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# DECODING RAUZY INDUCTION: AN ANSWER TO BUFETOV'S GENERAL QUESTION

by Jon Fickenscher

Abstract. — Given a typical interval exchange transformation, we may naturally associate to it an infinite sequence of matrices through Rauzy induction. These matrices encode visitations of the induced interval exchange transformations within the original. In 2010, W. A. Veech showed that these matrices suffice to recover the original interval exchange transformation, unique up to topological conjugacy, answering a question of A. Bufetov. In this work, we show that interval exchange transformation may be recovered and is unique modulo conjugacy when we instead only know consecutive products of these matrices. This answers another question of A. Bufetov. We also extend this result to any inductive scheme that produces square visitation matrices.

Résumé (*Décoder l'induction de Rauzy: Une réponse à la question générale de Bufetov*). — Etant donné une transformation d'échange d'intervalles typique, nous pouvons y associer naturellement une séquence infinie de matrices via l'induction de Rauzy. Ces matrices encodent les visites des transformations d'échanges d'intervalles induites dans l'intervalle original. En 2010, W. A. Veech a montré que ces matrices suffisent pour retrouver la transformation d'échange d'intervalles originale, unique à conjugaison topologique près, rèpondant à une question de A. Bufetov. Dans ce travail, nous montrons que la transformation d'échange d'intervalles peut être retrouvée et est unique à conjugaison près lorsque l'on connaît plutôt des produits consécutifs de ces matrices. Ceci répond à une autre question de A. Bufetov. Nous étendons également ce résultat à tout schéma inductif qui produit des matrices de visite carrées.

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## 1. Introduction

Interval exchange transformations (IET's) are invertible piece-wise translations on an interval $I$. They are typically defined by a permutation $\pi$ on $\{1,\dots,n\}$ and a choice of partitioning of $I$ into sub-intervals $I_1,\dots,I_n$ with respective lengths $\lambda_1,\dots,\lambda_n$. The sub-intervals are reordered by $T$ according to $\pi$.

For almost every IET $T$ (i.e., for every appropriate $\pi$ and Lebesgue almost every $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n_+$), Rauzy induction, as defined in [6], is a map that sends an IET $T$ on $I$ to its first return $T'$ on $I'\subset I$ for suitably chosen $I'$. For almost every[(1)] IET $T$, Rauzy induction may be applied infinitely often. This yields a sequence $T^{(k)}$, $k\geq 0$, of IET's so that each transition $T^{(k-1)}\mapsto T^{(k)}$ is the result of a Rauzy induction. To each step we may define a *visitation matrix* $A_k$ so that $(A_k)_{ij}$ counts the number of disjoint images of the intervals $I_j^{(k)}$ in $I_i^{(k-1)}$ before return to $I^{(k)}$. It is part of the general theory of IET's that the initial $\pi$ and the sequence of $A_k$'s define $T$ uniquely up to topological conjugacy. In preparation for [1], A. Bufetov posed the following.

QUESTION 1 (A. Bufetov). — *Given only the sequence of $A_k$'s, can the initial permutation $\pi$ be determined and is it unique?*

In response, W. A. Veech gave an affirmative answer in [10, Theorem 1.2]. This allowed A. Bufetov to ensure the injectivity of a map that intertwines the Kontsevich-Zorich cocycle with a renormalization cocycle (see the remark ending Section 4.3.1 in [1]).

However, if another induction scheme was used to get visitation matrices, we may not know each individual $A_k$. For instance, we may follow A. Zorich's acceleration of Rauzy induction (see [14]) or choose to induce on the first interval $I_1$. In either of these cases, our visitation matrix $B$ will actually be a product $A_1\cdots A_N$ of the $A_k$'s realized by Rauzy induction. Motivated by this, we say that a sequence $B_\ell$, $\ell\in\mathbb{N}$, is a product of the $A_k$'s if there exist an increasing sequence of integers $k_\ell$, $\ell\geq 0$, so that $k_0=0$ and $B_\ell=A_{k_{\ell-1}+1}A_{k_{\ell-1}+2}\cdots A_{k_\ell}$ for each $\ell\geq 1$. We now are able to pose Bufetov's second, more general, question.

QUESTION 2 (A. Bufetov). — *Given instead a sequence $B_\ell$, $\ell\in\mathbb{N}$, of products of the $A_k$'s, can the initial permutation $\pi$ still be determined and is it unique?*

This work is dedicated to answering this second question and its generalizations. We answer in the affirmative by our main results. Extended Rauzy induction is more general than regular Rauzy induction and is discussed in Section 2.5 before Lemma 2.15.

1. For every appropriate $\pi$ and Lebesgue almost every $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n_+$.

MAIN THEOREM 1. — *If $B_1, B_2, B_3, \dots$ are consecutive matrix products defined by an infinite sequence of steps of (extended) Rauzy induction, then the initial permutation $\pi$ is unique.*

Recently, J. Jenkins proved this result in [3] for the $3 \times 3$ matrix case. He then explored the $4 \times 4$ case numerically.

In the most general case, we call the inductions on $I' \supsetneq I'' \supsetneq I'' \supsetneq \cdots$ an *admissible induction sequence* if the $n \times n$ visitation matrices $A_k$ from $T^{(k-1)}$ to $T^{(k)}$ are well-defined. We then are able to answer Bufetov's question in a much broader setting.

MAIN THEOREM 2. — *If visitation matrices $B_1, B_2, B_3, \dots$ are defined by an admissible induction sequence, then the initial permutation $\pi$ is unique.*

*Outline of Paper.* — In Section 2 we establish our notation and provide known results concerning IET's and related objects as well as general linear algebra. In particular, the anti-symmetric matrix $L_\pi$ is defined given $\pi$, and this matrix plays a central role here. In Section 3 the Perron-Frobenius eigenvalue and eigenvector are discussed. The main argument of that section is Corollary 3.5, which says that the Perron-Frobenius eigenvector cannot be in the nullspace of any linear combination $L_\pi - cL_{\pi'}$ for permutations $\pi, \pi'$ and scalar $c$. Section 4 begins with a reduction of Main Theorem 1 to a special case, stated as the Main Lemma. The section ends with a proof of the Main Lemma. Section 5 reduces Main Theorem 2 to Main Theorem 1 by Lemma 5.1. This lemma states that any admissible induction sequence must arise from extended Rauzy induction. Appendix A provides further results concerning admissibility and induced maps which lead to the proof of Lemma 5.1 in Appendix B.

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## 2. Definitions

An *interval* or *sub-interval* is of the form $[a, b)$ for $a < b$, i.e., a non-empty subset of $\mathbb{R}$ that is closed on the left and open on the right. If $I = [a, b)$ is an interval, $|I| = b - a$ denotes its *length*. For a set $C$, $\#C$ denotes its cardinality. A *translation* $\phi : I \to J$ for intervals $I$ and $J$ is any function that may be expressed as $\phi(x) = x + c$ for constant $c$. If $\psi : C \to D$ is a function and $E \subseteq C$ we use the notation $\psi E$ to mean the *image of $E$ by $\psi$*, or $\psi E = \{\psi(c) : c \in E\} \subseteq D$. For $\lambda \in \mathbb{R}_+^n$, or a vector in $\mathbb{R}^n$ with all positive entries, $|\lambda| = \lambda_1 + \cdots + \lambda_n$ denotes the 1-norm of $\lambda$.

**2.1. Permutations and a Matrix**. — The notations in this section describe either standard definitions from algebra or standard literature on interval exchange transformations. Let $\mathfrak{S}_n$ be the set of all permutations on $\{1,\ldots,n\}$, i.e., bijections on $\{1,\ldots,n\}$.

Definition 2.1. — The *irreducible permutations* on $\{1,\ldots,n\}$, $\mathfrak{S}_n^0$, is the set of $\pi\in\mathfrak{S}_n$ so that $\pi\{1,\ldots,k\}=\{1,\ldots,k\}$ iff $k=n$.

