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# MULTI-MICROLOCALIZATION AND MICROSUPPORT

by Naofumi Honda, Luca Prelli & Susumu Yamazaki

Abstract. — The purpose of this paper is to establish the foundations of multi-microlocalization, in particular, to give the fiber formula for the multi-microlocalization functor and estimate of microsupport of a multi-microlocalized object. We also give some applications of these results.

Résumé (*Multi-microlocalisation et microsupport*). — Le but de cet article est d'établir les fondements de la multi-microlocalisation, en particulier, de donner une formule de fibre pour le foncteur de multi-microlocalisation et aussi une estimation du microsupport d'un objet multi-microlocalisé. Nous donnons également quelques applications de ces résultats.

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## Introduction

A microlocalized object of a sheaf $F$ along a closed submanifold $M$ was first introduced in M. Sato, T. Kawai and M. Kashiwara [9], which is locally described by, roughly speaking, local cohomology groups of $F$ with support in a dual cone of the edge $M$ (see also M. Kashiwara and P. Schapira [4]). As a result it can be tightly related with, via Čech cohomology groups, boundary values of local sections of $F$ defined on open cones of the edge $M$. It is well known that, for example, Sato's microfunctions, which are obtained by applying the microlocalization functor along a real analytic manifold $M$ to the sheaf of holomorphic functions, can be regarded as boundary values of holomorphic functions locally defined on wedges with the edge $M$.

We sometimes, in study of partial differential equations, need to consider a boundary value of a function defined on a cone along a family $\chi$ of several closed submanifolds. In such a study, J. M. Delort [1] had introduced *simultaneous microlocalization* along a normal crossing divisor $\chi$ which gives a boundary value of a function defined on a dual poly-sector. *Bi-microlocalization* along submanifolds $\chi = \{M_1, M_2\}$ with $M_1 \subset M_2$ was also introduced by P. Schapira and K. Takeuchi [10] and [11] which defines a different kind of a boundary value.

On the other hand, in the paper [3], the first and the second authors of this article established the notion of *the multi-normal cone* for a family $\chi$ of closed submanifolds with a suitable configuration, and they also constructed *the multi-specialization functor* along $\chi$. One can observe that cones appearing in simultaneous microlocalization and bi-microlocalization are characterized by using the multi-normal cone and that both microlocalization functors coincide with the *multi-microlocalization functor* along $\chi$ where the latter functor is obtained by repeated application of Sato's Fourier transformation to the multi-specialization functor. Hence multi-microlocalization gives us a uniform machinery for both simultaneous microlocalization and bi-microlocalization. The purpose of this paper is to establish the foundations of multi-microlocalization, in particular, to give the fiber formula for the multi-microlocalization functor and estimate of microsupport of a multi-microlocalized object. We briefly explain, in what follows, these two important results and their meanings.

The most fundamental question for the multi-microlocalization functor $\mu_\chi$ along closed submanifolds $\chi = \{M_1, \dots, M_\ell\}$ is a shape of a cone on which a boundary value given by $\mu_\chi$ is defined. The fiber formula gives us an explicit answer: A germ of $H^k(\mu_\chi(F))$ is isomorphic to local cohomology groups $\varinjlim\limits_{G} H^k_G(F)$ where $G$ is a vector sum of closed cones $G_i$'s and each $G_i$ is defined in the similar way as that in the fiber formula of the usual microlocalization functor along $M_i$. Therefore the multi-microlocalization functor can be understood as a natural extension of the usual microlocalization functor. Once we have grasped a geometrical aspect of multi-microlocalization, then the next fundamental problem to be considered is estimate of microsupport of $\mu_\chi(F)$ by that of $F$, for which the answer is quite simple and beautiful: The microsupport $SS(\mu_\chi(F))$ is contained in the multi-normal cone of $SS(F)$ along $\chi^*$. Here $\chi^*$ is a family of Lagrangian submanifolds $\{T^*_{M_1}X, \dots, T^*_{M_\ell}X\}$. This shows, in particular, soundness of our framework in the sense that the sharp estimate can be achieved by a geometrical tool (the multi-normal cone) already prepared in our framework. These two results have many applications, and some of them will be given in the last section of this paper.

The paper is organized as follows: We briefly recall, in Section 1, the theory of the multi-specialization developed in [3]. Then, in Section 2, we define the multi-microlocalization functor by repeatedly applying Sato's Fourier transformation to the multi-specialization functor. After showing several basic properties of the functor, we establish a fiber formula which explicitly describes a stalk of a multi-microlocalized object. In Section 3, after some geometrical preparations, we give an estimate of microsupport of a microlocalized object, that is our main result. Several applications of this result to $\mathscr{D}$-modules are studied in Section 4.

## 1. Multi-specialization: a review

In this section we recall some results of [3]. We first fix some notations, then we recall the notion of multi-normal deformation and the definition of the functor of multi-specialization with some basic properties.

**1.1. Notations. —** Let $X$ be a real analytic manifold with $\dim X = n$, and let $\chi = \{M_1, \dots, M_\ell\}$ be a family of closed submanifolds in $X$ ($\ell \geqslant 1$). Throughout the paper all the manifolds are always assumed to be countable at infinity.

We set, for $N \in \chi$ and $p \in N$,

$$\mathrm{NR}_p(N) := \{M_j \in \chi;\ p \in M_j,\ N \nsubseteq M_j \text{ and } M_j \nsubseteq N\}.$$

Let us consider the following conditions for $\chi$.

H1 Each $M_j \in \chi$ is connected and the submanifolds are mutually distinct, i.e., $M_j \neq M_{j'}$ for $j \neq j'$.

H2 For any $N \in \chi$ and $p \in N$ with $\mathrm{NR}_p(N) \neq \varnothing$, we have

$$\left(\bigcap_{M_j \in \mathrm{NR}_p(N)} T_pM_j\right) + T_pN = T_pX. \tag{1.1}$$

Note that, if $\chi$ satisfies the condition H2, the configuration of two submanifolds must be either 1. or 2. below.

1. Both submanifolds intersect transversely.
2. One of them contains the other.

If $\chi$ satisfies the condition H2, then for any $p \in X$, there exist a neighborhood $V$ of $p$ in $X$, a system of local coordinates $(x_1, \dots, x_n)$ in $V$ and a family of subsets $\{I_j\}_{j=1}^{\ell}$ of the set $\{1, 2, \dots, n\}$ for which the following conditions hold.

1. Either $I_k \subset I_j$, $I_j \subset I_k$ or $I_k \cap I_j = \varnothing$ holds $(k, j \in \{1, 2, \dots, \ell\})$.
2. A submanifold $M_j \in \chi$ with $p \in M_j$ $(j = 1, 2, \dots, \ell)$ is defined by $\{x_i = 0; i \in I_j\}$ in $V$.

We set, for $N \in \chi$,

$$\iota_\chi(N) := \bigcap_{N \subsetneqq M_j} M_j. \tag{1.2}$$

Here $\iota_\chi(N) := X$ for convention if there exists no $j$ with $N \subsetneqq M_j$. When there is no risk of confusion, we write for short $\iota(N)$ instead of $\iota_\chi(N)$. We also assume the condition H3 below for simplicity.

H3 $M_j \neq \iota(M_j)$ for any $j \in \{1, 2, \dots, \ell\}$.

In local coordinates let $I_1, \dots, I_\ell \subseteq \{1, \dots, n\}$ such that $M_i = \{x_k = 0\ ;\ k \in I_i\}$. Note that the family $\chi$ satisfies the conditions H1, H2 and H3 if and only if $I_1, \dots, I_\ell$ satisfy the corresponding conditions

$$\begin{aligned} &\text{(i) either } I_j \subsetneqq I_k,\ I_k \subsetneqq I_j \text{ or } I_j \cap I_k = \varnothing \text{ holds for any } j \neq k, \\ &\text{(ii) } \Big(\bigcup_{I_k \subsetneqq I_j} I_k\Big) \subsetneqq I_j \text{ for any } j. \end{aligned} \tag{1.3}$$

Hence, for any $j \in \{1, 2, \dots, \ell\}$, the set

$$\hat{I}_j := I_j \setminus \Big(\bigcup_{I_k \subsetneqq I_j} I_k\Big) \tag{1.4}$$

is not empty by the condition H3. Further it follows from the conditions H1, H2 and H3 that, for each $j \in \{1, \dots, \ell\}$, there exist unique $j_1, \dots, j_p \in \{1, \dots, \ell\}$ such that

$$I_j = \hat{I}_{j_1} \sqcup \cdots \sqcup \hat{I}_{j_p}$$

and $I_k \subseteq I_j$ for all $k \in \{j_1, \dots, j_p\}$ (equality holds only if $k = j$). In particular

$$\bigcup_{1 \leqslant j \leqslant \ell} I_j = \hat{I}_1 \sqcup \cdots \sqcup \hat{I}_\ell. \tag{1.5}$$

Set, for $i \in \{1, \dots, n\}$

$$J_i = \{j \in \{1, \dots, \ell\}\ ;\ i \in I_j\}. \tag{1.6}$$

It follows from the proof of Proposition 1.3 of [3] that

$$J_\alpha = J_\beta \quad \Leftrightarrow \quad \alpha, \beta \in \hat{I}_j \tag{1.7}$$

for some $j \in \{1, \ldots, \ell\}$.

LEMMA 1.1. — *Let $I_i \supseteq I_j$. Then for each $\alpha \in \hat{I}_i$ and $\beta \in \hat{I}_j$ we have $J_\alpha \subseteq J_\beta$.*

*Proof.* — Let $\alpha \in \hat{I}_i$ and $\beta \in \hat{I}_j$. By definition of $\hat{I}_i$ and condition H2 we have

$$k \in J_\alpha \quad \Leftrightarrow \quad I_k \supseteq I_i.$$

Since $I_i \supseteq I_j$ we have

$$I_k \supseteq I_j \quad \Leftrightarrow \quad k \in J_\beta.$$

Then $J_\alpha \subseteq J_\beta$. □

Thanks to the previous result we can introduce the following notation. Set for convenience

$$I_0 = \hat{I}_0 := \{1, \ldots, n\} \setminus \bigcup_{j=1}^{\ell} I_j. \tag{1.8}$$

Then, in local coordinates, we can write the coordinates $(x_1, \ldots, x_n)$ by

$$(x^{(0)}, x^{(1)}, \ldots, x^{(\ell)}), \tag{1.9}$$

where $x^{(j)}$ denotes the coordinates $(x_i)_{i \in \hat{I}_j}$ $(j = 0, \ldots, \ell)$. By taking (1.7) into account, we can also define, for $j \in \{0, 1, \ldots, \ell\}$

$$\hat{J}_j = \{k \in \{1, \ldots, \ell\}\ ;\ \hat{I}_j \subseteq I_k\} = \{k \in \{1, \ldots, \ell\}\ ;\ I_j \subseteq I_k\}. \tag{1.10}$$

Note that, with this notation, we have $\hat{J}_0 = \varnothing$. Moreover, by (1.7) we have $J_\alpha = J_\beta = \hat{J}_j$ for each $j \in \{1, \ldots, \ell\}$ and each $\alpha, \beta \in \hat{I}_j$. In particular, by Lemma 1.1

$$I_i \subseteq I_j \quad \Rightarrow \quad \hat{J}_j \subseteq \hat{J}_i. \tag{1.11}$$

**1.2. Multi-normal deformation. —** In [3] the notion of multi-normal deformation was introduced. Here we consider a slight generalization where we replace the condition H2 with the weaker one. Let $\chi = \{M_1, \ldots, M_\ell\}$ be a family of closed submanifolds of $X$. We say that $\chi$ is *simultaneously linearizable* on $M = M_1 \cap \cdots \cap M_\ell$ if for every $x \in M$ there exist a neighborhood $V$ of $x$ and a system of local coordinates $(x_1, \ldots, x_n)$ there for which we can find subsets $I_j$'s of $\{1, \ldots, n\}$ such that each $M_j \cap V$ is defined by equations $x_i = 0$ $(i \in I_j)$. Note that if $\chi$ satisfies the condition H2, then it is simultaneously linearizable. Now, through the section, we assume that $\chi$ is simultaneously linearizable on $M$.

First recall the classical construction of [4] of the normal deformation of $X$ along $M_1$. We denote it by $\widetilde{X}_{M_1}$ and we denote by $t_1 \in \mathbb{R}$ the deformation parameter. Let $\Omega_{M_1} = \{t_1 > 0\}$ and let us identify $s^{-1}(0)$ with $T_{M_1}X$. We have the commutative diagram

$$\begin{tikzcd} T_{M_1}X \arrow[r, "s_{M_1}"] \arrow[d, "\tau_{M_1}"] & \widetilde{X}_{M_1} \arrow[d, "p_{M_1}"] & \Omega_{M_1} \arrow[l, "i_{\Omega_{M_1}}"'] \arrow[dl, "\widetilde{p}_{M_1}"] \\ M \arrow[r, "i_{M_1}"] & X. & \end{tikzcd} \tag{1.12}$$

Set $\widetilde{\Omega}_{M_1} = \{(x;\, t_1)\ ;\ t_1 \neq 0\}$ and define

$$\widetilde{M}_2 := \overline{(p_{M_1}|_{\widetilde{\Omega}_{M_1}})^{-1} M_2}.$$

Then $\widetilde{M}_2$ is a closed smooth submanifold of $\widetilde{X}_{M_1}$.

Remark 1.2. — One cannot expect the smoothness of $\widetilde{M}_2$ without the simultaneously linearizable condition. For example, let $X = \mathbb{R}^2$ with $(x_1, x_2)$ and let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_1 - x_2^2 = 0\}$. Then in $\widetilde{X}_{M_1}$ with coordinates $(x_1, x_2;\, t_1)$ we have

$$\widetilde{M}_2 := \overline{\{t_1 x_1 - x_2^2 = 0,\, t_1 \neq 0\}} = \{t_1 x_1 - x_2^2 = 0\}$$

which is singular at $(0, 0;\, 0)$.

Now we can define the normal deformation along $M_1, M_2$ as

$$\widetilde{X}_{M_1,M_2} := (\widetilde{X}_{M_1})^{\sim}_{\widetilde{M}_2}.$$

Then we can define recursively the normal deformation along $\chi$ as

$$\widetilde{X} = \widetilde{X}_{M_1,\dots,M_\ell} := (\widetilde{X}_{M_1,\dots,M_{\ell-1}})^{\sim}_{\widetilde{M}_\ell}.$$

Set $S_\chi = \{t_1, \dots, t_\ell = 0\}$, $M = \bigcap_{i=1}^{\ell} M_i$ and $\Omega_\chi = \{t_1, \dots, t_\ell > 0\}$. Then we have the commutative diagram

$$\begin{tikzcd} S_\chi \arrow[r, "s"] \arrow[d, "\tau"] & \widetilde{X} \arrow[d, "p"] & \Omega_\chi \arrow[l, "i_\Omega"'] \arrow[dl, "\widetilde{p}"] \\ M \arrow[r, "i_M"] & X. & \end{tikzcd} \tag{1.13}$$

Let us consider the diagram (1.13). In local coordinates let $I_1, \dots, I_\ell \subseteq \{1, \dots, n\}$ such that $M_i = \{x_k = 0\ ;\ k \in I_i\}$. For $j \in \{0, \dots, \ell\}$ set

$$\hat{J}_j = \{k \in \{1, \dots, \ell\}\ ;\ \hat{I}_j \subseteq I_k\}, \qquad t_{\hat{J}_j} = \prod_{k \in \hat{J}_j} t_k,$$

where $t_1, \dots, t_\ell \in \mathbb{R}$ and $t_{\hat{J}_0} = 1$. Then $p : \widetilde{X} \to X$ is defined by

$$(x^{(0)}, x^{(1)} \dots, x^{(\ell)};\, t_1, \dots, t_\ell) \mapsto (t_{\hat{J}_0} x^{(0)}, t_{\hat{J}_1} x^{(1)}, \dots, t_{\hat{J}_\ell} x^{(\ell)}).$$

Definition 1.3. — Let $Z$ be a subset of $X$. The multi-normal cone to $Z$ along $\chi$ is the set $C_\chi(Z) = \overline{\widetilde{p}^{-1}(Z)} \cap S_\chi$.

