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ON THE SPECTRUMS OF ERGODIC SCHRODINGER OPERATORS WITH FINITELY VALUED POTENTIALS

BY ZHIYUAN ZHANG

ABSTRACT. — In this paper, we show that the Lebesgue measure of the spectrum of ergodic Schrödinger operators with potentials defined by non-constant function over any minimal aperiodic finite subshift tends to zero as the coupling constant tends to infinity. We also obtained a quantitative upper bound for the measure of the spectrum. This follows from a result we proved for ergodic Schrödinger operators with potentials generated by aperiodic subshift under a condition on the recurrence property of the subshift. We also show that such condition is necessary for such result.

RÉSUMÉ (*Sur les spectres des opérateurs de Schrödinger ergodique avec les potentiels des valeurs fini*). — Dans cet article, nous montrons que les mesures Lebesgue des spectres des opérateurs de Schrödinger avec les potentiels qui sont définis par les fonctions non-constantes sur un décalage de type fini, minimal et aperiodique tendent vers zero quand le constant du couplage tend vers l'infini. Ce résultat découle d'un résultat plus général dont nous montrons pour les opérateurs de Schrödinger avec les potentiels qui sont engendrés par un décalage de type fini avec certaine condition sous la récurrence. Nous montrons en même temps que cette condition est nécessaire pour obtenir ce résultat.

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1. Introduction

This paper is motivated by Simon's subshift conjecture (in [6], see also [2]) and the desire to get a better understanding of recently discovered counter-examples in [1].

Given a finite set $\mathcal{A}$, we define the shift transformation T on $\mathcal{A}^{\mathbb{Z}}$ by $T(\omega)_n = \omega_{n+1}$. Let Ω be a $T-$invariant compact subset of $\mathcal{A}^{\mathbb{Z}}$. Let $\mu \in \mathcal{P}(\Omega)$ be an ergodic $T-$invariant measure. Without loss of generality, in this paper we will always assume that $\Omega = \mathrm{supp}(\mu)$, for otherwise we can replace Ω by $\mathrm{supp}(\mu)$. We will assume that for any $\alpha \in \mathcal{A}$, we have $\mu(\{\omega \mid \omega_0 = \alpha\}) > 0$, for otherwise we can replace $\mathcal{A}$ by one of its subsets. To avoid triviality, a function $v : \mathcal{A} \to \mathbb{R}$ is said to be *admissible* if for any two distinct elements $\alpha, \beta \in \mathcal{A}$, we have $v(\alpha) \neq v(\beta)$. Given any admissible function v, we denote

$$\lambda_v := \min_{\alpha,\beta\in\mathcal{A},\alpha\neq\beta} |v(\alpha) - v(\beta)|.$$

For each $\omega \in \Omega$, we consider the Schrödinger operator H_ω on $\ell^2(\mathbb{Z})$ defined by

$$(H_\omega u)_n = u_{n+1} + u_{n-1} + v(\omega_n)u_n. \tag{1.1}$$

Let Σ_ω denote the spectrum of H_ω. By ergodicity, Σ_ω is the same for μ almost every ω. It is also well-known that when (Ω, T) is minimal, Σ_ω is the same for all $\omega \in \Omega$. In either case, we denote by Σ_v the (almost sure) common spectrum.

Consider an aperiodic minimal subshift as above, the common spectrum was suspected to be of zero Lebesgue measure. For CMV matrices, Barry Simon conjectured the following in [6].

Conjecture 1. — *Given a minimal subshift of Verblunsky coefficients which is not periodic, the common essential support of the associated measures has zero Lebesgue measure.*

There is also a Schrödinger version of the subshift conjecture (see [1]). We state it using our notations.

Conjecture 2. — *Given any admissible $v : \mathcal{A} \to \mathbb{R}$, and a minimal subshift $\Omega \subset \mathcal{A}^{\mathbb{Z}}$ which is not periodic, the associated common spectrum Σ_v has zero Lebesgue measure.*

It has been shown that for strictly ergodic subshifts satisfying the so-called Boshernitzan condition, the Schrödinger operators have zero-measure spectrum for any non-constant potentials [3], and for CMV matrices, one has zero-measure supports [4]. More results on subshifts associated operators can be found in [2].

