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A NOTE ON CRYSTALLINE LIFTINGS IN THE $\mathbb{Q}_p$ CASE

BY HUI GAO

ABSTRACT. — Let $p > 2$ be a prime. Let ρ be a crystalline representation of $G_{\mathbb{Q}_p}$ with distinct Hodge-Tate weights in $[0, p]$, such that its reduction $\overline{\rho}$ is upper triangular. Under certain conditions, we prove that $\overline{\rho}$ has an upper triangular crystalline lift ρ' such that $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$. The method is based on the author's previous work, combined with an inspiration from the work of Breuil-Herzig.

RÉSUMÉ (*Note sur les élévations cristallines dans le cas* $\mathbb{Q}_p$). — Soit $p > 2$ un premier. Soit ρ une représentation cristalline de $G_{\mathbb{Q}_p}$ avec des poids distincts de Hodge-Tate dans $[0, p]$, de telle sorte que sa réduction $\overline{\rho}$ soit triangulaire supérieure. Dans certaines conditions, nous prouvons que $\overline{\rho}$ a une élévation cristalline triangulaire supérieure ρ' telle que $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$. La méthode est basée sur le travail antérieur de l'auteur, combiné avec une inspiration de l'oeuvre de Breuil-Herzig.

1. Introduction

1.1. Overview. — Given (a lattice in) a crystalline representation, it is natural to study its reduction. Conversely, given a representation over an $\overline{\mathbb{F}}_p$-vector space, it is natural to consider its crystalline lifts. We are particularly interested with crystalline representations, because they will have applications to

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weight part of Serre's conjectures (see e.g., [6, 7, 3]). In general, both these questions are notoriously difficult. For example, given an $\overline{\mathbb{F}}_p$-representation, we do not even know if it has any crystalline lift. However, for applications to weight part of Serre's conjectures, we can *assume* at the beginning that certain $\overline{\mathbb{F}}_p$-representation already have at least one crystalline lift; the key point then is to show that it has some other *nicer* crystalline lift. And this is what we do in this paper.

To state our main result, we introduce some notations first. Let $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the Galois group of $\mathbb{Q}_p$. Let $E/\mathbb{Q}_p$ be a finite extension, $\mathcal{O}_E$ the ring of integers, ω_E a fixed uniformizer, and $k_E = \mathcal{O}_E/\omega_E\mathcal{O}_E$ the residue field. We will use the following notations often, (CRYS):

- Let $p > 2$ be an odd prime. Let V be a crystalline representation of $G_{\mathbb{Q}_p}$ of E-dimension d, such that the Hodge-Tate weights $\mathrm{HT}(V) = \{0 = r_1 < \ldots < r_d \leq p\}$.
- Let $\rho = T$ be a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattice in V, and $\hat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ the $(\varphi, \hat{G})$-module (with $\mathcal{O}_E$-coefficient) attached to T. Let $\overline{\rho} := T/\omega_E T$ be the reduction. Let $\overline{\hat{\mathfrak{M}}}$ be the reduction of $\hat{\mathfrak{M}}$, and $\overline{\mathfrak{M}}$ the reduction of $\mathfrak{M}$.

1.1.1. Theorem. — *With notations in (*CRYS*). Suppose that $\overline{\rho}$ is upper triangular, i.e., $\overline{\rho}$ is a successive extension of d characters: $\overline{\chi}_1, \ldots, \overline{\chi}_d$. Suppose $\overline{\chi}_i\overline{\chi}_j^{-1} \neq \overline{\varepsilon}_p, \forall i \neq j$, where $\overline{\varepsilon}_p$ is the reduction of the cyclotomic character. Then there exists an upper triangular crystalline representation ρ' such that $\overline{\rho}' \cong \overline{\rho}$, and* $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$ *as sets.*

Theorem 1.1.1 strengthens [3, Cor. 0.2(1)] in the $\mathbb{Q}_p$-case, and of course have direct application to weight part of Serre's conjectures as in *loc. cit.*. In our Theorem 1.1.1,

- we do not require the Condition (C-1) of [3, §3], and
- we only require a weaker version of Condition (C-2A) of [3, §6].
- Note that Condition (C-2B) of [3, §6] in general will never be satisfied in our current paper.

Let us also remark that Condition (C-1) seems to be the most difficult condition to remove in [3].

The proof of our theorem still uses results in [3] to study the possible shape of upper triangular reductions of crystalline representations. The difference in the current paper is a different crystalline lifting technique, which is inspired by some group theory developed in [1]. Roughly speaking, we can use the group theory to conjugate our upper triangular $\overline{\rho}$ to another upper triangular form, which can be lifted to an *ordinary* (in particular, upper triangular) crystalline representation via the result of [5]. The lifting process via *loc. cit.* is in some sense easier than those used in [3] (which is generalization of methods in [6, 7]). However, we can only apply this technique in the $\mathbb{Q}_p$-case, because it seems that we cannot apply the group theory in [1] to deal with general $K/\mathbb{Q}_p$ case for our problem. Let us remark that our current paper shows a much refined structure for upper triangular reductions of crystalline representations. It is also worth pointing out that our result gives a very *natural* example (see (4.1.2)) for some of the group theories in [1].

The paper is organized as follows. In Section 2, we review the theory of Kisin modules and $(\varphi, \hat{G})$-modules with $\mathcal{O}_E$-coefficients. In Section 3, we review the group theory in [1]. In Section 4, we study the shape of upper triangular torsion $(\varphi, \hat{G})$-modules, using results in [3], as well as techniques inspired by the group theory in Section 3. Finally in Section 5, we prove our crystalline lifting theorem.

1.2. Notations. — The notations in the following are taken directly from [3]. In particular, they are valid for any finite extension $K/\mathbb{Q}_p$ (and we use K_0 to denote the maximal unramified sub-extension of K, and k the residue field of K). See *loc. cit.* for any unfamiliar terms and more details.