Lemma 1.4. — *Let $p = (p^{(0)}, p^{(1)}, \dots, p^{(\ell)}; 0, \dots, 0) \in S_\chi$ and let $Z \subset X$. The following conditions are equivalent:*

1. $p \in C_\chi(Z)$.
2. *There exist sequences $\{(c_{1,m}, \dots, c_{\ell,m})\} \subset (\mathbb{R}^+)^\ell$ and $\{(q_m^{(0)}, q_m^{(1)}, \dots, q_m^{(\ell)})\} \subset Z$ such that $q_m^{(j)} c_{\hat{J}_j} \to p^{(j)}$, $j = 0, 1, \dots, \ell$ and $c_{m,j} \to +\infty$, $j = 1, \dots, \ell$.*

*Proof.* — Let us prove 1. $\Rightarrow$ 2. Since $p \in \overline{\widetilde{p}^{-1}(Z)} \cap S_\chi$ there exist sequences $\{(p_m^{(0)}, p_m^{(1)}, \dots, p_m^{(\ell)})\} \subset Z$, $\{(t_{1,m}, \dots, t_{\ell,m})\} \subset (\mathbb{R}^+)^\ell$ such that

$$\begin{cases} t_{j,m} \to 0, \quad j = 1, \dots, \ell, \\ (p_m^{(0)}, p_m^{(1)}, \dots, p_m^{(\ell)}) \to (p^{(0)}, p^{(1)}, \dots, p^{(\ell)}), \\ (p^{(0)}, p^{(1)} t_{\hat{J}_1,m}, \dots, p^{(\ell)} t_{\hat{J}_\ell,m}) \in Z. \end{cases}$$

Set $t_{j,m}^{-1} = c_{j,m}$, $j = 1, \dots, \ell$ and $p_m^{(j)} t_{\hat{J}_j,m} = q_m^{(j)}$, $j = 0, 1, \dots, \ell$. Then we have $\{(q_m^{(0)}, q_m^{(1)}, \dots, q_m^{(\ell)})\} \subset Z$, $q_m^{(j)} c_{\hat{J}_j} \to p^{(j)}$, $j = 0, 1, \dots, \ell$ and $c_{m,j} \to +\infty$, $j = 1, \dots, \ell$.

Let us prove 2. $\Rightarrow$ 1. Suppose that there exist sequences $\{(c_{1,m}, \dots, c_{\ell,m})\} \subset (\mathbb{R}^+)^\ell$ and $\{(q_m^{(0)}, q_m^{(1)}, \dots, q_m^{(\ell)})\} \subset Z$ such that $q_m^{(j)} c_{\hat{J}_j} \to p^{(j)}$, $j = 0, 1, \dots, \ell$ and $c_{m,j} \to +\infty$, $j = 1, \dots, \ell$. Define $p_m^{(j)} = q_m^{(j)} c_{\hat{J}_j}$, $j = 0, 1, \dots, \ell$ and $t_{j,m} = c_{m,j}^{-1}$. Then clearly

$$\begin{cases} t_{j,m} \to 0, \quad j = 1, \dots, \ell, \\ (p_m^{(0)}, p_m^{(1)}, \dots, p_m^{(\ell)}) \to (p^{(0)}, p^{(1)}, \dots, p^{(\ell)}), \\ (p^{(0)}, p^{(1)} t_{\hat{J}_1,m}, \dots, p^{(\ell)} t_{\hat{J}_\ell,m}) \in Z. \end{cases}$$

So $p \in C_\chi(Z)$. □

Let us consider the canonical map $T_{M_j}\iota(M_j) \to M_j \hookrightarrow X$, $j = 1, \dots, \ell$, we write for short

$$\underset{X, 1 \leqslant j \leqslant \ell}{\times} T_{M_j}\iota(M_j) := T_{M_1}\iota(M_1) \underset{X}{\times} T_{M_2}\iota(M_2) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_\ell}\iota(M_\ell).$$

When $\chi$ satisfies the conditions H1, H2 and H3 we have

$$S_\chi \simeq \underset{X, 1 \leqslant j \leqslant \ell}{\times} T_{M_j}\iota(M_j). \tag{1.14}$$

Remark 1.5. — When $\chi$ satisfies conditions H1, H2 and H3, the zero-section $S_\chi$ becomes a vector bundle over $M$. However, in general, the simultaneously linearizable condition is not enough to assure the existence of a vector bundle structure on $S_\chi$. The important exceptional case where $\chi$ does not satisfy H2 but $S_\chi$ has a vector bundle structure, the case of the family $\chi^*$ in $T^*X$, will be studied in § 3.1.

Example 1.6. — Let us see two typical examples of multi-normal deformations in the complex case. Let $X = \mathbb{C}^2$ ($\simeq \mathbb{R}^4$ as a real manifold) with coordinates $(z_1, z_2)$.

1. (Majima) Let $\chi = \{M_1, M_2\}$ with $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then $\chi$ satisfies H1, H2 and H3. We have $I_1 = \{1\}$, $I_2 = \{2\}$, $J_1 = \{1\}$, $J_2 = \{2\}$ (in $\mathbb{R}^4$, if $z_1 = (x_1, x_2)$ and $z_2 = (x_3, x_4)$ we have $I_1 = \{1, 2\}$, $I_2 = \{3, 4\}$, $J_1 = J_2 = \{1\}$, $J_3 = J_4 = \{2\}$). The map $p : \widetilde{X} \to X$ is defined by
$$(z_1, z_2;\, t_1, t_2) \mapsto (t_1 z_1, t_2 z_2).$$
Remark that the deformation is real though $X$ is complex. In particular $t_1, t_2 \in \mathbb{R}$. We have $\iota(M_1) = \iota(M_2) = X$ and then the zero section $S$ of $\widetilde{X}$ is isomorphic to $T_{M_1}X \underset{X}{\times} T_{M_2}X$.
2. (Takeuchi) Let $\chi = \{M_1, M_2\}$ with $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. Then $\chi$ satisfies H1, H2 and H3. We have $I_1 = \{1, 2\}$, $I_2 = \{2\}$, $J_1 = \{1\}$, $J_2 = \{1, 2\}$ (in $\mathbb{R}^4$, if $z_1 = (x_1, x_2)$ and $z_2 = (x_3, x_4)$ we have $I_1 = \{1, 2, 3, 4\}$, $I_2 = \{3, 4\}$, $J_1 = J_2 = \{1\}$, $J_3 = J_4 = \{1, 2\}$). The map $p : \widetilde{X} \to X$ is defined by
$$(z_1, z_2;\, t_1, t_2) \mapsto (t_1 z_1, t_1 t_2 z_2).$$
We have $\iota(M_1) = M_2$, $\iota(M_2) = X$ and then the zero section $S$ of $\widetilde{X}$ is isomorphic to $T_{M_1}M_2 \underset{X}{\times} T_{M_2}X$.

Example 1.7. — Let us see three typical examples of multi-normal deformations in the real case. Let $X = \mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$.

1. (Majima) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi$ satisfies H1, H2 and H3. We have $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3\}$, $J_1 = \{1\}$, $J_2 = \{2\}$, $J_3 = \{3\}$. The map $p : \widetilde{X} \to X$ is defined by
$$(x_1, x_2, x_3;\, t_1, t_2, t_3) \mapsto (t_1 x_1, t_2 x_2, t_3 x_3).$$
We have $\iota(M_1) = \iota(M_2) = \iota(M_3) = X$ and then the zero section $S$ of $\widetilde{X}$ is isomorphic to $T_{M_1}X \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X$.

2. (Takeuchi) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi$ satisfies H1, H2 and H3. We have $I_1 = \{1, 2, 3\}$, $I_2 = \{2, 3\}$, $I_3 = \{3\}$, $J_1 = \{1\}$, $J_2 = \{1, 2\}$, $J_3 = \{1, 2, 3\}$. The map $p : \widetilde{X} \to X$ is defined by

$$(x_1, x_2, x_3;\, t_1, t_2, t_3) \mapsto (t_1x_1, t_1t_2x_2, t_1t_2t_3x_3).$$

We have $\iota(M_1) = M_2$, $\iota(M_2) = M_3$, $\iota(M_3) = X$ and then the zero section $S$ of $\widetilde{X}$ is isomorphic to $T_{M_1}M_2 \underset{X}{\times} T_{M_2}M_3 \underset{X}{\times} T_{M_3}X$.

3. (Mixed) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi$ satisfies H1, H2 and H3. We have $I_1 = \{1, 2, 3\}$, $I_2 = \{2\}$, $I_3 = \{3\}$, $J_1 = \{1\}$, $J_2 = \{1, 2\}$, $J_3 = \{1, 3\}$. The map $p : \widetilde{X} \to X$ is defined by

$$(x_1, x_2, x_3;\, t_1, t_2, t_3) \mapsto (t_1x_1, t_1t_2x_2, t_1t_3x_3).$$

We have $\iota(M_1) = M_2 \cap M_3$, $\iota(M_2) = \iota(M_3) = X$ and then the zero section $S$ is isomorphic to $T_{M_1}(M_2 \cap M_3) \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X$.

Let $q \in \bigcap_{1 \leqslant j \leqslant \ell} M_j$ and $p_j = (q;\, \xi_j)$ be a point in $T_{M_j}\iota(M_j)$ $(j = 1, 2, \ldots, \ell)$. We set $p = p_1 \underset{X}{\times} \cdots \underset{X}{\times} p_\ell \in \underset{X, 1 \leqslant j \leqslant \ell}{\times} T_{M_j}\iota(M_j)$, and $\tilde{p}_j = (q;\, \tilde{\xi}_j) \in T_{M_j}X$ denotes the image of the point $p_j$ by the canonical embedding $T_{M_j}\iota(M_j) \hookrightarrow T_{M_j}X$. We denote by $\mathrm{Cone}_{\chi, j}(p)$ $(j = 1, 2, \ldots, \ell)$ the set of open conic cones in $(T_{M_j}X)_q \simeq \mathbb{R}^{n - \dim M_j}$ that contain the point $\tilde{\xi}_j \in (T_{M_j}X)_q \simeq \mathbb{R}^{n - \dim M_j}$.

Definition 1.8. — We say that an open set $G \subset (TX)_q$ is a multi-cone along $\chi$ with direction to $p \in \left(\underset{X, 1 \leqslant j \leqslant \ell}{\times} T_{M_j}\iota(M_j)\right)_q$ if $G$ is written in the form

$$G = \bigcap_{1 \leqslant j \leqslant \ell} \pi_{j,\,q}^{-1}(G_j) \qquad G_j \in \mathrm{Cone}_{\chi, j}(p)$$

where $\pi_{j,\,q} : (TX)_q \to (T_{M_j}X)_q$ is the canonical projection. We denote by $\mathrm{Cone}_\chi(p)$ the set of multi-cones along $\chi$ with direction to $p$.

For any $q \in X$, there exists an isomorphism $\psi : X \simeq (TX)_q$ near $q$ and $\psi(q) = (q;\, 0)$ that satisfies $\psi(M_j) = (TM_j)_q$ for any $j = 1, \ldots, \ell$.

Let $Z$ be a subset of $X$. When $\chi$ satisfies H1, H2 and H3 we also have the following equivalence: $p \notin C_\chi(Z)$ if and only if there exist an open subset $\psi(q) \in U \subset (TX)_q$ and a multi-cone $G \in \mathrm{Cone}_\chi(\psi_*(p))$ such that $\psi(Z) \cap G \cap U = \varnothing$ holds.

EXAMPLE 1.9. — We now give two examples of multi-cones in the complex case. Let $X = \mathbb{C}^2$ with coordinates $(z_1, z_2)$.

1. (Majima) Let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then $\mathrm{Cone}_\chi(p)$ for $p = (0,0;\, 1,1)$ is nothing but the set of multi sectors along $M_1 \cup M_2$ with their direction to $(1,1)$.
2. (Takeuchi) Let $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. For $p = (0,0;\, 1,1) \in T_{M_1}M_2 \underset{X}{\times} T_{M_2}X$, it is easy to see that a cofinal set of $\mathrm{Cone}_\chi(p)$ is, for example, given by the family of the sets
$$\{(\eta_1, \eta_2);\ |\eta_1| < \epsilon|\eta_2|,\ \eta_2 \in S\}_{S \ni 1, \epsilon > 0},$$
where $S$ is a sector in $\mathbb{C}$ containing the direction 1.

EXAMPLE 1.10. — We now give three examples of multi-cones in the real case. Let $X = \mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$.

1. (Majima) Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0,0,0;\, 1,1,1) \in T_{M_1}X \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X$, it is easy to see that $\mathrm{Cone}_\chi(p) = \{(\mathbb{R}^+)^3\}$.
2. (Takeuchi) Let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0,0,0;\, 1,1,1) \in T_{M_1}M_2 \underset{X}{\times} T_{M_2}M_3 \underset{X}{\times} T_{M_3}X$, it is easy to see that a cofinal set of $\mathrm{Cone}_\chi(p)$ is, for example, given by the family of the sets
$$\{(\xi_1, \xi_2, \xi_3);\ |\xi_2| + |\xi_3| < \epsilon\xi_1,\ |\xi_3| < \epsilon\xi_2,\ \xi_3 > 0\}_{\epsilon > 0}.$$
3. (Mixed) Let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0,0,0;\, 1,1,1) \in T_{M_1}(M_2 \cap M_3) \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X$, a cofinal set of $\mathrm{Cone}_\chi(p)$ is, for example, given by the family of the sets
$$\{(\xi_1, \xi_2, \xi_3);\ |\xi_2| + |\xi_3| < \epsilon\xi_1,\ \xi_2 > 0,\ \xi_3 > 0\}_{\epsilon > 0}.$$

This definition is also compatible with the restriction to a subfamily of $\chi$. Namely, let $k \leqslant \ell$ and $K = \{j_1, \dots, j_k\}$ be a subset of $\{1, 2, \dots, \ell\}$. Set $\chi_K = \{M_{j_1}, \dots, M_{j_k}\}$ and $S_K := T_{M_{j_1}}\iota_\chi(M_{j_1}) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_{j_k}}\iota_\chi(M_{j_k}) \underset{X}{\times} M$. Let $Z$ be a subset of $X$. Then we have
$$C_\chi(Z) \cap S_K = C_{\chi_K}(Z) \cap S_K.$$
In the following we will denote with the same symbol $C_{\chi_K}(Z)$ the normal cone with respect to $\chi_K$ and its inverse image via the map $\widetilde{X} \to \widetilde{X}_{M_{j_1}, \dots, M_{j_k}}$.

**1.3. Multi-specialization.** — Let $k$ be a field and denote by $\mathrm{Mod}(k_{X_{sa}})$ (resp. $D^b(k_{X_{sa}})$) the category (resp. bounded derived category) of sheaves on the subanalytic site $X_{sa}$. For the theory of sheaves on subanalytic sites we refer to [5, 7]. For the theory of multi-specialization we refer to [3].
Let $\chi$ be a family of submanifolds satisfying H1, H2 and H3.

Definition 1.11. — The multi-specialization along $\chi$ is the functor

$$\begin{aligned}\nu_\chi^{sa}\colon D^b(k_{X_{sa}}) &\to D^b(k_{S_{\chi sa}}),\\ F &\mapsto s^{-1}R\Gamma_{\Omega_\chi}p^{-1}F.\end{aligned}$$

Remark 1.12. — We can give a description of the sections of the multi-specialization of $F \in D^b(k_{X_{sa}})$: let $V$ be a conic subanalytic open subset of $S_\chi$. Then:

$$\mathrm{H}^j(V;\nu_M^{sa}F) \simeq \varinjlim_U \mathrm{H}^j(U;F),$$

where $U$ ranges through the family of open subanalytic subsets of $X$ such that $C_\chi(X\backslash U) \cap V = \varnothing$. Let $p = (q;\, \xi) \in \underset{X,1\leqslant j\leqslant \ell}{\times} T_{M_j}\iota(M_j)$, let $B_\epsilon \subset (TX)_q$ be an open ball of radius $\epsilon > 0$ with its center at the origin and set

$$\mathrm{Cone}_\chi(p,\, \epsilon) := \{G \cap B_\epsilon;\; G \in \mathrm{Cone}_\chi(p)\}.$$

Applying the functor $\rho^{-1}\colon D^b(k_{S_{\chi sa}}) \to D^b(k_{S_\chi})$ (see [7] for details) we can calculate the fibers at $p \in \underset{X,1\leqslant j\leqslant \ell}{\times} T_{M_j}\iota(M_j)$ which are given by

$$(\rho^{-1}H^j\nu_\chi^{sa}F)_p \simeq \varinjlim_W \mathrm{H}^j(W;F),$$

where $W$ ranges through the family $\mathrm{Cone}_\chi(p,\, \epsilon)$ for $\epsilon > 0$.

On the other hand, under ordinary topologies, we can define the multi-specialization functor

$$\begin{aligned}\nu_\chi\colon D^b(k_X) &\to D^b(k_{S_\chi}),\\ F &\mapsto s^{-1}R\Gamma_{\Omega_\chi}p^{-1}F.\end{aligned}$$

In the same way as the subanalytic case, for any $p \in \underset{X,1\leqslant j\leqslant \ell}{\times} T_{M_j}\iota(M_j)$, we have

$$(H^j\nu_\chi F)_p \simeq \varinjlim_W \mathrm{H}^j(W;F),$$

where $W$ ranges through the (not necessary subanalytic) family $\mathrm{Cone}_\chi(p,\, \epsilon)$ for $\epsilon > 0$.