In the recent work of Avila, Damanik and Zhang [1], the subshift conjecture is shown to be false, for both Schrödinger version and the orginal version for CMV matrices. In fact, the authors proved the following theorem for Schrödinger operators (Theorem 1 in [1]). We rephrase it using our notations.

THEOREM 1. — *Given any integer $p \geq 2$, there is an admissible function $v : \{1,\ldots,p\} \to \mathbb{R}$, and a minimal subshift $\Omega \subset \{1,\ldots,p\}^{\mathbb{Z}}$ which is not periodic, such that the associated spectrum Σ_v has strictly positive Lebesgue measure.*

They also proved a CMV matrices analog (Theorem 2 in [1]) which disproved the subshift conjecture in its original formulation.

In [1], the authors also proved a positive result which roughly says that when the system endowed with an ergodic invariant measure which is relatively simple, the associated density of states measure is purely singular. The precise conditions are formulated as being "almost surely polynomially transitive" and "almost surely of polynomial complexity". The positive result works for every subshift that is uniformly polynomially transitive and of polynomial complexity (see the remarks after Definition 1,2 in [1]). This theorem can be applied to subshifts generated by translations on tori with Diophantine frequencies, certain skew shifts and interval exchange transformations. Note that this theorem does not imply that the measure of the spectrum is zero.

Given this new phenomenon, namely that the minimal aperiodic subshift generated potentials can have positive-measure spectrum, the following question arises naturally.

QUESTION 1. — *Given a minimal aperiodic subshift and a non-constant potential function, how large can the Lebesgue measure of the spectrum be ?*

This paper is an attempt to study this question. One of our main results is the following.

THEOREM 2. — *Given any integer $p \geq 2$, a minimal aperiodic subshift $\Omega \subset \{1,\ldots,p\}^{\mathbb{Z}}$. Then for any $0 < \gamma < 1$ the following is true. For any admissible function $v : \{1,\ldots,p\} \to \mathbb{R}$, there exists $C > 0$, such that for any $\lambda > 0$, the Lebesgue measure of $\Sigma_{\lambda v}$ is smaller than $C\lambda^{-\gamma}$.*

We actually proved the following more general result for ergodic Schrödinger operators with subshift-generated potentials:

THEOREM 3. — *Given any integer $p \geq 2$, let μ be an ergodic shift invariant measure on $\{1,\ldots,p\}^{\mathbb{Z}}$, such that there exists an integer $K > 0$ satisfying $\mu(\{\omega \mid \omega_0 = \omega_1 = \cdots = \omega_{K-1}\}) = 0$. Then for any $0 < \gamma < 1$, there exists a constant $C > 0$, such that for any admissible function $v : \{1,\ldots,p\} \to \mathbb{R}$, we have $Leb(\Sigma_v) < C\lambda_v^{-\gamma}$.*

Since on any minimal subshift Ω, any ergodic shift invariant measure μ on Ω satisfies the condition of Theorem 3, Theorem 2 follows as an immediate corollary.

We note that assuming only the condition of Theorem 3, one cannot hope to prove zero-measure spectrum for all sufficiently sparse potentials. In fact we have the following theorem which is a slight modification of Theorem 1 in [1]. We will give a sketched proof for the convenience of the readers.

THEOREM 4. — *Given any integer $p \geq 2$, any countable subset B of the admissible functions from $\mathcal{A}$ to $\mathbb{R}$. There exists a minimal aperiodic subshift $\Omega \subset \{1,\dots,p\}^{\mathbb{Z}}$ such that for any $v \in B$, Σ_v has strictly positive Lebesgue measure.*

In fact, we can adapt the proof of Theorem 4 to further ensure that the subshift we cooked up is uniformly polynomially transitive and is of polynomial complexity. Thus we can apply Theorem 2 in [1] to show that, for any ergodic shift invariant measure, the associate density of sates measure is purely singular.

We also note that the condition in Theorem 3 is necessary to ensure that the measure of the spectrum tends to zero as the "sparseness" of the potential function grows to infinity. This is seen from the following theorem, which seems to be folklore.