Definition 2.2. — For $\pi\in\mathfrak{S}_n$, the anti-symmetric $n\times n$ matrix $L_\pi$ is given by

$$(L_\pi)_{ij}=\begin{cases}1, & i<j \text{ and } \pi(i)>\pi(j),\\ -1, & i>j \text{ and } \pi(i)<\pi(j),\\ 0, & \text{otherwise,}\end{cases}$$

$1\leq i,j,\leq n$.

The proof of the Main Theorem requires that no distinct $\pi,\pi'\in\mathfrak{S}_n^0$ satisfy $L_\pi=L_{\pi'}$. This is given by the next result.

Lemma 2.3. — *The map from $\mathfrak{S}_n$ to the set of $n\times n$ matrices given by*

$$\pi\mapsto L_\pi$$

*is injective.*

*Proof.* — The result follows immediately from the relationship

$$\pi(i)-i=\sum_{\pi(j)\leq\pi(i)}1-\sum_{k\leq i}1=\sum_{j=1}^{n}\chi_{\pi(i)\geq\pi(j)}(j)-\chi_{i\geq j}(j)=\sum_{j=1}^{n}(L_\pi)_{ij},$$

for all $i\in\{1,\ldots,n\}$, where $\chi$ is the indicator function. □

The final two definitions simply fix notation of established concepts from linear algebra and will be used without remark for what follows.

Definition 2.4. — If $L$ is an $n\times n$ matrix, $\mathcal{N}_L$ will denote its *nullspace*, i.e.,

$$\mathcal{N}_L=\{v\in\mathbb{C}^n: Lv=0\}.$$

For $\pi\in\mathfrak{S}_n^0$, $\mathcal{N}_\pi=\mathcal{N}_{L_\pi}$.

Definition 2.5. — If $L$ is an $n\times n$ anti-symmetric matrix, $(\cdot,\cdot)_L$ is the *bilinear form associated to $L$* given by

$$(u,v)_L=u^*Lv,$$

where the last value is treated as a scalar. For $\pi\in\mathfrak{S}_n^0$, $(\cdot,\cdot)_\pi=(\cdot,\cdot)_{L_\pi}$.

**2.2. Interval Exchange Transformations.** — An interval exchange transformation $T$ is an invertible transformation on an interval that divides the interval into sub-intervals of lengths $\lambda_1, \dots, \lambda_n$ and reorders them according to $\pi$. We will assume $n \geq 2$, as $T$ is the identity if $n = 1$.

More precisely, for fixed $\pi \in \mathfrak{S}_n^0$ and $\lambda \in \mathbb{R}_+^n$, let $\beta_j = \sum_{i \leq j} \lambda_i$ for $0 \leq j \leq n$ and $I = [0, \beta_n)$, where $\beta_0 = 0$ and $\beta_n = |\lambda|$. For each interval $I_j = [\beta_{j-1}, \beta_j)$, then $T$ restricted to $I_j$ is just translation by a value $\omega_j$. If the $j^{th}$ interval *is in position*[2] $\pi(j)$ after the application of $T$, then $\omega_j = \sum_{\pi(i) < \pi(j)} \lambda_i - \sum_{k<j} \lambda_k$. Note that many texts on interval exchange transformations instead let $\pi(j)$ describe the interval in position $j$ after the application of $T$. We see that $\omega = L_\pi \lambda$.

Definition 2.6. — The *interval exchange transformation (IET)* defined by $(\pi, \lambda)$, $T = \mathcal{I}_{\pi,\lambda}$, is a map $I \to I$ defined piece-wise by

$$T(x) = x + \omega_j, \text{ for } x \in I_j,$$

$1 \leq j \leq n$.

We restrict our attention to $\pi \in \mathfrak{S}_n^0$ when defining an IET. Indeed, if $\pi \in \mathfrak{S}_n \setminus \mathfrak{S}_n^0$ then there exists $k < n$ so that $T[0, \beta_k) = [0, \beta_k)$. In this case we may reduce to studying $T$ restricted to $[0, \beta_k)$ and $[\beta_k, \beta_n)$ separately.

Definition 2.7. — IET $T = \mathcal{I}_{\pi,\lambda}$ satisfies the *infinite distinct orbit condition or i.d.o.c.*[3] iff each orbit $\mathcal{O}_T(\beta_j) = \{T^k \beta_j : k \in \mathbb{Z}\}$, $1 \leq j < n$, is infinite and the orbits are pairwise distinct.

If $\lambda$ is rationally independent, meaning $c_1\lambda_1 + \cdots + c_n\lambda_n = 0$ has a solution with $c_i$'s integers iff $c_1 = \cdots = c_n = 0$, then $T = \mathcal{I}_{\pi,\lambda}$ is an i.d.o.c. IET. Therefore, $T$ is i.d.o.c. for fixed $\pi$ and Lebesgue almost every $\lambda \in \mathbb{R}_+^n$.

Lemma 2.8 (Keane [4]). — *If $T$ is an i.d.o.c. IET on $I$, then for any sub-interval $J \subseteq I$, $\bigcup_{k=0}^{\infty} T^k J = I$, and for any $x \in I$ the orbit $\mathcal{O}_T(x)$ is infinite and dense.*

**2.3. Admissible Inductions.** — Consider any interval $I' \subseteq I$ for IET $T = \mathcal{I}_{\pi,\lambda}$, $\pi \in \mathfrak{S}_n^0$ and $\lambda \in \mathbb{R}_+^n$. Informally, we will call $I'$ *admissible* if the induced map $T'$ is an IET on $n$ intervals and an $n \times n$ visitation matrix $A$ is well defined. Let $r(x) = \min\{k \in \mathbb{N} : T^k x \in I'\}$ be the *return time* of $x \in I'$. Then the *induced transformation* for $T$ on $I'$ is denoted by $T|_{I'}$ and is given by

$$T|_{I'}(x) = T^{r(x)}(x)$$

for each $x \in I'$.

2. Many texts on interval exchange transformations let $\pi(j)$ describe the interval in position $j$ after the application of $T$.

3. This is also known as the *Keane Condition.*

Definition 2.9. — Let $T = \mathcal{I}_{\pi,\lambda}$ be an i.d.o.c. $n$-IET on $I$. The sub-interval $I' \subseteq I$ is *admissible* for $T$ if there exists a partition of $I'$ into $n$ consecutive sub-intervals $I'_1, I'_2, \ldots, I'_n$ so that for each $1 \leq i \leq n$:

1. $r(x) = r(x')$ for each $x, y \in I'_i$, letting $r_i$ be this common value,
2. for each $0 \leq k < r_i$, $T$ restricted to $T^k I'_i$ is a translation,
3. for each $0 \leq k < r_i$, $T^k I'_i \subseteq I_j$ for some $1 \leq j \leq n$.

The $n \times n$ *visitation matrix* $A$ is given by $A_{ij} = \#\{0 \leq k < r_i : T^k I'_j \subseteq I_i\}$.

This definition of admissible is equivalent to the one given in [2, Section 3.3] for i.d.o.c. $T$. It follows that if $I'$ is admissible for $T$, then $T' = T|'_I$ is an $n$-IET, and

$$\lambda = A\lambda'$$

where $T' = \mathcal{I}_{\pi',\lambda'}$ and $\lambda'_j = |I'_j|$ for $1 \leq j \leq n$.

Remark 2.10. — Consider i.d.o.c. $T$ on $I$ with $I'' \subseteq I' \subseteq I$. It is a consequence that any of the two following statements imply the third:

1. $I'$ is admissible for $T$ on $I$,
2. $I''$ is admissible for $T$ on $I$,
3. $I''$ is admissible for $T'$ on $I'$, where $T'$ is the induction of $T$ on $I'$.

Also, if $A_1$ is the visitation matrix of the induction from $I$ to $I'$ and $A_2$ is the visitation matrix of the induction from $I'$ to $I''$, then the product $A_1 A_2$ is the visitation matrix of the induction from $I$ to $I''$.