In general, for a morphism $f\colon Y \to X$ between real analytic manifolds, we have

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\rho"] & X \arrow[d, "\rho"] \\ Y_{sa} \arrow[r, "f"] & X_{sa}. \end{tikzcd}$$

Then for any $G \in D^b(k_X)$ we have

$$f^{-1}R\rho_* G \to f^{-1}R\rho_* Rf_* f^{-1}G = f^{-1}Rf_* R\rho_* f^{-1}G \to R\rho_* f^{-1}G$$

(this is not an ismorphism in general). Thus, for any $F \in D^b(k_X)$ we have

$$\begin{aligned} \rho^{-1}\nu^{sa}_{\chi}(R\rho_* F) &= \rho^{-1}s^{-1}R\Gamma_{\Omega_\chi}(p^{-1}R\rho_* F) \\ &= \rho^{-1}s^{-1}R\mathcal{H}om(\rho_* k_{\Omega_\chi}, p^{-1}R\rho_* F) \\ &\to \rho^{-1}s^{-1}R\mathcal{H}om(R\rho_* k_{\Omega_\chi}, R\rho_* p^{-1}F) \\ &= \rho^{-1}s^{-1}R\rho_* R\mathcal{H}om(k_{\Omega_\chi}, p^{-1}F) \\ &\to \rho^{-1}R\rho_* s^{-1}R\Gamma_{\Omega_\chi}(p^{-1}F) \\ &= s^{-1}R\Gamma_{\Omega_\chi}(p^{-1}F) = \nu_\chi(F). \end{aligned}$$

Namely, we have a natural morphism $\rho^{-1}\nu^{sa}_{\chi}(R\rho_* F) \to \nu_\chi(F)$, and by the stalk fomulae, this gives an isomorphism. Thus we may identify $\nu_\chi$ with $\rho^{-1}\nu^{sa}_{\chi}R\rho_*$. Therefore, if there is no risk of confusion, in the rest of the paper we will also use the notation

$$\nu_\chi = \rho^{-1}\nu^{sa}_{\chi}\colon D^b(k_{X_{sa}}) \to D^b(k_{S_\chi})$$

under the identification $D^b(k_X) \ni F = R\rho_* F \in D^b(k_{X_{sa}})$.

## 2. Multi-microlocalization

In this section we introduce the functor of multi-microlocalization as the Fourier-Sato transformation of multi-specialization. We then compute its stalks as inductive limits of sections supported on convex subanalytic cones.

**2.1. Definition. —** Now we are going to apply the Fourier-Sato transformation to the multi-specialization. We refer to [4] for the classical Fourier-Sato transformation and to [8] for its generalization to subanalytic sheaves. First, we need a general result: Let $\tau_i\colon E_i \to Z$ $(1 \leqslant i \leqslant \ell)$ be vector bundles over $Z$, and let $E_i^*$ be the dual bundle of $E_i$. We denote by $\wedge_i$ and $\vee_i$ the Fourier-Sato and the inverse Fourier-Sato transformations on $E_i$ respectively. Moreover we denote by $\wedge_i^*$ and $\vee_i^*$ the Fourier-Sato and the inverse Fourier-Sato transformations on $E_i^*$ respectively. Recall that

$$G^{\wedge_i} = (G^{\vee_i})^a \otimes \omega_{E_i^*/Z}^{\otimes -1}.$$

Here $\omega_{E_i^*/Z}$ is the dualizing complex and $\omega_{E_i^*/Z}^{\otimes -1}$ its dual. Set $E := E_1 \underset{Z}{\times} \cdots \underset{Z}{\times} E_\ell$ and $E^* := E_1^* \underset{Z}{\times} \cdots \underset{Z}{\times} E_\ell^*$ for short. Let $\tau\colon E \to Z$ be the canonical projection. Set $P_i' := \{(\eta, \xi) \in E_i \underset{Z}{\times} E_i^*; \langle \eta, \xi \rangle \leqslant 0\}$. Further set

$$P' := P_1' \underset{Z}{\times} \cdots \underset{Z}{\times} P_\ell', \qquad P^+ := E \underset{Z}{\times} E^* \backslash P',$$

and denote by $p_1'\colon P' \to E$, $p_2'\colon P' \to E^*$, and $p_1^+\colon P^+ \to E_i$, $p_2^+\colon P^+ \to E^*$ the canonical projections respectively. Let $F$ and $G$ be a multi-conic object on $E$ and $E^*$ respectively. Then we set for short $\wedge_E$ (resp. $\vee_E^*$) the composition of the Fourier-Sato transformations $\wedge_i$ (resp. the composition of the inverse Fourier-Sato transformations $\vee_i^*$) on $E_i$ for each $i \in \{1, \ldots, \ell\}$.

Remark 2.1. — Let $X$, $Y$ be two real analytic manifolds, and $f\colon Y \to X$ a real analytic mapping. We have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\rho"] & X \arrow[d, "\rho"] \\ Y_{\mathrm{sa}} \arrow[r, "f"] & X_{\mathrm{sa}} \end{tikzcd}$$

For subanalytic sheaves we can also define the functor of proper direct image $f_{!!}$, and its derived functor $Rf_{!!}$. Note that $Rf_{!!} \circ R\rho_* \not\simeq R\rho_* \circ Rf_!$ in general. As in the notation above, let $i\colon Z \to E$ be the zero-section embedding. Then we have

$$R\tau_{!!} \circ R\rho_* = i^! \circ R\rho_* = R\rho_* \circ i^! = R\rho_* \circ R\tau_!.$$

Hence in what follows, we identify $R\tau_{!!}$ with $R\tau_!$.

Proposition 2.2. — *Let $F$ and $G$ be multi-conic objects on $E$ and $E^*$ respectively.*

(1) *$F^{\wedge_E}$ and $G^{\vee_E^*}$ are independent of the order of the Fourier-Sato transformations $\wedge_i$ and the inverse Fourier-Sato transformations $\vee_i^*$ respectively.*

(2) *It follows that*

$$G^{\vee_E^*} = Rp_{1*}' p_2'^! G.$$

*Proof.* — (1) By induction on $\ell$, we may assume that $\ell = 2$. We have the following commutative diagram:

$$\begin{tikzcd}
 & P' \arrow[r, "p'_{2,2}"'] \arrow[d, "p'_{1,1}"] \arrow[dl, bend right, "p'_1"'] \arrow[rr, bend left, "p'_2"] \arrow[dr, phantom, "\square"] & P'_1 \underset{Z}{\times} E^*_2 \arrow[r, "p'_{1,2}"'] \arrow[d, "p'_{1,1}"] & E^* \\
E & E_1 \underset{Z}{\times} P'_2 \arrow[l, "p'_{2,1}"] \arrow[r, "p'_{2,2}"'] & E_1 \underset{Z}{\times} E^*_2 .
\end{tikzcd}$$

Then we have

$$\begin{aligned}
G^{\vee_1^* \vee_2^*} &= Rp'_{2,1*}\, p'^{!}_{2,2}\, Rp'_{1,1*}\, p'^{!}_{1,2} G = Rp'_{2,1*}\, Rp'_{1,1*}\, p'^{!}_{2,2}\, p'^{!}_{1,2} G \\
&= Rp'_{1*} p'^{!}_{2} G.
\end{aligned} \tag{2.1}$$

For the same reason, we obtain

$$G^{\vee_1^* \vee_2^*} = Rp'_{1*} p'^{!}_{2} G = G^{\vee_2^* \vee_1^*}.$$

Therefore we have

$$\wedge_1^* \wedge_2^* = (\vee_2^* \vee_1^*)^{-1} = (\vee_1^* \vee_2^*)^{-1} = \wedge_2^* \wedge_1^* .$$

Replacing $(E_1^*, E_2^*)$ with $(E_1, E_2)$, we obtain

$$\wedge_1 \wedge_2 = \wedge_2 \wedge_1 .$$

(2) For $\ell = 2$, the result follows by (2.1). Next, assume that $\ell > 2$. Set $E' := \underset{Z, 2 \leqslant i \leqslant \ell}{\times} E_i$, $E'^* := \underset{Z, 2 \leqslant i \leqslant \ell}{\times} E'^*_i$, $P'_{E'} := \underset{Z, 2 \leqslant i \leqslant \ell}{\times} P'_i$, and $\vee^*_{E'}$ the composition of $\vee^*_i)$ for each $i \in \{2, \ldots, \ell\}$. We have the following commutative diagram:

$$\begin{tikzcd}
 & P' \arrow[r, "p'_{E',2}"'] \arrow[d, "p'_{1,1}"] \arrow[dl, bend right, "p'_1"'] \arrow[rr, bend left, "p'_2"] \arrow[dr, phantom, "\square"] & P'_1 \underset{Z}{\times} E'^* \arrow[r, "p'_{1,2}"'] \arrow[d, "p'_{1,1}"] & E^* \\
E & E_1 \underset{Z}{\times} P'_{E'} \arrow[l, "p'_{E',1}"] \arrow[r, "p'_{E',2}"'] & E_1 \underset{Z}{\times} E'^* .
\end{tikzcd}$$

Then by induction hypothesis, we have

$$\begin{aligned}
(G^{\vee_1^*})^{\vee^*_{E'}} &= Rp'_{E',1*}\, p'^{!}_{E',2}\, Rp'_{1,1*}\, p'^{!}_{1,2} G \\
&= Rp'_{E',1*}\, Rp'_{1,1*}\, p'^{!}_{E',2}\, p'^{!}_{1,2} G = Rp'_{1*} p'^{!}_{2} G.
\end{aligned}$$

Therefore, the induction proceeds. □

We shall need some notation. For a subset $K = \{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell\}$, set $\chi_K := \{M_i;\ i \in K\}$, $S_i := T_{M_i}\iota(M_i) \underset{X}{\times} M$ $(j = 1, \dots, \ell)$ and

$$S_K := T_{M_{i_1}}\iota(M_{i_1}) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_{i_k}}\iota(M_{i_k}) = S_{i_1} \underset{M}{\times} \cdots \underset{M}{\times} S_{i_k}\,.$$

Let $S_K^*$ be the dual of $S_K$:

$$S_K^* := T^*_{M_{i_1}}\iota(M_{i_1}) \underset{X}{\times} \cdots \underset{X}{\times} T^*_{M_{i_k}}\iota(M_{i_k}) = S^*_{i_1} \underset{M}{\times} \cdots \underset{M}{\times} S^*_{i_k}\,.$$

Given $C_{i_j} \subseteq S_{i_j}$, $j = 1, \dots, k$, we set for short $C_K := C_{i_1} \underset{X}{\times} \cdots \underset{X}{\times} C_{i_k} \subset S_K$. Define $\wedge_K$ as the composition of the Fourier-Sato transformation $\wedge_{i_k}$ on $S_{i_k}$ for each $i_k \in K$.

Let $I, J \subseteq \{1, \dots, \ell\}$ be such that $I \sqcup J = \{1, \dots, \ell\}$. We still denote by $\pi\colon S_I \underset{M}{\times} S_J^* \to M$ the projection. We define the functor $\nu^{\mathrm{sa}}_{\chi_I}\mu^{\mathrm{sa}}_{\chi_J}$ by

$$\nu^{\mathrm{sa}}_{\chi_I}\mu^{\mathrm{sa}}_{\chi_J}\colon D^b(k_{X_{\mathrm{sa}}}) \ni F \mapsto \nu^{\mathrm{sa}}_{\chi}(F)^{\wedge_J} \in D^b(k_{(S_I \underset{M}{\times} S_J^*)_{\mathrm{sa}}}).$$

By Proposition 2.2, this is well defined; that is, this definition does not depend on the order of the Fourier-Sato transformations. Composing with the functor $\rho^{-1}$, we set for short

$$\nu_{\chi_I}\mu_{\chi_J} := \rho^{-1}\nu^{\mathrm{sa}}_{\chi_I}\mu^{\mathrm{sa}}_{\chi_J}\colon D^b(k_{X_{\mathrm{sa}}}) \to D^b(k_{S_I \underset{M}{\times} S_J^*}).$$

When $I = \varnothing$ we obtain the functor of multi-microlocalization: Set $\wedge := \wedge_{\{1,\dots,\ell\}}$ for short.

Definition 2.3. — The multi-microlocalization along $\chi$ is the functor

$$\mu^{\mathrm{sa}}_{\chi}\colon D^b(k_{X_{\mathrm{sa}}}) \ni F \mapsto \nu^{\mathrm{sa}}_{\chi}(F)^{\wedge} \in D^b(k_{S^*_{\chi\mathrm{sa}}}).$$

As above, we set for short

$$\mu_\chi := \rho^{-1}\mu^{\mathrm{sa}}_{\chi}\colon D^b(k_{X_{\mathrm{sa}}}) \to D^b(k_{S^*_\chi}).$$

**2.2. Stalks.** — Let $X$ be a real analytic manifold and consider a family of submanifolds $\chi = \{M_1, \dots, M_\ell\}$ satisfying H1, H2 and H3. Let $S = T_{M_1}\iota(M_1) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_\ell}\iota(M_\ell)$. Locally $p \in S$ is given by $p = p_1 \times \cdots \times p_\ell = (q;\, \xi^{(1)}, \dots, \xi^{(\ell)})$, with $\xi^{(k)} \in T_{M_k}\iota(M_k)$. Set $M = \bigcap_{j=1}^{\ell} M_j$. Let $\tau_j : T_{M_j}\iota(M_j) \hookrightarrow T_{M_j}X$ denote the canonical injection and let $\pi_j : S \to T_{M_j}\iota(M_j)$ be the canonical projection.

Lemma 2.4. — *Let $F \in D^b(k_{X_{sa}})$.*

(i) *Let $A$ be a multi-conic closed subanalytic subset of $S$. Then* $\mathrm{H}^k_A(S;\nu^{sa}_\chi F) \simeq \varinjlim_{Z,U} \mathrm{H}^k_Z(U;F)$*, where $U$ ranges through the family of open subanalytic neighborhoods of $M$ and $Z$ is a closed subanalytic subset such that $C_\chi(Z) \subset A$.*

(ii) *Suppose that $A = \bigcap_{j=1}^{\ell} \pi_j^{-1} A_j$ with $A_j$ being a closed conic subanalytic subset in $T_{M_j}\iota(M_j) \underset{M_j}{\times} M$. Then* $\mathrm{H}^k_A(S;\nu^{sa}_\chi F) \simeq \varinjlim_{Z,U} \mathrm{H}^k_Z(U;F)$*, where $U$ ranges through the family of open subanalytic neighborhoods of $M$ and $Z = Z_1 \cap \cdots \cap Z_\ell$ with each $Z_j$ being a closed subanalytic subset in $X$ and $C_{M_j}(Z_j) \subset (T_{M_j}X \backslash \tau_j(T_{M_j}\iota(M_j)\backslash A_j))$.*

*Proof.* — (i) We have the exact sequence

$$\cdots \to \mathrm{H}^k_A(S;\nu^{sa}_\chi(F)) \to \mathrm{H}^k(S;\nu^{sa}_\chi F) \to \mathrm{H}^k(S\backslash A;\nu^{sa}_\chi F) \to \cdots$$

We have $\mathrm{H}^k(S;\nu^{sa}_\chi F) \simeq \varinjlim_{U} \mathrm{H}^k(U;F)$, where $U$ ranges through the family of subanalytic neighborhoods of $M$. Moreover

$$\mathrm{H}^k(S\backslash A;\nu^{sa}_\chi F) \simeq \varinjlim_{W} \mathrm{H}^k(W;F),$$

where $W \in \mathrm{Op}(X_{sa})$ is such that $C_\chi(X\backslash W) \cap (S\backslash A) = \varnothing$. Setting $Z = X\backslash W$ we obtain

$$\mathrm{H}^k(S\backslash A;\nu^{sa}_\chi F) \simeq \varinjlim_{U,Z} \mathrm{H}^k(U\backslash Z;F),$$

where $U$ ranges through the family of subanalytic neighborhoods of $M$ and $Z$ is closed subanalytic such that $C_\chi(Z) \subset A$. Then the result follows thanks to the five lemma applied to the exact sequence

$$\cdots \to \varinjlim_{U,Z} \mathrm{H}^k_Z(U;F) \to \varinjlim_{U} \mathrm{H}^k(U;F) \to \varinjlim_{U,Z} \mathrm{H}^k(U\backslash Z;F) \to \cdots$$

where $U$ ranges through the family of subanalytic neighborhoods of $M$ and $Z$ is closed subanalytic such that $C_\chi(Z) \subset A$.