THEOREM 5. — *Given any integer $p \geq 2$, an ergodic shift invariant measure μ on $\{1,\dots,p\}^{\mathbb{Z}}$ such that there exists $i \in \{1,\dots,p\}$ satisfying that for any integer $N > 0$, $\mu(\{\omega \mid \omega_0 = \omega_1 = \cdots = \omega_{N-1} = i\}) > 0$. Then for any function $v : \{1,\dots,p\} \to \mathbb{R}$, we have $[-2+v(i), 2+v(i)] \subset \bigcup_{\omega \in \mathrm{supp}\mu} \Sigma_\omega$.*

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After the earlier version is done, the referee suggested that major simplification could be done. This leads to the present paper. I thank the referee for his insight.

2. Preliminary

Let $\mathcal{A}$ be a finite set, and let $\Omega \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift. Let $v : \mathcal{A} \to \mathbb{R}$ be any function. For any $\alpha \in \mathcal{A}$, any $E \in \mathbb{R}$, we denote

$$A_\alpha^E = \begin{bmatrix} E - v(\alpha) & -1 \\ 1 & 0 \end{bmatrix}.$$

and define a map $A^E : \Omega \to SL(2, \mathbb{R})$ by $A^E(\omega) = A^E_{\omega_0}$. We define $A^E : \mathbb{Z} \times \Omega \to SL(2, \mathbb{R})$ by letting $A^E(0, \omega) \equiv Id$, and

$$A^E(k, \omega) = \begin{cases} A^E(T^{k-1}(\omega)) \cdots A^E(\omega), & \forall k > 0, \\ A^E(T^k(\omega))^{-1} \cdots A^E(T^{-1}(\omega))^{-1}, & \forall k < 0. \end{cases}$$

In particular, for any $n, m \geq 0$, any $\omega \in \Omega$ we have the following relation

$$A^E(n + m, \omega) = A^E(m, T^n(\omega))A^E(n, \omega). \tag{2.1}$$

For any finite word $\alpha = \omega_0\omega_1 \cdots \omega_{n-1} \in \mathcal{A}^n$, we define

$$A^E(\alpha) = A^E_{\omega_{n-1}} \cdots A^E_{\omega_0}.$$

We have the following notion.

Definition 1. — Given an admissible function $v : \mathcal{A} \to \mathbb{R}$ (defined in the introduction), for each $E \in \mathbb{R}$, we say $A^E : \Omega \to SL(2, \mathbb{R})$ is the *Schrödinger cocycle at energy E*. The map A^E is called *uniformly hyperbolic* if there are two continuous maps $u, s : \Omega \to \mathbb{P}(\mathbb{R}^2)$ such that

1. u, s are invariant under the cocycle dynamics in the sense that
$$A^E(\omega)u(\omega) = u(T(\omega)), \quad A^E(\omega)s(\omega) = s(T(\omega)).$$
2. there exists $C > 0$, $\lambda > 1$ such that $\|A^E(-n, \omega)w_1\|, \|A^E(n, \omega)w_2\| \leq C\lambda^{-n}$ for every $n \geq 1$, $\omega \in \Omega$ and all unit vectors $w_1 \in u(\omega), w_2 \in s(\omega)$.

The following result is contained in [5].

Lemma 1 (Johnson). — *For any function $v : \mathcal{A} \to \mathbb{R}$, we have*

$$\bigcup_{\omega \in \Omega} \Sigma_\omega = \{E \mid A^E \text{ is not uniformly hyperbolic}\}.$$

For any $c \geq 0$, we define the following cones,

$$\mathcal{C}_+(c) = \{u = (x, y) \in \mathbb{R}^2 \mid |y| \leq c|x|\},$$
$$\mathcal{C}_-(c) = \{u = (x, y) \in \mathbb{R}^2 \mid |x| \leq c|y|\}.$$

When $c = 0$, $\mathcal{C}_+(0)$ (resp. $\mathcal{C}_-(0)$) is reduced to the $x-$axis (resp. the $y-$axis).

