$ be two points of $Z(Y)$. Let γ be a path of Y from y to y' whose length $L(\gamma)$ is strictly smaller than $\pi\sinh\rho$. There exists a rectifiable path $\tilde\gamma : I \to Z(Y)\setminus\{v\}$ between x and x' such that $p\circ\tilde\gamma=\gamma$ and whose length satisfies

$$\cosh\left(L\left(\tilde\gamma\right)\right) \leq \cosh r\cosh r' - \sinh r\sinh r'\cos\left(\frac{L(\gamma)}{\sinh\rho}\right).$$

In particular, if Y is a length space, so is $Z(Y)$.

Group action. Let H be a group acting properly by isometries on Y. The action of H on Y extends to $Z(Y)$ by homogeneity: if $x=(y,r)$ is a point of $Z(Y)$ and $h\in H$, then $hx=(hy,r)$. Hence, H acts on $Z(Y)$ by isometries. If Y is not compact, this action may not be proper. The stabilizer of v (i.e. H) may indeed not be finite. Nevertheless, $Z(Y)/H$ still inherits a metric from $Z(Y)$. Let x and x' be two points of $Z(Y)$ and $\bar x$, $\bar x'$ their respective images in $Z(Y)/H$. The formula $|\bar x-\bar x'|=\inf_{h\in H}|x-hx'|$ defines a distance on $Z(Y)/H$. Moreover, the spaces $Z(Y)/H$ and $Z(Y/H)$ are isometric.

Lemma 2.3 ([8, Lemma 4.8]). — *Let $l\geq 2\pi\sinh\rho$. We assume that for every $h\in H\setminus\{1\}$, $[h]\geq l$. Let $x=(y,r)$ and $x'=(y',r')$ be two points of $Z(Y)$. If $|y-y'|\leq l-\pi\sinh\rho$ then $|\bar x-\bar x'|=|x-x'|$.*

2.2. Cone-off over a metric space. — For the rest of Section 2, X denotes a proper geodesic δ-hyperbolic space and $\mathcal{Y}$ a family of closed strongly quasi-convex subsets of X (see Definition 1.5). The goal is to define a new space $\dot X$ by attaching for every $Y\in\mathcal{Y}$ a cone over of Y on X. Note that all the arguments may be adapted if X is just a length space. However, the proofs would be rather technical.

Let $Y\in\mathcal{Y}$. We denote by $|\,.\,|_Y$ the length metric on Y induced by the restriction of $|\,.\,|_X$ to Y. We write $Z(Y)$ for the cone of radius ρ over $(Y,|\,.\,|_Y)$. It comes with a natural map $\iota : Y\hookrightarrow Z(Y)$ as defined in Section 2.1.

Definition 2.4. — The *cone-off of radius ρ over X relative to $\mathcal{Y}$* denoted by $\dot X_\rho(\mathcal{Y})$ (or simply $\dot X$) is obtained by attaching for every $Y\in\mathcal{Y}$, the cone $Z(Y)$ on X along Y according to ι.

In other words, the space $\dot X$ is the quotient of the disjoint union of X and all the $Z(Y)$ (where Y runs over $\mathcal{Y}$) by the equivalence relation which identifies every point $y\in Y$ with its image $\iota(y)\in Z(Y)$. By abuse of notation, we use the same letter to designate a point of this disjoint union and its image in $\dot X$. Let us denote by $v(\mathcal{Y})$ the set of all apices from the cones in $\dot X$. There exists a map $p\colon \dot X\setminus v(\mathcal{Y})\to X$, called the *radial projection*, with the following properties. The restriction of p to X is the identity. Let $Y\in\mathcal{Y}$. If v stands for the apex of $Z(Y)$, then p restricted to $Z(Y) \setminus \{v\}$ is the map $p\colon Z(Y) \setminus \{v\} \to Y$ defined in Section 2.1.

Metric on the cone-off. For the moment, $\dot{X}$ is just a set of points. We now define a metric on $\dot{X}$ and recall its main properties. Note that we did not require the attachment maps ι to be isometries. We endow the disjoint union of X and all the $Z(Y)$ (where Y runs over $\mathcal{Y}$) with the distance induced by $|\,.\,|_X$ and $|\,.\,|_{Z(Y)}$. This metric is not finite: the distance between two points in distinct components is infinite. Let x and x' be two points of $\dot{X}$. We define $\|x-x'\|$ to be the infimum over the distances between two points in the previous disjoint union whose images in $\dot{X}$ are x and x', respectively.

(i) Let $Y \in \mathcal{Y}$. If $x \in Z(Y) \setminus \iota(Y)$ and $x' \notin Z(Y)$, then $\|x-x'\| = +\infty$. In particular, $\|\,.\,\|$ is not a distance on $\dot{X}$ (it does not satisfy the triangle inequality).

(ii) Let x and x' be two points of X. Using the properties of μ (5) we get
$$\mu\left(|x-x'|_X\right) \leq \|x-x'\| \leq |x-x'|_X.$$
Moreover, if there is $Y \in \mathcal{Y}$ such that $x, x' \in Y$, then $\|x-x'\| \leq \mu\left(|x-x'|_X\right) + 8\delta$. This is a consequence of the strong quasi-convexity of Y.

Let x and x' be two points of $\dot{X}$. A *chain* between x and x' is a finite sequence $C = (z_1, \dots, z_m)$ of points of $\dot{X}$ such that $z_1 = x$ and $z_m = x'$. Its length, denoted by $l(C)$, is
$$l(C) = \|z_1 - z_2\| + \dots + \|z_{m-1} - z_m\|.$$
The following map endows $\dot{X}$ with a length metric [8, Proposition 5.10].
$$\begin{array}{rcl} \dot{X} \times \dot{X} & \to & \mathbf{R}_+ \\ (x,x') & \to & |x-x'|_{\dot{X}} = \inf\left\{l(C) \middle| C \text{ chain between } x \text{ and } x'\right\}. \end{array}$$

Note that given a chain between two points of X, one can always find a shorter chain joining the same extremities, whose points belong to X. Just apply the triangle inequality in the disjoint union of X and the cones $Z(Y)$. Therefore, in the rest of the section, we will always approximate the distance between two points of X by chains whose points lie in X.

For every $Y \in \mathcal{Y}$, the natural map $Z(Y) \to \dot{X}$ is a 1-Lipschitz embedding. The same holds for the map $X \to \dot{X}$. The next lemmas detail the relationship between the metrics of these spaces.

Lemma 2.5. — [8, Lemma 5.8] *For every $x, x' \in X$, $\mu\left(|x-x'|_X\right) \leq |x-x'|_{\dot{X}} \leq |x-x'|_X$.*

Lemma 2.6. — [8, Lemma 5.7] *Let $Y \in \mathcal{Y}$. Let $x \in Z(Y) \setminus \iota(Y)$. Let $d(x,Y)$ be the distance between x and $\iota(Y)$ computed with $|\,.\,|_{Z(Y)}$. For all $x' \in \dot{X}$, if $|x-x'|_{\dot{X}} < d(x,Y)$ then x' belongs to $Z(Y)$. Moreover, $|x-x'|_{\dot{X}} = |x-x'|_{Z(Y)}$.*

REMARK. — If v stands for the apex of the cone $Z(Y)$, then the previous lemma implies that $Z(Y)\setminus\iota(Y)$ is exactly the ball of $\dot{X}$ of center v and radius ρ.

LEMMA 2.7. — *Let $r \in (0,\rho]$. Let x and x' be two points of $\dot{X}$. Assume that for every $v \in v(\mathcal{Y})$, $\min\{|v-x|_{\dot{X}}, |v-x'|_{\dot{X}}\} \geq r$. If $|x-x'|_{\dot{X}} < 2r$, then*

$$|p(x)-p(x')| \leq \frac{\pi \sinh \rho}{\sinh r} \sinh\left(\frac{|x-x'|_{\dot{X}}}{2}\right).$$

Proof. — Let $\varepsilon > 0$ such that $|x-x'|_{\dot{X}} + \varepsilon < 2r$. Let $C = (z_0, \dots, z_m)$ be a chain of points between x and x' such that $l(C) \leq |x-x'|_{\dot{X}} + \varepsilon$. Without loss of generality we can assume that every point of C but z_0 and z_m belongs to X. In particular, every point of C is at a distance at least r from the apices of $\dot{X}$. Let $j \in [\![0, m-1]\!]$. The goal is to compare $|p(z_j)-p(z_{j+1})|$ and $\|z_j - z_{j+1}\|$. We distinguish two cases.

Let $\eta > 0$ such that $|x-x'|_{\dot{X}} + \varepsilon + \eta < 2r$. Assume first that z_j and z_{j+1} belong to X and $\|z_j - z_{j+1}\| = |z_j - z_{j+1}|$. Then, obviously, $\|z_j - z_{j+1}\| \leq |z_j - z_{j+1}|$. Otherwise there exists $Y \in \mathcal{Y}$ such that z_j and z_{j+1} belong to $Z(Y)$ and $|z_j - z_{+1}|_{Z(Y)} \leq \|z_j - z_{+1}\| + \eta$. In particular, $|z_j - z_{+1}|_{Z(Y)} < 2r$. If v stands for the apex of $Z(Y)$, then $|z_j - v|_{Z(Y)} + |v - z_{j+1}|_{Z(Y)} < |z_j - z_{j+1}|_{Z(Y)}$. By Lemma 2.2,

$$|p(z_j)-p(z_{j+1})| \leq |p(z_j)-p(z_{j+1})|_Y \leq \frac{\pi \sinh \rho}{\sinh r} \sinh\left(\frac{\|z_j - z_{+1}\| + \eta}{2}\right).$$

According to the triangle inequality

$$\begin{aligned}|p(x)-p(x')| \leq \sum_{j=0}^{m-1} |p(z_j)-p(z_{j+1})| &\leq \frac{\pi \sinh \rho}{\sinh r} \sum_{j=0}^{m-1} \sinh\left(\frac{\|z_j - z_{+1}\| + \eta}{2}\right) \\ &\leq \frac{\pi \sinh \rho}{\sinh r} \sinh\left(\frac{l(C)}{2} + \frac{m\eta}{2}\right).\end{aligned}$$

This inequality holds for every sufficiently small $\eta > 0$; hence

$$|p(x)-p(x')| \leq \frac{\pi \sinh \rho}{\sinh r} \sinh\left(\frac{l(C)}{2}\right) \leq \frac{\pi \sinh \rho}{\sinh r} \sinh\left(\frac{|x-x'|_{\dot{X}}}{2} + \frac{\varepsilon}{2}\right).$$

This last inequality holds again for every sufficiently small ε, which completes the proof. □

Large-scale geometry of the cone-off. In order to control the large-scale geometry of $\dot{X}$ we need to measure the length that two distinct elements of $\mathcal{Y}$ can fellow-travel. This can be achieved with the parameter $\Delta(\mathcal{Y})$ defined as follows.

$$\Delta(\mathcal{Y}) = \sup\left\{\operatorname{diam}\left(Y_1^{+5\delta} \cap Y_2^{+5\delta}\right) \,\middle|\, Y_1 \neq Y_2 \in \mathcal{Y}\right\}.$$

Theorem 2.8 ([8, Proposition 6.4]). — *There exist positive numbers δ_0, Δ_0 and ρ_0 with the following property. Let X be a δ-hyperbolic length space with $\delta \leq \delta_0$. Let $\mathcal{Y}$ be a family of strongly quasi-convex subsets of X with $\Delta(\mathcal{Y}) \leq \Delta_0$. Let $\rho \geq \rho_0$. Then the cone-off $\dot{X}_\rho(\mathcal{Y})$ of radius ρ over X relative to $\mathcal{Y}$ is $\dot{\delta}$-hyperbolic with $\dot{\delta} \leq 900\boldsymbol{\delta}$.*

Remark. — It is important to note that in this statement, the constants δ_0, Δ_0 and ρ_0 do not depend on X or $\mathcal{Y}$. Moreover δ_0 and Δ_0 (respectively ρ_0) can be chosen as arbitrarily small (respectively large).

2.3. Shortening chains. — Our goal is now to compare the geometry of $\dot{X}$ and X. In [12], T. Delzant and M. Gromov proved that the natural map $X \to \dot{X}$ restricted to any ball of radius 1000δ is a quasi-isometric embedding. For our purpose we need to compare X and $\dot{X}$ at a larger scale. In particular, we must take into account paths passing through the apices of $\dot{X}$.

Coarsely speaking, we show that the radial projection by $p\colon \dot{X} \setminus v(\mathcal{Y}) \to X$ preserves the shapes. Among other facts, we explain the following. Let x and x' be two points of X and γ a quasi-geodesic of $\dot{X}$ joining them. Then, p maps γ to a path contained in the neighborhood of any geodesic of X joining x and x' (see Proposition 2.15). To prove this statement we proceed in two steps. Let x, y, z and t be four points of X. If $\langle x, t\rangle_y$ or $\langle x, t\rangle_z$ is large (compared to $\Delta(\mathcal{Y})$ and δ), we first explain how to shorten the chain $C = (x, y, z, t)$ (see Proposition 2.12). Then we combine this fact with the stability of discrete quasi-geodesics to show that the points of a chain between x and x' whose length approximates $|x - x'|_{\dot{X}}$ lie in the neighborhood of any geodesic joining x to x' (see Proposition 2.13).

Lemma 2.9. — *Let $x, x' \in X$. Let p and p' be two points on a geodesic $[x, x']$. There exists a chain C joining p to p' such that $l(C) \leq \|x - x'\| + 16\delta$.*

Proof. — If $\|x - x'\| = |x - x'|$, then by triangle inequality, the chain $C = (p, p')$ works. Thus we can assume that there exists $Y \in \mathcal{Y}$ such that $x, x' \in Y$. The subset Y being 2δ-quasi-convex, there are $q, q' \in Y$ such that $|p - q| \leq 2\delta$ and $|p' - q'| \leq 2\delta$. We choose for C the chain $C = (p, q, q', p')$. Its length is bounded above by $l(C) \leq \mu(|q - q'|) + 12\delta$. However, by the triangle inequality, $|q - q'| \leq |x - x'| + 4\delta$. Consequently, we get

$$l(C) \leq \mu(|x - x'|) + 16\delta \leq \|x - x'\| + 16\delta. \qquad \square$$

Lemma 2.10. — *Let $x, y, z \in X$. Let p be a point on a geodesic $[x, y]$ and q a point on a geodesic $[y, z]$. We assume that there is $Y \in \mathcal{Y}$ such that $x, y \in Y$, but there is no $Y' \in \mathcal{Y}$ such that $x, y, z \in Y'$. Then there exists a chain C joining p to z satisfying*

$$l(C) \leq 2|p - q| + \|y - z\| - |y - q| + \Delta(\mathcal{Y}) + 49\delta.$$

Proof. — We distinguish two cases. Assume first that there exists $Y' \in \mathcal{Y}$ such that $y, z \in Y'$. According to our hypothesis we necessarily have $Y \neq Y'$. It follows from Lemma 1.12 that

$$\mathfrak{D}(\{x,y\},\{y,z\}) \leq \mathfrak{D}(Y,Y') \leq \Delta(Y) + 33\delta,$$

i.e., $\langle x,z\rangle_y \leq \Delta(Y) + 33\delta$. Using the triangle inequality we get

$$|y-q| \leq \langle x,z\rangle_y + |p-q| \leq |p-q| + \Delta(Y) + 33\delta. \tag{6}$$

By Lemma 2.9, there exists a chain C_0 joining q to z whose length is at most $\|y-z\| + 16\delta$. We obtain C by adding p at the beginning of C_0. It satisfies $l(C) \leq |p-q| + \|y-z\| + 16\delta$. Combined with (6) we get the required inequality.

Assume now that $\|y-z\| = |y-z|$. Then $\|q-z\| \leq \|y-z\| - |y-q|$. We choose for C the chain $C = (p,q,z)$ which satisfies $l(C) \leq |p-q| + \|y-z\| - |y-q|$. □

Lemma 2.11. — *Let $x,y,z,t \in X$. If there exists $Y \in \mathcal{Y}$ such that $x,t \in Y$, then*

$$\|x-t\| \leq \|x-y\| + \|y-z\| + \|z-t\| - \mu\left(\max\left\{\langle x,t\rangle_y, \langle x,t\rangle_z\right\}\right) + 8\delta.$$

Proof. — Since x and t are in Y, $\|x-t\| \leq \mu(|x-t|) + 8\delta$. Applying Lemma 2.1 (iii) we get

$$\mu(|x-t|) \leq \mu(|x-y|) + \mu(|y-t|) - \mu(\langle x,t\rangle_y).$$

However, by triangle inequality, $\mu(|y-t|) \leq \mu(|y-z|) + \mu(|z-t|)$. Consequently,

$$\|x-t\| \leq \|x-y\| + \|y-z\| + \|z-t\| - \mu(\langle x,t\rangle_y) + 8\delta$$

By symmetry, we have the same inequality with $\langle x,t\rangle_z$ instead of $\langle x,t\rangle_y$. □

Proposition 2.12. — *Let $x,y,z,t \in X$. There exists a chain C joining x to t such that*

$$l(C) \leq \|x-y\| + \|y-z\| + \|z-t\| - \mu\left(\max\left\{\langle x,t\rangle_y, \langle x,t\rangle_z\right\}\right) + 2\Delta(\mathcal{Y}) + 132\delta$$

Proof. — If there is $Y \in \mathcal{Y}$ such that $x,t \in Y$, Lemma 2.11 says that the chain $C = (x,t)$ works. Therefore, from now on we assume that there is no such $Y \in \mathcal{Y}$. By hyperbolicity,

$$|x-z| + |y-t| \leq \max\left\{|x-y| + |z-t|, |x-t| + |y-z|\right\} + 2\delta \tag{7}$$

Part 1. Assume first that the maximum is achieved by $|x-t| + |y-z|$ (see Figure 2.1). In particular, it follows that $\langle x,t\rangle_z \leq \langle y,t\rangle_z + \delta$ and $\langle x,t\rangle_y \leq \langle x,z\rangle_y + \delta$. Moreover, $|y-z| \geq \langle x,t\rangle_y + \langle x,t\rangle_z - \delta$. Let $p \in [x,y]$, $q,s \in [y,z]$ and $r \in [z,t]$ be four points such that

$$|y-p| = |y-q| = \max\{0, \langle x,t\rangle_y - \delta\},$$
$$|z-r| = |z-s| = \max\{0, \langle x,t\rangle_z - \delta\}.$$

By Lemma 1.1 (iii), $|p-q| \leq 4\delta$ and $|r-s| \leq 4\delta$. Furthermore, $|y-z| \geq |y-q| + |s-z|$. Hence, the points y, q, s and z are ordered in this way on $[y,z]$.