(ii) Let $A = \bigcap_{j=1}^{\ell} \pi_j^{-1} A_j$ with $A_j \subset T_{M_j}\iota(M_j) \times_{M_j} M$. Set for short $S_j := T_{M_j}\iota(M_j) \times_{M_j} M$. Then we have $S\backslash A = \bigcup_{j=1}^{\ell} \pi_j^{-1}(S_j\backslash A_j)$. Let $W \in \mathrm{Op}(X_{sa})$ be such that $C_\chi(X\backslash W) \cap (S\backslash A) = \varnothing$. Then $W = \bigcup_{j=1}^{\ell} W_j$ with $C_\chi(X\backslash W_j) \cap \pi_j^{-1}(S_j\backslash A_j) = \varnothing$. Let us find $W_1, \ldots, W_\ell$. Let $\widetilde{W}_j$ be an open neighborhood of $\pi_j^{-1}(S_j\backslash A_j)$ in the multi-normal deformation $\widetilde{X}$ of $X$. Then by Proposition 4.6 of [3] we have $C_\chi(X\backslash \widetilde{p}(\widetilde{W}_j \cap \Omega)) \cap \pi_j^{-1}(S_j\backslash A_j) = \varnothing$. Set

$W_j = \widetilde{p}(\widetilde{W_j} \cap \Omega) \cap W$. Up to shrinking $W$ we may suppose $W = \bigcup_{j=1}^{\ell} W_j$. We have

$$\begin{aligned} C_\chi(X \backslash W_j) \cap \pi_j^{-1}(S_j \backslash A_j) = \varnothing &\Leftrightarrow C_\chi(X \backslash W_j) \cap (S_j \backslash A_j) = \varnothing \\ &\Leftrightarrow C_{M_j}(X \backslash W_j) \cap \tau_j(S_j \backslash A_j) = \varnothing, \end{aligned}$$

where, for the second condition, $S_j \backslash A_j$ is regarded as a subset of

$$\{(q;\, \xi^{(1)}, \dots, \xi^{(\ell)}) \in S;\, q \in M,\, \xi^{(k)} = 0\,(k \neq j)\} \subset S.$$

The first equivalence follows from Lemma 4.2 of [3] and the second one from Corollary 4.3 of [3]. Setting $Z = X \backslash W$ and $Z_j = X \backslash W_j$, we obtain $Z = \bigcap_{j=1}^{\ell} Z_j$ with $C_{M_j}(Z_j) \cap \tau_j(S_j \backslash A_j) = \varnothing$ and the result follows. □

REMARK 2.5. — By Lemma 6.1 in [3], for any $F \in \mathbf{D}^{\mathrm{b}}(k_{X_{\mathrm{sa}}})$ there is a natural isomorphism

$$\nu_\chi^{sa}(F) = s^{-1} R\Gamma_{\Omega_\chi}(p^{-1}F) \simeq s^!(p^! F)_{\Omega_\chi}\,.$$

PROPOSITION 2.6. — *Assume that $\chi$ satisfies the conditions* H1, H2 *and* H3. *Let $\tau\colon S_\chi \simeq \underset{X, 1 \leqslant j \leqslant \ell}{\times} T_{M_j}\iota(M_j) \to M$ be the canonical projection, where $M := \bigcap_{j=1}^{\ell} M_j$. Then*

$$\nu_\chi^{sa}(F)|_M \simeq R\tau_* \nu_\chi^{sa}(F) \simeq F|_M\,, \tag{2.2}$$

$$R\Gamma_M(\nu_\chi^{sa}(F)) \simeq R\tau_! \nu_\chi^{sa}(F) \simeq R\Gamma_M(F). \tag{2.3}$$

*Proof.* — Let $k\colon M \to S_\chi$ be the zero-section embedding, and $i\colon M \to X$ the canonical embedding. Then

$$\begin{aligned} F|_M = k^{-1}s^{-1}p^{-1}F &\to k^{-1}s^{-1}Ri_{\Omega_\chi *}i_{\Omega_\chi}^{-1}p^{-1}(F) \\ &\simeq k^{-1}s^{-1}R\Gamma_{\Omega_\chi}(p^{-1}F) \simeq \nu_\chi^{sa}(F)|_M\,, \\ R\tau_!\nu_\chi^{sa}(F) = k^!\nu_\chi^{sa}(F) &\simeq k^!s^!(p^!F)_{\Omega_\chi} \to k^!s^!p^!F = R\Gamma_M(F). \end{aligned}$$

These morphisms are isomorphisms by the stalk formulae. □

Set $S^* := T^*_{M_1}\iota(M_1) \underset{X}{\times} \dots \underset{X}{\times} T^*_{M_\ell}\iota(M_\ell)$. Let $V = V_1 \underset{X}{\times} \dots \underset{X}{\times} V_\ell$ be a multi-conic open subanalytic subset in $S^*$, and let $\pi : S^* \to M$ denote the canonical projection. We set, for short, $V^\circ := V_1^\circ \underset{X}{\times} \dots \underset{X}{\times} V_\ell^\circ$ the multi-polar cone in $S$.

LEMMA 2.7. — *Let $V = V_1 \underset{X}{\times} \dots \underset{X}{\times} V_\ell$ be a multi-conic open subanalytic subset in $S^*$ such that $V \cap \pi^{-1}(q)$ is convex in $S^*_q$ for $q \in \pi(V)$. Then $\mathrm{H}^k(V; \mu_\chi^{sa}F) \simeq \varinjlim_{Z,U} \mathrm{H}^k_Z(U;F)$, where $U$ ranges through the family of open subanalytic subsets in $X$ with $U \cap M = \pi(V)$ and $Z = Z_1 \cap \dots \cap Z_\ell$ with $C_{M_j}(Z_j) \subset (T_{M_j}X \backslash \tau_j(\mathrm{Int}(V_j^{\circ a})))$. Here $(\cdot)^a$ denotes the inverse image of the antipodal map.*

*Proof.* — Since the fiber of $V_j$ is convex for each $j$ the Fourier-Sato transformation gives $\mathrm{H}^k(V;\mu_\chi^{sa}F) \simeq \mathrm{H}^k_{V^\circ}(S;\nu_\chi^{sa}F)$. Then the result follows from Lemma 2.4 (ii). $\square$

Let $p = p_1 \times \cdots \times p_\ell = (q;\, \xi^{(1)}, \ldots, \xi^{(\ell)}) \in S^*$. For any $p_k \in T^*_{M_k}\iota(M_k)$, we define the subset in $(T_{M_k}X)_q$

$$p_k^{\#} := (T_{M_k}X)_q \backslash \tau_k(p_k^{\circ a}). \tag{2.4}$$

Here the subset $p_k^{\circ a}$ in $(T_{M_k}\iota(M_k))_q$ denotes the antipodal polar set of the point $p_k$, i.e., $p_k^{\circ a} = \{\eta \in (T_{M_k}\iota(M_k))_q;\, \langle \eta, \xi^{(k)}\rangle \leqslant 0\}$. Note that $p_k^{\#}$ is an open subset. Set for short $\mu_\chi := \rho^{-1}\mu_\chi^{sa}$. As a consequence of Lemma 2.7 we have

Corollary 2.8. — *Let $p = p_1 \times \cdots \times p_\ell = (q;\, \xi^{(1)}, \ldots, \xi^{(\ell)}) \in S^*$, and let $F \in D^b(k_{X_{sa}})$. Then $H^k(\mu_\chi F)_p \simeq \varinjlim_{Z,U} \mathrm{H}^k_Z(U;F)$, where $U \in \mathrm{Op}(X_{sa})$ ranges through the family of open subanalytic neighborhoods of $q$ and $Z$ runs through a family of closed sets in the form $Z_1 \cap Z_2 \cap \cdots \cap Z_\ell$ with each $Z_k$ being closed subanalytic in $X$ and $C_{M_k}(Z_k)_q \subset p_k^{\#} \cup \{0\}$ $(k = 1, 2, \ldots, \ell)$.*

Now we are going to find a stalk formula for multi-microlocalization given by a limit of sections with support (locally) contained in closed convex cones. As the problem is local, we may assume that $X = \mathbb{R}^n$ and $q = 0$ with coordinates $(x_1, \ldots, x_n)$, and that there exists a subset $I_k$ $(k = 1, 2, \ldots, \ell)$ in $\{1, 2, \ldots, n\}$ with the conditions (1.3) such that each submanifold $M_k$ is given by $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\, x_i = 0\,(i \in I_k)\}$. Recall that $\hat{I}_k$ was defined by (1.4) and that we set $M = \cap_k M_k$ and $n_k = \sharp \hat{I}_k$. Then locally we have

$$X = M \times (N_1 \times N_2 \times \cdots \times N_\ell) = M \times N,$$

where $N_k$ is $\mathbb{R}^{n_k}$ with coordinates $x^{(k)} = (x_i)_{i \in \hat{I}_k}$. Set, for $k \in \{1, \ldots, \ell\}$,

$$\begin{aligned} J_{\prec k} &:= \{j \in \{1, \ldots, \ell\},\ I_j \subsetneqq I_k\}, \\ J_{\succ k} &:= \{j \in \{1, \ldots, \ell\},\ I_j \supsetneqq I_k\}, \\ J_{\nparallel k} &:= \{j \in \{1, \ldots, \ell\},\ I_j \cap I_k = \varnothing\}. \end{aligned} \tag{2.5}$$

Clearly we have

$$k \in J_{\prec j} \Leftrightarrow I_k \subsetneqq I_j \Leftrightarrow j \in J_{\succ k}, \tag{2.6}$$

and, by the conditions H1, H2 and H3, we also have

$$J_{\prec k} \sqcup \{k\} \sqcup J_{\succ k} \sqcup J_{\nparallel k} = \{1, 2, \ldots, \ell\}. \tag{2.7}$$

Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \ldots, \xi^{(\ell)}) \in T^*_{M_1}\iota(M_1) \underset{X}{\times} \ldots \underset{X}{\times} T^*_{M_\ell}\iota(M_\ell)$ and consider the following conic subset in $N$

$$\gamma_k := \left\{ (x^{(j)})_{j=1,\ldots,\ell} \in N;\ \begin{matrix} x^{(j)} = 0, & j \in J_{<k} \sqcup J_{\not\sharp k} \\ x^{(j)} \in \mathbb{R}^{n_j}, & j \in J_{>k} \\ \langle x^{(j)}, \xi^{(k)} \rangle > 0, j = k & \end{matrix} \right\}. \tag{2.8}$$

Note that, if $\xi^{(k)} = 0$, then $\gamma_k$ is empty.

Example 2.9. — We now compute $\gamma_k$ of (2.8) on the complex case in the following two typical situations. Let $X = \mathbb{C}^2$ with coordinates $(z_1, z_2)$.

1. (Majima) Let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then
$$\gamma_1 = \{(z_1, 0);\ \mathrm{Re}\langle z_1, \eta_1 \rangle > 0\},$$
$$\gamma_2 = \{(0, z_2);\ \mathrm{Re}\langle z_2, \eta_2 \rangle > 0\}.$$
2. (Takeuchi) Let $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. Then
$$\gamma_1 = \{(z_1, 0);\ \mathrm{Re}\langle z_1, \eta_1 \rangle > 0\},$$
$$\gamma_2 = \{(z_1, z_2);\ \mathrm{Re}\langle z_2, \eta_2 \rangle > 0\}.$$

Example 2.10. — We now compute $\gamma_k$ of (2.8) on the real case in three typical situations. Let $X = \mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$.

1. (Majima) Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then
$$\gamma_1 = \{(x_1, 0, 0);\ \langle x_1, \xi_1 \rangle > 0\},$$
$$\gamma_2 = \{(0, x_2, 0);\ \langle x_2, \xi_2 \rangle > 0\},$$
$$\gamma_3 = \{(0, 0, x_3);\ \langle x_3, \xi_3 \rangle > 0\}.$$
2. (Takeuchi) Let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then
$$\gamma_1 = \{(x_1, 0, 0);\ \langle x_1, \xi_1 \rangle > 0\},$$
$$\gamma_2 = \{(x_1, x_2, 0);\ \langle x_2, \xi_2 \rangle > 0\},$$
$$\gamma_3 = \{(x_1, x_2, x_3);\ \langle x_3, \xi_3 \rangle > 0\}.$$
3. (Mixed) Let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then
$$\gamma_1 = \{(x_1, 0, 0);\ \langle x_1, \xi_1 \rangle > 0\},$$
$$\gamma_2 = \{(x_1, x_2, 0);\ \langle x_2, \xi_2 \rangle > 0\},$$
$$\gamma_3 = \{(x_1, 0, x_3);\ \langle x_3, \xi_3 \rangle > 0\}.$$

Theorem 2.11. — *Let $p = p_1 \times \cdots \times p_\ell = (q;\, \xi^{(1)}, \ldots, \xi^{(\ell)}) \in S^*$, and let $F \in D^b(k_{X_{sa}})$. Then we have*

$$H^k(\mu_\chi F)_p \simeq \varinjlim_{G,U} \mathrm{H}^k_G(U; F). \tag{2.9}$$

*Here $U$ is an open subanalytic neighborhood of $q$ in $X$ and $G$ is a closed subanalytic subset in the form $M \times \left(\sum_{k=1}^{\ell} G_k\right)$ with $G_k$ being a closed subanalytic convex cone in $N$ satisfying $G_k \backslash \{0\} \subset \gamma_k$, where $\gamma_k$ is defined in* (2.8).

*Proof.* — As a point of the base manifold $M$ is irrelevant in the subsequent arguments, we may assume $M = \{0\}$ for simplicity. We also assume $q = 0$ and $|\xi^{(k)}| \leqslant 1$ for $k = 1, 2, \ldots, \ell$.

We first prove that, for any $Z = Z_1 \cap \cdots \cap Z_\ell$ with $Z_k$ being closed in $X$ and $C_{M_k}(Z_k)_q \subset p_k^{\#} \cup \{0\}$ $(k = 1, 2, \ldots, \ell)$, there exists $G$ described in the theorem with $G \supset Z$. As $G_k$ is convex and it contains the origin, we have

$$G_1^\circ \cap \cdots \cap G_\ell^\circ = (G_1 + \cdots + G_\ell)^\circ$$

and

$$(G_1^\circ \cap \cdots \cap G_\ell^\circ)^\circ = \overline{G_1 + \cdots + G_\ell}$$

where $G_k^\circ$ designates the usual polar cone of $G_k$ in $X$ as a vector space. We shall find a closed convex cone $V$ such that $V = V_1 \cap \cdots \cap V_\ell$ with $V_k$ convex for each $k = 1, \ldots, \ell$ and $V_k^\circ \backslash \{0\} \subset \gamma_k$ satisfying $V^\circ \supseteq Z$. Furthermore we choose $V_k$ so that every $V_k^\circ$ is proper with respect to the same direction $\widetilde{\xi} \neq 0$, i.e., $V^\circ \backslash \{0\} \subset \{x \in X;\, \langle x, \widetilde{\xi} \rangle > 0\}$. In this way

$$V^\circ = (V_1^{\circ\circ} \cap \cdots \cap V_\ell^{\circ\circ})^\circ = \overline{V_1^\circ + \cdots + V_\ell^\circ} = V_1^\circ + \cdots + V_\ell^\circ$$

and setting $G_k := V_k^\circ$ we obtain the claim. Here the last equality follows from the fact that every $V^\circ$ is closed and properly contained in the same half space in $X$.

It follows from the definition of $Z_k$ that there exists $\epsilon > 0$ and a closed convex cone $\Gamma_k \subset N_k$ with $\Gamma_k \backslash \{0\} \subset \{\langle x^{(k)},\, \xi^{(k)} \rangle > 0\}$ which satisfies

$$Z_k \subset \{x \in X;\, x^{(k)} \in \Gamma_k\} \cup \{x \in X;\, \epsilon |x^{(k)}| \leqslant \sum_{j \in J_{\prec k}} |x^{(j)}|\}.$$

Note that, for $k$ with $\xi^{(k)} = 0$, we always take $\Gamma_k = \{0\}$. The existence of such an $\epsilon$ and a $\Gamma_k$ is shown in the following way. We set

$$N' := \mathop{\times}_{j \in J_{\prec k}} N_j, \qquad N'' := \mathop{\times}_{j \in J_{\succ k} \cup J_{\not\sim k}} N_j,$$

for which we have $X = N' \times N_k \times N''$ with coordinates $(x',\, x^{(k)}, x'')$. Note that $M_k = \{0\}_{N' \times N_k} \times N''$ holds. We also define a closed subset $D$ in $N_k$ by $\{x^{(k)} \in N_k;\, \langle x^{(k)},\, \xi^{(k)} \rangle \leqslant 0\}$. Then $C_{M_k}(Z_k)_q \subset p_k^{\#} \cup \{0\}$ implies that, for any $\theta \in D\backslash\{0\}$, there exist an open cone $Q_\theta$ in $N' \times N_k$ with direction $(0_{N'}, \theta)$ and an open neighborhood $U_\theta$ of $q$ in $X$ satisfying

$$(Q_\theta \times N'') \cap U_\theta \subset X\backslash Z_k.$$

Then, as $\{0\}_{N'} \times D$ is a closed conic subset in $N' \times N_k$, we can find a finite subset $\Theta$ in $D\backslash\{0\}$ such that

$$\{0\}_{N'} \times (D\backslash\{0\}) \subset \bigcup_{\theta\in\Theta} Q_\theta,$$

$$\left(\left(\bigcup_{\theta\in\Theta} Q_\theta\right) \times N''\right) \cap \left(\bigcap_{\theta\in\Theta} U_\theta\right) \subset X\backslash Z_k.$$

Then, by taking $U$ in (2.9) sufficiently small so that $U \subset \bigcap_{\theta\in\Theta} U_\theta$, we may assume, from the beginning,

$$\left(\bigcup_{\theta\in\Theta} Q_\theta\right) \times N'' \subset X\backslash Z_k.$$

As $\bigcup_{\theta\in\Theta} Q_\theta$ is an open conic neighborhood of $\{0\}_{N'} \times (D\backslash\{0\})$ in $N' \times N_k$, there exist an open cone $T$ in $N_k$ with $D\backslash\{0\} \subset T$ and $\epsilon > 0$ satisfying

$$\left\{(x', x^{(k)}) \in N' \times N_k;\, x^{(k)} \in T,\, \sum_{j\in J_{\prec k}} |x^{(j)}| < \epsilon|x^{(k)}|\right\} \subset \bigcup_{\theta\in\Theta} Q_\theta.$$

Hence we have

$$\left\{(x', x^{(k)}) \in N' \times N_k;\, x^{(k)} \in T,\, \sum_{j\in J_{\prec k}} |x^{(j)}| < \epsilon|x^{(k)}|\right\} \times N'' \subset X\backslash Z_k,$$

which is equivalent to saying that

$$\left\{x \in X;\, x^{(k)} \in (N_k\backslash T)\right\} \cup \left\{x \in X;\, \sum_{j\in J_{\prec k}} |x^{(j)}| \geqslant \epsilon|x^{(k)}|\right\} \supset Z_k.$$

This shows the existence of $\epsilon > 0$ and $\Gamma_k := N_k\backslash T$.