For any $\alpha \in \mathcal{A}$, we set

$$I_{v(\alpha)} = [v(\alpha) - 4, v(\alpha) + 4].$$

By straightforward computation, we can see that for any $E \notin I_{v(\alpha)}$, $A^E(\alpha)\mathcal{C}_+(1)$ is in the interior of $\mathcal{C}_+(\frac{1}{2})$ except for the origin. By the invertibility of T, and the cone-field criterion for uniform hyperbolicity, it is standard to verify that A^E is uniformly hyperbolic for any $E \notin \bigcup_{\alpha\in\mathcal{A}} I_{v(\alpha)}$. Then by Lemma 1, we have

$$\bigcup_{\omega\in\Omega} \Sigma_\omega \subset \bigcup_{\alpha\in\mathcal{A}} I_{v(\alpha)}. \tag{2.2}$$

3. Proof of Theorem 3

In order to prove Theorem 3, it suffices to prove that for any $\alpha \in \{1, \dots, p\}$, we have an upper bound of this type for the Lebesgue measure of $\Sigma_v \bigcap I_{v(\alpha)}$. Then Theorem 3 will follow from (2.2).

In the rest of this section, we let p, Ω, K be given by Theorem 3. Denote $\mathcal{A} = \{1, \dots, p\}$. We fix $\alpha_0 \in \mathcal{A}$ and denote $E_0 = v(\alpha_0)$. Then Theorem 3 is reduced to the following.

PROPOSITION 1. — *Under the conditions of Theorem 3, for any $0 < \gamma < 1$, there exists a constant $C = C(K, \gamma) > 0$, such that for any admissible function $v : \{1, \dots, p\} \to \mathbb{R}$, we have $Leb(\Sigma_v \bigcap I_{E_0}) < C\lambda_v^{-\gamma}$.*

We first show the following lemma.

LEMMA 2. — *For any $\gamma \in (0, 1)$, there exists $\lambda_0 = \lambda_0(\gamma) > 0$ such that the following is true. Assume that $\lambda_v > \lambda_0$. Then for any integer $n \geq 1$, any $E \in I_{E_0}$, any finite word $\beta = \beta_0 \cdots \beta_{n-1}$ such that $\beta_k \in \mathcal{A} \setminus \{\alpha_0\}$ for all $0 \leq k \leq n-1$, we have*

1. $A^E(\beta)(\mathbb{R}^2 \setminus \mathcal{C}_-(\lambda_v^{-\gamma})) \subset \mathcal{C}_+(\lambda_v^{-\gamma})$,
2. *for any $w \in \mathbb{R}^2 \setminus \mathcal{C}_-(\lambda_v^{-\gamma})$, we have*

$$\|A^E(\beta)(w)\| \geq \lambda_v^{\frac{1-\gamma}{2}n}\|w\|.$$

Proof. — This follows from direct computations. Let $w = (x, y) \in \mathbb{R}^2 \setminus \mathcal{C}_-(\lambda_v^{-\gamma})$. Then by definition, $|x| > \lambda_v^{-\gamma}|y|$. Take an arbitrary $\alpha \in \mathcal{A} \setminus \{\alpha_0\}$. We have

$$A^E(\alpha)(w) = ((E - v(\alpha))x - y, x).$$

By $E \in I_{E_0}$, $\alpha \neq \alpha_0$, we have $|E - v(\alpha)| \geq \lambda_v - 4$. Then we have

$$|(E - v(\alpha))x - y| \geq (\lambda_v - 4)|x| - |y|. \tag{3.1}$$

By $\gamma \in (0, 1)$ and by letting λ_0 be sufficiently large, the following is true

$$\lambda - 4 > 2\lambda^\gamma + \lambda^{\frac{1-\gamma}{2}} + \lambda^{\frac{1+\gamma}{2}}, \quad \forall \lambda > \lambda_0. \tag{3.2}$$

Then by (3.2), $\lambda_v > \lambda_0$ and $|x| > \lambda_v^{-\gamma}|y|$, we have

$$(\lambda_v - 4)|x| - |y| > \lambda_v^\gamma|x|.$$

We then obtain (1) by (3.1). Similarly, by (3.2), $\lambda_v > \lambda_0$ and $|x| > \lambda_v^{-\gamma}|y|$, we have

$$(\lambda_v - 4)|x| - |y| > \lambda_v^{\frac{1-\gamma}{2}}(|x| + |y|).$$

We obtain (2) by (3.1). □

We denote

$$C_E = \begin{bmatrix} E - E_0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It is clear that $\|C_E\| \leq 5$ for any $E \in I_{E_0}$. For any $\varepsilon > 0$, we define

$$J(\varepsilon) = \{E \in I_{E_0} \mid \exists 1 \leq k \leq K, \text{ such that } C_E^k(\mathcal{C}_+(0)) \in \mathcal{C}_-(\varepsilon)\}.$$