We need to distinguish several cases depending on whether or not the points x, y, z and t lie in a quasi-convex $Y \in \mathcal{Y}$. In each case we implicitly exclude the previous ones.

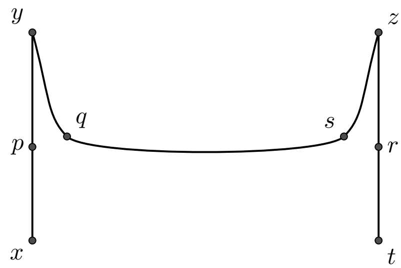


FIGURE 2.1. Shortening a four-point chain – Part 1

Case 1.1. There exist $Y, Y' \in \mathcal{Y}$ such that $x, y, z \in Y$ and $y, z, t \in Y'$. According to our assumption at the beginning of the proof $Y \neq Y'$. Since y and z belong to Y and Y', they satisfy $|y-z| \leq \Delta(\mathcal{Y})$. Consequently, $\langle x, t\rangle_y + \langle x, t\rangle_z \leq \Delta(\mathcal{Y}) + \delta$. We choose the chain $C = (x, y, z, t)$. Thus

$$l(C) \leq \|x-y\| + \|y-z\| + \|z-t\| - \langle x,t\rangle_y - \langle x,t\rangle_z + \Delta(\mathcal{Y}) + \delta.$$

Case 1.2. There exists $Y \in \mathcal{Y}$ such that $x, y, z \in Y$. The subset Y being 2δ-quasi-convex, there exists a point $s' \in Y$ such that $|s - s'| \leq 2\delta$. Hence, $\|x - s'\| \leq \mu(|x-s|) + 10\delta$. By Lemma 2.1 (iii), we get

$$\begin{aligned}\mu\left(|x-s|\right) &\leq \mu\left(|x-p| + |q-s|\right) + 4\delta \\ &\leq \mu\left(|x-p| + |p-y|\right) + \mu\left(|y-q| + |q-s|\right) - \mu(\langle x,t\rangle_y) + 5\delta.\end{aligned}$$

It follows that $\|x - s'\| \leq \|x-y\| + \|y-z\| - \mu(\langle x,t\rangle_y) + 15\delta$. On the other hand, by Lemma 2.10, there exists a chain C_0 joining s to t such that

$$l(C_0) \leq \|z-t\| - |z-r| + \Delta(\mathcal{Y}) + 57\delta \leq \|z-t\| - \langle x,t\rangle_z + \Delta(\mathcal{Y}) + 58\delta.$$

We obtain C by adding x and s' at the beginning of C_0. Its length satisfies

$$l(C) \leq \|x-y\| + \|y-z\| + \|z-t\| - \mu(\langle x,t\rangle_y) - \langle x,t\rangle_z + \Delta(\mathcal{Y}) + 75\delta.$$

Case 1.3. There exists $Y \in \mathcal{Y}$ such that $y, z, t \in Y$. This case is just the symmetric of the previous one.

Case 1.4. There exists $Y \in \mathcal{Y}$ such that $y, z \in Y$. By Lemma 2.9 there exists a chain C_0 joining q to s whose length is at most $\|y - z\| + 16\delta$. Applying Lemma 2.10, there is a chain C_- (respectively C_+) joining x to q (respectively

s to t) such that

$$l(C_-) \leq \|x-y\| - |y-p| + \Delta(\mathcal{Y}) + 57\delta \quad \leq \|x-y\| - \langle x,t\rangle_y + \Delta(\mathcal{Y}) + 58\delta,$$
$$l(C_+) \leq \|z-t\| - |z-r| + \Delta(\mathcal{Y}) + 57\delta \quad \leq \|z-t\| - \langle x,t\rangle_z + \Delta(\mathcal{Y}) + 58\delta.$$

Concatenating C_-, C_0 and C_+, we obtain a chain C such that

$$l(C) \leq \|x-y\| + \|y-z\| + \|z-t\| - \langle x,t\rangle_y - \langle x,t\rangle_z + 2\Delta(\mathcal{Y}) + 132\delta.$$

Case 1.5. This is the last case of Part 1. Negating the previous one, there is no $Y \in \mathcal{Y}$ such that $y,z \in Y$. In particular, $\|y-z\| = |y-z|$. Hence,

$$\|q-s\| \leq \|y-z\| - |y-q| - |z-s| \leq \|y-z\| - \langle x,t\rangle_y - \langle x,t\rangle_z + 2\delta.$$

We put $C_0 = (q,s)$. According to Lemma 2.9, there is a chain C_- (respectively C_+) joining x to p (respectively r to t) whose length is at most $\|x-y\| + 16\delta$ (respectively $\|t-z\| + 16\delta$). Concatenating C_-, C_0 and C_+, we obtain a chain C such that

$$l(C) \leq \|x-y\| + \|y-z\| + \|z-t\| - \langle x,t\rangle_y - \langle x,t\rangle_z + 42\delta.$$

Part 2. Assume now that the maximum in (7) is achieved by $|x-y|+|z-t|$ (see Figure 2.2). It follows that $\langle x,y\rangle_t \leq \langle y,z\rangle_t$. We assume that $\langle x,t\rangle_y \geq \langle x,t\rangle_z$ (the other case is symmetric). We denote by p and q two points of X lying on $[x,y]$ and $[t,y]$, respectively, such that $|y-p| = |y-q| = \langle x,t\rangle_y$. By Lemma 1.1 (iii) $|p-q| \leq 4\delta$. On the other hand, $|t-q| = \langle x,y\rangle_t \leq \langle y,z\rangle_t$. Consequently, if r is the point of $[z,t]$ such that $|t-r| = \langle x,y\rangle_t$, then $|q-r| \leq 4\delta$. Thus, $|p-r| \leq 8\delta$. Moreover, the triangle inequality leads to $\langle x,t\rangle_y \leq |z-y| + |z-t| - \langle x,y\rangle_t$; thus $\langle x,t\rangle_y \leq |y-z| + |z-r|$. According to Lemma 2.9, there exists a chain C_- (respectively C_+) joining x to p (respectively r to t) such that $l(C_-) \leq \|x-y\| + 16\delta$ (respectively $l(C_+) \leq \|z-t\| + 16\delta$). As previously, we must distinguish several cases.

Case 2.1. There exist $Y, Y' \in \mathcal{Y}$ such that $x,y \in Y$ and $z,t \in Y'$. According to our assumption at the beginning of the proof, $Y \neq Y'$. It follows from Lemma 1.12 that

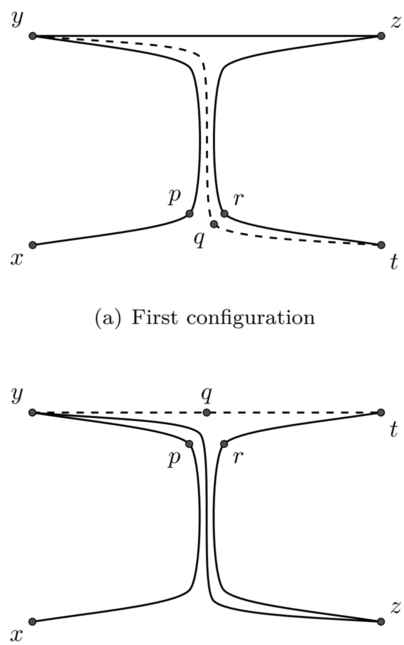
$$\mathfrak{D}(\{x,y\},\{z,t\}) \leq \mathfrak{D}(Y,Y') \leq \Delta(\mathcal{Y}) + 33\delta,$$

thus $\langle x,t\rangle_y \leq |y-z| + \Delta(\mathcal{Y}) + 33\delta$. It follows that $\mu(\langle x,t\rangle_y) \leq \|y-z\| + \Delta(\mathcal{Y}) + 33\delta$. By concatenating C_- and C_+, we obtain a chain whose length satisfies

$$l(C) \leq \|x-y\| + \|y-z\| + \|z-t\| - \mu(\langle x,t\rangle_y) + \Delta(\mathcal{Y}) + 73\delta.$$

Case 2.2. There exists $Y \in \mathcal{Y}$ such that $x,y \in Y$. In this case, $\|z-t\| = |z-t|$; thus $\|r-t\| \leq \|z-t\| - |z-r|$. We obtain C by adding r and t at the end of C_-. This new chain satisfies

$$l(C) \leq \|x-y\| + \|z-t\| - |z-r| + 24\delta.$$



(a) First configuration

(b) Second configuration

FIGURE 2.2. Shortening a four-point chain – Part 2

However, we proved that $\langle x,y\rangle_t \leq |y-z| + |z-r|$. In particular, $\mu(\langle x,y\rangle_t) \leq \|y-z\| + |z-r|$. Consequently,

$$l(C) \leq \|x-y\| + \|y-z\| + \|z-t\| - \mu(\langle x,t\rangle_y) + 24\delta.$$

Case 2.3. This is the last case of Part 2. In particular, $\|x-y\| = |x-y|$. It follows that $\|x-p\| \leq \|x-y\| - |y-p|$, i.e. $\|x-p\| \leq \|x-y\| - \langle x,t\rangle_y$. We obtain C by adding x and p at the beginning of C_+. It satisfies

$$l(C) \leq \|x-y\| + \|z-t\| - \langle x,y\rangle_t + 24\delta.$$

In all cases, we obtained the desired inequality. □

PROPOSITION 2.13. — *Let $\varepsilon > 0$. There exist positive numbers δ_0, Δ_0, ρ_0 and η which depend only on ε with the following property. Assume that $\rho \geq \rho_0$, $\delta \leq \delta_0$ and $\Delta(\mathcal{Y}) \leq \Delta_0$. Let $x, x' \in X$. Let C be a chain of points of X joining x to x'. If $l(C) \leq |x-x'|_{\dot X} + \eta$, then every point y of C satisfies $\langle x,x'\rangle_y \leq \varepsilon$.*

Proof. — We start by defining the constants δ_0, Δ_0, ρ_0 and η. Given $\rho \in \mathbf{R}_+^*$, the function μ defined in (5) satisfies

$$\forall t \in \mathbf{R}_+, \quad \mu(t) \geq t - \frac{1}{24}\left(1 + \frac{1}{\sinh^2 \rho}\right) t^3$$

Thus, there exist $\rho_0 \geq 0$, and $t_0 > 0$ with the following property: for every $\rho \geq \rho_0$, for every $t \in [0, t_0]$, $\mu(t) \geq t/2$. Up to increasing the value of ρ_0 we can also require that $\sinh \rho_0 \geq 100$. We now fix $\rho \geq \rho_0$. Since μ is non decreasing, for every $t \in \mathbf{R}_+$, if $\mu(t) < \mu(t_0)$, then $t \leq 2\mu(t)$. Recall that L_S is the parameter given by the stability of discrete quasi-geodesics (see the remark following Proposition 1.4). We put $l = 300$. We choose $\delta_0 > 0$, $\Delta_0 > 0$ and $\eta > 0$ such that

(i) $2\Delta_0 + 2(L_S + l)^3\delta_0^3 + 132\delta_0 + \eta < \mu(t_0)$,
(ii) $4\Delta_0 + 4(L_S + l)^3\delta_0^3 + 264\delta_0 + 2\eta \leq (l-1)\delta_0$,
(iii) $(2L_S + 3l + 5)\delta_0 \leq \varepsilon$.

From now on we assume that $\delta \leq \delta_0$ and $\Delta(\mathcal{Y}) \leq \Delta_0$. In particular, X is δ_0-hyperbolic. Let $x, x' \in X$ and $C = (z_0, \dots, z_n)$ be a chain of points of X joining x to x' such that $l(C) \leq |x - x'|_{\dot{X}} + \eta$. Note that for every $i \leq j$, the length of the subchain $(z_i, z_{i+1}, \dots, z_{j-1}, z_j)$ is at most $|z_j - z_i|_{\dot{X}} + \eta$.

We now extract a subchain $C' = (x_0, y_0, x_1, y_1, \dots y_{p-1}, x_p)$ of C. We proceed by induction as follows. First, put $x_0 = z_0$. Assume now that $x_j = z_k$ is defined. Let $k' \in [\![k+1, n]\!]$ be the smallest integer such that $|z_{k'} - z_k| > 2(L_S + l)\delta_0$. If such an integer does not exist, we simply let $k' = n$. We distinguish two cases

- *Assume that for every $j \in [\![k, k'-1]\!]$, $|z_j - z_k| \leq (L_S + l)\delta_0$.* Then we let $y_j = z_{k'-1}$ and $x_{j+1} = z_{k'}$.
- *Assume that there exists $i \in [\![k, k'-1]\!]$, $|z_i - z_k| > (L_S + l)\delta_0$.* In this case we take the largest i with this property. Then we let $y_j = z_k$ and $x_{j+1} = z_i$.

Note that for every $i \in [\![0, p-2]\!]$, we have $|x_{i+1} - y_i| > (L_S + l)\delta_0$. Moreover, every point of C is $2(L_S + l)\delta_0$-close to a point of $\{x_0, x_1, \dots, x_p\}$. We prove as in [8, Proposition 5.11] the following fact.

Claim 1. We have the following inequalities

(i) For every $j \in [\![0, p-1]\!]$,

$$\|x_j - y_j\| + \|y_j - x_{j+1}\| \leq |x_{j+1} - x_j|_{\dot{X}} + (L_S + l)^3\delta_0^3 + \eta.$$

(ii) For every $j \in [\![0, p-2]\!]$,

$$\|y_j - x_{j+1}\| + \|x_{j+1} - y_{j+1}\| + \|y_{j+1} - x_{j+2}\| \leq |x_{j+2} - y_j|_{\dot{X}} + 2(L_S + l)^3\delta_0^3 + \eta.$$

We continue with two claims regarding Gromov's products of consecutive points in C'.

Claim 2. For all $j \in [\![0, p-1]\!]$, we have $\langle x_j, x_{j+1}\rangle_{y_j} \leq l\delta_0$. Let $j \in [\![0, p-1]\!]$. Applying Proposition 2.12 with the points x_j, y_j, y_j and x_{j+1}, we obtain a chain joining x_j to x_{j+1} whose length is at most

$$\|x_j - y_j\| + \|y_j - x_{j+1}\| - \mu(\langle x_j, x_{j+1}\rangle_{y_j}) + 2\Delta(\mathcal{Y}) + 132\delta_0.$$

Combined with Claim 1, it yields

$$\mu(\langle x_j, x_{j+1}\rangle_{y_j}) \leq 2\Delta_0 + (L_S + l)^3\delta_0^3 + 132\delta_0 + \eta < \mu(t_0)$$

It follows from the definitions of t_0, δ_0, Δ_0 and η that

$$\langle x_j, x_{j+1}\rangle_{y_j} \leq 4\Delta_0 + 2(L_S + l)^3\delta_0^3 + 264\delta_0 + 2\eta \leq l\delta_0.$$

Claim 3. For all $j \in [[0, p-2]]$, we have $\langle x_j, x_{j+2}\rangle_{x_{j+1}} \leq l\delta_0$. Let $j \in [[0, p-2]]$. According to Proposition 2.12 applied to the points y_j, x_{j+1}, y_{j+1} and x_{j+2}, there exists a chain joining y_j to x_{j+2} whose length is at most

$$\|y_j - x_{j+1}\| + \|x_{j+1} - y_{j+1}\| + \|y_{j+1} - x_{j+2}\| - \mu(\langle y_j, x_{j+2}\rangle_{x_{j+1}}) + 2\Delta(\mathcal{Y}) + 132\delta_0.$$

Using the same argument as in Claim 2, we obtain that

$$\langle y_j, x_{j+2}\rangle_{x_{j+1}} \leq 4\Delta_0 + 4(L_S + l)^3\delta_0^3 + 264\delta_0 + 2\eta \leq (l-1)\delta_0.$$

It follows from the four-point inequality (1) that

$$\min\left\{\langle y_j, x_j\rangle_{x_{j+1}}, \langle x_j, x_{j+2}\rangle_{x_{j+1}}\right\} \leq \langle y_j, x_{j+2}\rangle_{x_{j+1}} + \delta_0 \leq l\delta_0$$

However, using Claim 2,

$$\langle y_j, x_j\rangle_{x_{j+1}} = |x_{j+1} - y_j| - \langle x_j, x_{j+1}\rangle_{y_j} > (L_S + l)\delta_0 - l\delta_0 > l\delta_0.$$

Consequently, $\langle x_j, x_{j+2}\rangle_{x_{j+1}} \leq l\delta_0$.

Claim 4. For all $j \in [\![0, p-2]\!]$ we have $|x_{j+1} - x_j| > L_S\delta_0$. The triangle inequality combined with Claim 2 gives

$$|x_{j+1} - x_j| \geq |x_{j+1} - y_j| - \langle x_j, x_{j+1}\rangle_{y_j} > (L_S + l)\delta_0 - l\delta_0.$$

Claims 3 and 4 exactly say that $x_0, x_1, \dots, x_p$ satisfies the assumptions of the stability of discrete quasi-geodesics (see Proposition 1.4). Therefore, for every $j \in [\![0, p]\!]$, $\langle x_0, x_p\rangle_{x_j} \leq (l+5)\delta_0$ i.e., $\langle x, x'\rangle_{x_j} \leq (l+5)\delta_0$. Nevertheless, we noted that every point of C is $2(L_S + l)\delta_0$-close to some x_j. Thus, for any point z_j of C, $\langle x, x'\rangle_{z_j} \leq (2L_S + 3l + 5)\delta_0 \leq \varepsilon$. □

2.4. Paths in a cone-off. — Let $x, x' \in X$. We now use the work of the previous section to produce a quasi-geodesic of $\dot{X}$ between x and x' whose radial projection stays close to any geodesic of X with the same endpoints.

Lemma 2.14. — *Let x and x' be two points of X. For all $\eta > 0$, there exists a path $\gamma : I \to \dot{X}$ between them with the following properties. Its length $L(\gamma)$ is smaller than $\|x - x'\| + \eta$. Moreover, for all $t \in I$, if $\gamma(t)$ does not belong to $v(\mathcal{Y})$, then $\langle x, x'\rangle_{p\circ\gamma(t)} \leq 5\delta$.*

Proof. — If $\|x-x'\| = |x-x'|_X$, then any geodesic of X joining x to x' works. Therefore, we can assume that $\|x-x'\| \neq |x-x'|_X$. Let $\varepsilon > 0$. By definition of $\| \, . \, \|$, there exists $Y \in \mathcal{Y}$ such that $x, x' \in Y$ and $|x-x'|_{Z(Y)} < \|x-x'\| + \varepsilon$. We distinguish two cases.