In this paper, we sometimes use boldface letters (e.g., $\boldsymbol{e}$) to mean a sequence of objects (e.g., $\boldsymbol{e} = (e_1, \ldots, e_d)$ a basis of some module). We use Mat(?) to mean the set of matrices with elements in ?. We use notations like $[u^{r_1}, \ldots, u^{r_d}]$ to mean a diagonal matrix with the diagonal elements in the bracket. We use Id to mean the identity matrix. For a matrix A, we use diag$A$ to mean the diagonal matrix formed by the diagonal of A.

In this paper, *upper triangular* always means successive extension of rank-1 objects. We use notations like $\mathcal{E}(m_d, \ldots, m_1)$ (note the order of objects) to mean the set of all upper triangular extensions of rank-1 objects in certain categories. That is, m is in $\mathcal{E}(m_d, \ldots, m_1)$ if there is an increasing filtration $0 = \mathrm{Fil}^0 m \subset \mathrm{Fil}^1 m \subset \ldots \subset \mathrm{Fil}^d m = m$ such that $\mathrm{Fil}^i m/\mathrm{Fil}^{i-1} m = m_i, \forall 1 \leq i \leq d$.

We normalize the Hodge-Tate weights so that $\mathrm{HT}_\kappa(\varepsilon_p) = 1$ for any $\kappa : K \to \overline{\mathbb{Q}_p}$, where ε_p is the p-adic cyclotomic character.

We fix a system of elements $\{\pi_n\}_{n=0}^\infty$ in $\overline{K}$, where $\pi_0 = \pi$ is a uniformizer of K, and $\pi_{n+1}^p = \pi_n, \forall n$. Let $K_n = K(\pi_n), K_\infty = \bigcup_{n=0}^\infty K(\pi_n)$, and $G_\infty :=$ $\mathrm{Gal}(\overline{K}/K_\infty)$. We fix a system of elements $\{\mu_{p^n}\}_{n=0}^\infty$ in $\overline{K}$, where $\mu_1 = 1$, μ_p is a primitive p-th root of unity, and $\mu_{p^{n+1}}^p = \mu_{p^n}, \forall n$. Let $K_{p^\infty} = \bigcup_{n=0}^\infty K(\mu_{p^n})$, and $\hat{K} = K_{\infty, p^\infty} = \bigcup_{n=0}^\infty K(\pi_n, \mu_{p^n})$. Note that $\hat{K}$ is the Galois closure of K_∞, and let $\hat{G} = \mathrm{Gal}(\hat{K}/K)$, $H_K = \mathrm{Gal}(\hat{K}/K_\infty)$, and $G_{p^\infty} = \mathrm{Gal}(\hat{K}/K_{p^\infty})$. When $p > 2$, then $\hat{G} \simeq G_{p^\infty} \rtimes H_K$ and $G_{p^\infty} \simeq \mathbb{Z}_p(1)$, and so we can (and do) fix a topological generator τ of G_{p^∞}. And we can furthermore assume that $\mu_{p^n} = \frac{\tau(\pi_n)}{\pi_n}$ for all n.

Let $C = \hat{\overline{K}}$ be the completion of $\overline{K}$, with ring of integers $\mathcal{O}_C$. Let $R :=$ $\varprojlim \mathcal{O}_C/p$ where the transition maps are p-th power map. R is a valuation ring with residue field $\bar{k}$ ($\bar{k}$ is the residue field of C). R is a perfect ring of characteristic p. Let $W(R)$ be the ring of Witt vectors. Let $\underline{\epsilon} := (\mu_{p^n})_{n=0}^{\infty} \in R$, $\underline{\pi} = (\pi_n)_{n=0}^{\infty} \in R$, and let $[\underline{\epsilon}], [\underline{\pi}]$ be their Teichmüller representatives respectively in $W(R)$. We normalize the valuation on R so that $v_R(\underline{\pi}) = \frac{1}{e}$, where e is the ramification index of $K/\mathbb{Q}_p$.

There is a map $\theta : W(R) \to \mathcal{O}_C$ which is the unique universal lift of the map $R \to \mathcal{O}_C/p$ (projection of R onto the its first factor), and $\operatorname{Ker}\theta$ is a principle ideal generated by $\xi = [\overline{\omega}] + p$, where $\overline{\omega} \in R$ with $\omega^{(0)} = -p$, and $[\overline{\omega}] \in W(R)$ its Teichmüller representative. Let $B_{\mathrm{dR}}^+ := \varprojlim_n W(R)[\frac{1}{p}]/(\xi)^n$, and $B_{\mathrm{dR}} := B_{\mathrm{dR}}^+[\frac{1}{\xi}]$. Let $t := \log([\underline{\epsilon}])$, which is an element in B_{dR}^+. Let A_{cris} denote the p-adic completion of the divided power envelope of $W(R)$ with respect to $\operatorname{Ker}(\theta)$. Let $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p]$ and $B_{\mathrm{cris}} := B_{\mathrm{cris}}^+[\frac{1}{t}]$. The projection from R to $\bar{k}$ induces a projection $\nu : W(R) \to W(\bar{k})$, since $\nu(\operatorname{Ker}\theta) = pW(\bar{k})$, the projection extends to $\nu : A_{\mathrm{cris}} \to W(\bar{k})$, and also $\nu : B_{\mathrm{cris}}^+ \to W(\bar{k})[\frac{1}{p}]$. Write $I_+B_{\mathrm{cris}}^+ := \operatorname{Ker}(\nu : B_{\mathrm{cris}}^+ \to W(\bar{k})[\frac{1}{p}])$, and for any subring $A \subseteq B_{\mathrm{cris}}^+$, write $I_+A = A \cap \operatorname{Ker}(\nu)$.

Let $\mathfrak{S} := W(k)[\![u]\!]$, $E(u) \in W(k)[u]$ the minimal polynomial of π over $W(k)$, and S the p-adic completion of the PD-envelope of $\mathfrak{S}$ with respect to the ideal $(E(u))$. We can embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ by mapping u to $[\underline{\pi}]$. The embedding extends to the embeddings $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$.