Lemma 3. — *There exists $C = C(K) > 0$, such that $J(\varepsilon) < C\varepsilon$ for any $\varepsilon > 0$.*

Proof. — This lemma is a well-known consequence of the monotonicity of the Schrödinger cocycles. Represent any vector $w \in \mathbb{R}^2 \setminus \{0\}$ in the polar coordinate by $w = (r(w)\cos\theta(w), r(w)\sin\theta(w))$. Then we have the following basic observations for any $E \in I_{E_0}$, which follows from direct computations,

1. for any $w \in \mathbb{R}^2 \setminus \{0\}$, we have $\partial_E\theta(C_E(w)) \leq 0$,
2. for any $w \in \mathbb{R}^2 \setminus \mathcal{C}_-(1)$, we have $C_E(w) \in \mathcal{C}_-(6) \setminus \{0\}$,
3. for any $w \in \mathcal{C}_-(6) \setminus \{0\}$, we have $\partial_E\theta(C_E(w)) < c$ for some constant $c > 0$.

This implies that for any $1 \leq k \leq K$, for any $\varepsilon' > 0$, the measure of the set of $E \in I_{E_0}$ such that $\theta(C_E^k(1,0)) \in (\frac{\pi}{2} - \varepsilon', \frac{\pi}{2} + \varepsilon')$ is bounded by $c'\varepsilon'$ for some constant $c' > 0$ depending only on K. This concludes the proof. □

Lemma 4. — *Under the conditions of Theorem 3, for any $\gamma \in (0,1)$, there exists $\lambda_1 = \lambda_1(K,\gamma) > 0$, $\eta = \eta(K,\gamma) > 0$, $n_0 = n_0(K,\gamma) > 0$ such that the following is true. Assume that $\lambda_v > \lambda_1$. Then for any $E \in I_{E_0} \setminus J(\lambda_v^{-\gamma})$, any $n \geq n_0$, any $\omega \in \operatorname{supp}(\mu)$, we have*

$$\|A^E(n,\omega)\| \geq e^{n\eta}. \tag{3.3}$$

Proof. — By our hypothesis in Theorem 3, we have

$$\{\omega \mid \omega_0 = \cdots = \omega_{K-1}\} \cap \operatorname{supp}(\mu) = \emptyset.$$

By the shift-invariance of $\operatorname{supp}(\mu)$, for any $\omega \in \operatorname{supp}(\mu)$, for any $n \in \mathbb{Z}$, the set $\{\omega_{n+1}, \ldots, \omega_{n+K}\}$ contains at least two distinct elements in $\mathcal{A}$.

For any $\omega = (\omega_i)_{i\in\mathbb{Z}} \in \Omega$, there exists $0 \leq k \leq K$ such that $\omega_k \neq \alpha_0$. Then by $\|A^E(\alpha_0)\| \leq 5$ for all $E \in I_{E_0}$, and by (2.1), it suffices to consider only $\omega \in \Omega$ such that $\omega_0 \neq \alpha_0$. Given any such ω, there exist integers $l \geq 1$ and $1 \leq a_1, a_2, \ldots, a_{l-1} \leq K$, $b_1, \ldots, b_l \geq 1$, $a_l \geq 0$, finite words $\beta^1, \beta^2, \ldots, \beta^l$ such that

(1) β^k is a word of length b_k in $\mathcal{A} \setminus \{\alpha_0\}$. We denote $\beta^k = \beta_1^k \cdots \beta_{b_k}^k$;

(2) $\omega_0\omega_1\cdots\omega_{n-1}$ is the ordered concatenation of $(\beta^1, \alpha_0[a_1], \dots, \beta^l, \alpha_0[a_l])$, i.e.,

$$\omega_0\omega_1\cdots\omega_{n-1} = \beta^1\alpha_0[a_1]\cdots\beta^l\alpha_0[a_l].$$

Here we denote by $\alpha_0[0]$ the empty word, and by $\alpha_0[k]$ the $k-$times concatenation of α_0 for any $k \geq 1$. It is clear that $A^E(\alpha_0[k]) = C_E^k$ for any $k \geq 0$.