Case 1. Assume that $|x-x'|_Y \geq \pi \sinh \rho$. Then $|x-x'|_{Z(Y)} = 2\rho$. Thus, x and x' are joined by a geodesic $\gamma : I \to Z(Y)$ that passes through the apex v of $Z(Y)$. Its length (as a path of $Z(Y)$) is 2ρ. Moreover, for all $t \in I$, if $\gamma(t) \neq v$, then $p \circ \gamma(t) \in \{x, x'\}$.

Case 2. Assume that $|x-x'|_Y < \pi \sinh \rho$. The space $(Y, | \, . \, |_Y)$ is a length space. Thus, there exists a path $\nu : I \to Y$ parametrized by arc length between x and x' whose length is less than $\min\{|x-x'|_Y + \varepsilon, \pi \sinh \rho\}$. Hence, there exists a path $\gamma : I \to Z(Y) \setminus \{v\}$ between x and x' such that $p \circ \gamma = \nu$ and its length $L(\gamma)$ (as a path of $Z(Y)$) satisfies

$$L(\gamma) \leq \mu\left(L\left(\nu\right)\right) \leq \mu\left(|x-x'|_Y + \varepsilon\right) \leq \|x-x'\| + 2\varepsilon$$

However, Y is strongly quasi-convex. It follows that for all $y, y' \in Y$, $|y-y'|_X \leq |y-y'|_Y \leq |y-y'|_X + 8\delta$. Consequently, as a path of X, ν is a $(1, 8\delta+\varepsilon)$-quasi-geodesic. In particular, for every $t \in I$, $\langle x, x' \rangle_{\nu(t)} \leq 4\delta + \varepsilon/2$.

Hence, we have build a path $\gamma : I \to Z(Y)$, whose length (as a path of $Z(Y)$) is smaller than $\|x-x'\| + 2\varepsilon$ and such that for all $t \in I$, if $\gamma(t) \neq v$, then $\langle x, x' \rangle_{\nu(t)} \leq 4\delta + \varepsilon/2$. However, the map $Z(Y) \to \dot{X}$ is 1-Lipschitz. It follows that the length of γ as a path of $\dot{X}$ is also smaller than $\|x-x'\| + 2\varepsilon$. By choosing ε small enough, we obtain the stated result. □

Proposition 2.15. — *Let $\varepsilon > 0$. There exist positive constants δ_0, Δ_0 and ρ_0 which do not depend on X or $\mathcal{Y}$ with the following property. Assume that $\rho \geq \rho_0$, $\delta \leq \delta_0$ and $\Delta(\mathcal{Y}) \leq \Delta_0$. Let x and x' be two points of $X \subset \dot{X}_\rho(\mathcal{Y})$. For all $\eta > 0$, there exists a $(1, \eta)$-quasi-geodesic $\gamma : I \to \dot{X}$ joining x and x' such that for all $t \in I$, if $\gamma(t)$ is not an apex of $\dot{X}$, then $\langle x, x' \rangle_{p \circ \gamma(t)} \leq \varepsilon$.*

Proof. — By Proposition 2.13, there exist positive constants δ_0, Δ_0, ρ_0 and η_0 which depend only on ε satisfying the following property. Assume that $\rho \geq \rho_0$, $\delta \leq \delta_0$ and $\Delta(\mathcal{Y}) \leq \Delta_0$. Let x and x' be two points of X and C a chain of X between them. If $l(C) \leq |x-x'|_{\dot{X}} + \eta_0$, then for every point z of C we have $\langle x, x' \rangle_z \leq \varepsilon/3$. By replacing δ_0 by a smaller constant if necessary, we may also assume that $6\delta_0 \leq \varepsilon/3$.

Consider now $\eta \in (0, \eta_0)$ and x and x' two points of X. By definition of $|x-x'|_{\dot{X}}$, there exists a chain $C = (z_0, \dots, z_m)$ of X between x and x' such that $l(C) \leq |x-x'|_{\dot{X}} + \eta/2$. By Proposition 2.13, for every $j \in [\![0, m]\!]$, we have $\langle x, x' \rangle_{z_j} \leq \varepsilon/3$. Let $k \in [\![0, m-1]\!]$. Applying Lemma 2.14, there exists a rectifiable path $\gamma_k : I_k \to \dot{X}$ joining z_k and z_{k+1} whose length is smaller than $\|z_k - z_{k+1}\| + \eta/2m$ and such that for all $t \in I_k$, if $\gamma_k(t)$ is not an apex of $\dot{X}$, then $\langle z_k, z_{k+1}\rangle_{p\circ\gamma_k(t)} \leq 5\delta$. It follows then from the four-point inequality (2) that

$$\langle x, x'\rangle_{p\circ\gamma_k(t)} \leq \langle x, x'\rangle_{z_k} + \langle x, x'\rangle_{z_{k+1}} + \langle z_k, z_{k+1}\rangle_{p\circ\gamma_k(t)} + \delta \leq 2\varepsilon/3 + 6\delta \leq \varepsilon.$$

We now choose for γ the concatenation of the γ_k. Its length is smaller than $l(C) + \eta/2 \leq |x - x'|_{\dot X} + \eta$. We reparametrize γ by arc length; hence γ becomes a $(1, \eta)$-quasi-geodesic. Moreover, it satisfies the stated property. □

Let x, y and z be three points of X. Throughout this section, we have kept the notation $\langle x, y\rangle_z$ for the Gromov product computed with the distance of X. Now, however, we will denote the Gromov product of these three points computed in $\dot X$ by

$$\langle x, y\rangle_z^{\dot X} = \frac{1}{2}\left(|x - z|_{\dot X} + |y - z|_{\dot X} - |x - y|_{\dot X}\right).$$

Proposition 2.16. — *There exist positive constants $\dot\delta \leq 900\boldsymbol{\delta}$, δ_0, Δ_0 and ρ_0 which do not depend on X or $\mathcal{Y}$, with the following property. Assume that $\rho \geq \rho_0$, $\delta \leq \delta_0$ and $\Delta(\mathcal{Y}) \leq \Delta_0$. Let $x, y, z \in X \subset \dot X_\rho(\mathcal{Y})$. If $\langle y, z\rangle_x^{\dot X} < \rho/2$ then*

$$\langle y, z\rangle_x \leq \frac{\pi \sinh\rho}{\sinh(\rho - \langle y, z\rangle_x^{\dot X} - 4\dot\delta)} \sinh\left(\frac{\langle y, z\rangle_x^{\dot X}}{2} + 4\dot\delta\right).$$

Remark. — The formula is not very elegant. The idea to keep in mind here is the following. If the Gromov product of three points is small when measured in $\dot X$, it is also small when measured in X.

Proof. — The constants $\dot\delta$, δ_0, Δ_0 and ρ_0 are given by Theorem 2.8. Without loss of generality we can assume that $\rho_0 \geq 10^{10}\dot\delta$. According to Proposition 2.15, by decreasing (respectively increasing), if necessary, δ_0, Δ_0 (respectively ρ_0), the following hold. Assume that $\rho \geq \rho_0$, $\delta \leq \delta_0$ and $\Delta(\mathcal{Y}) \leq \Delta_0$ then

(i) $\dot X$ is $\dot\delta$-hyperbolic,
(ii) for every $x, x' \in X$, for every $\eta > 0$ there is a $(1, \eta)$-quasi-geodesic $\gamma : I \to \dot X$ joining x and x' such that for all $t \in I$, if $\gamma(t)$ is not an apex of $\dot X$, then $\langle x, x'\rangle_{p\circ\gamma(t)} \leq \varepsilon$, where $\varepsilon = \pi\sinh(\dot\delta)$.

Note that we did not claim that the cone-off space $\dot X$ is geodesic. For this reason we have to work with quasi-geodesics during the rest of the proof.

Let $\eta > 0$. There exists a $(1, \eta)$-quasi-geodesic $\gamma : [0, a] \to \dot X$ (respectively $\nu : [0, b] \to \dot X$) joining y to z (respectively y to x) and satisfying (ii). For simplicity, let us put

$$t = \langle x, z\rangle_y^{\dot X} \quad \text{and} \quad s = \langle y, z\rangle_x^{\dot X}$$

By Lemma 1.1 (iii), $|\nu(t)-\gamma(t)|_{\dot{X}} \leq 4\dot{\delta}+2\eta$. On the other hand,

$$|x-\nu(t)|_{\dot{X}} \leq |x-y|_{\dot{X}} - |y-\nu(t)|_{\dot{X}} + \eta \leq \langle y,z\rangle_x^{\dot{X}} + 2\eta = s+2\eta$$

Recall that x belongs to X. Hence, for every apex $v \in v(\mathcal{Y})$, we have $|v-\nu(t)|_{\dot{X}} \geq \rho - s - 2\eta$ and $|v-\gamma(t)|_{\dot{X}} \geq \rho - s - 4\dot{\delta} - 4\eta$. However, $s<\rho/2$. If η is sufficiently small, Lemma 2.7 yields $|p\circ\gamma(t) - p\circ\nu(t)| \leq \alpha$, where

$$\alpha = \frac{\pi \sinh \rho}{\sinh(\rho - s - 4\dot{\delta} - 4\eta)} \sinh\left(2\dot{\delta}+\eta\right). \tag{8}$$

By construction of γ and ν, we know that $\langle y,z\rangle_{p\circ\gamma(t)} \leq \varepsilon$ and $\langle x,y\rangle_{p\circ\nu(t)} \leq \varepsilon$. It follows from the triangle inequality that

$$|p\circ\nu(t)-y| \leq \langle x,z\rangle_y + \langle x,y\rangle_{p\circ\nu(t)} + \langle y,z\rangle_{p\circ\gamma(t)} + |p\circ\nu(t)-p\circ\gamma(t)|.$$

Consequently, $|p\circ\nu(t)-y| \leq \langle x,z\rangle_y + \alpha + 2\varepsilon$ and

$$\langle y,z\rangle_x \leq |x-y| - |y-p\circ\nu(t)| + \alpha + 2\varepsilon \leq |x-p\circ\nu(t)| + \alpha + 2\varepsilon. \tag{9}$$

Applying again Lemma 2.7, with x and $\nu(t)$ we obtain

$$\begin{aligned} |x-p\circ\nu(t)| &\leq \frac{\pi\sinh\rho}{\sinh(\rho-s-2\eta)}\sinh\left(\frac{|x-\nu(t)|_{\dot{X}}}{2}\right) \\ &\leq \frac{\pi\sinh\rho}{\sinh(\rho-s-2\eta)}\sinh\left(\frac{s}{2}+\eta\right). \end{aligned} \tag{10}$$

Combining (8), (9) and (10), and letting η tend to zero, we get the desired inequality. □

3. Small cancellation theory

In this section we will be concerned with the small cancellation theory. We follow the geometrical point of view developed by T. Delzant and M. Gromov in [12] to study Burnside groups. This approach has been studied in [11] and [7]. We only recall the main steps of this construction. We follow here [8] and its generalization [9].

3.1. General framework. — Let X be a proper geodesic δ-hyperbolic space. Let G be a group acting properly co-compactly by isometries on X. In particular, G is hyperbolic. We assume that G satisfies the small centralizer assumption, i.e. that every elementary subgroup of G is cyclic. Let $\mathcal{Q}$ be a family of pairs (H,Y) with the following properties

- (i) H is a cyclic group generated by a loxodromic element h and Y the cylinder of h (see Section 1.3).
- (ii) $\mathcal{Q}$ is closed under the following action of G: for every $(H,Y)\in\mathcal{Q}$, for every $g\in G$, $g(H,Y) = (gHg^{-1}, gY)$.
- (iii) $\mathcal{Q}/G$ is finite.

We denote by K the (normal) subgroup generated by the subgroups H where $(H,Y) \in \mathcal{Q}$. The goal is to study the quotient $\bar{G} = G/K$. To that end, we define the following two small cancellation parameters

$$\Delta(\mathcal{Q},X) = \sup\left\{\operatorname{diam}\left(Y_1^{+5\delta} \cap Y_2^{+5\delta}\right) \mid (H_1,Y_1) \neq (H_2,Y_2) \in \mathcal{Q}\right\},$$
$$T(\mathcal{Q},X) = \inf\left\{[h] \mid h \in H \setminus \{1\},\ (H,Y) \in \mathcal{Q}\right\}.$$

If there is no ambiguity, we will omit the ambient space X in all the notations mentioned. Let $(H,Y) \in \mathcal{Q}$. By abuse of notation we write $T(H)$ for the quantity

$$T(H) = \inf\left\{[h] \mid h \in H \setminus \{1\}\right\}.$$

We are interested in situations where the ratios $\delta/T(\mathcal{Q})$ and $\Delta(\mathcal{Q})/T(\mathcal{Q})$ are very small. To that end, we build a space $\bar{X}$ with an action of $\bar{G}$.

Note that for every $(H,Y) \in \mathcal{Q}$, Y is strongly quasi-convex [8, Lemma 2.31]. Therefore, we can apply the cone-off construction described in Section 2 with the space X and the collection $\mathcal{Y} = \{Y \mid (H,Y) \in \mathcal{Q}\}$. As previously discussed, we fix a parameter $\rho > 0$. Its value will be made precise later. We denote by $\dot{X}$ the cone-off of radius ρ over X relative to the collection $\mathcal{Y}$. We write $v(\mathcal{Y})$ or $v(\mathcal{Q})$ for the set of all apices of $\dot{X}$. Let $p\colon \dot{X} \setminus v(\mathcal{Q}) \to X$ be the radial projection defined in Section 2.2.

The action of G on X extends by homogeneity to an action on $\dot{X}$. Let $(H,Y) \in \mathcal{Q}$. If $x = (y,r)$ is a point of the cone $Z(Y)$ and g an element of G, then gx is the point of $Z(gY)$ defined by $gx = (gy,r)$. It follows from the definition of the metric of $\dot{X}$ that this action is by isometries. However, it is not necessarily proper. The space $\bar{X}$ is defined as the quotient of $\dot{X}$ by K. It is endowed with an action on $\bar{G}$. We denote by $\zeta : \dot{X} \to \bar{X}$ the canonical map from $\dot{X}$ to $\bar{X}$. The image of $v(\mathcal{Q})$ in $\bar{X}$ is denoted by $\bar{v}(\mathcal{Q})$. The main theorem of small cancellation is stated below. Recall that $\boldsymbol{\delta}$ stands for the hyperbolicity constant of the hyperbolic plane $\mathbf{H}$.

Theorem 3.1 (Small cancellation theorem). — [12, Théorème 5.5.2], [8, Proposition 6.7] *There exist positive numbers δ_0, Δ_0 and ρ_0 which do not depend on X or $\mathcal{Q}$, with the following property. If $\rho \geq \rho_0$, $\delta \leq \delta_0$, $\Delta(\mathcal{Q}) \leq \Delta_0$ and $T(\mathcal{Q}) \geq \pi \sinh \rho$, then $\bar{X}$ is proper, geodesic and $\bar{\delta}$-hyperbolic, with $\bar{\delta} \leq 64 \cdot 10^4 \boldsymbol{\delta}$. Moreover, $\bar{G}$ acts properly, co-compactly, by isometries on it.*

According to Theorem 3.1, Theorem 2.8, Proposition 2.15 and Proposition 2.16, there exist positive numbers δ_0, Δ_0 and ρ_0 with the following property. Assume that $\delta \leq \delta_0$, $\Delta(\mathcal{Q}) \leq \Delta_0$, $\rho \geq \rho_0$ and $T(\mathcal{Q}) \geq 100\pi \sinh \rho$, then the following holds.

- (Theorem 2.8) The cone-off $\dot{X}$ is $\dot{\delta}$-hyperbolic with $\dot{\delta} = 64 \cdot 10^4 \boldsymbol{\delta}$.
- (Theorem 3.1) The space $\bar{X}$ is proper, geodesic and $\bar{\delta}$-hyperbolic, with $\bar{\delta} = 64 \cdot 10^4 \boldsymbol{\delta}$. Moreover, $\bar{G}$ acts properly, co-compactly, by isometries on it.

- (Proposition 2.16) Let $x, y, z \in X$. Recall that $\langle y, z\rangle_x^{\dot X}$ is the Gromov product measured in $\dot X$. If $\langle y, z\rangle_x^{\dot X} < \rho/2$, then
$$\langle y, z\rangle_x \leq \frac{\pi \sinh \rho}{\sinh(\rho - \langle y, z\rangle_x^{\dot X} - 4\dot\delta)} \sinh\left(\frac{\langle y, z\rangle_x^{\dot X}}{2} + 4\dot\delta\right).$$
- (Proposition 2.15) For all $x, y \in X$, for all $\eta > 0$, there exists a $(1, \eta)$-quasi-geodesic $\gamma : I \to \dot X$ between x and y such that for all $t \in I$, if $\gamma(t)$ is not an apex of $\dot X$, then $\langle x, y\rangle_{p\circ\gamma(t)} \leq \pi \sinh(\dot\delta)$.

Conventions and notations. Note that the constants δ_0 and Δ_0 (respectively ρ_0) can be chosen to be arbitrarily small (respectively large). From now on, we will always assume that $\rho_0 \geq 10^{100} \max\{L_S\boldsymbol{\delta}, \varepsilon\}$ and $\delta_0, \Delta_0 < 10^{-100} \min\{\boldsymbol{\delta}, \varepsilon\}$. Up to increasing again ρ_0, the statement of Proposition 2.16 can also be reformulated in a more convenient way: for every $x, y, z \in X$, if $\langle x, y\rangle_z^{\dot X} \leq 500\dot\delta$, then

$$\langle x, y\rangle_z \leq e^{510\dot\delta} \pi \sinh\left(\frac{\langle x, y\rangle_z^{\dot X}}{2} + 4\dot\delta\right). \tag{11}$$

In order to simplify the formulas later, we let

$$\varepsilon = e^{510\dot\delta} \pi \sinh(300\dot\delta).$$

These estimates are absolutely not optimal. We chose them very generously to be sure that all the inequalities that we might need later will be satisfied. What really matters is their orders of magnitude recalled below.

$$\delta_0, \Delta_0 \ll \boldsymbol{\delta} \ll \varepsilon \ll \rho_0 \ll \pi \sinh \rho_0.$$

In particular, in every statement below, we round the estimates in order to keep only the relevant term. For instance, if a quantity is bounded above by $\pi \sinh \rho + 3\varepsilon + 32\delta$, we will replace this estimate by $2\pi \sinh \rho$.