2. Kisin modules and $(\varphi, \hat{G})$-modules

In this section, we briefly review some facts in the theory of Kisin modules and $(\varphi, \hat{G})$-modules with $\mathcal{O}_E$-coefficients. The materials in this section are based on works of [8, 10, 2, 6, 9] etc.. But here we only cite them in the form as in [3, §1], where the readers can find more detailed attributions.

2.1. Kisin modules and $(\varphi, \hat{G})$-modules with coefficients. — In this subsection, all the definitions and results are valid for any finite extension $K/\mathbb{Q}_p$.

Recall that $\mathfrak{S} = W(k)[\![u]\!]$ with the Frobenius endomorphism $\varphi_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S}$ which acts on $W(k)$ via arithmetic Frobenius and sends u to u^p. Denote $\mathfrak{S}_{\mathcal{O}_E} := \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ and $\mathfrak{S}_{k_E} := \mathfrak{S} \otimes_{\mathbb{Z}_p} k_E = k[\![u]\!] \otimes_{\mathbb{F}_p} k_E$. We can extend $\varphi_{\mathfrak{S}}$ to $\mathfrak{S}_{\mathcal{O}_E}$ (resp. $\mathfrak{S}_{k_E}$) by acting on $\mathcal{O}_E$ (resp. k_E) trivially. Let r be any nonnegative integer.

- Let $'\mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ (called the category of Kisin modules of height r with $\mathcal{O}_E$-coefficients) be the category whose objects are $\mathfrak{S}_{\mathcal{O}_E}$-modules $\mathfrak{M}$, equipped with $\varphi : \mathfrak{M} \to \mathfrak{M}$ which is a $\varphi_{\mathfrak{S}_{\mathcal{O}_E}}$-semi-linear morphism such that the span of $\operatorname{Im}(\varphi)$ contains $E(u)^r\mathfrak{M}$. The morphisms in the category are $\mathfrak{S}_{\mathcal{O}_E}$-linear maps that commute with φ.

- Let $\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ be the full subcategory of $'\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ with $\mathfrak{M} \simeq \oplus_{i\in I}\mathfrak{S}_{\mathcal{O}_E}$ where I is a finite set. Let $\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{k_E}}$ be the full subcategory of $'\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ with $\mathfrak{M} \simeq \oplus_{i\in I}\mathfrak{S}_{k_E}$ where I is a finite set.

For any integer $n \geq 0$, write $n = (p-1)q(n) + r(n)$ with $q(n)$ and $r(n)$ the quotient and residue of n divided by $p-1$. Let $t^{\{n\}} = (p^{q(n)} \cdot q(n)!)^{-1} \cdot t^n$, we have $t^{\{n\}} \in A_{\mathrm{cris}}$.

We define a subring of B^+_{cris}, $\mathcal{R}_{K_0} := \left\{\sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S_{K_0}, f_i \to 0 \text{ as } i \to \infty\right\}$. Define $\hat{\mathcal{R}} := \mathcal{R}_{K_0} \cap W(R)$. Then $\hat{\mathcal{R}}$ is a φ-stable subring of $W(R)$, which is also G_K-stable, and the G_K-action factors through $\hat{G}$. Denote $\hat{\mathcal{R}}_{\mathcal{O}_E} := \hat{\mathcal{R}} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, $W(R)_{\mathcal{O}_E} := W(R) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, and extend the G_K-action and φ-action on them by acting on $\mathcal{O}_E$ trivially. Note that $\mathfrak{S}_{\mathcal{O}_E} \subset \hat{R}_{\mathcal{O}_E}$, and let $\varphi : \mathfrak{S}_{\mathcal{O}_E} \to \hat{R}_{\mathcal{O}_E}$ be the composite of $\varphi_{\mathfrak{S}_{\mathcal{O}_E}} : \mathfrak{S}_{\mathcal{O}_E} \to \mathfrak{S}_{\mathcal{O}_E}$ and the embedding $\mathfrak{S}_{\mathcal{O}_E} \to \hat{R}_{\mathcal{O}_E}$.

2.1.1. Definition. — Let $'\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ be the category (called the category of $(\varphi,\hat{G})$-modules of height r with $\mathcal{O}_E$-coefficients) consisting of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ where,

1. $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in {}'\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ is a Kisin module of height r;
2. $\hat{G}$ is a $\hat{\mathcal{R}}_{\mathcal{O}_E}$-semi-linear $\hat{G}$-action on $\hat{\mathfrak{M}} := \hat{\mathcal{R}}_{\mathcal{O}_E} \otimes_{\varphi, \mathfrak{S}_{\mathcal{O}_E}} \mathfrak{M}$;
3. $\hat{G}$ commutes with $\varphi_{\hat{\mathfrak{M}}} := \varphi_{\hat{\mathcal{R}}_{\mathcal{O}_E}} \otimes \varphi_{\mathfrak{M}}$;
4. Regarding $\mathfrak{M}$ as a $\varphi(\mathfrak{S}_{\mathcal{O}_E})$-submodule of $\hat{\mathfrak{M}}$, then $\mathfrak{M} \subseteq \hat{\mathfrak{M}}^{H_K}$;
5. $\hat{G}$ acts on the $\hat{\mathfrak{M}}/(I_+\hat{R})\hat{\mathfrak{M}}$ trivially.

A morphism between two $(\varphi,\hat{G})$-modules is a morphism in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ which commutes with $\hat{G}$-actions.

We denote $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ to be the full subcategory of $'\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ where $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$; and we denote $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{k_E}}$ for the full subcategory of $'\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ where $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}_{k_E}}$.

We can associate representations to $(\varphi,\hat{G})$-modules.

2.1.2. Theorem ([3, Thm. 1.2, Thm. 1.4]). —

1. *Suppose* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ *where* $\mathfrak{M}$ *is of* $\mathfrak{S}_{\mathcal{O}_E}$*-rank* d, *then*

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\hat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R))$$

is a finite free $\mathcal{O}_E$*-representation of* G_K *of rank* d.

2. *Suppose* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{k_E}}$ *where* $\mathfrak{M}$ *is of* $\mathfrak{S}_{k_E}$*-rank* d, *then*

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\hat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$$

is a finite free k_E*-representation of* G_K *of dimension* d.