Denote $\gamma' = \frac{1+\gamma}{2} \in (\gamma, 1)$, we have the following.

Claim. For sufficiently large λ_v, for any $1 \leq k \leq l$, we have

$$A^E(\beta^k\alpha_0[a_k])(\mathbb{R}^2 \setminus \mathcal{C}_-(\lambda_v^{-\gamma'})) \subset \mathbb{R}^2 \setminus \mathcal{C}_-(\lambda_v^{-\gamma'}).$$

Moreover, for any $w \in \mathbb{R}^2 \setminus \mathcal{C}_-(\lambda_v^{-\gamma'})$, we have

$$\|A^E(\beta^k\alpha_0[a_k])(w)\| > 5^{-K}\lambda_v^{\frac{1-\gamma'}{2}b_k}\|w\| > \lambda_v^{\frac{1-\gamma}{16K}(a_k+b_k)}\|w\|.$$

Proof. — By Lemma 2, for any $\lambda_v > \lambda_0(\gamma')$ we have

1. $A^E(\beta^k)(\mathbb{R}^2 \setminus \mathcal{C}_-(\lambda_v^{-\gamma'})) \subset \mathcal{C}_+(\lambda_v^{-\gamma'})$,
2. for any $w \in \mathbb{R}^2 \setminus \mathcal{C}_-(\lambda_v^{-\gamma'})$, we have

$$\|A^E(\beta^k)(w)\| \geq \lambda_v^{\frac{1-\gamma'}{2}b_k}\|w\|.$$

By $E \in I_{E_0} \setminus J(\lambda_v^{-\gamma})$, we have

$$\angle(A^E(\alpha_0[a_k]))\mathcal{C}_+(0), \mathcal{C}_-(0)) = \angle(C_E^{a_k}\mathcal{C}_+(0), \mathcal{C}_-(0)) \gtrsim \lambda_v^{-\gamma}.$$

Then by $\gamma < \gamma'$ and $5^K\lambda_v^{-\gamma'} \ll \lambda_v^{-\gamma}$ for sufficiently large λ_v (depending only on K, γ), we obtain

1. $A^E(\alpha_0[a_k])\mathcal{C}_+(\lambda_v^{-\gamma'}) \subset \mathbb{R}^2 \setminus \mathcal{C}_-(\lambda_v^{-\gamma'})$,
2. for any $w \in \mathbb{R}^2 \setminus \{0\}$, we have

$$\|A^E(\alpha_0[a_k])(w)\| \geq 5^{-K}\|w\|.$$

Our claim follows by

$$\lambda_v^{\frac{1-\gamma'}{4}b_k} > 5^K, \quad \lambda_v^{\frac{1-\gamma'}{4}b_k} \geq \lambda_v^{\frac{1-\gamma'}{8K}(b_k+K)} \geq \lambda_v^{\frac{1-\gamma}{16K}(b_k+a_k)}.$$

for sufficiently large λ_v, and by concatenating the above estimates. □

Our lemma follows from repeatedly using the above claim for $1 \leq k \leq l$. □

Proof of Proposition 1 : By Lemma 4 above and Proposition 2 in [7], we see that for any sufficiently large λ_v, any $E \in I_{E_0} \setminus J(\lambda_v^{-\gamma})$, the Schrödinger cocycle A^E is uniformly hyperbolic. By Lemma 1, this implies $I_{E_0} \bigcap \Sigma_v \subset J(\lambda_v^{-\gamma})$. Proposition 1 then follows from Lemma 3 and by letting C to be sufficiently large. □

4. Proof of Theorem 4 and Theorem 5

4.1. Proof of Theorem 4. — The construction of the required subshift follows closely the proof of Theorem 1 in [1]. We refer to [1] for some relevant lemmata. Without loss of generality, let us assume that B is a countably infinite set of potentials. We will inductively define collections of finite words S_n, subshifts Ω_n, closed subsets $\Sigma_{n,m}$ for $1 \le n \le m$.