Please refer to Appendix A for a more thorough discussion of admissible inductions. For example, Lemma A.2 proves the remark assuming the first statement and one other holds.

**2.4. Rauzy Induction.** — Rauzy induction was defined in [6], and we see that it is defined as an admissible induction over an appropriately chosen sub-interval $I'$. We recall the definition here and discuss some results relevant for our work.

For $\pi \in \mathfrak{S}^0_n$, let $m = \pi^{-1}(n)$ denote the interval placed last by $\pi$. Assume that $\lambda_n \neq \lambda_m$ and let $I' = [0, \beta_n - \min\{\lambda_m, \lambda_n\})$. The induced transformation $T' = T\big|_{I'}$ is also an IET and we give the description below for $T' = \mathcal{I}_{\pi',\lambda'}$. If $\lambda_n > \lambda_m$, then

$$\pi'(i) = \begin{cases} \pi(i), & \pi(i) < \pi(n), \\ \pi(n) + 1, & \pi(i) = n, \\ \pi(i) + 1, & \pi(i) > \pi(n), \end{cases} \quad \text{and } \lambda'_i = \begin{cases} \lambda_n - \lambda_m, & i = n, \\ \lambda_i, & i < n, \end{cases}$$

for $1 \leq i \leq n$. If $\lambda_n < \lambda_m$, then instead

$$\pi'(i) = \begin{cases} \pi(i), & i \leq m, \\ \pi(n), & i = m+1, \\ \pi(i-1), & i > m+1, \end{cases} \quad \text{and } \lambda'_i = \begin{cases} \lambda_i, & i < m, \\ \lambda_m - \lambda_n, & i = m, \\ \lambda_n, & i = m+1, \\ \lambda_{i-1}, & i > m+1, \end{cases}$$

for $1 \leq i \leq n$.

Definition 2.11. — Consider $T$ defined by $\pi$ and $\lambda$ with $m$ as above. If $\lambda_n > \lambda_m$, the change from $T$ to $T'$ is a move of *Rauzy induction of type* 0. If $\lambda_n < \lambda_m$, the change from $T$ to $T'$ is a move of *Rauzy induction of type* 1. If $\lambda_n = \lambda_m$, Rauzy Induction is not well defined.

For fixed $\pi \in \mathfrak{S}^0_n$, the condition $\lambda_n = \lambda_m$ is of zero Lebesgue measure in $\mathbb{R}^n_+$. Given $\pi$ and the type of Rauzy induction $\varepsilon$, we may define the visitation matrix $A = A_{(\pi,\varepsilon)}$ as given in Definition 2.9. If $\varepsilon = 0$, then

$$A_{ij} = \begin{cases} 1, i = j, \\ 1, i = n,\ j = m, \\ 0, \text{otherwise}, \end{cases}$$

and if $\varepsilon = 1$, then

$$A_{ij} = \begin{cases} 1, i = j < m, \\ 1, j = i + 1 > k, \\ 1, i = n,\ j = m, \\ 0, \text{otherwise}. \end{cases}$$

Suppose we may act by $N$ consecutive steps of Rauzy induction on $T = \mathcal{I}_{\pi,\lambda}$, and let $T, T', T'', \dots, T^{(N)}$ be the resulting IET's at each step where $T^{(k)} = \mathcal{I}_{\pi^{(k)},\lambda^{(k)}}$. Let $\varepsilon_k$ be the type of induction from $T^{(k-1)}$ to $T^{(k)}$. If

$$B = A_{(\pi,\varepsilon_1)} A_{(\pi',\varepsilon_2)} \cdots A_{(\pi^{(N-1)},\varepsilon_N)}$$

then $\lambda = B\lambda^{(N)}$.

We may verify that if $A = A_{\pi,\varepsilon}$ and $\pi'$ is the result of the type $\varepsilon$ induction on $\pi$, then $A^* L_\pi A = L_{\pi'}$. For a proof of this with different notation, see [11, Lemma 10.2]. It follows that if $B$ is defined by $N$ consecutive steps of induction with initial permutation $\pi$ and ending at $\pi^{(N)}$, then

$$B^* L_\pi B = L_{\pi^{(N)}}. \tag{1}$$

We finish this section by answering a question: for which IET's can Rauzy induction be applied infinitely many times? See Section 4 of [13] for a treatment of this result.

Lemma 2.12. — *If $\pi \in \mathfrak{S}^0_n$, $\lambda \in \mathbb{R}^n_+$ and $T = \mathcal{I}_{\pi,\lambda}$, then the following are equivalent:*

1. *$T$ is i.d.o.c., and*
2. *$T$ admits infinitely many steps of Rauzy induction.*

The following is shown in Sections 1.2.3–1.2.4 of [5].

LEMMA 2.13. — *If $T = \mathcal{I}_{\pi,\lambda}$ admits infinitely many steps of Rauzy induction and $A_k = A_{(\pi^{(k)},\lambda^{(k)})}$ are the corresponding matrices, then for each $j \in \mathbb{N}$ there exists $k_0 = k_0(j) \in \mathbb{N}$ so that for all $k > k_0$,*

$$A_{[j,j+k]} = A_k A_{k+1} \cdots A_{j+k}$$

*is a matrix with all positive entries.*

**2.5. Left Rauzy Induction.** — Left Rauzy induction was defined in [9] and describes inducing on $T = \mathcal{I}_{\pi,\lambda}$ by $I' = [\min\{\lambda_1, \lambda_{m'}\}, |\lambda|)$, where $m' = \pi^{-1}(1)$ for $\pi \in \mathfrak{S}_n^0$ and $\lambda \in \mathbb{R}_+^n$. In other words, instead of removing a sub-interval from the right as in (right) Rauzy induction, we remove one from the left. We will give the explicit definitions and then show how this type of induction relates to (right) Rauzy induction.

DEFINITION 2.14. — *Left Rauzy induction* is the result of taking the first return of $T = \mathcal{I}_{\pi,\lambda}$ on $I'$ as defined above. The induction is *type* $\tilde{0}$ iff $\lambda_1 > \lambda_{m'}$ and *type* $\tilde{1}$ iff $\lambda_1 < \lambda_{m'}$. The induction is not well defined if $\lambda_1 = \lambda_{m'}$.

Let $T' = \mathcal{I}_{\pi',\lambda'}$ be the resulting IET by this induction (up to a translation so that $I'$ begins at 0). If the induction is type $\tilde{0}$, then

$$\pi'(i) = \begin{cases} \pi(1) - 1, \, i = m', \\ \pi(i) - 1, \ 1 < \pi(i) < \pi(i), \\ \pi(i), \qquad \pi(i) \geq \pi(1), \end{cases} \text{ and } \lambda'_i = \begin{cases} \lambda_1 - \lambda_{m'}, \, i = 1, \\ \lambda_i, \qquad\quad i > 1. \end{cases}$$

Likewise, if the induction is type $\tilde{1}$, then

$$\pi'(i) = \begin{cases} \pi(i+1), \, i < m' - 1, \\ \pi(1), \qquad i = m' - 1, \\ \pi(i), \qquad i \geq m', \end{cases} \text{ and } \lambda'_i = \begin{cases} \lambda_{i+1}, \qquad i < m' - 1, \\ \lambda_1, \qquad\quad i = m' - 1, \\ \lambda_{m'} - \lambda_1, \, i = m', \\ \lambda_i, \qquad\quad i > m'. \end{cases}$$