From now on and until the end of Section 3, we fix $\rho \geq \rho_0$ and assume that X, G and $\mathcal{Q}$ satisfy the conditions stated above, i.e. $\delta \leq \delta_0$, $\Delta(\mathcal{Q}) \leq \Delta$, $\rho \geq \rho_0$ and $T(\mathcal{Q}) \geq 100\pi \sinh \rho$. In particular, $\dot X$ and $\bar X$ are $\dot\delta$- and $\bar\delta$-hyperbolic, respectively. Another important point to remember is the following. The constants δ_0, Δ_0, ε and $\pi \sinh \rho$ are used to describe the geometry of X, whereas $\dot\delta$, $\bar\delta$ and ρ refer to the that of $\dot X$ or $\bar X$.

In this section we work with three metric spaces, namely X, its cone-off $\dot X$ and the quotient $\bar X$. Since the map $X \hookrightarrow \dot X$ is an embedding, we use the same letter x to designate a point of X and its image in $\dot X$. We write $\bar x$ for its image by the natural map $\zeta : \dot X \to \bar X$. Unless stated otherwise, we keep the notation $| \,.\, |$ (without mentioning the space) for the distances in X or $\bar X$. The metric on $\dot X$ will be denoted by $| \,.\, |_{\dot X}$. Given $g \in G$, we write $\bar g$ for the image of g by the canonical projection $\pi : G \twoheadrightarrow \bar G$.

3.2. A Greendlinger Lemma. — In the usual context of small cancellation theory, the Greendlinger lemma states that if a reduced word (over the generators) represents the identity element, then it should contain a subword corresponding to a large portion of a relation. This is a major ingredient for solving the word problem. In [11], F. Dahmani, V. Guirardel and D. Osin state an analog of the Greendlinger lemma for the small cancellation quotient of a hyperbolic group (actually their framework is much more general). However, their statement takes place in $\dot{X}$: for every $x \in X$ and every $g \in K \setminus \{1\}$, there exists $(H, Y) \in \mathcal{Q}$, such that any geodesic of $\dot{X}$ between x and gx goes through the apex v of $Z(Y)$; moreover, the distance $|gx - x|_{\dot{X}}$ can be shortened using an element of H. For our purpose, we need another variation of the Greendlinger lemma that takes place directly in X (see Theorem 3.5). This is the goal of this section.

Lemma 3.2 ([8, Corollary 3.12]). — *The space $\dot{X} \setminus v(\mathcal{Q})$ is a covering space of $\bar{X} \setminus \bar{v}(\mathcal{Q})$. Let $r \geq 0$ and $x \in \dot{X}$. If for every $v \in v(\mathcal{Q})$, $|v - x|_{\dot{X}} \geq r$, then for every $g \in K \setminus \{1\}$, $|gx - x|_{\dot{X}} \geq \min\{2r, \rho/5\}$.*

Lemma 3.3 ([8, Proposition 3.15]). — *Let $r \in (0, \rho/20]$ and $x \in \dot{X}$. If for every $v \in v(\mathcal{Q})$, $|v - x|_{\dot{X}} \geq 2r$, then the map $\zeta : \dot{X} \to \bar{X}$ induces an isometry from $B(x, r)$ onto $B(\bar{x}, r)$.*

In Section 1.3, we saw that if a geodesic $[x, y]$ fellow-travels for a sufficiently long time with the axis of an element $g \in K$, then the distance between the images of x and y by the projection $X \to X/K$ is strictly shorter than $|x - y|$ (Lemma 1.16). The next statement can be seen as a strong converse of this observation: if $[x, y]$ does not fellow-travel with any of the quasi-convex Y, where $(H, Y) \in \mathcal{Q}$, then the projection $\zeta \colon \dot{X} \to \bar{X}$ maps a geodesic of $\dot{X}$ from x to y onto a local geodesic $\bar{\gamma}$. This is the key observation for proving the Greendlinger lemma (Theorem 3.5). Indeed, in such a situation, the stability of quasi-geodesics applied in $\bar{X}$ tells us that $\bar{\gamma}$ cannot close up. Hence, if $g \in G$ is an isometry sending x to y, then its image in $\bar{G}$ is nontrivial.

Proposition 3.4. — *Let x and y be two points of X. We assume that for all $(H, Y) \in \mathcal{Q}$,*

$$\mathfrak{D}(\{x, y\}, Y) \leq T(H) - 2\pi \sinh \rho.$$

Then, for all $\eta > 0$, there exists a $(1, \eta)$-quasi-geodesic $\gamma : I \to \dot{X}$ joining x to y such that the path $\bar{\gamma} : I \to \dot{X} \to \bar{X}$ is a $\rho/100$-local $(1, \eta)$-quasi-geodesic of $\bar{X}$.

Proof. — Let $\eta \in (0, \dot{\delta})$. Applying Proposition 2.15, there exists a $(1, \eta)$-quasi-geodesic $\gamma : I \to \dot{X}$ between x and y such that for all $t \in I$, if $\gamma(t)$ is not an apex of $\dot{X}$, then $\langle x, y \rangle_{p \circ \gamma(t)} \leq \varepsilon$. Let $s, t \in I$ such that $|s - t| \leq \rho/100$.

Since γ is a $(1,\eta)$-quasi-geodesic,

$$|\gamma(s)-\gamma(t)|_{\dot X} \leq |s-t| + \eta < \rho/50. \tag{12}$$

We claim that $|\bar\gamma(s)-\bar\gamma(t)| = |\gamma(s)-\gamma(t)|_{\dot X}$. To that end, we distinguish two cases. Assume first that for every $v \in v(\mathcal{Q})$, $|v-\gamma(s)| > \rho/2$. By Lemma 3.3 the map $\dot X \to \bar X$ restricted to the ball of center $\gamma(s)$ and radius $\rho/50$ preserves the distances. Hence, $|\bar\gamma(s)-\bar\gamma(t)| = |\gamma(s)-\gamma(t)|_{\dot X}$. Assume now that there exists $(H,Y)\in\mathcal{Q}$ such that $|v-\gamma(s)| \leq \rho/2$, where v stands for the apex of $Z(Y)$. If $\gamma(s)$ or $\gamma(t)$ is the apex v, then we directly obtain $|\bar\gamma(s)-\bar\gamma(t)| = |\gamma(s)-\gamma(t)|_{\dot X}$. Therefore, we can assume that $\gamma(s)$ and $\gamma(t)$ are distinct from v. By construction, $p\circ\gamma(s)$ and $p\circ\gamma(t)$ are two points of Y such that

$$\langle x,y\rangle_{p\circ\gamma(s)} \leq \varepsilon \quad\text{and}\quad \langle x,y\rangle_{p\circ\gamma(t)} \leq \varepsilon.$$

It follows from the four-point inequality (2) that

$$|p\circ\gamma(s)-p\circ\gamma(t)| \leq \mathfrak{D}(\{x,y\},Y) + \langle x,y\rangle_{p\circ\gamma(s)} + \langle x,y\rangle_{p\circ\gamma(t)} + \delta,$$

which leads to

$$|p\circ\gamma(s)-p\circ\gamma(t)|_Y \leq \mathfrak{D}(\{x,y\},Y) + 2\varepsilon + 9\delta \leq T(H) - \pi\sinh\rho$$

It follows from Lemma 2.3 that $|\bar\gamma(s)-\bar\gamma(t)| = |\gamma(s)-\gamma(t)|_{\dot X}$, which completes the proof of our claim. The result follows from the claim combined with (12). □

Theorem 3.5 (Greendlinger's Lemma). — *Let x be a point of X. Let g be an element of G. If g belongs to $K\setminus\{1\}$, then there exists $(H,Y)\in\mathcal{Q}$ such that*

$$\mathfrak{D}(\{x,gx\},Y) > T(H) - 2\pi\sinh\rho.$$

Proof. — We prove the theorem by contradiction. Assume that for all $(H,Y)\in\mathcal{Q}$, $\mathfrak{D}(\{x,gx\},Y)$ is bounded above by $T(H)-\pi\sinh\rho$. Let $\eta\in(0,\bar\delta)$. Applying Proposition 3.4, there exists a $(1,\eta)$-quasi-geodesic $\gamma\colon I\to\dot X$ between x and gx such that the path $\bar\gamma\colon I\to\dot X\to\bar X$ is a $\rho/100$-local $(1,\eta)$-quasi-geodesic of $\bar X$. However, $\rho/100 \geq L_S\bar\delta$. By Corollary 1.3, $\bar\gamma$ is a $(2,\eta)$-quasi-geodesic. Hence, $|gx-x|_{\dot X} \leq 2|\bar g\bar x-\bar x| + \eta = \eta$. This inequality holds for all $\eta>0$. It implies $gx=x$. However, K acts freely on X (see Lemma 3.2); thus $g=1$. Contradiction. □

We saw in Proposition 2.16 how to control the Gromov product $\langle y,z\rangle_x$ computed in X in terms of the same product computed in $\dot X$. We would now like to bound $\langle y,z\rangle_x$ from above using only $\langle\bar y,\bar z\rangle_{\bar x}$. To that end we need an extra assumption, namely that the sides $[x,y]$ and $[x,z]$ of the triangle $[x,y,z]$ do not fellow-travel too much with any of the quasi-convex Y, where $(H,Y)\in\mathcal{Q}$. This proposition is a key statement to lift figures from $\bar X$ to X (see Section 3.6).

Proposition 3.6 (Preserving shape Lemma). — *Let x, y and z be three points of X such that for all $(H,Y) \in \mathcal{Q}$,*

$$\max\{\mathfrak{D}(\{x,y\},Y), \mathfrak{D}(\{x,z\},Y)\} \leq T(H) - 2\pi\sinh\rho.$$

If $\langle \bar y, \bar z\rangle_{\bar x} \leq 450\bar\delta$, then $\langle y,z\rangle_x \leq \varepsilon$

Proof. — We would like to apply Proposition 2.16 with the points x, y and z. To that end we first need to have an estimate of the Gromov product $\langle y,z\rangle_x^{\dot X}$ (computed in $\dot X$). We claim that $\langle y,z\rangle_x^{\dot X} \leq \langle \bar y,\bar z\rangle_{\bar x} + 14\dot\delta$. Without loss of generality we can assume that $\min\{|x-y|_{\dot X}, |x-z|_{\dot X}\} > \langle \bar y,\bar z\rangle_{\bar x} + 12\dot\delta$. Let $\eta \in (0,\dot\delta)$ such that $\min\{|x-y|_{\dot X}, |x-z|_{\dot X}\} > \langle \bar y,\bar z\rangle_{\bar x} + 12\dot\delta + 2\eta$. According to Proposition 3.4, there exists a $(1,\eta)$-quasi-geodesic $\gamma : [0,a] \to \dot X$ joining x to y whose image $\bar\gamma : [0,a] \to \dot X \to \bar X$ in $\bar X$ is a $\rho/100$-local $(1,\eta)$-quasi-geodesic. However, $\rho/100 \geq L_S\dot\delta$. By Corollary 1.3, for every $t \in [0,a]$, $\langle \bar x,\bar y\rangle_{\bar\gamma(t)} \leq \eta/2 + 5\dot\delta$. We also construct a path $\sigma : [0,b] \to \dot X$ between x and z having the same properties. Recall that $\langle \bar y,\bar z\rangle_{\bar x} \leq 450\bar\delta$. Let $s \leq \min\{a,b,\rho/100\}$ such that $s > \langle \bar y,\bar z\rangle_{\bar x} + 12\dot\delta + 2\eta$. Let us denote by p and q the points $\gamma(s)$ and $\sigma(s)$. Applying Lemma 1.1 (iii) in the space $\dot X$ we get

$$|p-q|_{\dot X} \leq \max\left\{2\eta, 2s - 2\langle y,z\rangle_x^{\dot X}\right\} + 4\dot\delta. \tag{13}$$

Our next step is to give a lower bound for $|p-q|_{\dot X}$. Recall that $s \leq \rho/100$. Thus, p and q are contained in the ball of $\dot X$ of center x and radius $\rho/50$. However, the map $\dot X \to \bar X$ restricted to this ball is an isometry; hence $|\bar x-\bar p|+|\bar x-\bar q| \geq 2s-2\eta$. According to the triangle inequality,

$$|\bar p-\bar q| \geq |\bar x-\bar p| + |\bar x-\bar q| - 2\langle \bar y,\bar z\rangle_{\bar x} - 2\langle \bar x,\bar y\rangle_{\bar p} - 2\langle \bar x,\bar z\rangle_{\bar q}$$

We noted that $\langle \bar x,\bar y\rangle_{\bar p} \leq \eta/2 + 5\bar\delta$ and $\langle \bar x,\bar z\rangle_{\bar q} \leq \eta/2 + 5\bar\delta$, hence we have

$$|p-q|_{\dot X} \geq |\bar p-\bar q| \geq 2s - 2\langle \bar y,\bar z\rangle_{\bar x} - 20\dot\delta - 2\eta > 4\dot\delta + 2\eta$$

It follows then from (13) that $\langle y,z\rangle_x^{\dot X} \leq s + 2\dot\delta$. This inequality holds for every sufficiently small η and for every $s > \langle \bar y,\bar z\rangle_{\bar x} + 12\dot\delta + 2\eta$ thus $\langle y,z\rangle_x^{\dot X} \leq \langle \bar y,\bar z\rangle_{\bar x} + 14\dot\delta$, which completes the proof of our claim. In particular, $\langle y,z\rangle_x^{\dot X} < 500\dot\delta$. Therefore, we can apply Proposition 2.16. It gives $\langle y,z\rangle_x \leq \varepsilon$. □

3.3. $\mathcal{Q}$-close points. —

Definition 3.7. — Two points x and y of X are said to be $\mathcal{Q}$-*close* if for every $(H,Y) \in \mathcal{Q}$, we have $\mathfrak{D}(\{x,y\},Y) \leq T(H)/2 + 10\varepsilon$.

Remark. — There is a very simple way to obtain $\mathcal{Q}$-close points. Let x and y be two points of X. Let $u \in K$. If $|x-uy| \leq \inf_{u'\in K}|x-u'y| + \delta$, then x and uy are $\mathcal{Q}$-close. Indeed, if this were not the case, according to Lemma 1.16, one could reduce the distance between x and uy.

PROPOSITION 3.8. — *Let $A \in [0, \pi \sinh \rho]$. Let x and y be two $\mathcal{Q}$-close points of X. Let $z \in X$ such that for all $u \in K$, $\langle x, y \rangle_z \leq \langle x, y \rangle_{uz} + A$. Then, for all $(H, Y) \in \mathcal{Q}$, we have $\mathfrak{D}(\{x, z\}, Y) \leq T(H) - C$ where $C = 4\pi \sinh \rho - A/2$.*

Proof. — We prove this result by contradiction. Assume that there exists $(H, Y) \in \mathcal{Q}$ such that $\mathfrak{D}(\{x, z\}, Y) > T(H) - C$. Recall that by assumption, H is a cyclic group generated by an element $h \in G$. Since $T(H) \geq 10\pi \sinh \rho$, one observes that $T(H) = [h]$. Let $\gamma \colon \mathbf{R} \to X$ be a δ-nerve of h and T its fundamental length. We denote by $x' = \gamma(a)$ and $z' = \gamma(b)$ the respective projections of x and z on γ. We denote by I the closed interval whose endpoints are a and b. Let $y' = \gamma(c)$ be a projection of y on $\gamma(I)$. It follows from Proposition 1.18 that

(i) $|x' - z'| \geq [h] - C - 73\delta$,
$|y' - z'| \geq [h]/2 - C - 10\varepsilon - 156\delta$,

(ii) $\langle x, y \rangle_z \geq \langle x, y \rangle_{y'} + |y' - z'| + |z' - z| - 62\delta$.

Up to replacing h by h^{-1}, we may assume that a and $b + T$ belong to the same component of $\mathbf{R} \setminus \{b\}$. Recall that by definition, $hz' = \gamma(b + T)$. We want to compare $\langle x, y \rangle_z$ and $\langle x, y \rangle_{hz}$. To that end, we distinguish two cases depending on the relative positions of x', y', and hz' on γ.