Now we set

$$Z_{k,\Gamma} := \left\{x \in X;\, x^{(k)} \in \Gamma_k\right\}$$

$$Z_{k,\epsilon} := \left\{x \in X;\, \epsilon|x^{(k)}| \leqslant \sum_{j\in J_{\prec k}} |x^{(j)}|\right\}.$$

Note that, for $k \in \{1, 2, \ldots, \ell\}$ with $\hat{I}_k = I_k$, we have $Z_k \subset Z_{k,\Gamma}$ and no $Z_{k,\epsilon}$ appears.

Define $V = V_1 \cap \cdots \cap V_\ell$. Here each $V_k$ is given by, if $\xi^{(k)} \neq 0$,

$$\left\{ x \in X;\, x^{(k)} \in T_k,\, \delta |\langle x^{(k)}, \xi^{(k)} \rangle| \geqslant \sum_{j \in J_{>k}} |x^{(j)}| \right\}$$

where $\delta > 0$ and $T_k$ is a proper closed convex cone in $N_k$ with $T_k \subset \{x^{(k)} \in N_k;\, \langle x^{(k)},\, \xi^{(k)} \rangle \geqslant 0\}$ and $\xi^{(k)} \in \mathrm{Int}_{N_k} T_k$. And if $\xi^{(k)} = 0$, then $V_k$ is the whole $X$. Note that $V_k$ is a convex set in any case, i.e., $V_k^{\circ\circ} = V_k$. Then such a $V$ satisfies the desired properties. Indeed it is easy to see $V_k^{\circ} \backslash \{0\} \subset \gamma_k$. We will show that

$$V^{\circ} \supseteq \bigcap_k (Z_{k,\,\Gamma} \cup Z_{k,\,\epsilon}) \supseteq Z.$$

Here we emphasize that the inequality appearing in $Z_{k,\epsilon}$

$$\epsilon |x^{(k)}| \leqslant \sum_{j \in J_{<k}} |x^{(j)}| \tag{2.10}$$

and that in $V_k$ for $k$ with $\xi^{(k)} \neq 0$

$$\delta |\langle x^{(k)},\, \xi^{(k)} \rangle| \geqslant \sum_{j \in J_{>k}} |x^{(j)}| \tag{2.11}$$

play an important role below.

Let $\sigma = \sigma_1 \sigma_2 \ldots \sigma_\ell$ be an $\ell$-length sequence where $\sigma_k$ is either the symbols $\Gamma$ or $\epsilon$ ($k = 1, 2, \ldots, \ell$), and let us define

$$K_\sigma := Z_{1,\,\sigma_1} \cap Z_{2,\,\sigma_2} \cap \cdots \cap Z_{\ell,\,\sigma_\ell}.$$

Then we have

$$\bigcap_k (Z_{k,\,\Gamma} \cup Z_{k,\,\epsilon}) = \bigcup_\sigma K_\sigma.$$

We now show that, for each sequence $\sigma$, we obtain $V^{\circ} \supset K_\sigma$ if we take $T_k$ ($k = 1, 2, \ldots, \ell$) and $\delta > 0$ sufficiently small.

Set

$$J_\Gamma(\sigma) := \{j \in \{1, 2, \ldots, \ell\};\, \sigma_j = \Gamma\}$$

and

$$J_\epsilon(\sigma) := \{j \in \{1, 2, \ldots, \ell\};\, \sigma_j = \epsilon\}.$$

Note that, for $k \in \{1, 2, \ldots, \ell\}$ with $\hat{I}_k = I_k$, we have $k \in J_\Gamma(\sigma)$ and $k \notin J_\epsilon(\sigma)$, which implies, in particular, $J_\Gamma(\sigma)$ is non-empty. As both $V_k$ and $\Gamma_k$ are proper cones with direction $\xi^{(k)}$ in $N_k$ if $\xi^{(k)} \neq 0$, and as $\Gamma_k = \{0\}$ if $\xi^{(k)} = 0$, there exists a constant $M > 0$ such that

$$M |x^{(j)}||y^{(j)}| \leqslant \quad \langle x^{(j)},\, y^{(j)} \rangle \quad \leqslant |x^{(j)}||y^{(j)}|$$

holds for $j \in J_\Gamma(\sigma)$ and $x = (x^{(1)}, \dots, x^{(\ell)}) \in K_\sigma$ and $y = (y^{(1)}, \dots, y^{(\ell)}) \in V$. Furthermore, by (2.10), there exists a constant $N > 0$ such that, for any $j \in J_\epsilon(\sigma)$, we have

$$|x^{(j)}| \leqslant \frac{N}{\epsilon^N} \sum_{\alpha \in J_\Gamma(\sigma) \cap J_{<j}} |x^{(\alpha)}|$$

for $x = (x^{(1)}, \dots, x^{(\ell)}) \in K_\sigma$. By noticing these facts, we obtain, for $x = (x^{(1)}, \dots, x^{(\ell)}) \in K_\sigma$ and $y = (y^{(1)}, \dots, y^{(\ell)}) \in V$.

$$\begin{aligned}
\langle x, y\rangle = \sum_j \langle x^{(j)}, y^{(j)}\rangle \quad &= \sum_{j \in J_\Gamma(\sigma)} \langle x^{(j)}, y^{(j)}\rangle + \sum_{j \in J_\epsilon(\sigma)} \langle x^{(j)}, y^{(j)}\rangle \\
&\geqslant M \sum_{j \in J_\Gamma(\sigma)} |x^{(j)}||y^{(j)}| - \sum_{j \in J_\epsilon(\sigma)} |x^{(j)}||y^{(j)}| \\
&\geqslant M \sum_{j \in J_\Gamma(\sigma)} |x^{(j)}||y^{(j)}| - \frac{N}{\epsilon^N} \sum_{j \in J_\epsilon(\sigma)} \Bigg( \sum_{\alpha \in J_\Gamma(\sigma) \cap J_{<j}} |x^{(\alpha)}| \Bigg) |y^{(j)}|.
\end{aligned}$$

Here, as $\Gamma_\alpha = \{0\}$ for $\alpha$ with $\xi^{(\alpha)} = 0$, we have $|x^{(\alpha)}| = 0$ for such an $\alpha \in J_\Gamma(\sigma)$ and the last term in the above inequalities is equal to

$$M \sum_{j \in J_\Gamma(\sigma)} |x^{(j)}||y^{(j)}| - \frac{N}{\epsilon^N} \sum_{j \in J_\epsilon(\sigma)} \Bigg( \sum_{\alpha \in J_\Gamma(\sigma) \cap J_{<j},\, \xi^{(\alpha)} \neq 0} |x^{(\alpha)}| \Bigg) |y^{(j)}|. \tag{2.12}$$

It follows from (2.11) that we get $\delta |y^{(\alpha)}| \geqslant \delta |\langle y^{(\alpha)}, \xi^{(\alpha)}\rangle| \geqslant |y^{(j)}|$ for $\alpha$ with $\xi^{(\alpha)} \neq 0$ and for $j \in J_{>\alpha}$ ($\Leftrightarrow \alpha \in J_{<j}$). Hence the (2.12) is lower bounded by

$$\begin{aligned}
M &\sum_{j \in J_\Gamma(\sigma)} |x^{(j)}||y^{(j)}| - \frac{\delta N}{\epsilon^N} \sum_{j \in J_\epsilon(\sigma)} \sum_{\alpha \in J_\Gamma(\sigma) \cap J_{<j},\, \xi^{(\alpha)} \neq 0} |x^{(\alpha)}||y^{(\alpha)}| \\
&\geqslant M \sum_{j \in J_\Gamma(\sigma)} |x^{(j)}||y^{(j)}| - \frac{\delta N \# J_\epsilon(\sigma)}{\epsilon^N} \sum_{\alpha \in J_\Gamma(\sigma)} |x^{(\alpha)}||y^{(\alpha)}| \\
&= \left( M - \frac{\delta N \# J_\epsilon(\sigma)}{\epsilon^N} \right) \sum_{\alpha \in J_\Gamma(\sigma)} |x^{(\alpha)}||y^{(\alpha)}|.
\end{aligned}$$

Note that the set $J_\Gamma(\sigma)$ is non-empty as noted above. Hence, if we take $\delta$ sufficiently small, $\langle x, y\rangle$ takes non-negative values for $x \in K_\sigma$ and $y \in V$, which implies $K_\sigma \subset V^\circ$. Hence we have shown the existence of $G$ described in the theorem with $Z \subset G$.

Now we show that $G$ described in the theorem satisfies $C_{M_k}(G)_q \subset p_k^{\#} \cup \{0\}$ for any $k$. We may assume $G = V^\circ$ where $V$ was defined in the first part of the proof. Suppose that there exists a non-zero vector $\eta \in (T_{M_k} X)_q = N' \times N_k$ such that

$$0 \neq \eta \in C_{M_k}(V^\circ)_q \cap \tau_k(p_k^{\circ a}).$$

Note that $\eta \in \tau_k(p_k^{\circ a})$ implies the existence of $0 \neq \eta^{(k)} \in N_k$ such that $\eta = (0_{N'}, \eta^{(k)})$ and $\langle \eta^{(k)}, \xi^{(k)} \rangle \leqslant 0$. Set, for any $\epsilon > 0$,

$$Q_\epsilon := \left\{ x = (x', x^{(k)}, x'') \in X; \begin{array}{l} |x'| < \epsilon\langle \eta^{(k)}, x^{(k)}\rangle, \, |x''| < \epsilon \\ |x^{(k)}| < \epsilon, \, x^{(k)} \in Q_\epsilon^{(k)} \end{array} \right\},$$

where $\{Q_\epsilon^{(k)}\}_{\epsilon>0}$ is a family of open cone neighborhoods of the direction $\eta^{(k)}$ in $N_k$. Then $\eta \in C_{M_k}(V^\circ)_q$ means $Q_\epsilon \cap V^\circ \neq \varnothing$ for any $\epsilon > 0$. By noticing $\langle \eta^{(k)}, \xi^{(k)} \rangle \leqslant 0$, it follows from the definition of $V$ that there exists a vector $v = (v', v^{(k)}, 0_{N''}) \in V$ such that we can find a positive constant $C > 0$ with

$$\langle x^{(k)}, v^{(k)} \rangle < -C|x^{(k)}| \quad (x^{(k)} \in Q_\epsilon^{(k)})$$

for any sufficiently small $\epsilon > 0$. Hence we have, for $x = (x', x^{(k)}, x'') \in Q_\epsilon$,

$$\langle x, v \rangle = \langle x', v' \rangle + \langle x^{(k)}, v^{(k)} \rangle \leqslant (\epsilon|\eta^{(k)}||v'| - C)|x^{(k)}|.$$

As a result, if we take a sufficiently small $\epsilon > 0$, we have $\langle x, v \rangle < 0$ for any $x \in Q_\epsilon$, and thus, we get $Q_\epsilon \cap V^\circ = \varnothing$ which contradicts $Q_\epsilon \cap V^\circ \neq \varnothing$. Therefore we have obtained the conclusion. This completes the proof. □

REMARK 2.12. — In the case $\ell = 2$ with $M_1 \subset M_2 \subset X$ we obtain the stalk formula computed in [11].

Now let us consider the mixed cases between specialization and microlocalization. We shall need some notations. Given a subset $K = \{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell\}$, let $\chi_K = \{M_i, \ i \in K\}$, set $S_{i_j} = T_{M_{i_j}}\iota(M_{i_j}) \underset{M_{i_j}}{\times} M$ $(j = 1, \dots, k)$ and $S_K = T_{M_{i_1}}\iota(M_{i_1}) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_{i_k}}\iota(M_{i_k})$ and let $S_K^*$ be its dual. Given $C_{i_j} \subseteq S_{i_j}$, $j = 1, \dots, k$, we set for short $C_K := C_{i_1} \underset{X}{\times} \cdots \underset{X}{\times} C_{i_k} \subset S_K$. Define $\wedge_K$ as the composition of the Fourier-Sato transformations $\wedge_{i_k}$ on $S_{i_k}$ for each $i_k \in K$.

Let $I, J \subseteq \{1, \dots, \ell\}$ be such that $I \sqcup J = \{1, \dots, \ell\}$.

LEMMA 2.13. — *Let $V = V_I \underset{X}{\times} V_J$ be a multi-conic open subanalytic subset in $S_I \underset{X}{\times} S_J^*$ such that $V \cap \pi^{-1}(q)$ is convex for $q \in \pi(V)$. Then $\mathrm{H}^k(V; \nu_{\chi_I}^{sa}\mu_{\chi_J}^{sa}F) \simeq \varinjlim_{Z,U} \mathrm{H}_Z^k(U; F)$, where $U$ ranges through the family of open subanalytic subsets in $X$ with $C_{\chi_I}(X \backslash U) \cap \bigcup_{i \in I} \pi_i^{-1}(V_i) = \varnothing$ and $Z = \bigcap_{j \in J} Z_j$ with $C_{M_j}(Z_j) \subset (T_{M_j}X \backslash \tau_j(\mathrm{Int}(V_j^{\circ a})))$. Here $(\cdot)^a$ denotes the inverse image of the antipodal map.*

*Proof.* — We write $\times$ instead of $\underset{X}{\times}$ for short. Since $V$ is convex the Fourier-Sato transformation gives $\mathrm{H}^k(V; \nu_{\chi_I}^{sa}\mu_{\chi_J}^{sa}F) \simeq \mathrm{H}_{V_I \times V_J^\circ}^k(S; \nu_\chi^{sa}F)$. Consider the distinguished triangle

$$(2.13) \qquad R\Gamma_{(S_I \backslash V_I) \times V_J^\circ}\nu_\chi^{sa}F \to R\Gamma_{S_I \times V_J^\circ}\nu_\chi^{sa}F \to R\Gamma_{V_I \times V_J^\circ}\nu_\chi^{sa}F \xrightarrow{+}$$

By Lemma 2.7 we have $\mathrm{H}^k_{S_I \times V_J^\circ}(S; \nu^{sa}_\chi F) \simeq \varinjlim_{Z,U} \mathrm{H}^k_Z(U; F)$, where $U$ ranges through the family of open subanalytic subsets of $X$ such that $U \cap M = \pi(V)$ and $Z = \bigcap_{j\in J} Z_j$ with $C_{M_j}(Z_j) \subset (T_{M_j}X \backslash \tau_j(\mathrm{Int}(V_j^{\circ a})))$. By Lemma 2.4 we also have $\mathrm{H}^k_{(S_I \backslash V_I)\times V_J^\circ}(S; \nu^{sa}_\chi F) \simeq \varinjlim_{Z,U} \mathrm{H}^k_Z(U; F)$, where $U$ ranges through the family of open subanalytic subsets of $X$ such that $U \cap M = \pi(V)$ and $Z = \bigcap_{j=1}^{\ell} Z_j$ with $C_{M_j}(Z_j) \subset (T_{M_j}X \backslash \tau_j(\mathrm{Int}(V_j^{\circ a})))$ if $j \in J$ and $C_{M_j}(Z_j) \subset (T_{M_j}X \backslash \tau_j(V_j))$ if $j \in I$. Thanks to the long exact sequence associated to (2.13) we obtain $\mathrm{H}^k_{V_I \times V_J^\circ}(S; \nu^{sa}_\chi F) \simeq \varinjlim_{Z,U} \mathrm{H}^k_Z(U \cap W; F)$, where $U$ ranges through the family of open subanalytic subsets of $X$ such that $U \cap M = \pi(V)$, $Z = \bigcap_{j\in J} Z_j$ with $C_{M_j}(Z_j) \subset (T_{M_j}X \backslash \tau_j(\mathrm{Int}(V_j^{\circ a})))$ and $W = \bigcup_{i\in I}(X \backslash Z_i)$ such that $C_{M_i}(Z_i) \subset (T_{M_i}X \backslash \tau_i(V_i))$. Then the result follows since, as in Lemma 2.4

$$C_{M_i}(Z_i) \subset (T_{M_i}X \backslash \tau_i(V_i)) \Leftrightarrow C_\chi(Z_i) \cap \pi_i^{-1}(V_i) = \varnothing. \qquad \square$$