3. *For* $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi,\hat{G}}$, *we have* $\hat{T}(\hat{\mathfrak{M}}/\omega_E\hat{\mathfrak{M}}) \simeq \hat{T}(\hat{\mathfrak{M}})/\omega_E\hat{T}(\hat{\mathfrak{M}})$.

When $p > 2$, the theory of $(\varphi, \hat{G})$-modules becomes simpler.

2.1.3. Lemma ([3, Lem. 1.6]). — *Suppose* $p > 2$. *Let* $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi,\hat{G}}$. *Then* $\hat{\mathfrak{M}}$ *is uniquely determined up to isomorphism by the following information:*

1. *A matrix* $A_\varphi \in \mathrm{Mat}(\mathfrak{S}_{\mathcal{O}_E})$ *for the Frobenius* $\varphi : \mathfrak{M} \to \mathfrak{M}$, *such that there exist* $B \in \mathrm{Mat}(\mathfrak{S}_{\mathcal{O}_E})$ *with* $A_\varphi B = E(u)^r Id$.
2. *A matrix* $A_\tau \in \mathrm{Mat}(\hat{R}_{\mathcal{O}_E})$ *(for the* τ*-action* $\tau : \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}$*) such that*
   - $A_\tau - Id \in \mathrm{Mat}(I_+\hat{\mathcal{R}}_{\mathcal{O}_E})$,
   - $A_\tau \tau(\varphi(A_\varphi)) = \varphi(A_\varphi)\varphi(A_\tau)$.
   - $g(A_\tau) = \prod_{k=0}^{\varepsilon_p(g)-1} \tau^k(A_\tau)$ *for all* $g \in G_\infty$ *such that* $\varepsilon_p(g) \in \mathbb{Z}^{\geq 0}$.

For $\overline{\hat{\mathfrak{M}}} \in \mathrm{Mod}_{\mathfrak{S}_{k_E}}^{\varphi,\hat{G}}$, *it is also uniquely determined up to isomorphism by its matrix* A_φ *and* A_τ *satisfying similar conditions as above.*

2.2. Rank 1 Kisin modules and $(\varphi, \hat{G})$-modules.

— We only recall the following definitions and results in the $\mathbb{Q}_p$ case.

2.2.1. Definition. —

1. Suppose t is a non-negative integer, $a \in k_E^\times$. Let $\overline{\mathfrak{M}}(t; a)$ be the rank-1 module in $\mathrm{Mod}_{\mathfrak{S}_{k_E}}^{\varphi}$ such that $\overline{\mathfrak{M}}(t; a)$ is generated by some basis e, and $\varphi(e) = au^t e$.
2. Suppose t is a non-negative integer, $\hat{a} \in \mathcal{O}_E^\times$. Let $\mathfrak{M}(t; \hat{a})$ be the rank-1 module in $\mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ such that $\mathfrak{M}(t; \hat{a})$ is generated by some basis $\tilde{e}$, and $\varphi(\tilde{e}) = \hat{a}(u-p)^t\tilde{e}$.

2.2.2. Lemma ([3, Lem. 1.11]). —

1. *Any rank* 1 *module in* $\mathrm{Mod}_{\mathfrak{S}_{k_E}}^{\varphi}$ *is of the form* $\overline{\mathfrak{M}}(t; a)$ *for some* t *and* a.
2. *When* $\hat{a}$ *is a lift of* a, $\mathfrak{M}(t; \hat{a})/\omega_E\mathfrak{M}(t; \hat{a}) \simeq \overline{\mathfrak{M}}(t; a)$.
3. *There is a unique* $\hat{\mathfrak{M}}(t; \hat{a}) \in \mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi,\hat{G}}$ *such that the ambient Kisin module of* $\hat{\mathfrak{M}}(t; \hat{a})$ *is* $\mathfrak{M}(t; \hat{a})$, *and* $\hat{T}(\hat{\mathfrak{M}}(t; \hat{a}))$ *is a crystalline character. In fact,* $\hat{T}(\hat{\mathfrak{M}}(t; \hat{a})) = \lambda_{\hat{a}}\psi^t$, *where* ψ *is a certain crystalline character such that* $\mathrm{HT}(\psi) = 1$, *and* $\lambda_{\hat{a}}$ *is the unramified character of* $G_{\mathbb{Q}_p}$ *which sends the arithmetic Frobenius to* $\hat{a}$.
4. *There is a unique* $\overline{\hat{\mathfrak{M}}}(t; a) \in \mathrm{Mod}_{\mathfrak{S}_{k_E}}^{\varphi,\hat{G}}$ *such that the ambient Kisin module is* $\overline{\mathfrak{M}}(t; a)$. *Furthermore,* $\hat{T}(\overline{\hat{\mathfrak{M}}}(t; a))$ *is the reduction of* $\hat{T}(\hat{\mathfrak{M}}(t; \hat{a}))$ *for any lift* $\hat{a} \in \mathcal{O}_E$ *of* a.

3. Some group theory

We recall some group theory, which will be useful for our work. All the materials in this section are developed in [1, §2.3], for general split connected reductive groups. But we will only need it for GL_d, which we recall.

Let H be the algebraic group GL_d, T the torus consisting of diagonal matrices, B the Borel consisting of upper triangular matrices, and U the unipotent radical consisting of unipotent matrices.

We have $X(T) := \mathrm{Hom}_{\mathrm{alg}}(T, \mathbb{G}_m) = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_d$, where ϵ_i is the character sending the diagonal matrix $[x_1, \ldots, x_d]$ to x_i. Let $S = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq d-1\}$ be the simple roots, and let $R^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq d\}$ be the positive roots. Denote W the Weyl group of H, which is isomorphic to the permutation group S_d. If $\alpha = \epsilon_i - \epsilon_j \in R^+$, let $U_\alpha \subset H$ be the root subgroup, which corresponds to the unipotent upper triangular matrices where the only nonzero element above the diagonal is at the (i, j)-position.