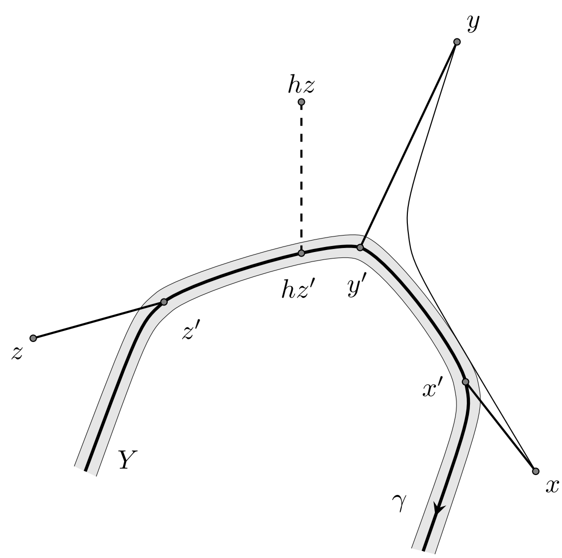
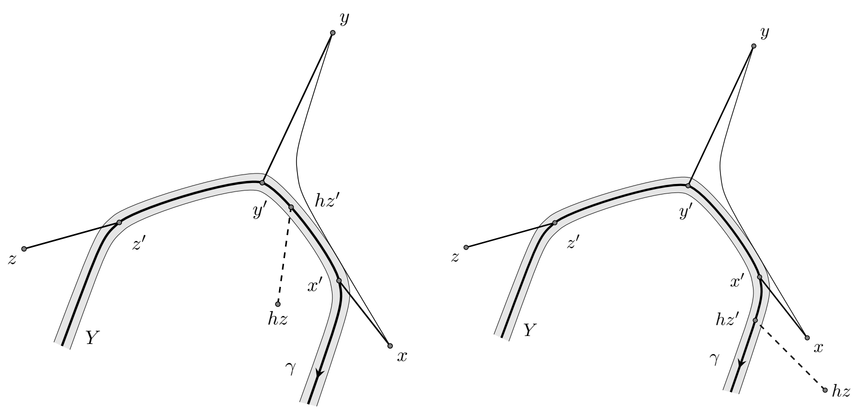


FIGURE 3.1. Case 1

Case 1. Assume that $b+T$ is between b and c. What we have in mind is that hz' is between z' y' (see Figure 3.1). However, the path γ is not necessarily embedded in X; therefore, we need to state the condition using the parametrization of γ. Since z' is a projection of z on γ, we have $|y'-z| \geq |y'-hz| + [h] - 29\delta$. Combined with the lower bound of $\langle x,y\rangle_z$ provided by (ii), we get

$$\langle x,y\rangle_z \geq \langle x,y\rangle_{y'} + |y'-hz| + [h] - 91\delta \geq \langle x,y\rangle_{hz} + [h] - 91\delta.$$



(a) hz' between y' and x' (b) x' between y' and hz'

FIGURE 3.2. Case 2

Case 2. Assume that $b+T$ is not between b and c. We claim that $\langle x,y'\rangle_{hz'} \leq C + 84\delta$. If $b+T$ is between c and a (see Figure 3.2(a)), it follows from the definition of γ. If not (see Figure 3.2(b)), γ being a T-local $(1,\delta)$-quasi-geodesic, we have

$$\langle x,y'\rangle_{hz'} \leq |x'-hz'| + \langle x,y'\rangle_{x'} \leq [h] - |x'-z'| + \langle x,y'\rangle_{x'} + 2\delta.$$

The point x' is a projection of x on γ; thus $\langle x,y'\rangle_{x'} \leq 9\delta$. Moreover, by (i) we have $[h]-|x'-z'| \leq C+73\delta$, which completes the proof of our claim. Applying the triangle inequality we get

$$\langle x,y\rangle_{hz'} \leq \langle x,y\rangle_{y'} + \langle x,y'\rangle_{hz'} \leq \langle x,y\rangle_{y'} + C + 84\delta.$$

Combined with (i) and (ii), it gives

$$\langle x,y\rangle_z \geq \langle x,y\rangle_{hz'} + |hz'-hz| + [h]/2 - 2C - 10\varepsilon - 302\delta.$$

In both cases $\langle x,y\rangle_{hz} \leq \langle x,y\rangle_z + 2C - 10\pi\sinh\rho + 10\varepsilon + 302\delta < \langle x,y\rangle_z - A$, which contradicts our assumption on z. $\square$

3.4. $\mathcal{Q}$-reduced isometries. —

DEFINITION 3.9. — Let g be an element of G. The isometry g is $\mathcal{Q}$-reduced if its image $\bar{g}$ in $\bar{G}$ is loxodromic, and for all $(H,Y) \in \mathcal{Q}$ we have $\mathfrak{D}(Y_g, Y) \leq T(H)/2 + 2\varepsilon$.

REMARK. — Since $\mathcal{Q}$ is invariant under conjugation, all conjugates of a $\mathcal{Q}$-reduced isometry are also $\mathcal{Q}$-reduced.

The next proposition explains how to construct $\mathcal{Q}$-reduced elements of G. To that end we need to assume that the elements of $\mathcal{Q}$ are powers of small isometries.

PROPOSITION 3.10. — *We assume that for all $(H,Y) \in \mathcal{Q}$, there exists $h \in G$ such that $[h] \leq L_S\delta$ and $H \subset \langle h \rangle$. In addition, we require that $A(G,X) \leq \varepsilon$. Let $g \in G$ such that its image $\bar{g}$ in $\bar{G}$ is loxodromic. Then there exists $u \in K$ such that ug is $\mathcal{Q}$-reduced.*

Proof. — We choose $u \in K$ such that for all $u' \in K$, $[ug] \leq [u'g] + \delta$. Since $\bar{g} = \bar{u}\bar{g}$ is a loxodromic element of $\bar{X}$, so is ug in X. We now suppose that the isometry ug is not $\mathcal{Q}$-reduced. There is $(H,Y) \in \mathcal{Q}$ such that $\mathfrak{D}(Y_{ug}, Y) > T(H)/2 + 2\varepsilon$. By assumption, there exists $h \in G$ together with a positive integer n such that $[h] \leq L_S\delta$ and $H = \langle h^n \rangle$. Since $T(H) \geq 100\pi \sinh\rho$, one observes that $T(H) = [h^n]$. According to Proposition 1.14, Y (respectively Y_{ug}) lies in the 52-neighborhood of A_h and (respectively A_{ug}). Combining Lemma 1.12 and Lemma 1.9 we get

$$\begin{aligned}
[h^n]/2 + 2\varepsilon < \mathfrak{D}(Y_{ug}, Y) &\leq \operatorname{diam}\left(Y_{ug}^{+5\delta} \cap Y^{+5\delta}\right) + 33\delta \\
&\leq \operatorname{diam}\left(A_{ug}^{+57\delta} \cap A_h^{+57\delta}\right) + 33\delta \\
&\leq \operatorname{diam}\left(A_{ug}^{+13\delta} \cap A_h^{+13\delta}\right) + 151\delta
\end{aligned}$$

Recall that G satisfies the small centralizer property. Thus, ug and h do not generate an elementary subgroup. Indeed if they did, then $\bar{g}$ would have finite order. Consequently, Proposition 1.22 leads to $[ug] > [h^n]/2 + 2\varepsilon - A(G,X) - 310\delta$. It follows from our assumptions and Lemma 1.17 that there exists $m \in \mathbf{Z}$ such that $[h^{nm}ug] < [ug] + A(G,X) - 2\varepsilon + 638\delta$. However, $h^{nm}u$ belongs to K. The last inequality contradicts the definition of u. Consequently, ug is $\mathcal{Q}$-reduced. □

LEMMA 3.11. — *Let g be a $\mathcal{Q}$-reduced element of G. Let x and y be two points of X. For all $(H,Y) \in \mathcal{Q}$ we have*

$$\mathfrak{D}(\{x,y\}, Y) < T(H)/2 + d(x, Y_g) + d(y, Y_g) + 2\varepsilon.$$

In particular, if $d(x, Y_g) + d(y, Y_g) \leq 8\varepsilon$, then x and y are $\mathcal{Q}$-closed.

Proof. — Let (H, Y) be an element of $\mathcal{Q}$. Let x' and y' be respective projections of x and y on Y_g. One knows by (4) that

$$\mathfrak{D}(\{x,y\},Y) \leq \mathfrak{D}(\{x',y'\},Y) + \langle x',y'\rangle_x + \langle x',y'\rangle_y + \delta.$$

However, g is $\mathcal{Q}$-reduced; therefore, $\mathfrak{D}(\{x',y'\},Y) \leq \mathfrak{D}(Y_g,Y) < T(H)/2 + \varepsilon$. On the other hand, $\langle x',y'\rangle_x \leq |x-x'| \leq d(x,Y_g)$. Similarly, $\langle x',y'\rangle_y \leq d(y,Y_g)$. □

Proposition 3.12. — *Let $A \geq 0$. Let g be a $\mathcal{Q}$-reduced element of G. Let x be a point of X such that for all $u \in K$, $d(x,Y_g) \leq d(ux,Y_g) + A$. Then there exists k_0 such that for all $k \geq k_0$, for all $(H,Y) \in \mathcal{Q}$, $\mathfrak{D}(\{x,g^kx\},Y) \leq T(H) - C$ where $C = 3\pi\sinh\rho - A/2$.*

Proof. — Let x' be a projection of x on Y_g. Recall that $\mathcal{Q}/G$ is finite. Since g is loxodromic, there exists k_0 such that for all $k \geq k_0$, for all $(H,Y) \in \mathcal{Q}$, $|x' - g^kx'| > T(H)/2 + 2\varepsilon + 21\delta$. Assume now that our proposition is false, i.e. there exists $k \geq k_0$ and $(H,Y) \in \mathcal{Q}$ such that $\mathfrak{D}(\{x,g^kx\},Y) > T(H) - C$. The point x' is a projection of x on Y_g; thus $\langle x', g^kx'\rangle_x \leq d(x,Y_g)$. Moreover, Y_g is 2δ-quasi-convex. It follows from our assumption on x that for all $u \in K$, $\langle x',g^kx'\rangle_x \leq \langle x',g^kx'\rangle_{ux} + A + 2\delta$. On the other hand, g is $\mathcal{Q}$-reduced. By Lemma 3.11, x' and g^kx' are $\mathcal{Q}$-close. According to Proposition 3.8 we have $\mathfrak{D}(\{x,g^kx'\},Y) \leq T(H) - C'$, where $C' = 3\pi\sinh\rho - A/2 - \delta$. The same inequality holds if one replaces $\{x,g^kx'\}$ by $\{x',g^kx\}$. We now denote by p and q the respective projections of x and g^kx on Y. According to Proposition 1.11

$$|p-q| \geq \mathfrak{D}(\{x,g^kx\},Y) - 5\delta > T(H) - C - 5\delta. \tag{14}$$

Claim. The point x' is an 8δ-projection of p on Y_g. Thanks to Lemma 1.8, it is sufficient to show that $\langle x,x'\rangle_p \leq 3\delta$. Assume that this statement is false. Let $z \in Y$. By hyperbolicity we have

$$\min\left\{\langle x,x'\rangle_p, \langle x',z\rangle_p\right\} \leq \langle x,z\rangle_p + \delta \leq 3\delta.$$

Thus, for every $z \in Y$, $\langle x',z\rangle_p \leq 3\delta$. In particular, p is a 3δ-projection of x' on Y. Using Proposition 1.11 we obtain that

$$|p-q| \leq \mathfrak{D}(\{x',g^kx\},Y) + 8\delta \leq T(H) - C' + 8\delta,$$

which contradicts (14). In the same way, we prove that g^kx' is an 8δ-projection of q on Y_g. It then follows from Proposition 1.11 that

$$\left|x' - g^kx'\right| \leq \mathfrak{D}(\{p,q\},Y_g) + 21\delta \leq \mathfrak{D}(Y,Y_g) + 21\delta.$$

By assumption, g is $\mathcal{Q}$-reduced. Consequently, $|x'-g^kx'| \leq T(H)/2 + 2\varepsilon + 21\delta$, which contradicts our assumption on k. Thus, the proposition is true. □

3.5. Foldable configurations. — In this section, we are interested in the following situation. Let x, p and q be three points of X such that x and p (respectively x and q) are $\mathcal{Q}$-close. We assume that p and q have the same image $\bar{p} = \bar{q}$ in $\bar{X}$, but are distinct as points of X. We would like to understand the reason why $p \neq q$ in X and which transformation could move p closer to q.

Let us explain the main ideas. Recall that δ and ε are negligible compared to $\pi \sinh \rho$. For simplicity, we omit them in the following explanation. Since $\bar{p} = \bar{q}$, there exists $g \in K \setminus \{1\}$ such that $q = gp$. By the Greendlinger Lemma (Theorem 3.5), there exists $(H, Y) \in \mathcal{Q}$ such that

$$\mathfrak{D}(\{p, q\}, Y) \geq T(H) - 2\pi \sinh \rho.$$

However, x and p (respectively x and q) are $\mathcal{Q}$-closed. Hence, approximatively half of the overlap between Y and $[p, q]$ is covered by $[x, p]$, and the other half by $[x, q]$ (see Figure 3.3). Using an element of H, we translate the point p. In particular, there exists $h \in H$ such that

$$\langle hp, q \rangle_x \geq \langle p, q \rangle_x + T(H)/2 - 2\pi \sinh \rho.$$

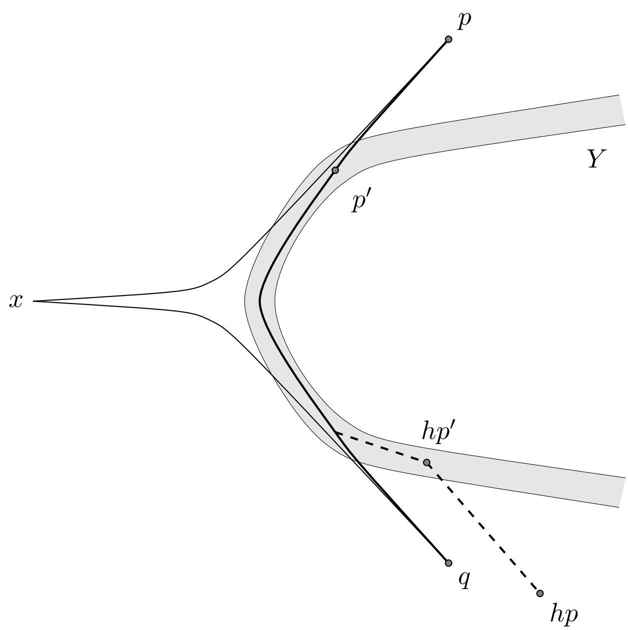


Figure 3.3. Folding a geodesic

By replacing p with hp, we increase $\langle p, q \rangle_x$. Note that $\langle p, q \rangle_x$ is bounded above by $|x - q|$. Thus, after iterating the same process finitely many times, we get $p = q$. However, to induce the argument, we need the points x and hp to be $\mathcal{Q}$-close, which is unfortunately not exactly the case: we might have approximatively

$$\mathfrak{D}(\{x, hp\}, Y) = T(H)/2 + 2\pi \sinh \rho$$

The definition of a *foldable configuration* gives a set of conditions on x, p and q which are sufficient to detail the previous discussion and which will still be satisfied by x, hp and q.

Definition 3.13 (Foldable configuration). — Let x, p, q and y be four points of X. We say that the configuration (x, p, q, y) is *foldable* if there exist $s, t \in X$ satisfying the following conditions (see Figure 3.4).

(C1) s and y are $\mathcal{Q}$-close and $|x - s| \leq \langle p, q \rangle_x + 3\pi \sinh \rho$,
(C2) t and q are $\mathcal{Q}$-close and $|x - t| \leq \langle p, q \rangle_x + 3\pi \sinh \rho$.
(C3) $\langle y, s \rangle_p = 0$.

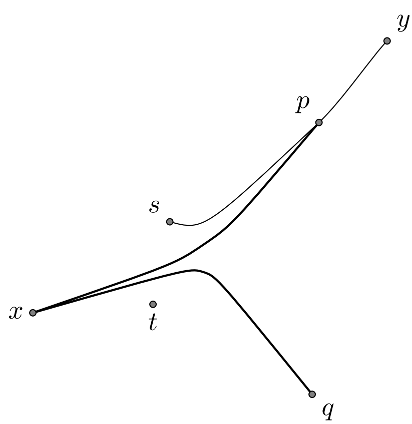


Figure 3.4. Definition of a foldable configuration

Remark. — This framework is a little more general than the one presented above. It naturally arises later when folding a geodesic along the axis of a relation (see Proposition 3.18). The reason to keep track of the point y will appear in Lemma 4.12. Note that Condition (C3) implies not only that s and y are $\mathcal{Q}$-close, but also that s and p. Indeed, for every $(H, Y) \in \mathcal{Q}$, Inequality (3) yields

$$\mathfrak{D}(\{s, p\}, Y) \leq \mathfrak{D}(\{s, y\}, Y) + \langle y, s \rangle_p \leq \mathfrak{D}(\{s, y\}, Y) \leq T(H)/2 + 10\varepsilon.$$

In the situation described above, the configuration (x, p, q, p) is foldable. Indeed, one can choose $s = t = x$ in Definition 3.13 (actually any point s on a geodesic $[x, p]$ such that $|x - s| \leq \langle p, q \rangle_x + 3\pi \sinh \rho$ would work). After folding the geodesic $[x, p]$ on $[x, q]$ using h, we will prove that the configuration (x, hp, q, hp) is still foldable, where the points hp' and x (see Figure 3.3) roughly play the role of s and t in Definition 3.13.

Proposition 3.14. — *Let (x, p, q, y) be a foldable configuration such that $\bar{p} = \bar{q}$ but $p \neq q$. There exists $(H, Y) \in \mathcal{Q}$ and a generator h of H satisfying the following.*

(i) $\mathfrak{D}(\{x, y\}, Y) \geq T(H)/2 - 6\pi \sinh \rho$,
(ii) $\langle hp, q \rangle_x \geq \langle p, q \rangle_x + T(H)/2 - 6\pi \sinh \rho$,
(iii) *the configuration (x, hp, q, hy) is foldable.*
(iv) $\langle x, y \rangle_p \leq \delta$ *and* $\langle x, hy \rangle_{hp} \leq 9\pi \sinh \rho$

Proof. — The points s and t are those given by the definition of a foldable configuration. We assumed that $\bar{p} = \bar{q}$ but $p \neq q$. By Greendlinger's Lemma (Theorem 3.5), there exists $(H, Y) \in \mathcal{Q}$ such that $\mathfrak{D}(\{p, q\}, Y) \geq T(H) - 2\pi \sinh \rho$. By assumption, H is a cyclic group generated by some $h \in G$. Since $T(H) \geq 100\pi \sinh \rho$, we observe that $T(H) = [h]$. We denote by γ a δ-nerve of h, and T its fundamental length. Let $x' = \gamma(a)$, $p' = \gamma(b)$, $q' = \gamma(c)$ and $y' = \gamma(d)$ be respective projections of x, p, q and y on γ. According to Proposition 1.19, a is between b and c (see Figure 3.5). Moreover, we have

(F1) $|p' - q'| \geq [h] - 2\pi \sinh \rho - 73\delta$,
(F2) $[h]/2 - 5\pi \sinh \rho - 10\varepsilon - 156\delta \leq |x' - p'| \leq [h]/2 + 3\pi \sinh \rho + 10\varepsilon + 83\delta$,
(F3) $[h]/2 - 5\pi \sinh \rho - 10\varepsilon - 156\delta \leq |x' - q'| \leq [h]/2 + 3\pi \sinh \rho + 10\varepsilon + 83\delta$,
(F4) $||x - x'| - \langle p, q \rangle_x| \leq 25\delta$,
(F5) $\langle s, p \rangle_{p'} \leq 20\delta$,

On the configuration (x, p, q, y). Proposition 1.19 told us how the projections p', q' and x' are arranged on Figure 3.5. We now explain how the point y' appears in this picture.