Let $\tau_n$ be given by

$$\tau_n(i) = n - i \text{ for } 0 \leq i \leq n.$$

For $\pi \in \mathfrak{S}_n^0$, let $\pi_\tau$ be given by

$$\pi_\tau = \tau_{n+1} \circ \pi \circ \tau_{n+1},$$

noting that $\pi_\tau \in \mathfrak{S}_n^0$ as well. If $\tilde{\varepsilon}\pi$ is the result of type $\tilde{\varepsilon}$ induction on $\pi$ and $\varepsilon\pi_\tau$ is the result of type $\varepsilon$ induction on $\pi_\tau$, then

$$\tilde{\varepsilon}\pi = (\varepsilon\pi_\tau)_\tau.$$

For $\lambda \in \mathbb{R}_+^n$, let $\lambda_\tau$ be given by

$$(\lambda_\tau)_i = \lambda_{\tau_{n+1}(i)}.$$

If $\tilde{\varepsilon}\lambda$ and $\varepsilon\lambda_\tau$ are defined analogously to $\tilde{\varepsilon}\pi$ and $\varepsilon\pi_\tau$, then

$$\tilde{\varepsilon}\lambda = (\varepsilon\lambda_\tau)_\tau.$$

We define the $n \times n$ permutation matrix $P_n$ by

$$(P_n)_{ij} = \begin{cases} 1, \, i = \tau_{n+1}(j), \\ 0, \, \text{otherwise}, \end{cases}$$

then we see that $A_{\pi,\tilde{\varepsilon}} = P_n A_{\pi_\tau,\varepsilon} P_n$, where $A_{\pi,\tilde{\varepsilon}}$ is the visitation matrix that satisfies $\lambda = A_{\pi,\tilde{\varepsilon}} \cdot \tilde{\varepsilon}\lambda$. Furthermore,

$$L_\pi = P_n L_{\pi_\tau} P_n,$$

and so $A^*_{\pi,\tilde{\varepsilon}} L_\pi A_{\pi,\tilde{\varepsilon}} = L_{\tilde{\varepsilon}\pi}$ as a direct consequence.

Therefore, if $A_1, \dots, A_N$ are visitation matrices given by consecutive steps of *extended Rauzy induction*(4), i.e., left and/or right Rauzy induction, and $B = A_1 \cdots A_N$ is the product, then

$$B^* L_\pi B = L_{\pi^{(N)}}$$

where $\pi$ is the initial permutation and $\pi^{(N)}$ is the resulting permutation after the $N$ steps.

The proof of the following is a modification of Lemma 2.13 and has a similar proof. However, the notation from [5] is significantly different and will not be included here.

LEMMA 2.15. — *If $T = \mathcal{I}_{\pi,\lambda}$ admits infinitely many steps of extended Rauzy induction and $A_k$ are the corresponding matrices, then for each $j \in \mathbb{N}$ there exists $k_0 = k_0(j) \in \mathbb{N}$ so that for all $k > k_0$,*

$$A_{[j,j+k]} = A_k A_{k+1} \cdots A_{j+k}$$

*is a matrix with all positive entries.*

**2.6. Veech's result for $\mathcal{N}_\pi$.** — The main result in this section is shown in [8, Lemma 5.7]. Please refer to that work as well as [7] for the original definitions and proofs.

Consider each $\pi \in \mathfrak{S}^0_n$ to be extended so that $\pi(0) = 0$ and $\pi(n+1) = n+1$. Note that $\pi_\tau$ respects this extension as well. Let $\sigma_\pi$ be a function on $\{0, \dots, n\}$ given by

$$\sigma_\pi(i) = \pi^{-1}(\pi(i) + 1) - 1,$$

as in [7]. Let $\Sigma(\pi)$ be the partition of $\{0, 1, \dots, n\}$ given by orbits of $\sigma_\pi$. For each $S \in \Sigma(\pi)$, let $b_S \in \mathbb{Z}^n$ be given by

$$(b_S)_i = \chi_S(i-1) - \chi_S(i).$$

It was shown in [8, Lemma 5.3] that

$$\#\Sigma(\pi) = \dim \mathcal{N}_\pi + 1,$$

4. This is also known as two-sided Rauzy induction.

and [8, Proposition 5.2] states that

$$\operatorname{span}\{b_S : S \in \Sigma(\pi)\} = \mathcal{N}_\pi.$$

LEMMA 2.16 (Veech [8]). — *For $\pi \in \mathfrak{S}_n^0$ and $\varepsilon \in \{0,1\}$, there exists a bijection $\varepsilon : \Sigma(\pi) \to \Sigma(\varepsilon\pi)$ so that*

$$A_{(\pi,\varepsilon)} b_S = b_{\varepsilon S}$$

*for each $S \in \Sigma(\pi)$.*

Recall $\tau_n$, $\tau_{n+1}$ and $P = P_n$ from the previous section. By direct computation, we see that

$$\sigma_{\pi_\tau} = \tau_n \circ \sigma_\pi^{-1} \circ \tau_n,$$

and so $\Sigma(\pi_\tau) = \tau_n \Sigma(\pi)$.

COROLLARY 2.17. — *For $\pi \in \mathfrak{S}_n^0$ and $\varepsilon \in \{0,1\}$, there exists a bijection $\tilde{\varepsilon} : \Sigma(\pi) \to \Sigma(\varepsilon\pi)$ so that*

$$A_{(\pi,\tilde{\varepsilon})} b_S = b_{\tilde{\varepsilon} S}$$

*for each $S \in \Sigma(\pi)$.*

*Proof.* — For each $S \in \Sigma(\pi)$ and $1 \le i \le n$,

$$\begin{aligned}(Pb_S)_i &= (b_S)_{n+1-i} \\ &= \chi_S(n-i) - \chi_S(n+1-i) \\ &= \chi_{\tau_n S}(\tau_n(n-i)) - \chi_{\tau_n S}(\tau_n(n+1-i)) \\ &= \chi_{\tau_n S}(i) - \chi_{\tau_n S}(i-1) \\ &= -(b_{\tau_n S})_i.\end{aligned}$$

And so

$$A_{(\pi,\tilde{\varepsilon})} b_S = P A_{(\pi_\tau,\varepsilon)} P b_S = -P A_{(\pi_\tau,\varepsilon)} b_{\tau_n S}.$$

By Lemma 2.16 we continue,

$$-P A_{(\pi_\tau,\varepsilon)} b_{\tau_n S} = -P b_{\varepsilon(\tau_n S)} = b_{\tau_n(\varepsilon(\tau_n S))}.$$

Therefore the desired bijection is $\tilde{\varepsilon} = \tau_n \circ \varepsilon \circ \tau_n$. □

**2.7. Invariant Spaces.** — For $n \times n$ matrix $B$, a subspace $V \subseteq \mathbb{C}^n$ is *$B$-invariant* if $BV \subseteq V$. If $B$ is invertible then $V$ is $B$-invariant iff $BV = V$. An *eigenbasis* of $B$ for $V$ is a basis $\{u_1, \ldots, u_m\}$ of $V$ so that each $u_j$ is an eigenvector for $B$. Recall that an *eigenvector* for $B$ is a non-zero vector $u$ with a corresponding *eigenvalue* $\alpha$ such that $u \in \mathcal{N}_{B_\alpha^p}$ for some $p \in \mathbb{N}$ where $B_\alpha = B - \alpha I$ for identity matrix $I$. The lemma and corollary in this section allow us to find an eigenbasis for $\mathbb{C}^n$ that includes bases of invariant subspaces. The definition that follows then correctly associates to a $B$-invariant subspace eigenvalues.