3.0.1. Definition. — A subset $C \subseteq R^+$ is called *closed* if the following condition is satisfied: if $\alpha \in C, \beta \in C$ and $\alpha + \beta \in R^+$, then $\alpha + \beta \in C$.

For a closed subset $C \subseteq R^+$, let $U_C \subseteq U$ be the Zariski closed subgroup of B generated by the subgroups U_α for all $\alpha \in C$. Let $B_C = TU_C \subseteq B$. If $C = \{\epsilon_{i_1} - \epsilon_{j_1}, \ldots, \epsilon_{i_m} - \epsilon_{j_m}\}$ is a closed subset of R^+, then it is easy to see that B_C corresponds to the matrices where the only nonzero elements above the diagonal are at the positions (i_ℓ, j_ℓ) for all $1 \leq \ell \leq m$.

Recall that if we let $N_H(T)$ be the normalizer of T in H, then $N_H(T)/T$ is isomorphic to W. For each $\sigma \in W$ which is a permutation sending $(1, \ldots, d)$ to $(\sigma(1), \ldots, \sigma(d))$, we fix a representative of σ in H to be the $d \times d$ matrix $w_\sigma := (\delta_{i,\sigma(j)})_{1 \leq i,j \leq d} = (\delta_{\sigma^{-1}(i),j})$ where the notation is $\delta_{x,y} = 0$ if $x \neq y$, and $\delta_{x,y} = 1$ if $x = y$. Note that if we have another $d \times d$ matrix $A = (a_{k,l})$, we have the matrix multiplication:

$$(\delta_{i,\sigma(j)})(a_{k,l}) = (a_{\sigma^{-1}(k),l}), \quad (a_{k,l})(\delta_{i,\sigma(j)}) = (a_{k,\sigma(l)}),$$

and so in particular $\sigma^{-1}(a_{i,j})\sigma = (a_{\sigma(i),\sigma(j)})$.

Let $C \subseteq R^+$ closed, we define the following subset of W:

$$W_C := \{\sigma \in W : \sigma^{-1}(C) \subseteq R^+\}.$$

3.0.2. Lemma ([1, Lem. 2.3.6]). — *With notations as above, we have*

$$W_C = \{\sigma \in W : w_\sigma^{-1} B_C w_\sigma \subseteq B\}.$$

The above lemma says that conjugations of a matrix in B_C by permutations in W_C will stay upper triangular. In the following, we will sometimes simply use σ to mean the matrix w_σ. If ? is a ring, we will use $B_C(?)$ to mean the subring of $\mathrm{Mat}_d(?)$ corresponding to the algebraic group B_C. That is, if $C = \{\epsilon_{i_1} - \epsilon_{j_1}, \dots, \epsilon_{i_m} - \epsilon_{j_m}\}$ is closed in R^+, then we let

$$B_C(?) := \{A = (a_{i,j}) \in \mathrm{Mat}_d(?) : \\ A \text{ is upper triangular, and } a_{i,j} = 0 \text{ if } \epsilon_i - \epsilon_j \notin C\}.$$

It is clear that for any $A \in B_C(?)$ and any $\sigma \in W_C$, $\sigma^{-1}A\sigma$ is an upper triangular matrix.

4. Shape of upper triangular $(\varphi, \hat{G})$-modules with k_E-coefficients

In this section, we study the shape of upper triangular torsion $(\varphi, \hat{G})$-modules, using results in [3], as well as ideas in Section 3.

4.1. Shape of φ

4.1.1. PROPOSITION. — *With notations from (*CRYS*). Suppose that $\overline{\rho}$ is upper triangular. Then $\overline{\mathfrak{M}} \in \mathcal{E}(\overline{\mathfrak{N}}_d, \dots, \overline{\mathfrak{N}}_1)$, where $\overline{\mathfrak{N}}_i = \overline{\mathfrak{M}}(t_i; a_i)$ for some $a_i \in k_E^\times$, and $\{t_1, \dots, t_d\} = \{r_1, \dots, r_d\}$ as sets.*

Furthermore, there exists a basis $\boldsymbol{e}$ of $\overline{\mathfrak{M}}$, such that the matrix A_φ of φ with respect to this basis can be decomposed as $A_\varphi = \widetilde{A_\varphi} + u^p N$ where

1. *$\widetilde{A_\varphi}$ is upper triangular, with diagonal equal to $[a_1 u^{t_1}, \dots, a_d u^{t_d}]$, and $(\widetilde{A_\varphi})_{i,j} = u^{t_i} y_{i,j}$ for $i < j$ (here $(\widetilde{A_\varphi})_{i,j}$ is the element of $\widetilde{A_\varphi}$ in the (i,j)-position), where*
   - *$y_{i,j} = 0$ if $t_j < t_i$.*
   - *$y_{i,j} \in k_E$ if $t_j > t_i$.*
2. *$N \in \mathrm{Mat}_d(k_E[u])$ is strictly upper triangular (i.e., the diagonal is 0).*

Proof. — This is a slight generalization of [3, Prop. 4.1] (using, in particular, [3, Prop. 2.2, Prop. 2.3]). The novelty here is that we can allow the existence of nonzero morphisms $\overline{\mathfrak{N}}_j \to \overline{\mathfrak{N}}_i$ for some $j > i$, i.e., the situation in Statement (3) of [3, Prop. 2.2] is allowed.

Step 1. — First of all, the existence and shape of $\overline{\mathfrak{N}}_i$ is proved in [3, Prop. 2.3]. To construct the basis $\boldsymbol{e}$ and the upper triangular matrix A_φ, we will apply [3, Prop. 2.2]. For the convenience of the reader, let us give some more explanation of *loc. cit.*, in the $\mathbb{Q}_p$-case (the general unramified case is similar).

The statement of [3, Prop. 2.2] is correct. A minor imperfection is that in the proof of *loc. cit.*, we cited [6, Prop. 7.4]. Indeed, to be more precise, we should have cited [7, Prop. 5.1.3] instead (although as mentioned in [7, Prop. 5.1.3], their proof are almost identical). The difference between [6, Prop. 7.4] and [7, Prop. 5.1.3] is that in the latter situation, we can allow all the Hodge-Tate numbers to be nonzero (we thank one of the referees for pointing this out).