The points x' and p' are projections of x and p, respectively, on γ; thus $|x - p| \geq |x - x'| + |x' - p'| - 38\delta$. Combined with Points (F2) and (F4), we get

$$|x - p| > \langle p, q \rangle_x + 3\pi \sinh \rho + \delta \geq |x - s| + \delta. \tag{15}$$

Applying the four-point inequality (1) we get

$$\min \left\{ \langle x, y \rangle_p , \langle x, s \rangle_p \right\} \leq \langle s, y \rangle_p + \delta \leq \delta$$

According to (15), $\langle x, s \rangle_p \geq |x - p| - |x - s| > \delta$. Hence, we necessarily have $\langle x, y \rangle_p \leq \delta$, which proves the first part of Point (iv). By definition, the nerve

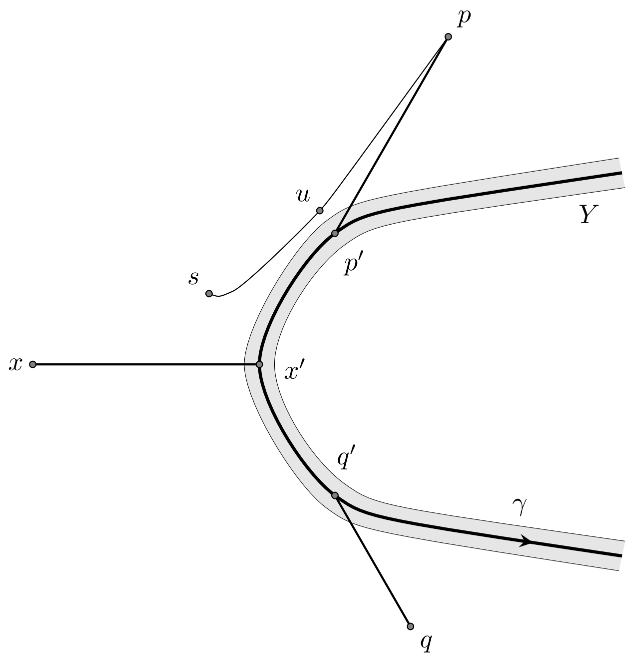


FIGURE 3.5. Positions of the points x', p', q' and u

γ is contained in Y. Combining Proposition 1.11 with (3) and (F2), we get

$$\begin{aligned}\mathfrak{D}(\{x,y\},Y) \geq \mathfrak{D}(\{x,y\},\gamma) &\geq \mathfrak{D}(\{x,p\},\gamma) - \langle x,y\rangle_p \\ &\geq [h]/2 - 5\pi\sinh\rho - 10\varepsilon - 176\delta,\end{aligned}$$

which provides to Point (i)

Claim 1. $|x'-y'| \leq [h]/2 + 3\pi\sinh\rho + 10\varepsilon + 85\delta$. The triangle inequality combined with Lemma 1.1 (i) leads to

$$\langle s,y\rangle_{x'} \leq \max\left\{|x-s| - |x-x'| + 2\langle x,y\rangle_{x'}, \langle x,y\rangle_{x'}\right\} + \delta.$$

By (F4), we know that $|x-s| \leq \langle p,q\rangle_x + 3\pi\sinh\rho \leq |x-x'| + 3\pi\sinh\rho + 25\delta$. On the other hand, x' and p' being projections of x and p on γ, we have $\langle x,p\rangle_{x'} \leq 19\delta$ (Lemma 1.7). Thus, the triangle inequality leads to $\langle x,y\rangle_{x'} \leq \langle x,y\rangle_p + \langle x,p\rangle_{x'} \leq 20\delta$. It follows that $\langle s,y\rangle_{x'} \leq 3\pi\sinh\rho + 66\delta$. However, y' is a projection of y on γ. The points s and y being $\mathcal{Q}$-close, Proposition 1.11 combined with (3) yields

$$\begin{aligned}|x'-y'| \leq \mathfrak{D}(\{x',y\},Y) + 19\delta &\leq \mathfrak{D}(\{s,y\},Y) + \langle s,y\rangle_{x'} + 19\delta \\ &\leq [h]/2 + 3\pi\sinh\rho + 10\varepsilon + 85\delta.\end{aligned}$$

Claim 2. $\langle y', y\rangle_p \leq 8\pi \sinh\rho + 20\varepsilon + 272\delta$. By triangle inequality, $\langle y', y\rangle_p \leq \langle x, y\rangle_p + \langle x, p\rangle_{p'} + |p' - y'|$. The Gromov products on the right-hand side of the inequality are small – $\langle x, y\rangle_p \leq \delta$ and $\langle x, p\rangle_{x'} \leq 19\delta$; therefore, it is sufficient to find an upper bound for $|p' - y'|$. In particular, we can assume that $|p' - y'| > \pi \sinh\rho$. Note that, since $\langle x, y\rangle_p \leq \delta$, the parameters a and d cannot belong to the same component of $\mathbf{R} \setminus \{b\}$. Roughly speaking, the points y' and x' cannot belong to the same component of $\gamma \setminus \{p'\}$. Thus, b lies between a and d. It follows from Claim 1 and Point (F2) that

$$|p' - y'| = |x' - y'| - |x' - p'| + 2\,\langle x', y'\rangle_{p'} \leq 8\pi \sinh\rho + 20\varepsilon + 252\delta.$$

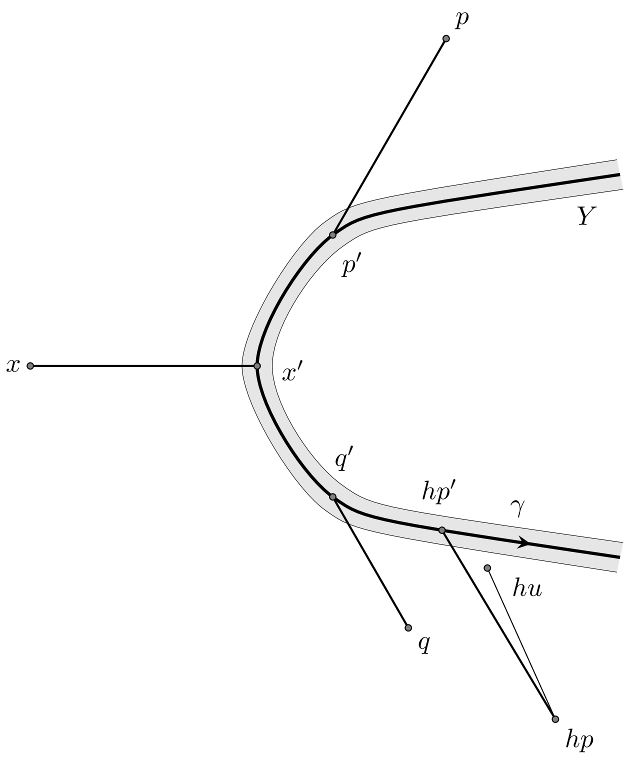


FIGURE 3.6. Positions of the point hp', hu and hp

Translation by h. We now use h to make a fold (see Figure 3.6). The isometry h acts on γ by translating the parameter by a length T. By replacing, if necessary, h by h^{-1}, we can assume that $b + T$ and c belong to the same component of $\mathbf{R} \setminus \{b\}$ (see Figure 3.6). Thus, hp' and q' lie on γ "on the same component" of $\gamma \setminus \{p'\}$.

Claim 3. $|x - hp'| \leq \langle hp, q \rangle_x + 2\pi \sinh \rho + 101\delta$. Note first that

$$|a - b| \leq |x' - p'| + \delta \leq [h] \leq |hp' - p'| \leq T$$

Thus a is between b and $b + T$. By triangle inequality we have

$$|hp' - x'| \geq [h] - |x' - p'| \geq [h]/2 - 3\pi \sinh \rho - 10\varepsilon - 83\delta \tag{16}$$

We now distinguish two cases. If $b + T$ is between a and c (i.e. hp' lies on γ between x' and q'), then $\langle x, q \rangle_{hp'} \leq 25\delta$ and $\langle x, hp \rangle_{hp'} \leq 19\delta$. By hyperbolicity we obtain

$$|x - hp'| \leq \langle hp, q \rangle_x + \max \left\{ \langle x, hp \rangle_{hp'}, \langle x, q \rangle_{hp'} \right\} + \delta \leq \langle hp, q \rangle_x + 26\delta.$$

Assume now that c is between a and $b + T$ (i.e. q' lies on γ between x' and hp'). As previously, we show that $|x - q'| \leq \langle hp, q \rangle_x + 26\delta$. On the other hand, γ is a T-local $(1, \delta)$-quasi-geodesic; thus, using Point (F1), we get

$$|q' - hp'| \leq [h] - |p' - q'| + 2\delta \leq 2\pi \sinh \rho + 75\delta.$$

It follows from the triangle inequality that

$$|x - hp'| \leq |x - q'| + |q' - hp'| \leq \langle hp, q \rangle_x + 2\pi \sinh \rho + 101\delta,$$

which completes the proof of Claim 3.

As we already observed, a is between b and $b + T$, i.e. the point x' lies on γ between p' and hp'. Since x' is a projection of x on γ, we have $|x - hp'| \geq |x - x'| + |x' - hp'| - 18\delta$. Consequently,

$$\begin{aligned} \langle hp, q \rangle_x &\geq |x - x'| + |x' - hp'| - 2\pi \sinh \rho - 119\delta \\ &\geq \langle p, q \rangle_x + [h]/2 - 5\pi \sinh \rho - 10\varepsilon - 227\delta, \end{aligned}$$

The first inequality is indeed a consequence of Claim 3, while the second follows from (F4) and (16). This completes the proof of Point (ii).

We now prove that (x, hp, q, hy) is foldable. We denote by $[s, p]$ a geodesic joining s to p and u a projection of p' on $[s, p]$. The points hu and t will play the role of s and t, respectively, in the definition of foldable configuration. Note that the point t already satisfies Condition (C2). Indeed, the points x and q did not move during the folding; hence they are still $\mathcal{Q}$-close. Moreover, Point (ii) implies $\langle p, q \rangle_x \leq \langle hp, q \rangle_x$. Consequently,

$$|x - q| \leq \langle p, q \rangle_x + 3\pi \sinh \rho \leq \langle hp, q \rangle_x + 3\pi \sinh \rho.$$

We now check Conditions (C1) and (C3). Recall that $\langle s, y \rangle_p = 0$. Thus, the points y, p, u and s can be ordered in this way along a geodesic of X. Hence, $\langle y, u \rangle_p = 0$ and $\langle s, y \rangle_u = 0$. Moreover, for every $(H', Y') \in \mathcal{Q}$, Inequality (3) yields

$$\mathfrak{D}(\{u, y\}, Y') \leq \mathfrak{D}(\{s, y\}, Y') + \langle s, y \rangle_u \leq \mathfrak{D}(\{s, y\}, Y') \leq T(H')/2 + 10\varepsilon.$$

Translating these properties by h we get that $\langle hy, hu\rangle_{hp} = 0$ and for every $(H', Y') \in \mathcal{Q}$,

$$\mathfrak{D}(\{hu, hy\}, Y') \leq T(H')/2 + 10\varepsilon,$$

which gives the first half of Condition (C1) and Condition (C3). As a geodesic, $[s, p]$ is 4δ-quasi-convex. Point (F5) yields $|p' - u| \leq 24\delta$. Using Claim 3 we obtain

$$|x - hu| \leq |x - hp'| + |p' - u| \leq \langle hp, q\rangle_x + 2\pi \sinh \rho + 125\delta \leq \langle hp, q\rangle_x + 3\pi \sinh \rho.$$

Consequently, hu satisfies the second half of Condition (C1). Finally, (x, hp, q, hy) is foldable, which is exactly Point (iii).

In only remains to prove that $\langle x, hy\rangle_{hp} \leq 9\pi \sinh \rho$. By Claim 1, $|x' - y'| \leq [h]/2 + 3\pi \sinh \rho + 10\varepsilon + 85\delta$. Thus, the triangle inequality gives $|x' - hy'| \geq [h]/2 - 3\pi \sinh \rho - 10\varepsilon - 85\delta$. Since x' and hy' are respective projections of x and hy on γ, we get $\langle x, hy\rangle_{hy'} \leq 19\delta$. Claim 2 leads to $\langle x, hy\rangle_{hp} \leq \langle x, hy\rangle_{hy'} + \langle y', y\rangle_p \leq 8\pi \sinh \rho + 20\varepsilon + 291\delta$, which completes the proof of Point (iv) and of the proposition. ☐

3.6. Lifting figures of $\bar{X}$ in X. — In this section we try to find the best way to lift in X a figure of $\bar{X}$. Lemma 3.15 explains the following phenomenon. The inclusion $X \to \dot{X}$ distorts the metric. In particular, $X \to \bar{X}$ does not map geodesics to geodesics. Nevertheless, given two $\mathcal{Q}$-close points $x, y \in X$, we prove that if $\bar{z} \in \bar{X}$ is a point *in the image of* X, lying on a geodesic $[\bar{x}, \bar{y}] \subset \bar{X}$, then it admits a pre-image (in X), which is roughly on a geodesic of X from x to y. Lemma 3.16 has a similar flavor. Consider a $\mathcal{Q}$-reduced isometry $g \in G$. In particular, g and its image $\bar{g} \in \bar{G}$ are loxodromic. If $\bar{x} \in \bar{X}$ is a point *in the image of* X, which belongs to the axis of $\bar{g}$ (seen as an isometry of $\bar{X}$), then it admits a pre-image in X which roughly lies on the axis of g (seen as an isometry of X). In Proposition 3.18 we are interested in the following situation. Let x and y be two $\mathcal{Q}$-close points of X and g a $\mathcal{Q}$-reduced isometry of G. We assume that $[\bar{x}, \bar{y}]$ and $Y_{\bar{g}}$ have a large overlap in $\bar{X}$ – for instance larger than $[\bar{g}^k]$ with $k \gg 1$ – and would like to “lift” this overlap. By replacing if necessary g by a conjugate of g we may translate Y_g such that $[x, y]$ and Y_g have more or less a non-empty intersection. However, there is no reason that their overlap should be as large in X as in $\bar{X}$. We face the same kind of problem exposed at the beginning of Section 3.5. Nevertheless, lifting the endpoints of $[\bar{x}, \bar{y}] \cap Y_{\bar{g}}$, one can build a foldable configuration. In the same way as explained in Section 3.5, we will use this configuration in Section 4 in order to translate y by elements of $\mathcal{Q}$ and fold the geodesic $[x, y]$ onto Y_g.

LEMMA 3.15. — *Let x and y be two $\mathcal{Q}$-close points of X. Let $z \in X$ such that for all $u \in K$, $\langle x, y\rangle_z \leq \langle x, y\rangle_{uz} + \delta$. Moreover, we assume that $\langle \bar{x}, \bar{y}\rangle_{\bar{z}} \leq 450\bar{\delta}$, Then $\langle x, y\rangle_z \leq \varepsilon$.*

Proof. — The points x and y are $\mathcal{Q}$-close. Hence, by Proposition 3.8, for all $(H,Y) \in \mathcal{Q}$, the quantities $\mathfrak{D}(\{x,z\},Y)$ and $\mathfrak{D}(\{y,z\},Y)$ are bounded above by $T(H) - 2\pi \sinh \rho$. The result follows from Proposition 3.6. □

LEMMA 3.16. — *Let g be a $\mathcal{Q}$-reduced element of G. Let $x \in X$ such that for all $u \in K$, $d(x, Y_g) \leq d(ux, Y_g) + \delta$. We assume also that $d(\bar{x}, Y_{\bar{g}}) \leq 350\bar{\delta}$. Then $d(x, Y_g) \leq 2\varepsilon$.*

Proof. — By Proposition 3.12, there exists $k_0 \in \mathbf{N}$ such that for all $k \geq k_0$, for all $(H,Y) \in \mathcal{Q}$, $\mathfrak{D}(\{x, g^k x\}, Y) \leq T(H) - 2\pi \sinh \rho$. However, $\bar{g}$ is a loxodromic isometry. Therefore, there exists $k \geq k_0$ such that $[g^k] > L_S \delta$ and $[\bar{g}^k] > L_S \bar{\delta}$. It follows from Lemma 1.15 that the distance from x to Y_g is approximatively given by $\langle g^{-k}x, g^k x\rangle_x$. The same works for $\bar{x}$ and $Y_{\bar{g}}$. More precisely,

$$\left\langle \bar{g}^{-k}\bar{x}, \bar{g}^k \bar{x} \right\rangle_{\bar{x}} \leq d(\bar{x}, Y_{\bar{g}}) + 52\bar{\delta} \leq 450\bar{\delta}.$$

Applying Proposition 3.6 we get

$$d(x, Y_g) \leq \left\langle g^{-k}x, g^k x \right\rangle_x + 52\delta \leq \varepsilon + 52\delta \leq 2\varepsilon. \qquad \square$$

PROPOSITION 3.17. — *Let $k \in \mathbf{N}$. Let $D > 2\rho$. Let g be a $\mathcal{Q}$-reduced element of G such that $[\bar{g}^k] > L_S \bar{\delta}$. Let p and q be two points of X satisfying the followings*

(i) *$d(\bar{p}, Y_{\bar{g}}), d(\bar{q}, Y_{\bar{g}}) \leq 350\bar{\delta}$,*
(ii) *for all $u \in K$, $d(p, Y_g) \leq d(up, Y_g) + \delta$ and $d(q, Y_g) \leq d(uq, Y_g) + \delta$,*
(iii) *$|\bar{p} - \bar{q}| \geq [\bar{g}^k] + D$.*

Then $|p - q| \geq [g^k] + D - 3\varepsilon$.