LEMMA 2.18. — *Let $B$ be an $n \times n$ matrix and $V, W \subseteq \mathbb{C}^n$ be subspaces such that $W \subseteq V$ and $V, W$ are each $B$-invariant. There exists an* eigenbasis *$\{u_1, \ldots, u_m\}$ of $B$ for $V$ so that $\{u_1, \ldots, u_{m'}\}$ is a basis for $W$, $m' = \dim W$.*

COROLLARY 2.19. — *If $V, W \subseteq \mathbb{C}^n$ are $B$-invariant subspaces, $m = \dim V$ and $m' = \dim W$, then there exists an eigenbasis $\{u_1, \ldots, u_n\}$ of $B$ for $\mathbb{C}^n$ such that*

- $\{u_{n-m-m'+\ell+1}, \ldots, u_{n-m'+\ell}\}$ *is a basis for $W$,*
- $\{u_{n-m'+1}, \ldots, u_{n-m'+\ell}\}$ *is a basis for $V \cap W$,*
- $\{u_{n-m'+1}, \ldots, u_n\}$ *is a basis for $W$,*

*where $\ell = \dim(V \cap W)$.*

DEFINITION 2.20. — If $V \subseteq \mathbb{C}^n$ is $B$-invariant, and $\alpha_1, \ldots, \alpha_m$ are the respective eigenvalues for the eigenbasis in the previous lemma or corollary, then they are the *eigenvalues of $B$ over $V$.*

## 3. The Perron-Frobenius Eigenvalue

We begin with a specific case of a fundamental result. See [12, Theorem 0.16] for a more general version of this theorem.

THEOREM (Perron-Frobenius Theorem). — *If $B$ is a positive matrix, then there exists positive eigenvalue $\alpha$ for $B$ so that for all other eigenvalues $\alpha'$ of $B$, $\alpha > |\alpha'|$. Furthermore, there exists a positive eigenvector $u$ for $B$ with eigenvalue $\alpha$ and any eigenvector $u'$ for $B$ with eigenvalue $\alpha$ is a scalar multiple of $u$.*

We call $\alpha$ the *Perron-Frobenius eigenvalue* such a positive vector $u$ a *Perron-Frobenius eigenvector.* If $B$ is a positive integer matrix, then $\alpha > 1$. Corollary 3.3 tells us that $u$ is not in $\mathcal{N}_\pi$ for any $\pi \in \mathfrak{S}_n^0$. Then Corollary 3.5 forbids $u$ from being in $\mathcal{N}_L$ for any non-zero matrix $L = L_\pi - cL_{\pi'}$. Finally, Corollary 3.7 tells us that for a fixed eigenbasis $\{u_1, \ldots, u_n\}$ of $B$ with $u_1 = u$ there exists a unique $u_j$ so that $(u_1, u_j)_\pi \neq 0$.

DEFINITION 3.1. — An *extended Rauzy cycle* at $\pi$ is a finite sequence of consecutive steps of extended Rauzy induction that begins and ends at $\pi$.

LEMMA 3.2. — *If $B$ is described by an extended Rauzy cycle at $\pi \in \mathfrak{S}_n^0$, then there exists a basis $\{b_1, \ldots, b_m\}$ of $\mathcal{N}_\pi$ and $p \in \mathbb{N}$ so that*

$$B^p b_i = b_i,$$

*for each $i \in \{1, \ldots, m\}$.*

*Proof.* — Let $b_S$ for $S \in \Sigma(\pi)$ be as in Section 2.6. By applying Lemma 2.16 and Corollary 2.17 to the product $B$, we have a bijection $d$ on $\Sigma(\pi)$ so that $Bb_S = b_{dS}$ for each $S \in \Sigma(\pi)$.

Let $p$ be any power such that $d^p$ is the identity on $\Sigma(\pi)$, and choose the $b_1, \ldots, b_m$ as a subset of the $b_S$'s that form a basis of $\mathcal{N}_\pi$. Then for any $i \in \{1, \ldots, m\}$,

$$B^p b_i = B^p b_S = b_{d^p S} = b_S = b_i,$$

where $b_i = b_S$. □

COROLLARY 3.3. — *Let $B$ be given by an extended Rauzy cycle at $\pi$. If $\beta_1, \ldots, \beta_m$ are the eigenvalues of $B$ over $\mathcal{N}_\pi$, $\pi \in \mathfrak{S}_n^0$, then each $\beta_j$ is a root of unity. Furthermore, if $B$ is a positive integer matrix with Perron-Frobenius eigenvalue $\alpha > 1$, then $\{\beta_1, \ldots, \beta_m\} \cap \{\alpha, 1/\alpha\} = \emptyset$.*

LEMMA 3.4. — *If $L = L_\pi - cL_{\pi'}$ for distinct $\pi, \pi' \in \mathfrak{S}_n^0$ and real $c$, then there exists $i$ such that row $i$ of $L$ is non-zero and either non-positive or non-negative.*

*Proof.* — We consider the value of $c$. If $c \leq 0$, then the first row of $L$ is at least the first row of $L_\pi$. The claim then holds for $i = 1$. If $c > 1$ and $L' = L_{\pi'} - \frac{1}{c} L_\pi$ then $L = -cL'$. If we find a non-zero row for $L'$, then the same row satisfies the claim for $L$ (but with the opposite sign). We therefore consider two remaining cases: $0 < c < 1$ and $c = 1$.

If $0 < c < 1$, let $i$ satisfy $\pi(i) = n$. Because $\pi \in \mathfrak{S}_n^0$, $i < n$. Then if $i > j$,

$$L_{ij} = (L_\pi)_{ij} - c(L_{\pi'})_{ij} = -c(L_{\pi'})_{ij} \geq 0.$$

If $i < j$,

$$L_{ij} = (L_\pi)_{ij} - c(L_{\pi'})_{ij} = 1 - c(L_{\pi'})_{ij} > 0.$$

Therefore row $i$ is non-zero and is non-negative.

If $c = 1$, then let $i$ be such that $\pi(i) \neq \pi'(i)$ and that maximizes $\pi(i)$. Let $k \leq n$ be the position of $i$ in $\pi$, i.e., $k = \pi(i)$. Note that $\pi'(i) < k$. If $\pi(j) > k$, then $\pi(j) = \pi'(j)$ and so

$$(L_\pi)_{ij} = (L_{\pi'})_{ij} \Rightarrow L_{ij} = 0.$$

If $\pi(j) < k$ and $i > j$, then

$$L_{ij} = (L_\pi)_{ij} - (L_{\pi'})_{ij} = -(L_{\pi'})_{ij} \geq 0.$$

If $\pi(j) < k$ and $i < j$, then

$$L_{ij} = (L_\pi)_{ij} - (L_{\pi'})_{ij} = 1 - (L_{\pi'})_{ij} \geq 0.$$

Row $i$ of $L$ is non-negative, and we must verify that it is non-zero. Let $i' \neq i$ satisfy $\pi'(i') = k$. By definition, $\pi(i') < \pi(i)$ and $\pi'(i') > \pi'(i)$. Either

- $i < i'$ and so $(L_\pi)_{ii'} = 1$ and $(L_{\pi'})_{ii'} = 0$, or
- $i > i'$ and so $(L_\pi)_{ii'} = 0$ and $(L_{\pi'})_{ii'} = -1$.

Therefore, $L_{ii'} = 1$ and row $i$ of $L$ is non-zero. □

COROLLARY 3.5. — *If $v$ is a positive vector, then it does not belong to the nullspace of*

$$L = L_\pi - cL_{\pi'}$$

*for distinct $\pi, \pi' \in \mathfrak{S}_n^0$ and complex $c$.*

*Proof.* — Suppose $c$ is real. By Lemma 3.4, there exists a row $i$ that is non-zero and non-negative (resp. non-positive). Therefore $(Lv)_i > 0$ (resp. $(Lv)_i < 0$) and so $Lv \neq 0$. If $c$ is non-real, then by Lemma 3.4 the real vector $L_{\pi'}v$ is non-zero. Therefore

$$Lv = (L_\pi - cL_{\pi'})v$$

must have a non-zero imaginary component and cannot be zero. □

LEMMA 3.6. — *Let $B^*LB = L$ for anti-symmetric matrix $L$ and matrix $B$. If $u, u'$ are eigenvectors of $B$ with corresponding eigenvalues $\alpha, \alpha'$, and $(u, u')_L \neq 0$, then $\overline{\alpha}\alpha' = 1$.*

*Proof.* — Recall that the *order* of eigenvalue $u$ with eigenvalue $\alpha$ is the minimum $p \in \mathbb{N}$ so that $u \in \mathcal{N}_{B_\alpha^p}$, $B_\alpha = B - \alpha I$. We proceed by induction, first on the order of $u$ and then on the sum of the orders of $u$ and $u'$.