The construction of the basis $\boldsymbol{e}$ (in [3, Prop. 2.2]) when $d > 2$ is really an easy inductive process from that of [7, Prop. 5.1.3] (where $d = 2$). Let us only sketch the case when $d = 3$. That is, suppose now we have $\overline{\mathfrak{M}} \in \mathcal{E}(\overline{\mathfrak{N}}_3, \overline{\mathfrak{N}}_2, \overline{\mathfrak{N}}_1)$. So we have a basis $\{f_1, f_2, f_3\}$ such that

$$\varphi(f_1, f_2, f_3) = (f_1, f_2, f_3)\begin{pmatrix} a_1u^{t_1} & x & z \\ 0 & a_2u^{t_2} & y \\ 0 & 0 & a_3u^{t_3} \end{pmatrix}$$

The key point then is to make change of bases so that x, y, z will satisfy the conditions in [3, Prop. 2.2]. By [7, Prop. 5.1.3] (the $d = 2$ case), we can and do assume that x already satisfies all the conditions in [3, Prop. 2.2]. That is: x is a polynomial in $k_E[u]$ of degree less than t_2, unless if there exists nonzero morphism $\overline{\mathfrak{N}}_2 \to \overline{\mathfrak{N}}_1$, then x can have an extra term of degree $t_2 + \frac{t_2 - t_1}{p-1}$.

The next step is to alter f_3 in order to make y, z satisfy [3, Prop. 2.2]. We can first change f_3 to $f_3' = f_3 + \alpha f_2$ as in the proof of [7, Prop. 5.1.3] to make y to some y' that satisfy [3, Prop. 2.2]. Note that this process will not have any effect on x, but it will alter z. So now we are in the situation

$$\varphi(f_1, f_2, f_3') = (f_1, f_2, f_3')\begin{pmatrix} a_1u^{t_1} & x & z' \\ 0 & a_2u^{t_2} & y' \\ 0 & 0 & 0a_3u^{t_3} \end{pmatrix}$$

where both x, y' satisfy [3, Prop. 2.2]. Now we only need to change f_3' to some $f_3'' = f_3' + \beta f_1$ in order to make z' satisfy [3, Prop. 2.2]. Note that there is no extension between $\overline{\mathfrak{N}}_3$ and $\overline{\mathfrak{N}}_1$, so we can not directly apply [7, Prop. 5.1.3] to get f_3''. However, the "f_2-parts" of $\varphi(f_3'') = \varphi(f_3') + \varphi(\beta)a_1u^{t_1}f_1$ and $\varphi(f_3')$ are the same. So we can "forget" about f_2 and pretend that there is an extension between $\overline{\mathfrak{N}}_3$ and $\overline{\mathfrak{N}}_1$. The same process as in [7, Prop. 5.1.3] will in the end produce our desired basis $\boldsymbol{e}$.

Step 2. — Now let us discuss about the "extra terms". Recall that in [3, Prop. 2.2], when there exists nonzero morphisms $\overline{\mathfrak{N}}_j \to \overline{\mathfrak{N}}_i$ for some $j > i$, then A_φ can have *extra* terms as described in Statement (3) of *loc. cit.*, and this extra term has degree $t_j + \frac{t_j - t_i}{p-1}$. Note that in order to have $\overline{\mathfrak{N}}_j \to \overline{\mathfrak{N}}_i$ for $j > i$, the only possibility is to have $t_j - t_i = p - 1$ and $a_i = a_j$ (easy by [3, Lem. 1.13] since we are in the $\mathbb{Q}_p$ situation). So the extra terms are always of degree p or $p + 1$, i.e., the extra terms are always divisible by u^p. (In fact, clearly we can only have at most two extra terms). Decompose A_φ as $\widetilde{A_\varphi} + u^pN$ where u^pN are the extra terms.

Step 3. — Finally, we only need to prove the properties regarding $y_{i,j}$. We argue similarly as in [3, Prop. 2.3], let $\boldsymbol{e}'$ be another basis of $\overline{\mathfrak{M}}$ such that $\varphi(\boldsymbol{e}') = \boldsymbol{e}'X[u^{r_1}, \dots, u^{r_d}]$ where $X \in \mathrm{Mat}_d(k_E[\![u]\!])$ as in [3, Thm. 2.1]. Let $\boldsymbol{e}' = \boldsymbol{e}T$ for some matrix $T \in \mathrm{GL}_d(k_E[\![u]\!])$, then $A_\varphi = TX[u^{r_1}, \dots, u^{r_d}]\varphi(T^{-1})$. Similarly as in [3, Prop. 4.1], let $\varphi(T) = P + u^pQ$ for some $P \in \mathrm{GL}_d(k_E)$,

$Q \in \mathrm{Mat}_d(k_E[\![u]\!])$, and let $R \in \mathrm{GL}_d(k_E)$ such that $R^{-1}[u^{r_1}, \ldots, u^{r_d}]R = [u^{t_1}, \ldots, u^{t_d}]$, then we have

$$(\widetilde{A_\varphi} + u^p N)(P + u^p Q)R = TXR[u^{t_1}, \ldots, u^{t_d}].$$

So we have $u^{t_i} \mid \mathrm{col}_i(\widetilde{A_\varphi} PR)$. Now we can again apply [3, Lem. 4.3] to conclude (note that $\widetilde{A_\varphi}$ satisfies property (DEG) of *loc. cit.*, since we removed the extra terms $u^p N$ from A_φ). □

With notations in Proposition 4.1.1, we can define the following subset C of R^+:

$$C := \{\epsilon_i - \epsilon_j : i < j, t_i < t_j\}. \tag{4.1.2}$$

It is easy to see that C is closed in R^+, and $\widetilde{A_\varphi}$ is a matrix in the subring $B_C(k_E[u])$. But in fact, we also have $A_\varphi \in B_C(k_E[u])$, because the extra terms in $u^p N$ only show up in positions (i, j) where $t_i < t_j$.