Proof. — Let $\bar{\gamma}$ be a $\bar{\delta}$-nerve of $\bar{g}^k$ (in $\bar{X}$) and T its fundamental length. We denote by $\bar{p}' = \bar{\gamma}(a)$ and $\bar{q}' = \bar{\gamma}(b)$ respective projections of $\bar{p}$ and $\bar{q}$ on $\bar{\gamma}$. The isometry $\bar{g}^k$ acts on $\bar{\gamma}$ by translating the parameter by T. By replacing if necessary g by g^{-1}, we can assume that b and $a + T$ belong to the same component of $\mathbf{R} \setminus \{a\}$, which, roughly speaking, means that $\bar{q}'$ and $\bar{g}^k \bar{p}'$ belong to the same component of $\bar{\gamma} \setminus \{\bar{p}'\}$. Since $[\bar{g}^k] > L_S \delta$, $Y_{\bar{g}}$ is contained in the $27\bar{\delta}$-neighborhood of $\bar{\gamma}$. In particular $|\bar{p} - \bar{p}'|$ and $|\bar{q} - \bar{q}'|$ are bounded above by $377\bar{\delta}$. It follows from the triangle inequality that

$$|a - b| \geq |\bar{p}' - \bar{q}'| \geq |\bar{p} - \bar{q}| - 754\bar{\delta} \geq [\bar{g}^k] + \bar{\delta} \geq T.$$

It follows that $a \leq a + T \leq b$. In other words, $\bar{g}^k \bar{p}'$ lies on $\bar{\gamma}$ between $\bar{p}'$ and $\bar{q}'$. Moreover, $\langle \bar{p}, \bar{q} \rangle_{\bar{g}^k \bar{p}'} \leq 25\bar{\delta}$. Hence, $\langle \bar{p}, \bar{q} \rangle_{\bar{g}^k \bar{p}} \leq \langle \bar{p}, \bar{q} \rangle_{\bar{g}^k \bar{p}'} + |\bar{p}' - \bar{p}| \leq 450\bar{\delta}$. According to Point (ii), p and q are the respective lifts of $\bar{p}$ and $\bar{q}$, which are almost the closest to Y_g. Hence, by Lemma 3.16, p and q belongs to the 2ε-neighborhood of Y_g. It follows from Lemma 3.11 that for all $(H,Y) \in$

$\mathcal{Q}$, the quantities $\mathfrak{D}(\{p, g^k p\}, Y)$ and $\mathfrak{D}(\{q, g^k p\}, Y)$ are bounded above by $T(H) - 2\pi \sinh \rho$. Consequently, by Proposition 3.6 $\langle p, q \rangle_{g^k p} \leq \varepsilon$. In particular,

$$|p - q| \geq \left|p - g^k p\right| + \left|g^k p - q\right| - 2\varepsilon \geq [g^k] + \left|g^k p - q\right| - 2\varepsilon. \tag{17}$$

Recall that $Y_{\bar{g}}$ (and thus $\bar{\gamma}$) lies in the $52\bar{\delta}$-neighborhood of the axis of $\bar{g}^k$ (Proposition 1.14). Consequently, we have $|\bar{g}^k \bar{p} - \bar{p}| \leq 2|\bar{p} - \bar{p}'| + [\bar{g}^k] + 112\bar{\delta}$. Since the map $X \to \bar{X}$ shortens the distances, we get

$$\left|g^k p - q\right| \geq \left|\bar{g}^k \bar{p} - \bar{q}\right| \geq |\bar{p} - \bar{q}| - \left|\bar{g}^k \bar{p} - \bar{p}\right| \geq |\bar{p} - \bar{q}| - [\bar{g}^k] - 2\left|\bar{p} - \bar{p}'\right| - 112\bar{\delta}$$

Using Point (iii) we deduced that $|g^k p - q| \geq D - 866\bar{\delta}$, which together with (17) leads to the result. ☐

Proposition 3.18. — *Let x and y be two $\mathcal{Q}$-close points of X. Let g be a $\mathcal{Q}$-reduced element of G. Let $k \in \mathbf{N}$ such that $[\bar{g}^k] > L_S \bar{\delta}$. Let $D > 10\rho$ such that $\mathfrak{D}(\{\bar{x}, \bar{y}\}, Y_{\bar{g}}) \geq [\bar{g}^k] + D$. There exist three points $r, p, q \in X$ and $v \in K$ satisfying the following properties*

(i) $\bar{p} = \bar{q}$.
(ii) $d(r, vY_g) \leq 2\varepsilon$, $d(q, vY_g) \leq 4\varepsilon$, $\langle x, q \rangle_r \leq 3\varepsilon$ *and* $\langle x, y \rangle_p = 0$,
(iii) $|r - q| \geq [g^k] + D - \pi \sinh \rho$.
(iv) *The configuration (x, p, q, y) is foldable.*

Proof. — Let us denote by $\bar{x}'$ and $\bar{y}'$ respective projections of $\bar{x}$ and $\bar{y}$ on $Y_{\bar{g}} \subset \bar{X}$. By Proposition 1.11, $|\bar{x}' - \bar{y}'| \geq [\bar{g}^k] + D - 5\bar{\delta}$. Recall that $\bar{X}$ is obtained by attaching cones on X/K. Hence, $\bar{x}'$ and $\bar{y}'$ may not belong to $\zeta(X)$, the image of X in $\bar{X}$. However, these cones have diameter 2ρ. Thus there exist two points $\bar{r}$ and $\bar{z}$ in $\zeta(X)$ on a geodesic $[\bar{x}', \bar{y}']$ such that $|\bar{x}' - \bar{r}|, |\bar{x}' - \bar{z}| \leq 2\rho$. In particular, $|\bar{r} - \bar{z}| \geq [\bar{g}^k] + D - 4\rho - 5\bar{\delta}$. Since $Y_{\bar{g}}$ is $2\bar{\delta}$-quasi-convex, $\bar{r}$ and $\bar{z}$ are in the $2\bar{\delta}$-neighborhood of $Y_{\bar{g}}$. Moreover, $\langle \bar{x}, \bar{y} \rangle_{\bar{r}}, \langle \bar{x}, \bar{y} \rangle_{\bar{z}} \leq 5\bar{\delta}$ and $\langle \bar{x}, \bar{z} \rangle_{\bar{r}} \leq 2\bar{\delta}$. The next step of the proof consists in lifting this figure in X. Let $r, z \in X$ be respective pre-images of $\bar{r}$ and $\bar{z}$ such that for all $u \in K$, we have in X $\langle x, y \rangle_r \leq \langle x, y \rangle_{ur} + \delta$ and $\langle x, y \rangle_z \leq \langle x, y \rangle_{uz} + \delta$. Since x and y are $\mathcal{Q}$-close, Lemma 3.15 leads to $\langle x, y \rangle_r, \langle x, y \rangle_z \leq \varepsilon$. In particular, there is a point $p \in X$ on a geodesic $[x, y]$ such that $|p - z| \leq \varepsilon + 4\delta$.

We now choose a conjugate of g whose axis in X approximatively passes through r. To that end, we fix $v \in K$ such that for all $u \in K$, we have $d(r, vY_g) \leq d(ur, vY_g) + \delta$. By assumption, g is $\mathcal{Q}$-reduced. Hence, vY_g is the cylinder of vgv^{-1} which is $\mathcal{Q}$-reduced as well. By Lemma 3.16, $d(r, vY_g) \leq 2\varepsilon$. We choose for z a lift of $\bar{z}$ such that $\langle x, y \rangle_z \leq \varepsilon$. Unfortunately, z is not necessarily in the neighborhood of vY_g. This is why we have to introduce a second pre-image of $\bar{z}$. Let $w \in K$ such that for all $u \in K$, $d(wz, vY_g) \leq d(uwz, vY_g) + \delta$. By Lemma 3.16, $d(wz, vY_g) \leq 2\varepsilon$. We finally put $q = wp$. In particular, $d(q, vY_g) \leq 3\varepsilon + 4\delta$. Moreover, $\bar{p} = \bar{q}$, which proves Point (i).

By construction $\langle x, y\rangle_r \leq \varepsilon$. However, x and y are $\mathcal{Q}$-close. Hence, for all $(H, Y) \in \mathcal{Q}$, we have

$$\mathfrak{D}(\{x, r\}, Y) \leq \mathfrak{D}(\{x, y\}, Y) + \langle x, y\rangle_r \leq T(H) - 2\pi \sinh \rho.$$

On the other hand, $d(r, vY_g)$ and $d(wz, vY_g)$ are bounded above by 2ε. The isometry vgv^{-1} being $\mathcal{Q}$-reduced, Lemma 3.11 implies that for all $(H, Y) \in \mathcal{Q}$, we have

$$\mathfrak{D}(\{r, wz\}, Y) \leq T(H) - 2\pi \sinh \rho.$$

Since $\langle \bar{x}, \bar{z}\rangle_{\bar{r}} \leq 2\bar{\delta}$, Proposition 3.6 yields

$$\langle x, q\rangle_r \leq \langle x, wz\rangle_r + |p - z| \leq 2\varepsilon + 4\delta,$$

which completes the proof of Point (ii). In the same way, we can prove that $\langle x, p\rangle_r \leq 2\varepsilon + 4\delta$. Note that vgv^{-1}, r and wz satisfy the assumptions of Proposition 3.17. Therefore, $|r - wz| \geq [g^k] + D - 4\rho - 3\varepsilon - 5\bar{\delta}$. Thus, $|r - q| \geq [g^k] + D - \pi \sinh \rho$, which gives Point (iii).

It only remains to prove that (x, p, q, y) is foldable. In the definition of foldable configuration, we choose $s = x$ and $t = r$. By construction $\langle x, y\rangle_p = 0$. Moreover, x and y are $\mathcal{Q}$-closed; thus Conditions (C1) and (C3) are fulfilled. We proved that $d(r, vY_g) \leq 2\varepsilon$ and $d(q, vY_g) \leq 3\varepsilon + 4\delta$. Moreover, vgv^{-1} is $\mathcal{Q}$-reduced. By Lemma 3.11, r and q are $\mathcal{Q}$-close. On the other hand, $\langle x, q\rangle_r \leq 2\varepsilon + 4\delta$ and $\langle x, p\rangle_r \leq 2\varepsilon + 4\delta$. Therefore, by hyperbolicity, $|x - r| \leq \langle p, q\rangle_x + 2\varepsilon + 5\delta$. Thus, Condition (C2) holds. ☐

4. Periodic groups

4.1. General framework. — This section is dedicated to the proof of our main theorem. Our goal is to provide a criterion to decide whether an element of $\mathbf{F}_r$ induces a trivial element of $\mathbf{B}_r(n)$. The tools that we developed also work for periodic quotients of hyperbolic groups. A.Y. Ol'shanskĭ proved that hyperbolic groups indeed allow infinite periodic quotients.

THEOREM 4.1 (Ol'shanskĭ [21]). — *Let G be a non-elementary torsion-free hyperbolic group. There exists a critical exponent n_0 such that for every odd integer $n \geq n_0$, the quotient G/G^n is infinite.*

In [12] T. Delzant and M. Gromov provide an alternative proof of this result. In order to study periodic quotients of G, they use small cancellation theory to build a sequence of hyperbolic groups (G_k), which approximate G/G^n. We recall here the main steps of the construction following the exposition given by the author in [8].

Let (X, x_0) be a pointed proper geodesic δ-hyperbolic space. Let G be a group acting properly co-compactly by isometries on X. Recall that L_S is the constant given by the stability of quasi-geodesics. Let $\delta_1 = 64.10^4\boldsymbol{\delta}$. The constants ρ_0, ε, δ_0 and Δ_0 are the ones defined in Section 3.1. Note that ρ_0,

δ_1 and L_S are also the constants that appear in [8, Proposition 7.1]. We fix $\rho = \rho_0$. Observe that none of these constants depends on G or X. For every positive integer n, define a rescaling parameter λ_n by

$$\lambda_n = \frac{8\pi \sinh \rho}{\sqrt{n\rho L_S \delta_1}}.$$

The integer n_0 is chosen so that it is large enough that for every $n \geq n_0$, the parameter λ_n satisfies a set of inequalities. The exact statement of the inequalities it should satisfy is not important here. Basically, they are chosen in such a way that one can iterate the small cancellation process explained below. The conditions to fulfill coarsely say that $\lambda_n \delta_1 \ll \delta_0$ and $\lambda_n \pi \sinh(L_S \delta_1) \ll \Delta_0$. For more details, see the proof of [8, Proposition 7.1]. We build by induction two sequences (X_k) and (G_k) as follows.

The base of induction. Among other things, we can assume, by rescaling X if necessary, that X is δ-hyperbolic, with $\delta \leq \delta_1$ and $A(G,X) \leq 6\pi \sinh(2L_S\delta_1)$, where the invariant A is the one given in Definition 1.21. Recall that $r_{inj}(G,X) > 0$; thus there exists $n_1 \geq n_0$ such that $r_{inj}(G,X) \geq \sqrt{\rho L_S \delta_1 / 4n_1}$. For our purpose we also ask that

$$\lambda_{n_1} \leq 1/12, \quad 6\lambda_{n_1}\pi \sinh(2L_S\delta_1) \leq \varepsilon \quad \text{and} \quad 10\sqrt{n_1 \rho L_S \delta_1} \geq L_S \delta_1. \tag{18}$$

We now fix ξ such that

$$\xi = n_1 + 1$$

We let $n_2 = \max\{100, 50n_1\}$ and fix an odd integer $n \geq n_2$. We put $G_0 = G$, and $X_0 = X$. Using the vocabulary introduced in [8, Section 7.1], we can say that (G_0, X_0) *satisfies the induction hypotheses for exponent* n. For simplicity of notation, we write λ instead of λ_{n_1}.

The inductive step. We now assume that we constructed the group G_k together with the space X_k such that (G_k, X_k) satisfies the induction hypotheses for exponent n. In particular, we have the following properties.

(i) X_k is a proper geodesic δ_1-hyperbolic space.
(ii) G_k acts properly co-compactly by isometries on X_k.
(iii) $A(G_k, X_k) \leq 6\pi \sinh(2L_S\delta_1)$.
(iv) $r_{inj}(G_k, X_k) \geq \sqrt{\rho L_S \delta_1 / 4n_1}$.

We denote by P_k the set of elements $h \in N_k$ which are not a proper power such that $[h]_{X_k} \leq L_S\delta_1$. We define $\mathcal{Q}_k$ by

$$\mathcal{Q}_k = \left\{ \left(\langle h^n \rangle, Y_h \right) \mid h \in P_k \right\}.$$

It turns out that $\mathcal{Q}_k$ satisfies (among others) the assumptions of the small cancellation theorem (Theorem 3.1) in the rescaled space λX_k [8, Proposition 7.1]. More precisely, λX_k is δ-hyperbolic with $\delta \leq \delta_0$ and $r_{inj}(G_k, \lambda X_k) \geq 4\pi \sinh \rho / n_1$. In addition, (18) yields $A(G_k, \lambda X_k) \leq \varepsilon$. It follows from our choice of n that $T(\mathcal{Q}_k) \geq 100\pi \sinh \rho$ (measured in λX_k) and $(\xi - 1) r_{inj}(G_k, \lambda X_k) \geq 4\pi \sinh \rho$.

Let K_k be the (normal) subgroup of G_k generated by $\{h^n \mid h \in P_k\}$ and G_{k+1} the quotient G_k/K_k. The space $\bar{X}_{k+1}$ is the one constructed from λX_k by small cancellation (see Section 3). It follows from [8, Proposition 7.1] that (G_{k+1}, X_{k+1}) satisfies the induction hypothesis for exponent n. In particular, the assumptions (i)–(iv) holds at rank $k+1$. Moreover, the canonical map $\zeta_k : X_k \to X_{k+1}$ has the following property: for all $x, x' \in X_k$, $|\zeta_k(x) - \zeta_k(x')|_{X_{k+1}} \leq \lambda |x - x'|_{X_k}$.

Remark. — The set P_k could be empty. This would mean that $r_{inj}(G_k, X_k) \geq (L_S - 32)\delta_1$ (Proposition 1.13). In this situation, we let $G_{k+1} = G_k$ and take for X_{k+1} the rescaled space $X_{k+1} = \lambda X_k$. It is easy to check that still (G_{k+1}, X_{k+1}) satisfies the induction hypothesis for exponent n. Moreover, the canonical map $\zeta_k \colon X_k \to X_{k+1}$ is obviously λ-Lipschitz.

The group G/G^n studied in Theorem 4.1 is isomorphic to the direct limit of the sequence (G_k).

Notations.

(i) For all $k \in \mathbf{N}$ the kernel of the projection $G \twoheadrightarrow G_k$ is denoted by N_k. In particular, for all $k \in \mathbf{N}$, $N_k \lhd N_{k+1}$. Note that G^n is the ascending union of all the N_k.

(ii) Let x be a point of X (respectively g be an element of G). For simplicity of notation, we still write x (respectively g) for its image by the natural map $X \to X_k$ (respectively $G \twoheadrightarrow G_k$).

4.2. Close points, reduced elements of rank k. —

Remark. — From now on, unless otherwise stated, all the metric objects (distances, diameters, Gromov's products) are measured with the distance of the rescaled space λX_k (and never with the one of X_k).

Definition 4.2. — Let $k \in \mathbf{N}$. Two points x and x' of X are close of rank k if for all $j < k$, for all $(H, Y) \in \mathcal{Q}_j$, we have $\mathfrak{D}(\{x, x'\}, Y) \leq T(H)/2 + 10\varepsilon$ in the space λX_j.

Definition 4.3. — Let $k \in \mathbf{N}$. An element g of G is reduced of rank k if g is loxodromic as element of G_k (acting on X_k) and for all $j < k$, for all $(H, Y) \in \mathcal{Q}_j$, we have $\mathfrak{D}(Y_g, Y) < T(H)/2 + 2\varepsilon$ in the space λX_j.

Remark. — Note that *being close* (respectively *reduced*) *of rank 0* is an empty condition. Any two points of X are close of rank 0. Any loxodromic element of G is reduced of rank 0.

Proposition 4.4. — *Let $k \in \mathbf{N}$. Let $g \in G$. If g is loxodromic in G_k then there exists $u \in N_k$ such that ug is reduced of rank k.*

Proof. — The proof is by induction on k. Since every loxodromic element of G is reduced of rank 0, the proposition is true for $k = 0$. Assume now that the proposition holds for $k \in \mathbf{N}$. Let $g \in G$ such that g is loxodromic in G_{k+1}. Recall that for every $h \in P_k$, $[h] \leq L_S\delta_1$ and $A(G_k, \lambda X_k) \leq \varepsilon$. Hence, by Proposition 3.10 there exists $u \in N_{k+1}$ such that ug is $\mathcal{Q}_k$-reduced, i.e. for all $(H, Y) \in \mathcal{Q}_k$, $\mathfrak{D}(Y_{ug}, Y) \leq T(H)/2 + 2\varepsilon$ in the space λX_k. Note that $g = ug$ in G_{k+1}. Thus, ug is loxodromic in G_{k+1} and therefore in G_k. We apply the induction hypothesis on ug: there exists $v \in N_k$ such that vug is reduced of rank k. However, $vug = ug$ in G_k. Hence, for all $j \leq k$, for all $(H, Y) \in \mathcal{Q}_j$, $\mathfrak{D}(Y_{vug}, Y) \leq T(H)/2 + 2\varepsilon$ in the space λX_j, which means that vug is reduced of rank $k + 1$. Moreover, since $N_k \lhd N_{k+1}$, $vu \in N_{k+1}$. Consequently, the proposition holds for $k + 1$. □

4.3. Elementary moves in X. — Recall that x_0 is a base point of X.

DEFINITION 4.5. — Let y and z be two points of X.

- We say that z is the image of y by a (n, ξ)-*elementary move* (or simply an *elementary move*), if there exist $g \in G$ satisfying the following
  (i) $\mathfrak{D}(\{x_0, y\}, Y_g) \geq [g^m]$ in the space X, with $m \geq n/2 - \xi$.
  (ii) $z = g^n y$ in X.
- We say that z is the image of y by a *sequence of elementary moves*, and we write $y \to z$, if there exists a finite sequence of points $y = y_0, y_1, \dots, y_\ell = z$ of X, such that for all $j \in [\![0, \ell - 1]\!]$, the point y_{j+1} is the image of y_j by an elementary move.