Let $p = p_1 \times \cdots \times p_\ell = (q;\, \xi^{(1)}, \dots, \xi^{(\ell)}) = (q;\, \xi_I, \xi_J) \in S_I \underset{X}{\times} S_J^*$. Locally we may identify $S_J$ with its dual. Set for short $\nu_{\chi_I}\mu_{\chi_J} := \rho^{-1}\nu^{sa}_{\chi_I}\mu^{sa}_{\chi_J}$. As a consequence of Lemma 2.13 we have

Corollary 2.14. — *Let $p = p_1 \times \cdots \times p_\ell = (q;\, \xi^{(1)}, \dots, \xi^{(\ell)}) \in S_I \underset{X}{\times} S_J^*$, and let $F \in D^b(k_{X_{sa}})$. Then $H^k(\nu_{\chi_I}\mu_{\chi_J} F)_p \simeq \varinjlim_{Z, W_\epsilon} \mathrm{H}^k_Z(W_\epsilon; F)$, where $W_\epsilon = W \cap B_\epsilon$, with $W \in \mathrm{Cone}_\chi(q;\, \xi_I, 0_J)$, $B_\epsilon$ is an open ball containing $q$ of radius $\epsilon > 0$ and $Z$ runs through a family of closed sets in the form $Z_1 \cap Z_2 \cap \cdots \cap Z_\ell$ with each $Z_k$ being closed subanalytic in $X$ and $C_{M_k}(Z_k)_q \subset p_k^{\#} \cup \{0\}$ $(k = 1, 2, \dots, \ell)$.*

*Proof.* — The result follows since for any subanalytic conic neighborhood $V$ of $(q;\, \xi_I, 0_J)$, any $U \in \mathrm{Op}(X_{sa})$ such that $C_\chi(X \backslash U) \cap V = \varnothing$ contains $W \cap B_\epsilon$, $q \in B_\epsilon$, $\epsilon > 0$, $W \in \mathrm{Cone}(q;\, \xi_I, 0_J)$. Moreover, by definition of multi-cone we may assume $W = \bigcap_{j=1}^{\ell} W_j$ such that $C_{M_j}(\overline{W_j})_q \subset p_j^{\#} \cup \{0\}$ if $j \in I$ and $W_j = X$ if $j \in J$. $\square$

As in Theorem 2.11 we can find a cofinal family to the family of closed subsets defining the stalk formula in Corollary 2.14 which (locally) consists of convex cones and we can formulate the stalk formula in the mixed case.

Theorem 2.15. — *Let $p = p_1 \times \cdots \times p_\ell = (q;\, \xi^{(1)}, \dots, \xi^{(\ell)}) \in S_I \underset{X}{\times} S_J^*$, and let $F \in D^b(k_{X_{sa}})$. Then we have*

$$H^k(\nu_{\chi_I}\mu_{\chi_J} F)_p \simeq \varinjlim_{G, W_\epsilon} \mathrm{H}^k_G(W_\epsilon; F). \tag{2.14}$$

*Here* $W_\epsilon = W \cap B_\epsilon$, *with* $W \in \mathrm{Cone}_\chi(q;\, \xi_I, 0_J)$, $B_\epsilon$ *is an open ball of radius* $\epsilon > 0$ *containing* $q$ *and* $G$ *is a closed subanalytic subset in the form* $M \times \left(\sum_{k=1}^{\ell} G_k\right)$ *with* $G_k$ *being a closed subanalytic convex cone in* $N$ *satisfying* $G_k \backslash \{0\} \subset \gamma_k$*, where* $\gamma_k$ *is defined in* (2.8).

## 3. Multi-microlocalization and microsupport

In this section we give an estimate of the microsupport of multi-microlocalization. The main point is to find a suitable ambient space: this is done (via Hamiltonian isomorphism) by identifying $T^*S_\chi$ with the normal deformation of $T^*X$ with respect to a suitable family of submanifolds $\chi^*$.

**3.1. Geometry.** — Let $X$ be a real analytic manifold and consider a family of submanifolds $\chi = \{M_1, \ldots, M_\ell\}$ satisfying H1, H2 and H3. We consider the conormal bundle $T^*X$ with local coordinates $(x^{(0)}, x^{(1)}, \ldots, x^{(\ell)};\, \xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(\ell)})$, where $x^{(j)} = (x_{j_1}, \ldots, x_{j_p})$ with $\hat{I}_j = \{j_1, \ldots, j_p\}$ etc. We use the notations in § 2.1; for example, we set $S_i := T_{M_i}\iota(M_i) \underset{X}{\times} M$. Let $I, J \subseteq \{1, \ldots, \ell\}$ be such that $I \sqcup J = \{1, \ldots, \ell\}$. Recall that

$$\begin{aligned} S_\chi &= S_1 \underset{X}{\times} \cdots \underset{X}{\times} S_\ell, \\ S^*_\chi &= S^*_1 \underset{X}{\times} \cdots \underset{X}{\times} S^*_\ell, \\ S_I \underset{M}{\times} S^*_J &= \big(\underset{M, i \in I}{\times} S_i\big) \underset{M}{\times} \big(\underset{M, j \in J}{\times} S^*_j\big). \end{aligned}$$

Then we consider a mapping

$$\begin{aligned} H_{IJ} \colon T^*S_\chi &\ni (x^{(0)}, x^{(1)}, \ldots, x^{(\ell)};\, \xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(\ell)}) \\ &\mapsto (x^{(0)}, (x^{(i)})_{i\in I}, (\xi^{(j)})_{j \in J};\, \eta^{(0)}, (\xi^{(i)})_{i \in I}, (-x^{(j)})_{j\in J}) \in T^*(S_I \underset{M}{\times} S^*_J). \end{aligned}$$

Note that $H_{IJ}$ is induced by the Hamiltonian isomorphisms $T^*S_J \xrightarrow{\sim} T^*S^*_J$.

Proposition 3.1. — *$H_{IJ}$ gives a bundle isomorphism over $M$; that is, $H_{IJ}$ does not depend on the choice of local coordinates.*

*Proof.* — Let $\varphi \colon X \to X$ be a local coordinate transformation near any $x \in X$. We may assume that $X = \mathbb{R}^n$ with coordinates $x = (x^{(0)}, x^{(1)} \ldots, x^{(\ell)})$, where $M$ is given by $(x^{(0)}, 0, \ldots, 0)$, and $\varphi$ is given by

$$y^{(j)} = \varphi^{(j)}(x^{(0)}, x^{(1)}, \ldots, x^{(\ell)}) \quad (j = 0, 1, \ldots, \ell).$$

Here $\varphi^{(j)}(x) = (\varphi_{j_1}(x), \dots, \varphi_{j_{p(j)}}(x))$ with $\hat{I}_j = \{j_1, \dots, j_{p(j)}\}$. This induces a coordinate transformation

$$T^*X \ni (x;\, \xi) \mapsto (y;\, \eta) \in T^*X$$

defined by

$$\begin{cases} y^{(j)} = \varphi^{(j)}(x), \\ \xi^{(j)} = \displaystyle\sum_{i=0}^{\ell} {}^{\mathrm{t}}\Big[\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(x)\Big]\eta^{(i)}, \end{cases}$$

where

$$\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(x) = \begin{bmatrix} \frac{\partial \varphi_{i_1}}{\partial x_{j_1}}(x) & \cdots & \frac{\partial \varphi_{i_1}}{\partial x_{j_{p(j)}}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_{i_{p(i)}}}{\partial x_{j_1}}(x) & \cdots & \frac{\partial \varphi_{i_{p(i)}}}{\partial x_{j_{p(j)}}}(x) \end{bmatrix}$$

is a $p(i) \times p(j)$-matrix, and ${}^{\mathrm{t}}$ means the transpose of a matrix. Set $J_j(x^{(0)}) := \frac{\partial \varphi^{(j)}}{\partial x^{(j)}}(x^{(0)}, 0)$ for short. Then the coordinate transformation

$$(x^{(0)}, x^{(1)}, \dots, x^{(\ell)}) \mapsto (y^{(0)}, y^{(1)}, \dots, y^{(\ell)})$$

on $S_\chi$ is given by

$$\begin{cases} y^{(0)} = \varphi^{(0)}(x^{(0)}, 0), \\ y^{(j)} = J_j(x^{(0)})\, x^{(j)} \quad (j = 1, \dots, \ell). \end{cases} \tag{3.1}$$

The Jacobian matrix of (3.1) is

$$\begin{bmatrix} J_0(x^{(0)}) & 0 & \cdots & 0 \\ \frac{\partial J_1}{\partial x^{(0)}}(x^{(0)})\, x^{(1)} & J_1(x^{(0)}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial J_\ell(x^{(0)})}{\partial x^{(0)}}(x^{(0)})\, x^{(\ell)} & 0 & \cdots & J_\ell(x^{(0)}) \end{bmatrix}.$$

Thus the coordinate transformation

$$(x^{(0)}, x^{(1)}, \dots, x^{(\ell)};\, \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)}) \\ \mapsto (y^{(0)}, y^{(1)}, \dots, y^{(\ell)};\, \eta^{(0)}, \eta^{(1)}, \dots, \eta^{(\ell)})$$

on $T^*S_\chi$ is given by (3.1) and

$$\begin{cases} \xi^{(0)} = {}^{\mathrm{t}}J_0(x^{(0)})\, \eta^{(0)} + \displaystyle\sum_{i=1}^{\ell} {}^{\mathrm{t}}x^{(i)} \frac{\partial\, {}^{\mathrm{t}}J_i}{\partial x^{(0)}}(x^{(0)})\, \eta^{(i)}, \\ \xi^{(j)} = {}^{\mathrm{t}}J_j(x^{(0)})\, \eta^{(j)} \quad (j = 1, \dots, \ell). \end{cases} \tag{3.2}$$

Next, consider the coordinate transformation on $S_I \underset{M}{\times} S_J^*$. After a permutation, we may assume that $I = \{1, \dots, p\}$, $J = \{p+1, \dots, \ell\}$. Then the coordinate transformation

$$(x^{(0)}, (x^{(i)})_{i=1}^p, (\xi^{(j)})_{j=p+1}^{\ell}) \mapsto (y^{(0)}, (y^{(i)})_{i=1}^p, (\eta^{(j)})_{j=p+1}^{\ell})$$

on $S_I \underset{M}{\times} S_J^*$ is given by

$$\begin{cases} y^{(0)} = \varphi^{(0)}(x^{(0)}, 0), \\ y^{(i)} = J_i(x^{(0)})\, x^{(i)} \quad (i = 1, \dots, p), \\ \eta^{(j)} = {}^{\mathrm{t}}J_j^{-1}(x^{(0)})\, \xi^{(j)} \quad (j = p+1, \dots, \ell). \end{cases} \tag{3.3}$$

The Jacobian matrix of (3.3) is

$$\begin{bmatrix} J_0(x^{(0)}) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \frac{\partial J_1}{\partial x^{(0)}}(x^{(0)})\, x^{(1)} & J_1(x^{(0)}) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \frac{\partial J_p}{\partial x^{(0)}}(x^{(0)})\, x^{(p)} & 0 & \cdots & J_p(x^{(0)}) & 0 & \cdots & 0 \\ \frac{\partial\, {}^{\mathrm{t}}J_{p+1}^{-1}}{\partial x^{(0)}}(x^{(0)})\, \xi^{(p+1)} & 0 & \cdots & 0 & {}^{\mathrm{t}}J_{p+1}^{-1}(x^{(0)}) & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \vdots \\ \frac{\partial\, {}^{\mathrm{t}}J_{\ell}^{-1}}{\partial x^{(0)}}(x^{(0)})\, \xi^{(\ell)} & 0 & \cdots & 0 & 0 & \cdots & {}^{\mathrm{t}}J_{\ell}^{-1}(x^{(0)}) \end{bmatrix}.$$

Thus the coordinate transformation

$$\begin{aligned} &(x^{(0)}, (x^{(i)})_{i=1}^p, (\xi^{(j)})_{j=p+1}^{\ell}; \xi^{(0)}, (\xi^{(i)})_{i=1}^p, (-x^{(j)})_{j=p+1}^{\ell}) \\ &\qquad \mapsto (y^{(0)}, (y^{(i)})_{i=1}^p, (\eta^{(j)})_{j=p+1}^{\ell}; \eta^{(0)}, (\eta^{(i)})_{i=1}^p, (-y^{(j)})_{j=p+1}^{\ell}) \end{aligned}$$

on $T^*S_\chi^*$ is given by (3.3) and

$$\begin{cases} \xi^{(0)} = {}^{\mathrm{t}}J_0(x^{(0)})\, \eta^{(0)} + \displaystyle\sum_{i=1}^{p} {}^{\mathrm{t}}x^{(i)} \frac{\partial\, {}^{\mathrm{t}}J_i}{\partial x^{(0)}}(x^{(0)})\, \eta^{(i)} - \sum_{j=p+1}^{\ell} {}^{\mathrm{t}}\xi^{(j)} \frac{\partial J_j^{-1}}{\partial x^{(0)}}(x^{(0)})\, y^{(j)}, \\ \xi^{(i)} = {}^{\mathrm{t}}J_i(x^{(0)})\, \eta^{(i)} \quad (i = 1, \dots, p), \\ x^{(j)} = J_j^{-1}(x^{(0)})\, y^{(j)} \quad (j = p+1, \dots, \ell). \end{cases}$$

Since

$$\frac{\partial J_j^{-1}}{\partial x^{(0)}}(x^{(0)}) = -J_j^{-1}(x^{(0)}) \frac{\partial J_j}{\partial x^{(0)}}(x^{(0)})\, J_j^{-1}(x^{(0)}),$$

we have

$$\begin{aligned}
-({}^{\mathrm{t}}\xi^{(j)}\frac{\partial J_j^{-1}}{\partial x^{(0)}}(x^{(0)})\,y^{(j)})_k &= ({}^{\mathrm{t}}\xi^{(j)}J_j^{-1}(x^{(0)})\frac{\partial J_j}{\partial x^{(0)}}(x^{(0)})\,J_j^{-1}(x^{(0)})\,y^{(j)})_k \\
&= ({}^{\mathrm{t}}\eta^{(j)}\frac{\partial J_j}{\partial x^{(0)}}(x^{(0)})\,x^{(j)})_k = \sum_{\mu,\nu\in\hat{I}_j}\eta_\mu^{(j)}\Big(\frac{\partial J_j}{\partial x_k^{(0)}}(x^{(0)})\Big)_{\mu,\nu}x_\nu^{(j)} \\
&= ({}^{\mathrm{t}}x^{(j)}\frac{\partial\,{}^{\mathrm{t}}J_j}{\partial x^{(0)}}(x^{(0)})\,\eta^{(j)})_k\,.
\end{aligned}$$

Therefore

$$-\sum_{j=p+1}^{\ell}{}^{\mathrm{t}}\xi^{(j)}\frac{\partial J_p^{-1}}{\partial x^{(0)}}(x^{(0)})\,y^{(j)} = \sum_{j=p+1}^{\ell}{}^{\mathrm{t}}x^{(j)}\frac{\partial\,{}^{\mathrm{t}}J_j}{\partial x^{(0)}}(x^{(0)})\,\eta^{(j)}.$$

Thus we can prove that

$$H_{IJ}\colon T^*S_\chi \xrightarrow{\sim} T^*(S_I \underset{M}{\times} S_J^*). \tag{3.4}$$

□

Hence, using Proposition 5.5.5 of [4] repeatedly, we obtain:

Proposition 3.2. — *Let $I, J \subseteq \{1,\dots,\ell\}$ be such that $I \sqcup J = \{1,\dots,\ell\}$. Then, under the identification by (3.4), for any $F \in D^b(k_X)$ it follows that*

$$\begin{array}{ccc}
T^*S_\chi & = & T^*(S_I \underset{M}{\times} S_J^*) \\
\cup & & \cup \\
\mathrm{SS}(\nu_\chi(F)) & = & \mathrm{SS}(\nu_{\chi_I}\mu_{\chi_J}(F)).
\end{array}$$

*In particular, it follows that*

$$\begin{array}{ccc}
T^*S_\chi & = & T^*S_\chi^* \\
\cup & & \cup \\
\mathrm{SS}(\nu_\chi(F)) & = & \mathrm{SS}(\mu_\chi(F)).
\end{array}$$

Next, we study the relation between the normal deformations of $T^*X$ with respect to $\chi^* := \{T^*_{M_1}X,\dots,T^*_{M_\ell}X\}$ and of $X$ with respect to $\chi$. We denote by $\widetilde{T^*X}_{\chi^*} := \widetilde{T^*X}_{T^*_{M_1}X,\dots,T^*_{M_\ell}X}$ the normal deformation of $T^*X$ with respect to $\chi^*$ and by $S_{\chi^*}$ its zero-section. Set $x := (x^{(0)}, x^{(1)},\dots,x^{(\ell)})$, $\xi := (\xi^{(0)},\xi^{(1)},\dots,\xi^{(\ell)})$ and $t := (t_1,\dots,t_\ell)$. We have a mapping

$$\widetilde{T^*X}_{\chi^*} \ni (x;\,\xi;\,t) \mapsto (\mu_x(x;\,t);\,\mu_\xi(\xi;\,t)) \in T^*X$$

defined by

$$\begin{aligned}
\mu_x(x;\,t) &:= (t_{\hat{J}_0}x^{(0)}, t_{\hat{J}_1}x^{(1)},\dots,t_{\hat{J}_\ell}x^{(\ell)}),\\
\mu_\xi(\xi;\,t) &:= (t_{\hat{J}_0^c}\xi^{(0)}, t_{\hat{J}_1^c}\xi^{(1)},\dots,t_{\hat{J}_\ell^c}\xi^{(\ell)}),
\end{aligned}$$

where $\hat{J}_j^c := \{1,\ldots,\ell\}\backslash\hat{J}_j$ $(j = 0,1,\ldots,\ell)$. In particular $\hat{J}_0^c = \{1,\ldots,\ell\}$ since $\hat{J}_0 = \varnothing$. In particular $t_{\hat{J}_0} = 1$ and $t_{\hat{J}_0^c} = t_1 \cdots t_\ell$.