4.1.3. Proposition. — *There exists a unique $\sigma \in W_C$ such that $\sigma^{-1} A_\varphi \sigma$ is still upper triangular, and* $\mathrm{diag}(\sigma^{-1} A_\varphi \sigma) = [a_{\sigma(1)} u^{r_1}, \ldots, a_{\sigma(d)} u^{r_d}]$.

Proof. — The uniqueness of σ is determined since we have $t_{\sigma(i)} = r_i, \forall i$, that is,

$$t_{\sigma(1)} < \ldots < t_{\sigma(d)}. \tag{4.1.4}$$

It suffices to show that $\sigma \in W_C$ ($\Leftrightarrow \sigma^{-1}(W_C) \subseteq R^+$), i.e., if $\epsilon_i - \epsilon_j \in C$, then $\sigma^{-1}(i) < \sigma^{-1}(j)$. Let $x = \sigma^{-1}(i)$ and $\sigma^{-1}(j) = y$. Then $t_i = t_{\sigma(x)} < t_{\sigma(j)} = t_j$. So by (4.1.4), we must have $x < y$. □

4.1.5. Remark. — The following remark is suggested by one of the referees. Since we have already shown that $(A_\varphi)_{i,j} = 0$ when $t_j < t_i$ (where $j > i$), we could use an elementary "swapping" process to obtain the above proposition. Namely, suppose for example $t_{i+1} < t_i$, we could simply change the basis $(e_1, \ldots, e_i, e_{i+1}, \ldots, e_d)$ to $(e_1, \ldots, e_{i+1}, e_i, \ldots, e_d)$; the matrix for φ will remain upper triangular. After all these possible two by two swappings, the u-power on the diagonal will become eventually increasing.

As the readers can see, this elementary swapping process is precisely the key idea in the Breuil-Herzig group theory that we reviewed in Section 3. Indeed, the "ordinary part" of the p-adic Langlands in [1] is precisely built out from GL_2! It is also interesting to point out in the paper [4], a similar Weyl group element played a similar useful role in determining the locally algebraic vectors in the "ordinary part" of [1] (see the remarks following [4, Thm 1.2]).

We have chosen to keep the Breuil-Herzig theory in our paper (instead of the more elementary swapping process), because it does make the argument cleaner. Also, as we mentioned in the Introduction, this indeed provides a natural example of the Breuil-Herzig group theory.

4.2. Shape of $\boldsymbol{\tau}$. — Our following lemma (Lemma 4.2.1) is valid for any $K/\mathbb{Q}_p$. So we use notations introduced in Section 1. Recall that $u = (\pi_n)_{n=0}^{\infty} \in R$, and we normalize the valuation on R so that $v_R(u) = \frac{1}{e}$ where e is the ramification degree of $K/\mathbb{Q}_p$. For $\zeta \in R \otimes_{\mathbb{F}_p} k_E$, write it as $\zeta = \sum_{i=1}^{m} y_i \otimes a_i$ where $y_i \in R$, and $a_i \in k_E$ are independent over $\mathbb{F}_p$. Let

$$v_R(\zeta) := \min\{v_R(y_i)\}.$$

Then by [3, Lem. 5.6], v_R is a well-defined valuation on $R \otimes_{\mathbb{F}_p} k_E$ (so in particular, it does not depend on the sum representing ζ). In particular, $v_R(\varphi(\zeta)) = p v_R(\zeta)$. We also use the convention that $v_R(0) = +\infty$.

4.2.1. Lemma. — *Let $\zeta \in R \otimes_{\mathbb{F}_p} k_E$ with $v_R(\zeta) > 0$, such that*

$$\zeta\tau(\varphi(u^b)) = \varphi(u^a)\varphi(\zeta)$$

for some $a > b \geq 0$, then $\zeta = 0$.

Proof. — Note that $\tau(u) = u\underline{\epsilon}$, where $\underline{\epsilon} = (\mu_{p^n})_{n=0}^{\infty} \in R$. Consider the valuation on both side of the equation, then $v_R(\zeta) + \frac{pb}{e} = \frac{pa}{e} + p v_R(\zeta)$. The only possibility is when $v_R(\zeta) = +\infty$. □

Now we return to the $\mathbb{Q}_p$ case.

4.2.2. Proposition. — *With notations as in Proposition 4.1.1, let $A_\tau \in \mathrm{Mat}(R \otimes_{\mathbb{F}_p} k_E)$ be the matrix of τ with respect to the basis $1 \otimes_\varphi \boldsymbol{e}$. Then A_τ is in the subring $B_C(R \otimes_{\mathbb{F}_p} k_E)$ defined by (4.1.2), i.e., if $i < j$ and $t_i > t_j$, then $(A_\tau)_{i,j} = 0$.*

Proof. — This is easy consequence of the following lemma. Note that for any $i < j$, $v_R((A_\tau)_{i,j}) > 0$ by [3, Lem. 5.7]. □

4.2.3. Lemma. — *Let $F = (f_{i,j}) \in \mathrm{Mat}_d(k_E[\![u]\!])$, $M = (m_{i,j}) \in \mathrm{Mat}_d(R \otimes_{\mathbb{F}_p} k_E)$ two upper triangular matrices. Suppose* $\mathrm{diag}(F) = [a_1 u^{t_1}, \dots, a_d u^{t_d}]$ *where $a_i \in k_E^{\times}$ and t_i are* distinct *non-negative integers. Suppose that*

- *If $i < j$ and $t_i > t_j$, then $f_{i,j} = 0$,*
- *$v_R(m_{i,j}) > 0, \forall i < j$, and*
- *$M\tau(\varphi(F)) = \varphi(F)\varphi(M)$.*

Then $M \in B_C(R \otimes_{\mathbb{F}_p} k_E)$, where $C := \{\epsilon_i - \epsilon_j : i < j, t_i < t_j\}$ the closed subset of R^+.

Proof. — We prove by induction on the dimension d. When $d = 1$, there is nothing to prove. Suppose the lemma is true for dimension less than d, and consider it for d.