If $u, u'$ are both *true eigenvectors*, i.e., order 1, then

$$(u, u')_L = (Bu, Bu')_L = \overline{\alpha}\alpha'(u, u')_L,$$

and $\overline{\alpha}\alpha' = 1$ as $(u, u')_L \neq 0$ by assumption.

If $u$ is higher order and $u'$ is order 1, let $w$ be defined by $Au = \alpha u + w$ and note that $w$ is one order lower than $u$ for the same eigenvalue $\alpha$. Then

$$(u, u')_L = (Bu, Bu')_L = \overline{\alpha}\alpha'(u, u')_L + \alpha'(w, u')_L.$$

If $(w, u')_L \neq 0$, then the claim follows by induction. If $(w, u')_L = 0$, then the claim follows just as in the base case.

If $u, u'$ are both of higher order, let $w$ be defined as before and let $w'$ be defined by $Bu' = \alpha' u' + w'$. Then $(u, u')_L = (Bu, Bu')_L$ which is the sum

$$\overline{\alpha}\alpha'(u, u')_L + \alpha'(w, u')_L + \overline{\alpha}(u, w')_L + (w, w')_L.$$

If any term but the first is non-zero, then the claim follows by induction. Note that $(v, v')_L = \overline{(v', v)_L}$. If only the first term is non-zero, then the claim is verified as before. □

COROLLARY 3.7. — *Let $B^*LB = L$ for anti-symmetric matrix $L$ and positive matrix $B$. Let $u_1, \ldots, u_n$ an eigenbasis for $B$ with respective eigenvalues $\alpha_1, \ldots, \alpha_n$ such that $u_{n-m+1}, \ldots, u_n$ forms a basis of $\mathcal{N}_L$ of dimension $m < n$. If $\alpha_1 > 0$ is the Perron-Frobenius eigenvalue with positive eigenvector $u_1$ (in particular $u_1 \notin \mathcal{N}_L$), then there is a unique $j \leq n - m$ so that $(u_1, u_j)_L \neq 0$. This is also the unique $j \leq n - m$ so that $\alpha_j = 1/\alpha_1$.*

*Proof.* — The final claim follows from the first by Lemma 3.6. Because $u_1 \notin \mathcal{N}_L$, there must exist $u_j$ so that $(u_1, u_j)_L = c \neq 0$. By Lemma 3.6, $\alpha_j = 1/\alpha_1$. Because $u_j \notin \mathcal{N}_L$, $j \leq n - m$.

Suppose by contradiction there exists $j' \neq j$ so that $(u_1, u_{j'})_L = c' \neq 0$. Again $j' \leq n - m$ and by Lemma 3.6, $\alpha_{j'} = \alpha_j$. If there exists $i \neq 1$ so that either $(u_i, u_{j'})_L \neq 0$ or $(u_i, u_j)_L \neq 0$, then $\alpha_i = \alpha_1$, a contradiction to the simplicity of the Perron-Frobenius eigenvalue. Therefore $(u_k, u_j)_L$ is non-zero iff $k = 1$ and the same statement holds for $(u_k, u_{j'})_L$.

Let $u = c'u_j - cu_{j'}$ and note that

$$(u_1, u)_L = c'(u_1, u_j)_L - c(u_1, u_{j'})_L = c'c - cc' = 0.$$

Because $(u_k, u)_L = 0$ for $k \neq 1$, $c \in \mathcal{N}_L$. This implies a linear dependence between $u_j$, $u_{j'}$ and $u_{n-m+1}, \ldots, u_n$, a contradiction. □

## 4. Proof of Main Theorem 1

Main Lemma. — *If $\tilde{B}$ is a positive matrix defined by an extended Rauzy cycle, then the initial permutation $\pi \in \mathfrak{S}_n^0$ is unique.*

*Proof of Main Theorem 1.* — Let $B_1, B_2, \ldots$ be the matrix products defined by an infinite sequence of extended Rauzy induction steps, and assume by contradiction that there exist distinct $\pi, \pi' \in \mathfrak{S}_n^0$ such that infinite induction steps beginning at $\pi$ and $\pi'$ each exist and define the $B_k$'s. There exist $\pi_k$'s and $\pi'_k$'s, $k \in \mathbb{N}_0$, so that $\pi_0 = \pi$, $\pi'_0 = \pi'$ and for each $k \in \mathbb{N}$,

$$B_k^* L_{\pi_{k-1}} B_k = L_{\pi_k} \text{ and } B_k^* L_{\pi'_{k-1}} B_k = L_{\pi'_k}.$$

Because the $B_k$'s are invertible and by Lemma 2.3, $L_{\pi_k} \neq L_{\pi'_k}$ by induction on $k$ and so $\pi_k \neq \pi'_k$ for all $k \in \mathbb{N}_0$ and are uniquely determined by $\pi$, $\pi'$ and the $B_k$'s. There exist distinct $\tilde{\pi}, \tilde{\pi}' \in \mathfrak{S}_n^0$ so that $\pi_k = \tilde{\pi}$ and $\pi'_k = \tilde{\pi}'$ simultaneously for infinitely many $k$. By Lemma 2.13 we may choose such $k_0, k_1$ so that $k_0 < k_1$ and $\tilde{B} = B_{k_0+1} B_{k_0+2} \cdots B_{k_1}$ is positive. Therefore $\tilde{B}$ is a positive matrix defined by an extended Rauzy cycle at $\tilde{\pi}$ and also an extended Rauzy cycle at $\tilde{\pi}'$. By the Main Lemma $\tilde{\pi} = \tilde{\pi}'$, a contradiction. □

*Proof of Main Lemma.* — Suppose $\tilde{B}$ is a positive integer matrix that is described by two extended Rauzy cycles: one each at distinct $\tilde{\pi}, \tilde{\pi}' \in \mathfrak{S}_n^0$. Then by Equation (1),

$$\tilde{B}^* L_{\tilde{\pi}} \tilde{B} = L_{\tilde{\pi}} \text{ and } \tilde{B}^* L_{\tilde{\pi}'} \tilde{B} = L_{\tilde{\pi}'}.$$

Let $m = \dim(\mathcal{N}_{\tilde{\pi}})$, $m' = \dim(\mathcal{N}_{\tilde{\pi}'})$ and $\ell = \dim(\mathcal{N}_{\tilde{\pi}} \cap \mathcal{N}_{\tilde{\pi}'})$. Using Corollary 2.19, let $\{u_1, \ldots, u_n\}$ be an eigenbasis for $B$ with respective eigenvalues $\alpha_1, \ldots, \alpha_n$ such that

- $\alpha_1 > 1$ is the Perron-Frobenius eigenvalue and $u_1$ is positive,
- $\{u_{n-m-m'+\ell+1}, \ldots, u_{n-m'+\ell}\}$ is a basis for $\mathcal{N}_{\tilde{\pi}}$,

- $\{u_{n-m'+1},\ldots,u_{n-m'+\ell}\}$ is a basis for $\mathcal{N}_{\tilde{\pi}}\cap\mathcal{N}_{\tilde{\pi}'}$, and
- $\{u_{n-m'+1},\ldots,u_{n}\}$ is a basis for $\mathcal{N}_{\tilde{\pi}'}$.

Because $u_1$ is not in $\mathcal{N}_{\tilde{\pi}}$ there must exist a unique $j\leq n+\ell-m-m'$ so that $(u_1,u_j)_{\tilde{\pi}}\neq 0$ by Corollary 3.7. By Corollary 3.3 this is also the unique $1\leq j\leq n$ so that $\alpha_j=1/\alpha_1$. It follows that $(u_1,u_i)_{\tilde{\pi}'}\neq 0$ iff $i=j$ as well, and $j\leq n+\ell-m-m'$.