THEOREM 3.3. — *As vector bundles, there exist the following canonical isomorphism*:

$$S_{\chi^*} \simeq T^*S_\chi \simeq T^*S_\chi^*.$$

*Proof.* — Let $\varphi\colon X \to X$ be a local coordinate transformation near any $x \in X$, and retain the notation of the proof of Proposition 3.1. The coordinate transformation $(x;\,\xi;\,t) \mapsto (y;\,\eta;\,t)$ on $\widetilde{T^*X}_{\chi^*}\backslash S_{\chi^*}$ is given by

$$\begin{cases} y^{(j)} = \dfrac{1}{t_{\hat{J}_j}}\,\varphi^{(j)}(t_{\hat{J}}x), \\[2ex] \xi^{(j)} = \displaystyle\sum_{i=0}^{\ell} {}^{\mathrm{t}}[\frac{\partial\varphi^{(i)}}{\partial x^{(j)}}(t_{\hat{J}}x)]\frac{t_{J_i^c}}{t_{J_j^c}}\eta^{(i)}, \end{cases}$$

where $t_{\hat{J}}x := \mu_x(x;\,t) = (t_{\hat{J}_0}x^{(0)}, t_{\hat{J}_1}x^{(1)},\ldots,t_{\hat{J}_\ell}x^{(\ell)})$. Let us consider the coordinate transformation on $S_{\chi^*}$. We write for short $t \to 0$ instead of $(t_1,\ldots,t_\ell) \to (0,\ldots,0)$. Set $J_j(x^{(0)}) := \frac{\partial\varphi^{(j)}}{\partial x^{(j)}}(x^{(0)},0)$ for short. Then, by Proposition 1.5 of [3] on $S_{\chi^*}$

$$\begin{cases} y^{(0)} = \varphi^{(0)}(x^{(0)},0), \\ y^{(j)} = J_j(x^{(0)})\,x^{(j)}, \quad (j = 1,\ldots,\ell), \end{cases}$$

that means $y_k = \sum_{p\in\hat{I}_k}\frac{\partial\varphi_k}{\partial x_p}(x^{(0)},0)\,x_p$ for all $k \in \hat{I}_j$. Concerning the variable $\xi^{(0)}$, as in the proof of Proposition 3.1 we get

$$\xi^{(0)} = {}^{\mathrm{t}}[\frac{\partial\varphi^{(0)}}{\partial x^{(0)}}(t_{\hat{J}}x)]\eta^{(0)} + \sum_{i=1}^{\ell}{}^{\mathrm{t}}[\frac{\partial\varphi^{(i)}}{\partial x^{(0)}}(t_{\hat{J}}x)]\frac{t_{\hat{J}_i^c}}{t_{\hat{J}_0^c}}\,\eta^{(i)}.$$

Let $M_i = \{x_k = 0;\ k \in I_i\}$ and $I_i = \hat{I}_{j_1} \sqcup \cdots \sqcup \hat{I}_{j_p}$. By expanding ${}^{\mathrm{t}}[\frac{\partial\varphi^{(i)}}{\partial x^{(0)}}(t_{\hat{J}}x)]$ along the submanifold $M_i$, we obtain

$$\begin{aligned} {}^{\mathrm{t}}[\frac{\partial\varphi^{(i)}}{\partial x^{(0)}}(t_{\hat{J}}x)]\frac{t_{\hat{J}_i^c}}{t_{\hat{J}_0^c}}\,\eta^{(i)} &= {}^{\mathrm{t}}[\frac{\partial\varphi^{(i)}}{\partial x^{(0)}}(t_{\hat{J}}x)]\Big|_{M_i}\frac{t_{\hat{J}_i^c}}{t_{\hat{J}_0^c}}\eta^{(i)} \\ &\quad + \sum_{k\in I_i} x_k\,{}^{\mathrm{t}}[\frac{\partial^2\varphi^{(i)}}{\partial x_k\partial x^{(0)}}(t_{\hat{J}}x)]\Big|_{M_i}\frac{t_{\hat{J}_i^c}t_{\hat{J}_k}}{t_{\hat{J}_0^c}}\eta^{(i)} + \cdots \end{aligned}$$

Since $\varphi^{(i)}(t_{\hat{J}}x)\big|_{M_i} = 0$ and $\hat{I}_0 \cap I_i = \varnothing$ we have $\frac{\partial\varphi^{(i)}}{\partial x^{(0)}}(t_{\hat{J}}x)\Big|_{M_i} = 0$. Moreover

$$\begin{cases} \dfrac{t_{\hat{J}_i^c} t_{\hat{J}_i}}{t_{\hat{J}_0^c}} = 1, \\ \dfrac{t_{\hat{J}_i^c} t_{\hat{J}_k}}{t_{\hat{J}_0^c}} \to 0, \quad (k \neq i), \end{cases}$$

when $t \to 0$. This is because $\hat{J}_k \subsetneqq \hat{J}_i$ when $k \in \{j_1, \dots, j_p\}$, $k \neq i$ by Lemma 1.1 and (1.7). In a similar way the higher order terms vanish when $t \to 0$. Hence on $S_{\chi *}$

$$\begin{aligned} \xi^{(0)} &= {}^{\mathrm{t}}[\frac{\partial\varphi^{(0)}}{\partial x^{(0)}}(x^{(0)}, 0)]\,\eta^{(0)} + \sum_{i=1}^{\ell} {}^{\mathrm{t}}x^{(i)}\,{}^{\mathrm{t}}[\frac{\partial^2\varphi^{(i)}}{\partial x^{(i)}\partial x^{(0)}}(x^{(0)}, 0)]\eta^{(i)} \\ &= {}^{\mathrm{t}}J_0(x^{(0)})\,\eta^{(0)} + \sum_{i=1}^{\ell} {}^{\mathrm{t}}x^{(i)}\,\frac{\partial\,{}^{\mathrm{t}}J_i}{\partial x^{(0)}}(x^{(0)})\,\eta^{(i)}. \end{aligned}$$

Concerning the variable $\xi^{(j)}$ $(j \neq 0)$, we get

$$\xi^{(j)} = \sum_{i=0}^{\ell} {}^{\mathrm{t}}[\frac{\partial\varphi^{(i)}}{\partial x^{(j)}}(t_{\hat{J}}x)]\,\frac{t_{\hat{J}_i^c}}{t_{\hat{J}_j^c}}\,\eta^{(i)}.$$

(i) If $\hat{J}_j^c \subsetneqq \hat{J}_i^c \Leftrightarrow \hat{J}_i \subsetneqq \hat{J}_j$ we have $\frac{t_{\hat{J}_i^c}}{t_{\hat{J}_j^c}} \to 0$ $(t \to 0)$.

(ii) If $\hat{J}_i \supsetneqq \hat{J}_j$ or $\hat{J}_i \cap \hat{J}_j = \varnothing$, we have $\hat{I}_j \cap I_i = \varnothing$.

By expanding $\frac{\partial\varphi^{(i)}}{\partial x^{(j)}}(t_{\hat{J}}x)$ along the submanifold $M_i$, we obtain

$$\begin{aligned} {}^{\mathrm{t}}[\frac{\partial\varphi^{(i)}}{\partial x^{(j)}}(t_{\hat{J}}x)]\,\frac{t_{\hat{J}_i^c}}{t_{\hat{J}_j^c}}\eta^{(i)} &= {}^{\mathrm{t}}[\frac{\partial\varphi^{(i)}}{\partial x^{(j)}}(t_{\hat{J}}x)]\Big|_{M_i}\,\frac{t_{\hat{J}_i^c}}{t_{\hat{J}_j^c}}\,\eta^{(i)} \\ &\quad + \sum_{k \in I_i} x_k\,{}^{\mathrm{t}}[\frac{\partial^2\varphi^{(i)}}{\partial x_k \partial x^{(j)}}(t_{\hat{J}}x)]\Big|_{M_i}\,\frac{t_{\hat{J}_i^c} t_{\hat{J}_k}}{t_{\hat{J}_j^c}}\eta^{(i)} + \cdots \end{aligned}$$

Since $\varphi^{(i)}(t_{\hat{J}}x)\big|_{M_i} = 0$ and $\hat{I}_j \cap I_i = \varnothing$ we have $\frac{\partial\varphi^{(i)}}{\partial x^{(j)}}(t_{\hat{J}}x)\Big|_{M_i} = 0$. Moreover

$$\frac{t_{\hat{J}_i^c} t_{\hat{J}_k}}{t_{\hat{J}_j^c}} \to 0$$

when $t \to 0$ since $\hat{I}_k \subseteq I_i \Rightarrow \hat{J}_k \supseteq \hat{J}_i$ by Lemma 1.1 and $\hat{J}_i^c \cup \hat{J}_k = \{1, \dots, \ell\} \supsetneqq \hat{J}_j^c$ when $j \neq 0$. Hence on $S_{\chi *}$

$$\xi^{(j)} = {}^{\mathrm{t}}[\frac{\partial\varphi^{(j)}}{\partial x^{(j)}}(x^{(0)}, 0)]\,\eta^{(j)} = {}^{\mathrm{t}}J_j(x^{(0)})\,\eta^{(j)}.$$

Summarizing, the coordinate transformation on $S_{\chi^*}$ is given by

$$\begin{cases} y^{(0)} = \varphi^{(0)}(x^{(0)}, 0), \\ y^{(i)} = J_i(x^{(0)})\, x^{(i)} \quad (1 \leqslant i \leqslant \ell), \end{cases}$$

$$\begin{cases} \xi^{(0)} = {}^{\mathrm{t}}J_0(x^{(0)})\, \eta^{(0)} + \displaystyle\sum_{i=1}^{\ell} {}^{\mathrm{t}}x^{(i)} \frac{\partial\, {}^{\mathrm{t}}J_i}{\partial x^{(0)}}(x^{(0)})\, \eta^{(i)}, \\ \xi^{(i)} = {}^{\mathrm{t}}J_i(x^{(0)})\, \eta^{(i)} \quad (1 \leqslant i \leqslant \ell). \end{cases}$$

This is nothing but (3.1), (3.2). □

EXAMPLE 3.4. — Let $X = \mathbb{C}^2$ with coordinates $(z_1, z_2)$ and consider $T^*X$ with coordinates $(z;\, \eta) = (z_1, z_2;\, \eta_1, \eta_2)$. Set $t = (t_1, t_2) \in (\mathbb{R}^+)^2$.

1. (Majima) Let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X\}$ and we have a map

$$\begin{aligned} \widetilde{T^*X} &\to T^*X, \\ (z;\, \eta;\, t) &\mapsto (\mu_z(z;\, t);\, \mu_\eta(\eta;\, t)), \end{aligned}$$

which is defined by

$$\begin{aligned} \mu_z(z;\, t) &= (t_1 z_1, t_2 z_2), \\ \mu_\eta(\eta;\, t) &= (t_2 \eta_1, t_1 \eta_2). \end{aligned}$$

By Theorem 3.3 we have $S^*_\chi \simeq T^*(T_{M_1}X \underset{X}{\times} T_{M_2}X) \simeq T^*(T^*_{M_1}X \underset{X}{\times} T^*_{M_2}X)$.

2. (Takeuchi) Let $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X\}$ and we have a map

$$\begin{aligned} \widetilde{T^*X} &\to T^*X, \\ (z;\, \eta;\, t) &\mapsto (\mu_z(z;\, t);\, \mu_\eta(\eta;\, t)), \end{aligned}$$

which is defined by

$$\begin{aligned} \mu_z(z;\, t) &= (t_1 z_1, t_1 t_2 z_2), \\ \mu_\eta(\eta;\, t) &= (t_2 \eta_1, \eta_2). \end{aligned}$$

By Theorem 3.3 we have $S^*_\chi \simeq T^*(T_{M_1}M_2 \underset{X}{\times} T_{M_2}X) \simeq T^*(T^*_{M_1}M_2 \underset{X}{\times} T^*_{M_2}X)$.

EXAMPLE 3.5. — Let $X = \mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$ and consider $T^*X$ with coordinates $(x;\, \xi) = (x_1, x_2, x_3;\, \xi_1, \xi_2, \xi_3)$. Set $t = (t_1, t_2, t_3) \in (\mathbb{R}^+)^3$.

1. (Majima) Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X, T^*_{M_3}X\}$ and we have a map

$$\begin{aligned} \widetilde{T^*X} &\to T^*X, \\ (x;\, \xi;\, t) &\mapsto (\mu_x(x;\, t);\, \mu_\xi(\xi;\, t)), \end{aligned}$$

which is defined by

$$\begin{aligned} \mu_x(x;\, t) &= (t_1x_1, t_2x_2, t_3x_3), \\ \mu_\xi(\xi;\, t) &= (t_2t_3\xi_1, t_1t_3\xi_2, t_2t_3\xi_3). \end{aligned}$$

By Theorem 3.3 we have $S^*_\chi \simeq T^*(T_{M_1}X \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_2}X) \simeq T^*(T^*_{M_1}X \underset{X}{\times} T^*_{M_2}X \underset{X}{\times} T^*_{M_3}X)$.

2. (Takeuchi) Let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X, T^*_{M_3}X\}$ and we have a map

$$\begin{aligned} \widetilde{T^*X} &\to T^*X, \\ (x;\, \xi;\, t) &\mapsto (\mu_x(x;\, t);\, \mu_\xi(\xi;\, t)), \end{aligned}$$

which is defined by

$$\begin{aligned} \mu_x(x;\, t) &= (t_1x_1, t_1t_2x_2, t_1t_2t_3x_3), \\ \mu_\xi(\xi;\, t) &= (t_2t_3\xi_1, t_3\xi_2, \xi_3). \end{aligned}$$

By Theorem 3.3 we have $S^*_\chi \simeq T^*(T_{M_1}M_2 \underset{X}{\times} T_{M_2}M_3 \underset{X}{\times} T_{M_2}X) \simeq T^*(T^*_{M_1}M_2 \underset{X}{\times} T^*_{M_2}M_3 \underset{X}{\times} T^*_{M_3}X)$.

3. (Mixed) Let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X, T^*_{M_3}X\}$ and we have a map

$$\begin{aligned} \widetilde{T^*X} &\to T^*X, \\ (x;\, \xi;\, t) &\mapsto (\mu_x(x;\, t);\, \mu_\xi(\xi;\, t)), \end{aligned}$$

which is defined by

$$\begin{aligned} \mu_x(x;\, t) &= (t_1x_1, t_1t_2x_2, t_1t_3x_3), \\ \mu_\xi(\xi;\, t) &= (t_2t_3\xi_1, t_3\xi_2, t_2\xi_3). \end{aligned}$$

By Theorem 3.3 we have $S^*_\chi \simeq T^*(T_{M_1}(M_2 \cap M_3) \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_2}X) \simeq T^*(T^*_{M_1}(M_2 \cap M_3) \underset{X}{\times} T^*_{M_2}X \underset{X}{\times} T^*_{M_3}X)$.

**3.2. Estimate of microsupport.** — In this section we shall prove an estimate for the microsupport of the multi-specialization and multi-microlocalization of a sheaf on $X$. We refer to [4] for the theory of microsupport of sheaves.