Note that the condition $\mathfrak{D}(\{x_0, y\}, Y_g) \geq [g^m]$ does not change if we rescale the space X. Therefore, we can use it either in X or $\lambda X_0 = \lambda X$. Our goal is to characterize the elements in the kernel N of the map $G \twoheadrightarrow G/G^n$ using elementary moves. To that end, we describe by induction on k the elements in the subgroups N_k. More precisely, we prove the following statement.

PROPOSITION 4.6. — *Let $k \in \mathbf{N}$.*

(i) *Let $y \in X$. There exists $u \in N_k$ such that x_0 and uy are close of rank k and uy is the image of y by a sequence of elementary moves.*
(ii) *Let $y, z \in X$ such that x_0 and y (respectively x_0 and z) are close of rank k. If $y = z$ in X_k, then z is the image of y by a sequence of elementary moves.*
(iii) *Let $y \in X$ such that x_0 and y are close of rank k. Let g be an element of G which is reduced of rank k. We assume that there exists an integer $m \geq n/2 - \xi$ such that*

$$\mathfrak{D}(\{x_0, y\}, Y_g) \geq [g^m] + \pi \sinh \rho \text{ in } \lambda X_k.$$

Then there exist $u, v \in N_k$ such that uy is the image of y by a sequence of elementary moves and

$$\mathfrak{D}(\{x_0, uy\}, vY_g) \geq [g^m] \text{ in } X.$$

Proof. — Most of the rest of this section is devoted to the proof of this proposition. The proof is by induction of k. If $k = 0$, all the conclusions are already contained in the hypothesis (take $u = v = 1$). Hence, the proposition is true for $k = 0$. Assume now that the proposition holds for $k \in \mathbf{N}$.

Lemma 4.7. — *Let $y \in X$ such that x_0 and y are close of rank k, but not close of rank $k+1$. There exists $u \in N_{k+1}$ such that*

(i) *x_0 and uy are close of rank k,*
(ii) *uy is the image of y by a sequence of elementary moves,*
(iii) *$|x_0 - uy| < |x_0 - y| - 19\varepsilon$ in λX_k.*

Proof. — By assumption, there exists $h \in P_k$ such that

$$\mathfrak{D}(\{x_0, y\}, Y_h) > [h^n]/2 + 10\varepsilon \text{ in } \lambda X_k.$$

We apply Lemma 1.16 (ii). Up to replacing h by its inverse we get that $|x_0 - h^n y| < |x_0 - y| - 19\varepsilon$ in λX_k. However, h is loxodromic in G_k. By Proposition 4.4, there exists $g \in G$, which is reduced of rank k such that $g = h$ in G_k. In particular, g^n belongs to N_{k+1} and $\mathfrak{D}(\{x_0, y\}, Y_g) \geq [g^n]/2 + 10\varepsilon$ in λX_k. We put $m = \lceil n/2 - \xi \rceil$. According to our choice of ξ, we have $(\xi - 1) r_{inj}(G_k, \lambda X_k) \geq 4\pi \sinh \rho$. It follows from Proposition 1.13 that in λX_k

$$[g^n] \geq 2[g^m]^\infty + 2(\xi - 1)[g]^\infty_{X_k} \geq 2[g^m] + 7\pi \sinh \rho.$$

Consequently, we have in λX_k, $\mathfrak{D}(\{x_0, y\}, Y_g) \geq [g^m] + \pi \sinh \rho$, with $m \geq n/2 - \xi$. By assumption x_0 and y are close of rank k and g is reduced of rank k. Applying the induction hypothesis (Proposition 4.6 (C)), there exist $u, v \in N_k$ such that uy is the image of y by a sequence of elementary moves and

$$\mathfrak{D}(\{x_0, uy\}, vY_g) \geq [g^m] \geq [vg^m v^{-1}] \text{ in } X.$$

Therefore, $(vg^n v^{-1})uy$ is the image of uy by an elementary move. However, by induction hypothesis (Proposition 4.6 (A)), there exists $w \in N_k$ such that x_0 and $w(vg^n v^{-1})uy$ are close of rank k and $w(vg^n v^{-1})uy$ is the image of $(vg^n v^{-1})uy$ by a sequence of elementary moves.

Let us now summarize. Using a finite number of elementary moves, we have done the following transformations:

$$y \to uy \to (vg^n v^{-1})uy \to w(vg^n v^{-1})uy.$$

On the other hand $u, v, w \in N_k$ and $g^n \in N_{k+1}$. Thus, $w(vg^n v^{-1})u$ belongs to N_{k+1} and $w(vg^n v^{-1})u = g^n = h^n$ in G_k. Hence, we have in λX_k

$$\left| x_0 - w(vg^n v^{-1})uy \right| = |x_0 - h^n y| < |x_0 - y| - 19\varepsilon. \qquad \square$$

Lemma 4.8. — *Let $y \in X$. There exists $u \in N_{k+1}$ such that x_0 and uy are close of rank $k+1$ and uy is the image of y by a sequence of elementary moves.*

Remark. — This lemma proves Proposition 4.6 (A) for $k+1$.

Proof. — Let $\mathcal{U}$ be the set of elements of $u \in N_{k+1}$ such that x_0 and uy are close of rank k and uy is the image of y by a sequence of elementary moves. According to the induction hypothesis (Proposition 4.6 (A)), $\mathcal{U}$ is non-empty (more precisely $\mathcal{U} \cap N_k \neq \emptyset$). Hence, we can choose $u \in \mathcal{U}$ such that for all $u' \in \mathcal{U}$, $|x_0 - uy| \leq |x_0 - u'y| + \delta$ in λX_k. We claim that x_0 and uy are close of rank $k+1$. On the contrary, suppose that this assertion is false. By construction of $\mathcal{U}$, x_0 and uy are close of rank k. By Lemma 4.7, there exists v in N_{k+1} such that vu belongs to $\mathcal{U}$ and $|x_0 - vuy| < |x_0 - uy| - 19\varepsilon$ in λX_k, which contradicts the definition of u. □

Lemma 4.9. — *Let $y \in X$ such that x_0 and y are close of rank k. Let $p, q \in X_k$ such that the configuration (x_0, p, q, y) is foldable in λX_k. We assume that p and q are equal in X_{k+1} but not in X_k. There exists $u \in N_{k+1}$ such that in λX_k*

(i) *x_0 and uy are close of rank k,*
(ii) *uy is the image of y by a sequence of elementary moves,*
(iii) *$\langle up, q\rangle_{x_0} \geq \langle p, q\rangle_{x_0} + \pi \sinh \rho$,*
(iv) *the configuration (x_0, up, q, uy) is foldable (in λX_k) and $\langle x_0, uy\rangle_{up} \leq 9\pi \sinh \rho$.*

Proof. — Let us apply Proposition 3.14 in X_k with (x_0, p, q, y). There exists $h \in P_k$ satisfying the following in λX_k.

- $\mathfrak{D}(\{x_0, y\}, Y_h) \geq [h^n]/2 - 6\pi \sinh \rho$.
- $\langle h^n p, q\rangle_{x_0} \geq \langle p, q\rangle_{x_0} + [h^n]/2 - 6\pi \sinh \rho$.
- The configuration $(x_0, h^n p, q, h^n y)$ is foldable.
  Furthermore, $\langle x_0, h^n y\rangle_{h^n p} \leq 9\pi \sinh \rho$.

However, h is loxodromic in G_k. By Proposition 4.4, there exists $g \in G$, which is reduced of rank k such that $g = h$ in G_k. In particular, g^n belongs to N_{k+1}. Moreover, we have $\mathfrak{D}(\{x_0, y\}, Y_g) \geq [g^n]/2 - 6\pi \sinh \rho$ in λX_k. We put $m = \lceil n/2 - \xi \rceil$. Just as in Lemma 4.7, we have in λX_k $[g^n] \geq 2[g^m] + 7\pi \sinh \rho$. Consequently, we get in λX_k, $\mathfrak{D}(\{x_0, y\}, Y_g) \geq [g^m] + \pi \sinh \rho$, with $m \geq n/2 - \xi$. By construction x_0 and y are close of rank k and g is reduced of rank k. Applying the induction hypothesis (Proposition 4.6 (C)), there exist $u, v \in N_k$ such that uy is the image of y by a sequence of elementary moves and

$$\mathfrak{D}(\{x_0, uy\}, vY_g) \geq [g^m] \geq [vg^m v^{-1}] \text{ in } X.$$

Therefore, $(vg^n v^{-1})uy$ is the image of uy by an elementary move. By induction hypothesis (Proposition 4.6 (A)), there exists $w \in N_k$ such that x_0 and

$w(vg^nv^{-1})uy$ are close of rank k and $w(vg^nv^{-1})uy$ is the image of $(vg^nv^{-1})uy$ by a sequence of elementary moves.

Let us now summarize. Using a finite number of elementary moves, we have done the following transformations:

$$y \to uy \to (vg^nv^{-1})uy \to w(vg^nv^{-1})uy.$$

On the other hand $u, v, w \in N_k$ and $g^n \in N_{k+1}$. Thus, $w(vg^nv^{-1})u$ belongs to N_{k+1} and

$$w(vg^nv^{-1})u = g^n = h^n \quad \text{in } G_k.$$

Consequently, in λX_k, $\langle w(vg^nv^{-1})up, q\rangle_{x_0} \geq \langle p, q\rangle_{x_0} + \pi \sinh \rho$, the configuration

$$(x_0, w(vg^nv^{-1})up, q, w(vg^nv^{-1})uy)$$

is foldable (indeed its coincide with the configuration (x_0, h^np, q, h^ny) in λX_k) and

$$\left\langle x_0, w(vg^nv^{-1})uy\right\rangle_{w(vg^nv^{-1})up} \leq 9\pi \sinh \rho.$$ □

Lemma 4.10. — *Let $y \in X$ such that x_0 and y are close of rank k. Let $p, q \in X_k$ such that the configuration (x_0, p, q, y) is foldable in X_k and $\langle x_0, y\rangle_p \leq 9\pi \sinh \rho$. We assume that p and q are equal in X_{k+1}. There exists $u \in N_{k+1}$ such that*

(i) *x_0 and uy are close of rank k,*
(ii) *uy is the image of y by a sequence of elementary moves,*
(iii) *in λX_k, $up = q$ and $\langle x_0, uy\rangle_q \leq 9\pi \sinh \rho$.*

Proof. — Let us denote by $\mathcal{U}$ the set of elements $u \in N_{k+1}$ such that

- x_0 and uy are close of rank k,
- uy is the image of y by a sequence of elementary moves,
- in λX_k, the configuration (x_0, up, q, uy) is foldable and $\langle x_0, uy\rangle_{up} \leq 9\pi \sinh \rho$.

The set $\mathcal{U}$ is non-empty ($1 \in \mathcal{U}$). On the other hand, for all $u \in \mathcal{U}$, $\langle up, q\rangle_{x_0}$ is bounded above by $|q - x_0|$ in λX_k. Hence, we can choose $u \in \mathcal{U}$ such that for all $u' \in \mathcal{U}$, $\langle up, q\rangle_{x_0} \geq \langle u'p, q\rangle_{x_0} - \delta$ in λX_k. We claim that $up = q$. On the contrary, suppose that this assertion is false. By definition of $\mathcal{U}$, the configuration (x_0, up, q, uy) is foldable in X_k. Therefore, applying Lemma 4.9, there exists $v \in N_{k+1}$ such that vu belongs to $\mathcal{U}$ and $\langle vup, q\rangle_{x_0} \geq \langle up, q\rangle_{x_0} + \pi \sinh \rho$ in λX_k, which contradicts the definition of u. Consequently, $up = q$ in X_k. It follows from the definition of $\mathcal{U}$ that $\langle x_0, uy\rangle_q \leq 9\pi \sinh \rho$ in λX_k. □

Lemma 4.11. — *Let $y, z \in X$ such that x_0 and y (respectively x_0 and z) are close of rank $k+1$. If $y = z$ in X_{k+1} then z is the image of y by a sequence of elementary moves.*

Remark. — This lemma proves Proposition 4.6 (B) for $k+1$.

Proof. — By assumption x_0 and y are close of rank k. Moreover, x_0 and y (respectively x_0 and z) are $\mathcal{Q}_k$-close in λX_k. Thus, the configuration (x_0, y, z, y) is foldable in λX_k (take $s = t = x_0$ in Definition 3.13) and $\langle x_0, y\rangle_y = 0$. Applying Lemma 4.10, there exists $u \in N_{k+1}$ such that uy is the image of y by a sequence of elementary moves, $uy = z$ in λX_k and x_0 and uy are close of rank k. By assumption, x_0 and z are also close of rank k. According to the induction hypothesis (Proposition 4.6 (B)), z is the image of uy by a sequence of elementary moves. Hence, z is the image of y by a sequence of elementary moves. □

Lemma 4.12. — *Let $y \in X$ such that x_0 and y are close of rank $k+1$. Let $g \in G$, which is reduced of rank $k+1$. We assume that there exists an integer $m \geq n/2 - \xi$ such that*

$$\mathfrak{D}(\{x_0, y\}, Y_g) \geq [g^m] + \pi \sinh \rho, \text{ in } \lambda X_{k+1}.$$

Then there exist $u, v \in N_{k+1}$ such that uy is the image of y by a sequence of elementary moves and

$$\mathfrak{D}(\{x_0, uy\}, vY_g) \geq [g^m] \text{ in } X.$$

Remark. — This lemma proves Proposition 4.6 (C) for $k+1$.

Proof. — Exceptionally we begin the proof by working in X_{k+1} (instead of λX_{k+1}). Written in X_{k+1}, our assumption says that $\mathfrak{D}(\{x_0, y\}, Y_g) \geq [g^m] + \lambda^{-1}\pi \sinh \rho$. We want to apply Proposition 3.18 with $D = \lambda^{-1}\pi \sinh \rho$. Let us check the assumptions of this proposition. First, recall that we have chosen n_1 so that $\lambda < 1/12$; see (18). Hence, $\lambda^{-1}\pi \sinh \rho > 10\rho$. On the other hand, we assumed that $n \geq 50 n_1$, while $\xi = n_1 + 1$. Thus, $m \geq 23 n_1$. According to the construction of X_{k+1},

$$r_{inj}(G_{k+1}, \lambda X_{k+1}) \geq \sqrt{\rho L_S \delta_1 / 4 n_1}$$

Combined with (18) it tells us that the length of g^m in X_{k+1} satisfies

$$[g^m] \geq 10\sqrt{n_1 \rho L_S \delta_1} \geq L_S \delta_1.$$

Hence, the assumptions of Proposition 3.18 are fulfilled. Consequently, there exist $r, p, q \in X_k$ and $v \in N_{k+1}$ satisfying the following

(i) $d(r, vY_g) \leq 2\varepsilon$, $d(q, vY_g) \leq 4\varepsilon$, $\langle x_0, q\rangle_r \leq 3\varepsilon$ and $\langle x_0, y\rangle_p = 0$ in λX_k,
(ii) $|r - q| \geq [g^m] + (1/\lambda - 1)\pi \sinh \rho$ in λX_k.
(iii) $\bar{p} = \bar{q}$ in X_{k+1}. Moreover, the configuration (x_0, p, q, y) is foldable in λX_k.

Applying Lemma 4.10, there exists $u \in N_{k+1}$ such that

- x_0 and uy are close of rank k,
- uy is the image of y by a sequence of elementary moves,
- in X_k, $up = q$ and $\langle x_0, uy\rangle_q \leq 9\pi \sinh \rho$.

It follows from the triangle inequality applied in λX_k that

$$\mathfrak{D}(\{x_0, uy\}, vY_g) \geq \mathfrak{D}(\{r, q\}, vY_g) - \langle x_0, uy\rangle_q - \langle x_0, q\rangle_r .$$

On the other hand, r and q are both in the 4ε-neighborhood of vY_g; thus we have in λX_k

$$\mathfrak{D}(\{r, q\}, vY_g) \geq |r - q| - 8\varepsilon \geq [g^m] + (1/\lambda - 11)\pi \sinh \rho.$$

Recall that $1/\lambda \geq 12$ (Equation (18)). It follows that $\mathfrak{D}(\{x_0, uy\}, vY_g) \geq [g^m] + \pi \sinh \rho$ in λX_k. According to the induction hypothesis (Proposition 4.6 (C)), there exist $u', v' \in N_k$ such that $u'uy$ is the image of uy by a sequence of elementary moves and $\mathfrak{D}(\{x_0, u'uy\}, v'vY_g) \geq [g^m]$ in X. In particular, $u'u, v'v \in N_{k+1}$ and $u'uy$ is the image of y by a sequence of elementary moves, which ends the proof of the lemma. □

Lemma 4.8, Lemma 4.11 and Lemma 4.12 prove that Proposition 4.6 holds for $k + 1$. □

We now have all the tools to prove our main theorem.

Theorem 4.13. — *Let (X, x_0) be a pointed geodesic proper hyperbolic space. Let G be a non-elementary torsion-free group acting properly co-compactly by isometries on X. There exist numbers n_2 and ξ such that for every odd integer $n \geq n_2$, the following holds. Let y be a point of X. An element $g \in G$, belongs to G^n if and only if there exist two sequences of elementary moves which send y and gy, respectively, to the same point.*

Proof. — The numbers n_2 and ξ are defined as explained in Section 4.1. Let $n \geq n_2$ be an odd integer. Let $y \in X$ and $g \in G$. Assume first that there are two sequences of elementary moves which send y and gy, respectively, to the same point. By definition, this common point can be written $uy = vgy$, where u and v belong to G^n. In particular, $u^{-1}vg$ has a fixed point and thus is elliptic. However, G is torsion-free. Consequently, $u^{-1}vg = 1$; hence $g \in G^n$. Suppose now that $g \in G^n$. By construction, the direct limit of the sequence (G_k) is G/G^n. Thus, there exists $k \in \mathbf{N}$ such that g is trivial in G_k. In particular, $y = gy$ in X_k. By Proposition 4.6 (A), there exist $u, v \in N_k$ such that x_0 and uy (respectively x_0 and vgy) are close of rank k. Moreover, uy (respectively vgy) is the image of y (respectively gy) by a sequence of elementary moves. However, u and v belong to N_k; thus $uy = vgy$ in X_k. Applying Proposition 4.6 (B), vgy is the image of uy by a sequence of elementary moves. □

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