We can apply the induction hypothesis to $F_{1,1}$ and $M_{1,1}$ (resp. $F_{d,d}$ and $M_{d,d}$), where $F_{1,1}$ is the co-matrix of F by deleting the 1st row and 1st column (and similarly for $M_{1,1}, F_{d,d}$ and $M_{d,d}$). So we only need to deal with the element on the most upper right corner. That is, we only need to prove that if $t_1 > t_d$, then $m_{1,d} = 0$.

For any $2 \leq i \leq d$, we have

- (Case 1) If $t_i > t_1 > t_d$, then $f_{i,d} = 0$ (property of F), and $m_{i,d} = 0$ (induction hypothesis).
- (Case 2) If $t_1 > t_i$, then $f_{1,i} = 0$ (property of F), and $m_{1,i} = 0$ (induction hypothesis).

By the condition that $M\tau(\varphi(F)) = \varphi(F)\varphi(M)$, we must have

$$\sum_{i=1}^{d} m_{1,i}\tau(\varphi(f_{i,d})) = \sum_{i=1}^{d} \varphi(f_{1,i})\varphi(m_{i,d}).$$

So we will always have $m_{1,d}\tau(\varphi(u^{t_d})) = \varphi(u^{t_1})\varphi(m_{1,d})$, because all the other terms vanish. Now we can conclude $m_{1,d} = 0$ by Lemma 4.2.1. □

5. Crystalline lifting theorem

5.0.1. THEOREM. — *With notations in (*CRYS*), and suppose that $\overline{\rho}$ is upper triangular. Suppose $\overline{\rho} \in \mathcal{E}(\overline{\chi}_1, \ldots, \overline{\chi}_d)$ such that $\overline{\chi}_i\overline{\chi}_j^{-1} \neq \overline{\varepsilon}_p, \forall i \neq j$. Then there exists an upper triangular crystalline representation ρ' such that $\overline{\rho}' \cong \overline{\rho}$, and* $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$ *as sets.*

Proof. — Recall that $\boldsymbol{e} = (e_1, \ldots, e_d)$ is the basis of $\overline{\mathfrak{M}}$ in Proposition 4.1.1. Let $\sigma \in W_C$ be the unique element as in Proposition 4.1.3, and denote $\boldsymbol{e}^\sigma := (e_{\sigma(1)}, \ldots, e_{\sigma(d)})$. By *loc. cit.*, the matrix of φ for $\overline{\mathfrak{M}}$ with respect to the basis $\boldsymbol{e}^\sigma$ (which is $\sigma^{-1}A_\varphi\sigma$) is still upper triangular. By Proposition 4.2.2 and Lemma 3.0.2, the matrix of τ for $\overline{\hat{\mathfrak{M}}}$ with respect to the basis $1 \otimes_\varphi \boldsymbol{e}^\sigma$ (which is $\sigma^{-1}A_\tau\sigma$) is also upper triangular. That is to say (by Lemma 2.1.3), $\overline{\hat{\mathfrak{M}}} \in \mathcal{E}(\overline{\hat{\mathfrak{N}}}_{\sigma(d)}, \ldots, \overline{\hat{\mathfrak{N}}}_{\sigma(1)})$, where $\overline{\hat{\mathfrak{N}}}_{\sigma(i)} := \overline{\hat{\mathfrak{M}}}(r_i; a_{\sigma(i)})$. And so $\overline{\rho} = \hat{T}(\overline{\hat{\mathfrak{M}}}) \in \mathcal{E}(\overline{\chi}_{\sigma(1)}, \ldots, \overline{\chi}_{\sigma(d)})$.

By Lemma 2.2.2(3), each $\overline{\chi}_{\sigma(i)}$ has a crystalline lift $\chi_{\sigma(i)} =: \hat{T}(\hat{\mathfrak{M}}(r_i; \hat{a}_{\sigma(i)}))$, where $\hat{a}_{\sigma(i)} \in \mathcal{O}_E^\times$ is any lift of $a_{\sigma(i)}$. Since $r_1 < \ldots < r_d$, by [5, Lem. 3.1.5] (note that our convention of Hodge-Tate weights is the opposite of *loc. cit.*), $\overline{\rho}$ has an upper triangular crystalline lift ρ' such that $\rho' \in \mathcal{E}(\chi_{\sigma(1)}, \ldots, \chi_{\sigma(d)})$. Let us remark here that ρ' is in fact *ordinary* in the sense of [5, Def. 3.1.3]. □

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BIBLIOGRAPHY

[1] C. Breuil & F. Herzig – "Ordinary representations of $G(\mathbb{Q}_p)$ and fundamental algebraic representations", *Duke Math. J.* **164** (2015), p. 1271–1352.
[2] X. Caruso & T. Liu – "Some bounds for ramification of p^n-torsion semi-stable representations", *J. Algebra* **325** (2011), p. 70–96.
[3] H. Gao – "Crystalline liftings and the weight part of Serre's conjecture", *Israel J. Math.* **221** (2017), p. 117–164.
[4] H. Gao & C. Sorensen – "Locally algebraic vectors in the Breuil-Herzig ordinary part", *Manuscripta Math.* **151** (2016), p. 113–131.
[5] T. Gee & D. Geraghty – "Companion forms for unitary and symplectic groups", *Duke Math. J.* **161** (2012), p. 247–303.
[6] T. Gee, T. Liu & D. Savitt – "The Buzzard-Diamond-Jarvis conjecture for unitary groups", *J. Amer. Math. Soc.* **27** (2014), p. 389–435.
[7] ———, "The weight part of Serre's conjecture for GL(2)", *Forum Math. Pi* **3** (2015), p. e2.
[8] M. Kisin – "Crystalline representations and F-crystals", in *Algebraic geometry and number theory*, Progr. Math., vol. 253, Birkhäuser, 2006, p. 459–496.
[9] B. Levin – "G-valued crystalline representations with minuscule p-adic Hodge type", *Algebra Number Theory* **9** (2015), p. 1741–1792.
[10] T. Liu – "A note on lattices in semi-stable representations", *Math. Ann.* **346** (2010), p. 117–138.