Let $c_1=(u_1,u_j)_{\tilde{\pi}}$, $c_2=(u_1,u_j)_{\tilde{\pi}'}$ and $L=L_{\tilde{\pi}}-\frac{c_1}{c_2}L_{\tilde{\pi}'}$. For $i\neq j$,

$$(u_1,u_i)_L=(u_1,u_i)_{\tilde{\pi}}-\frac{c_1}{c_2}(u_1,u_i)_{\tilde{\pi}'}=0-\frac{c_1}{c_2}0=0$$

and

$$(u_1,u_j)_L=(u_1,u_j)_{\tilde{\pi}}-\frac{c_1}{c_2}(u_1,u_j)_{\tilde{\pi}'}=c_1-\frac{c_1}{c_2}c_2=0.$$

This implies that $u_1\in\mathcal{N}_L$, a contradiction to Corollary 3.5. □

## 5. Proof of Main Theorem 2

The following result is proven in [2, Theorem 4.3] under different notation. A proof is provided in Appendix B.

Lemma 5.1. — *Let $T:I\to I$ be an i.d.o.c. $n$-IET. Then $J\subsetneq I$ is admissible iff the induction realized by $J$ is given by consecutive steps of extended Rauzy induction.*

Definition 5.2. — An *admissible induction sequence* for $T=\mathcal{I}_{\pi,\lambda}:I\to I$ is a sequence $I',I'',\ldots,I^{(k)},\ldots$ so that the induction from $I^{(k-1)}$ to $I^{(k)}$ is admissible for each $k$.

By Lemma A.3, it follows that $|I^{(k)}|\to 0$ as $k\to\infty$ for any admissible induction sequence. We say that $B_1,B_2,\ldots$ are defined by an admissible induction sequence $I',I'',\ldots$ if for each $k$ $B_k$ is the visitation matrix the induction of $T^{(k-1)}$ on $I^{(k)}$, where $T^{(k)}$ is the induction of $T$ on $I^{(k)}$, or equivalently $T^{(k)}$ is the induction of $T^{(k-1)}$ on $I^{(k)}$.

*Proof of Main Theorem 2.* — A sequence of matrices $B_1,B_2,\ldots$ defined by an admissible induction sequence must be given by products of $A_k$'s given by extended Rauzy induction by Lemma 5.1. The result then follows from Main Theorem 1. □

## Appendix A. Admissible Inductions

We will give a construction of the induced map given by $T:I\to I$ over sub-interval $I'\subsetneq I$. Let $T=\mathcal{I}_{\pi,\lambda}$, $\pi\in\mathfrak{S}_n^0$ and $\lambda\in\mathbb{R}_+^n$, be an i.d.o.c. $n$-IET with sub-intervals $I_i=[\beta_{i-i},\beta_i)$, $1\leq i\leq n$, where we recall $\beta_i=\sum_{j\leq i}\lambda_j$ for $0\leq i\leq n$. Also, recall the return time $r(x)$ of $x\in I'$ to $I'=[a',b')$ by $T$.

Given $x \in I'$, we describe how to construct the sub-interval $I'_x \subseteq I'$ so that

1. $x \in I'_x$,
2. for each $z, z' \in I'_x$, $r(z) = r(z')$,
3. for each $0 \leq k < r(x)$, $T$ restricted to $T^k I'_x$ is a translation,
4. for each $0 \leq k < r(x)$, there exists $j = j(k)$ so that $T^k I'_x \subseteq I_j$, and
5. $I'_x$ is the maximal sub-interval with these properties.

Compare properties 2–4 with those of intervals $I'_i$ for admissibility in Definition 2.9.

Observe that for any sub-interval $[c, d)$ such that $[c, d) \subseteq I_j$ for some $j$, $T$ restricted to $[c, d)$ is a translation. Also, $T[c, d) \subseteq I_{j'}$ for some $j'$ iff $(c, d) \cap T^{-1}\mathcal{D} = \emptyset$ where

$$\mathcal{D} = \{\beta_1, \ldots, \beta_{n-1}\}.$$

Let $a(x)$ be the minimum $y \in \mathcal{I}' \cap [0, x]$ so that

- $r(z) = r(x)$ for all $z \in [y, x]$, and
- for each $0 \leq k < r(x)$, $T^k[y, x] \subset I_j$ for some $j = j(k)$.

It is an exercise to see that

$$x - a(x) = \min\{T^k x - z : 0 \leq k < r(x),\ z \in \mathcal{D}' \cap [0, T^k x]\},$$

where $\mathcal{D}' = \{a', b', 0\} \cup \{\beta_1, \ldots, \beta_{n-1}\}$. Therefore, either $a(x) = a'$ or $a(x) = T^{-r_-(z)}z$ for some $z \in \mathcal{D}'$, where $r_-(z) = \min\{k \in \mathbb{N}_0 : T^{-k}z \in (a', b')\}$. Likewise, let $b(x)$ be defined by

$$b(x) - x = \min\{z - T^k x : 0 \leq k < r(x),\ z \in (\{|\lambda|\} \cup \mathcal{D}') \cap [T^k x, |\lambda|]\},$$

and note that $b(x) \in (I' \cup \{b'\}) \cap [x, |\lambda|]$ is the maximum value $y$ so that

- $r(z) = r(x)$ for all $z \in [x, y)$, and
- for each $0 \leq k < r(x)$, $T^k[x, y) \subset I_j$ for some $j = j(k)$.

Either $b(x) = b'$ or $b(x) = a(x')$ for some $x' > x$.

Let $I'_x = [a(x), b(x))$ for each $x \in I'$. Because $I'_y = I'_x$ for each $y \in I'_x$,

$$\mathcal{P}_{I'} = \{I'_x : x \in I'\}$$

is a partition of $I'$. If $\mathcal{P}_{I'} = \{I'\}$, then $T$ is periodic and cannot be and i.d.o.c. IET. Therefore $\#\mathcal{P}_{I'} \geq 2$.

Lemma A.1. — *For i.d.o.c. $n$-IET $T : I \to I$ and sub-interval $I' \subseteq I$, if $m = \#\mathcal{P}_{I'}$ then $n \leq m \leq n + 2$.*

*Proof.* — Let $\mathcal{D}_{I'} = \{a'\} \cup \{T^{-r_-(z)}z : z \in \mathcal{D}'\}$ and $\gamma_j = T^{-r_-(\beta_j)}\beta_j$ for $0 \leq j < n$. As discussed in the previous paragraphs, $a(x) \in \mathcal{D}_{I'}$ for each $x$. Furthermore, $a(z) = z$ for each $a \in \mathcal{D}_{I'}$. Therefore $\#\mathcal{P}_{I'} = \#\mathcal{D}_{I'}$. Because $T$ is i.d.o.c., $T^\ell \beta_j = \beta_{j'}$ iff $j = \pi^{-1}(1)$, $j' = 0$ and $\ell = 1$. It follows that $\gamma_0 = \gamma_{\pi^{-1}(1)}$ and $\gamma_j \neq \gamma_{j'}$ for distinct $j, j' > 0$. Therefore $\#(\mathcal{D}_{I'} \setminus \{a'\}) \geq n - 1$, or $\#\mathcal{P}_{I'} \geq n$, and $\#(\mathcal{D}_{I'} \setminus \{a'\}) \leq n + 1$ or $\#\mathcal{P}_{I'} \leq n + 2$. ☐

We order and name the sub-intervals in $\mathcal{P}_{I'}$ as $I'_1, \dots, I'_m$. We call this the *natural decomposition* of $I'$ by $T' = T|_{I'}$, and $I'$ is admissible iff $m = n$. The next statement proves that $T|_{I''} = (T|_{I'})|_{I''}$, under appropriate choices of $T$, $I'$ and $I''$, and the natural decompositions agree with this identity.