For $n = 1$, we define

$$S_1 = \{1, \dots, p\}. \tag{4.1}$$

We define Ω_1 to be the two-sided infinite concatenations of the words in S_1. We now pick any element $v_1 \in B$. For each word $w \in S_1$, we denote the spectrum of the periodic potential associated to v_1 and w by $\Sigma_{1,1}(w)$, and define

$$\Sigma_{1,1} = \bigcup_{w \in S_1} \Sigma_{1,1}(w). \tag{4.2}$$

Assume S_n, Ω_n, $\Sigma_{i,n}, \forall 1 \le i \le n$ are constructed. We denote

$$S_n = \{w_{n,1}, w_{n,2}, \dots, w_{n,k_n}\}.$$

For any given integer $N_n \ge 1$, we define

$$S_{n+1} = \{w_{n,1}w_{n,2} \cdots w_{n,k_n} w_{n,k}^l \mid 1 \le k \le k_n, 1 \le l \le N_n\}.$$

and define Ω_{n+1} to be the two-sided infinite concatenation of the words in S_{n+1}. It is direct to see that $\Omega_{n+1} \subset \Omega_n$.

We pick any element $v_{n+1} \in B \setminus \{v_1, \dots, v_n\}$. For each $1 \le i \le n+1$, for each $w \in S_{n+1}$, we denote the spectrum of the periodic potential associated to v_i and w by $\Sigma_{i,n+1}(w)$, and denote

$$\Sigma_{i,n+1} = \bigcup_{w \in S_{n+1}} \Sigma_{i,n+1}(w). \tag{4.3}$$

It is clear that $Leb(\Sigma_{n+1}) > 0$. By Lemma 1 in [1], we can choose a positive integer N_n depending only on S_n, Ω_n, $\Sigma_{i,n}$ such that the following is true.

$$Leb(\Sigma_{i,n} \setminus \Sigma_{i,n+1}) < Leb(\Sigma_{i,i})2^{-(n+1)}. \tag{4.4}$$

for any $1 \le i \le n$. We define $\Omega = \bigcap_n \Omega_n$. For each $v \in B$, denote the spectrum associated to Ω and v by Σ. For some $i \in \mathbb{N}$, we have $v = v_i$. Then following [1], we have

$$\Sigma \supseteq \limsup_{n \to \infty} \Sigma_{i,n}. \tag{4.5}$$

Then by the same reasoning in [1], we have $Leb(\Sigma) > \frac{1}{2} Leb(\Sigma_{i,i}) > 0$. By Lemma 2 in [1], we can show that Ω is minimal and aperiodic. This completes the proof.

4.2. Proof of Theorem 5. — Fix $E \in (-2+v(i), 2+v(i))$, then A_i^E is an elliptic matrix. Assume to the contrary that $E \notin \bigcup_{\omega\in \operatorname{supp}\mu} \Sigma_\omega$. Then we can take an open interval neighborhood of E, denoted by J, such that $J \subset (-2+v(i), 2+v(i)) \bigcap \Sigma_v^c$. By Lemma 1 the cocycle A^E over Ω is uniformly hyperbolic. Thus we can define stable, unstable directions, denoted respectively by $s, u : \Omega \to \mathbb{P}(\mathbb{R}^2)$. After possibly reducing J, we can assume that for any $E' \in J$, there exists $s(E'), u(E') : \Omega \to \mathbb{P}(\mathbb{R}^2)$ so that for any $\omega \in \Omega$, the function $s(\cdot,\omega), u(\cdot,\omega) : J \to \mathbb{P}(\mathbb{R}^2)$ are C^1 (in fact analytic) and the C^1 norm of these functions are bounded uniformly in $\omega \in \Omega$. We take any $\omega \in \Omega$ such that $\omega_0 = \cdots = \omega_{N-1} = i$, where N will be chosen to be large. Denote $\omega' = T^N(\omega)$. Then $s(E', \omega') = (A_i^{E'})^N s(E', \omega)$ for all $E' \in J$. Straightforward calculation shows that the C^1 norm of $s(\cdot, \omega')$ will be $\Theta(N)$. When N is large, we have a contradiction. Hence $E \in \bigcup_{\omega\in \operatorname{supp}\mu} \Sigma_\omega$. This proves the theorem.

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