LEMMA A.2. — *Let $T = \mathcal{I}_{\pi,\lambda}$ be an $n$-IET defined on $I$ with sub-intervals $I_1, \dots, I_n$ and let $\emptyset \subsetneq I'' \subsetneq I' \subsetneq I$ be sub-intervals. If $T' = T|_{I'}$ with natural decomposition $I'_1, \dots, I'_m$ of $I'$, $T'' = T|_{I''}$ with natural decomposition $I''_1, \dots, I''_{m'}$ of $I''$ and $S = T'|_{I''}$ with the natural decomposition $J'_1, \dots, J'_{m''}$ of $I''$, then $m'' = m'$ and $J'_i = I''_i$ for all $1 \leq i \leq m'$.*

For $x \in I'$, let $q(x) = (j_0, \dots, j_{r(x)-1})$ be the ordered $r(x)$-tuple given by $T^k x \in I_{j_k}$ for $0 \leq k < r(x)$. Note that $I'_1, \dots, I'_m$ is the natural decomposition of $I'$ by $T'$ iff the following statements hold:

1. $q(x) = q(y)$ for all $x, y \in I'_i$, $1 \leq i \leq m$, and
2. $q(x) \neq q(y)$ if $x \in I'_i$ and $y \in I'_{i'}$ for $i \neq i'$.

The definition of $q(x)$ coincides with a *return word* when considering an IET by its natural symbolic coding. See [4] for an introduction to this coding. Because these tuples are constant on each $I'_i$, let $q_i = q(x)$ for any $x \in I'_i$.

*Proof of Lemma A.2.* — Let $q_i$'s be the return words of $I'$ in $I$ of $T$, $q'_i$'s be the return words of $I''$ in $I$ of $T'$ and $q''_i$'s the return words of $I''$ in $I'$ of $S$. Let $q_i q_{i'}$ denote the concatenation of the tuples $q_i$ and $q_{i'}$, meaning

$$q_i q_{i'} = (j_0, \dots, j_{r-1}, j'_0, \dots, j'_{r'-1})$$

where $q_i = (j_0, \dots, j_{r-1})$ and $q_{i'} = (j'_0, \dots, j'_{r'-1})$.

Let $\tilde{q}_i$, $1 \leq i \leq m''$, be given by

$$\tilde{q}_i = q_{j_0} q_{j_1} \cdots q_{j_{r-1}}$$

where $q''_i = (j_0, \dots, j_{r-1})$. These are the return words for the $J'_i$'s in $I$ by $T$. The lemma follows from two claims: each $\tilde{q}_i$ equals $q'_j$ for some $j = j(i)$ and $\tilde{q}_i \neq \tilde{q}_{i'}$ for $i \neq i'$. The first claim follows as for any $x \in J'_i$ and $k \in \mathbb{N}_0$, if $T^k x \in I'_{j'}$ then $T^{k+k'} x \in I_{j''_{k'}}$ for each $0 \leq k' < r_{j'}$, where $q_{j'} = (j''_0, \dots, j''_{r_{j'}-1})$. The second claim holds because $q''_i \neq q''_{i'}$ and $q_j \neq q_{j'}$ for $i \neq i'$ and $j \neq j'$. $\square$

LEMMA A.3. — *Let $T = \mathcal{I}_{\pi,\lambda}$, $\pi \in \mathfrak{S}^0_n$ and $\lambda \in \mathbb{R}^n_+$, be an i.d.o.c. $n$-IET. If $I', I'', I''', \dots, I^{(k)}, \dots$, $k \in \mathbb{N}$, are admissible sub-intervals so that $\emptyset \subsetneq I^{(k)} \subsetneq I^{(k-1)}$ for each $k \in \mathbb{N}$, then either $J = \bigcap_{k=1}^{\infty} I^{(k)}$ is empty or consists of one point.*

*Proof.* — Suppose by contradiction that $J$ is contains an open interval and therefore is a sub-interval of $I$ by possibly removing the right endpoint. Let $J_1, \dots, J_m$ be the natural decomposition of $T|_J$. By Lemma A.1, $m \leq n + 2$. Because $m$ is finite, $\varepsilon = \min\{|J_i| : 1 \leq i \leq m\} > 0$ is well-defined. Fix $k_0 \in \mathbb{N}$

so that $|I^{(k_0)}| < |J| + \varepsilon/2$ and let $T' = T|_{I^{(k_0)}}$. By Lemma A.2, $J_1, \ldots, J_m$ is the natural decomposition of $T'$ restricted to $J$. By Lemma 2.8, because $J \subsetneq I^{(k_0)}$ there must exist $J_i$ with return time in $I^{(k)}$ greater than 1. In this case $T^{(k)}$ acts on $J_i$ by translation and $T^{(k)} J_i \subseteq I^{(k)} \setminus J$. However, this implies that $|J_i| < \varepsilon/2$, a contradiction. □

## Appendix B. Admissibility and Extended Rauzy Induction

*Proof of Lemma 5.1.* — Suppose that $J = [a, b)$ is not given by extended Rauzy induction, and let $S = T|_J$ be the induced map with natural decomposition $J_1, \ldots, J_m$. We will construct a sub-interval $I'$ of $I$ with induced $T' = T|_{I'}$ so that $I' = [a', b')$ is given by steps of extended Rauzy induction and if $T' = \mathcal{I}_{\pi', \lambda'}$ then

$$a' \leq a < a' + \min\{\lambda'_1, \lambda'_{i_0}\} \text{ and } b' - \min\{\lambda'_{i_1}, \lambda'_n\} < b \leq b', \tag{2}$$

where $i_0, i_1$ satisfy $\pi'(i_0) = 1$ and $\pi'(i_1) = n$. Because $J$ is not realized by extended Rauzy induction, at most one equality may occur. We then will show that $J$ is not admissible for $I'$, which shows our claim by Lemma A.2.

Initially, let $I' = I$ and so $J \subseteq I'$ trivially. Given our definitions, suppose the inequalities for $a$ and $a'$ do not hold for $I'$. We then perform left Rauzy induction on $T|_{I'}$ and replace $I'$ by this new sub-interval, noting that $a'$ will increase. If the inequalities between $b$ and $b'$ do not hold, then we act by right Rauzy induction and replace $I'$ with this new sub-interval. Note that $J \subseteq I'$ still holds after each step. This process must terminate at our desired $I'$, otherwise we have constructed an infinite properly nested sequence of admissible intervals that each contain $J$, a contradiction to Lemma A.3.

Given $I'$ and $J$ so that inequalities (2) hold, $J_1, \ldots, J_m$ is the natural decomposition from $T'$ on $J$ and the natural decomposition from $T$ on $J$ as well by Lemma A.2. If $a = a'$, then $m = n + 1$ and

$$J_i = \begin{cases} I'_i, & i \leq i_1, \\ I'_{i_1} \setminus (T')^{-1} I'_n, & i = i_1, \\ (T')^{-1} I'_n, & i = i_1 + 1, \\ I'_{i-1}, & i_1 < i \leq m, \end{cases}$$

by direct computation. Similarly, $m = n + 1$ if $b = b'$. If both $a > a'$ and $b < b'$, then analogous computations will hold unless $i_0 = n$ and $i_1 = 1$, or $\pi'$ is *standard.*

In this case, each $I'_i$ is a sub-interval in the natural decomposition of $J$ for $1 < i < n$ with return time 1. Let $J'_1 = I'_1 \cap J$ and $J'_n = I'_n \cap J$. Both $J'_1 \cap (T')^{-2} J'_n$ and $J'_n \cap (T')^{-2} J'_1$ are sub-intervals in the decomposition of $J$ with return time 2. Because $T'$ is i.d.o.c., $(T')^2 J'_1 \neq J'_n$ and $(T')^2 J'_n \neq J'_1$.

Therefore, there must be at least one more sub-interval of $J$ in $(J'_1 \setminus (T')^{-2} J'_n) \cup (J'_n \setminus (T')^{-2} J'_1)$ and so $m > n$. □

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