THEOREM 3.6. — *Let $F \in D^b(k_X)$. Then*

$$\mathrm{SS}(\nu_\chi(F)) = \mathrm{SS}(\mu_\chi(F)) \subseteq C_{\chi^*}(\mathrm{SS}(F)).$$

Since the problem is local, we may assume that $X = \mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$. Let $I_k$ ($k = 1, 2, \ldots, \ell$) in $\{1, 2, \ldots, n\}$ such that each submanifold $M_k$ is given by

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ x_i = 0\, (i \in I_k)\}\,.$$

Then Theorem 3.6 follows from Lemma 1.4 and the following theorem:

THEOREM 3.7. — *Let $F \in D^b(k_X)$ and take a point*

$$p_0 = (x_0^{(0)}, x_0^{(1)}, \ldots, x_0^{(\ell)};\ \xi_0^{(0)}, \xi_0^{(1)}, \ldots, \xi_0^{(\ell)}) \in T^* S_\chi.$$

*Assume that $p_0 \in \mathrm{SS}(\nu_\chi(F))$. Then there exist sequences*

$$\{(c_{1,k}, \ldots, c_{\ell,k})\}_{k=1}^{\infty} \subset (\mathbb{R}^+)^\ell,$$
$$\{(x_k^{(0)}, x_k^{(1)}, \ldots, x_k^{(\ell)};\ \xi_k^{(0)}, \xi_k^{(1)}, \ldots, \xi_k^{(\ell)})\}_{k=1}^{\infty} \subset \mathrm{SS}(F),$$

*such that*

$$\begin{cases} \lim\limits_{k\to\infty} c_{j,k} = \infty, \qquad (j = 1, \ldots, \ell), \\ \lim\limits_{k\to\infty} (x_k^{(0)}, x_k^{(1)} c_{\hat{J}_1,k}, \ldots, x_k^{(\ell)} c_{\hat{J}_\ell,k};\ \xi_k^{(0)} c_k, \xi_k^{(1)} c_{\hat{J}_1^c,k}, \ldots, \xi_k^{(\ell)} c_{\hat{J}_\ell^c,k}) \\ \qquad = (x_0^{(0)}, x_0^{(1)}, \ldots, x_0^{(\ell)};\ \xi_0^{(0)}, \xi_0^{(1)}, \ldots, \xi_0^{(\ell)}), \end{cases}$$

*where $c_k := \prod\limits_{j=1}^{\ell} c_{j,k}$, $\hat{J}_j^c := \{1, \ldots, \ell\} \setminus \hat{J}_j$, and $c_{J,k} := \prod\limits_{j\in J} c_{j,k}$ for any $J \subseteq \{1, \ldots, \ell\}$.*

*Proof.* — Let $(x; t) = (x^{(0)}, x^{(1)}, \ldots, x^{(\ell)};\ t_1, \ldots, t_\ell)$ be the coordinates in $\widetilde{X}$. It follows from the estimate of the microsupport of the inverse image of a closed embedding (Lemma 6.2.1 (ii) and Proposition 6.2.4 (iii) of [4]) that there exists a sequence

$$\{(x_k^{(0)}, x_k^{(1)}, \ldots, x_k^{(\ell)};\ t_{1,k}, \ldots, t_{\ell,k};\ \xi_k^{(0)}, \xi_k^{(1)}, \ldots, \xi_k^{(\ell)};\ \tau_{1,k}, \ldots, \tau_{\ell,k})\}_{k=1}^{\infty}$$

in $\mathrm{SS}(Rj_{\Omega*}\widetilde{p}^{-1}F)$ such that for any $j = 1, \ldots, \ell$

$$\begin{cases} \lim\limits_{k\to\infty} x_k^{(j)} = x_0^{(j)}, \quad \lim\limits_{k\to\infty} \xi_k^{(j)} = \xi_0^{(j)}, \quad \lim\limits_{k\to\infty} t_{j,k} = 0, \\ \lim\limits_{k\to\infty} |(t_{1,k}, \ldots, t_{\ell,k})| \cdot |(\tau_{1,k}, \ldots, \tau_{\ell,k})| = 0. \end{cases}$$

By Theorem 6.3.1 of [4] we have

$$\mathrm{SS}(Rj_{\Omega*}\widetilde{p}^{-1}F) \subseteq \mathrm{SS}(\widetilde{p}^{-1}F) \,\widehat{+}\, N^*(\Omega).$$

By Proposition 5.4.5 of [4], we have

$$
\begin{aligned}
\mathrm{SS}(\widetilde{p}^{-1}F) &= \widetilde{p}_d(\widetilde{p}_\pi^{-1}(\mathrm{SS}(F)) \\
&= \{(x^{(0)}, \frac{x^{(1)}}{t_{\hat{J}_1}}, \ldots, \frac{x^{(\ell)}}{t_{\hat{J}_\ell}}; t_1, \ldots, t_\ell; \xi^{(0)}, t_{\hat{J}_1}\xi^{(1)}, \ldots, t_{\hat{J}_\ell}\xi^{(\ell)}; \tau_1, \ldots, \tau_\ell); \\
&\qquad t_j > 0\ (j = 1, \ldots, \ell),\ (x^{(0)}, x^{(1)}, \ldots, x^{(\ell)}; \xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(\ell)}) \in \mathrm{SS}(F)\},
\end{aligned}
$$

where we did not calculate the terms in the variables $\tau_j$ $(j = 1, \ldots, \ell)$ since we are not going to use them. Thanks to Remark 6.2.8 (ii) of [4] and the fact that $N^*(\Omega) \subset \{(x; t; 0; \tau)\}$, for each $k \in \mathbb{N}$ we get sequences

$$
\begin{aligned}
\{(x^{(0)}_{k,m}, x^{(1)}_{k,m}, \ldots, x^{(\ell)}_{k,m}; \xi^{(0)}_{k,m}, \xi^{(1)}_{k,m}, \ldots, \xi^{(\ell)}_{k,m})\}_{m=1}^{\infty} &\subset \mathrm{SS}(F), \\
\{(t_{1,k,m}, \ldots, t_{\ell,k,m})\} &\subset (\mathbb{R}^+)^\ell,
\end{aligned}
$$

such that

$$
\begin{aligned}
&\lim_{m\to\infty} (x^{(0)}_{k,m}, \frac{x^{(1)}_{k,m}}{t_{\hat{J}_1,k,m}}, \ldots, \frac{x^{(\ell)}_{k,m}}{t_{\hat{J}_\ell,k,m}}; t_{1,k,m}, \ldots, t_{\ell,k,m}) \\
&\qquad = (x^{(0)}_k, x^{(1)}_k, \ldots, x^{(\ell)}_k; t_{1,k}, \ldots, t_{\ell,k}), \\
&\lim_{m\to\infty} (\xi^{(0)}_{k,m}, t_{\hat{J}_1,k,m}\xi^{(1)}_{k,m}, \ldots, t_{\hat{J}_\ell,k,m}\xi^{(\ell)}_{k,m}) = (\xi^{(0)}_k, \xi^{(1)}_k, \ldots, \xi^{(\ell)}_k).
\end{aligned}
$$

Then extracting a subsequence, we can find

$$
\{(x^{(0)}_k, x^{(1)}_k, \ldots, x^{(\ell)}_k; \xi^{(0)}_k, \xi^{(1)}_k, \ldots, \xi^{(\ell)}_k)\}_{k=1}^{\infty} \subset \mathrm{SS}(F)
$$

and $\{(t_{1,k}, \ldots, t_{\ell,k})\}_{k=1}^{\infty} \subset (\mathbb{R}^+)^\ell$ such that

$$
\begin{cases}
\lim_{k\to\infty} (x^{(0)}_k, \frac{x^{(1)}_k}{t_{\hat{J}_1,k}}, \ldots, \frac{x^{(\ell)}_k}{t_{\hat{J}_\ell,k}}) = (x^{(0)}, x^{(1)}, \ldots, x^{(\ell)}), \\
\lim_{k\to\infty} (\xi^{(0)}_k, t_{\hat{J}_1,k}\xi^{(1)}_k, \ldots, t_{\hat{J}_\ell,k}\xi^{(\ell)}_k) = (\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(\ell)}), \\
\lim_{k\to\infty} (t_{1,k}, \ldots, t_{\ell,k}) = (0, \ldots, 0).
\end{cases}
$$

Since $\mathrm{SS}(F)$ is conic, we have

$$
(x^{(0)}_k, x^{(1)}_k, \ldots, x^{(\ell)}_k; t_k\xi^{(0)}_k, t_k\xi^{(1)}_k, \ldots, t_k\xi^{(\ell)}_k) \in \mathrm{SS}(F),
$$

where $t_k := \prod_{j=1}^{\ell} t_{j,k}$. Setting $c_{j,k} := \frac{1}{t_{j,k}}$ $(j = 1, \ldots, \ell)$ we obtain the desired result. □

Remark 3.8. — Theorem 3.6 extends the estimate of microsupport computed in [6].

## 4. Applications to $\mathscr{D}$-modules

In this section, we consider applications of multi-microlocalizations to $\mathscr{D}$-module theory.

**4.1. Uchida's Triangle.** — First, recall the notations of § 2.1; for example, let $\tau_i\colon E_i \to Z$ ($1 \leqslant i \leqslant \ell$) be vector bundles over $Z$, and let $E_i^*$ be the dual bundle of $E_i$. Set $E := E_1 \underset{Z}{\times} \cdots \underset{Z}{\times} E_\ell$. Let $\tau\colon E \to Z$ be the canonical projection. We set $\dot{E} := E \backslash Z$. Similar notations shall be adopted for $E^*$.

Theorem 4.1 (cf. [12]). — *Let $F$ be a multi-conic object on $E$. Then there exists a natural isomorphism*

$$\tau^! R\tau_! F \xrightarrow{\sim} Rp_{1*}\, p_2^!(F^{\wedge_E}),$$

*and the natural morphism $F \to \tau^! R\tau_!\, F$ is embedded to the following distinguished triangle*:

$$F \to \tau^! R\tau_! F \to Rp_{1*}^{+} p_2^{+!}(F^{\wedge_E}) \xrightarrow{+1} .$$

*Proof.* — If $\ell = 1$, the result follows by Lemma A.2 of [12]. Assume $\ell > 1$, and set $E' := \times_{Z,i=2}^{\ell} E_i$, $E'^* := \times_{Z,i=2}^{\ell} E_i^*$, and $P'_{E'} := \times_{Z,i=2}^{\ell} P'_i$. Moreover, let $\wedge_{E'}$ (resp $\vee_{E'}^*$) be the composition of $\wedge_i$ (resp. $\vee_i^*$) for $i = 2, \ldots, \ell$. Consider:

$$\begin{tikzcd}
 & E \underset{Z}{\times} E^* \arrow[dl, bend right, "p_1"'] \arrow[r, "p_{E',2}"'] \arrow[d, "p_{1,1}"] \arrow[rr, bend left, "p_2"] & E_1 \underset{Z}{\times} E^* \arrow[r, "p_{1,2}"'] \arrow[d, "p_{1,1}"] & E^* \\
E & E_1 \underset{Z}{\times} E' \underset{Z}{\times} E'^* \arrow[l, "p_{E',1}"] \arrow[r, "p_{E',2}"'] \arrow[ur, phantom, "\square"] & E_1 \underset{Z}{\times} E'^*. &
\end{tikzcd}$$

By the commutative diagram

$$\begin{tikzcd}
E \arrow[r, "\tau_1"] \arrow[d, "\tau_{E'}"] \arrow[dr, phantom, "\square"] & E' \arrow[d, "\tau_{E'}"] \\
E_1 \arrow[r, "\tau_1"'] & Z
\end{tikzcd}$$

we have

$$\begin{aligned}
\tau_{E'}^!\, R\tau_{E'!}\, \tau_1^!\, R\tau_{1!}\, F &= \tau_{E'}^!\, R\tau_{E'!}\, \tau_1^{-1} R\tau_{1!}\, F \otimes \omega_{E/Z} \\
&= \tau_{E'}^!\, \tau_1^{-1} R\tau_{E'!}\, R\tau_{1!}\, F \otimes \omega_{E/Z} = \tau_{E'}^!\, \tau_1^! R\tau_{E'!}\, R\tau_{1!}\, F = \tau^! R\tau_!\, F.
\end{aligned}$$

Hence, by Lemma A.2 of [12] and induction hypothesis, we obtain:

$$
\begin{tikzcd}
F \arrow[r, "\sim"] \arrow[d] & F^{\wedge_{E'}\vee^{*}_{E'}} \arrow[r, "\sim"] & F^{\wedge_{E'}\vee^{*}_{E'}\wedge_1\vee^{*}_1} \arrow[d] \\
\tau^{!}_{E'}R\tau_{E'!}F \arrow[rr, "\sim"] \arrow[d] & & Rp_{E',1*}\,p^{!}_{E',2}(F^{\wedge_{E'}}) \arrow[d] \\
\tau^{!}_{1}R\tau_{1!}\tau^{!}_{E'}R\tau_{E'!}F \arrow[rr, "\sim"] \arrow[d, equal] & & \tau^{!}_{1}R\tau_{1!}Rp_{E',1*}\,p^{!}_{E',2}(F^{\wedge_{E'}}) \arrow[d, "\wr"] \\
\tau^{!}R\tau_{!}F \arrow[rr, "\sim"] & & Rp_{1,1*}\,p^{!}_{1,2}((Rp_{E',1*}\,p^{!}_{E',2}(F^{\wedge_{E'}}))^{\wedge_1}).
\end{tikzcd}
\tag{4.1}
$$

For the same reasoning as in (2.1), for any multi-conic object $H$ on $E_1 \underset{Z}{\times} E'^{*}$ we have

$$H^{\vee^{*}_{i}\vee_1} = H^{\vee_1\vee^{*}_{i}}.$$

Therefore, we have

$$H^{\vee^{*}_{i}\wedge_1} = (H^{\vee^{*}_{i}\vee_1})^{\mathrm{id}\times a}\otimes\omega^{\otimes -1}_{E_1/Z} = (H^{\vee_1\vee^{*}_{i}})^{\mathrm{id}\times a}\otimes\omega^{\otimes -1}_{E_1/Z} = H^{\wedge_1\vee^{*}_{i}}. \tag{4.2}$$

Thus, by Proposition 2.2 (2) and (4.2) we have

$$F^{\wedge_{E'}\vee^{*}_{E'}\wedge_1\vee^{*}_1} = Rp'_{1*}p'^{!}_{2}(F^{\wedge_E}). \tag{4.3}$$

Lastly, by Proposition 3.7.13 of [4] we have

$$
\begin{aligned}
&Rp_{1,1*}\,p^{!}_{1,2}((Rp_{E',1*}\,p^{!}_{E',2}(F^{\wedge_{E'}}))^{\wedge_1}) \\
&\qquad = Rp_{1,1*}\,p^{!}_{1,2}\,Rp_{E',1*}((p^{!}_{E',2}(F^{\wedge_{E'}}))^{\wedge_1}) \\
&\qquad = Rp_{1,1*}\,p^{!}_{1,2}\,Rp_{E',1*}\,p^{!}_{E',2}(F^{\wedge_{E'}\wedge_1}) = Rp_{1*}p^{!}_{2}(F^{\wedge_E}).
\end{aligned}
\tag{4.4}
$$

Therefore, by (4.1), (4.3), (4.4) and the definition of $P^{+}$, we have

$$
\begin{tikzcd}
F \arrow[r] \arrow[d, "\wr"] & \tau^{!}\,R\tau_{!}F \arrow[d, "\wr"] & \\
F^{\wedge_{E'}\vee^{*}_{E'}\wedge_1\vee^{*}_1} \arrow[r] \arrow[d, dash, "\wr"] & Rp_{1,1*}\,p^{!}_{1,2}((Rp_{E',1*}\,p^{!}_{E',2}(F^{\wedge_{E'}}))^{\wedge_1}) \arrow[d, dash, "\wr"] & \\
Rp'_{1*}\,p'^{!}_{2}(F^{\wedge_E}) \arrow[r] & Rp_{1*}\,p^{!}_{2}(F^{\wedge_E}) \arrow[r] & Rp^{+}_{1*}\,p^{+\,!}_{2}(F^{\wedge_E}) \xrightarrow{+1}.
\end{tikzcd}
$$

The commutativity follows from the constructions. □

Therefore, we obtain the following:

Theorem 4.2. — *Let $X$ be a real analytic manifold, and assume that the family $\chi = \{M_i\}_{i=1}^{\ell}$ of submanifolds in $X$ satisfies conditions* H1, H2 *and* H3. *Set $M := \bigcap_{i=1}^{\ell} M_i$. Then for any $F \in D^b(k_{X_{\mathrm{sa}}})$, there exists the following distinguished triangle*:

$$\nu_\chi(F) \to \tau^{-1} R\Gamma_M(F) \otimes \omega_{M/X}^{\otimes -1} \to Rp_{1*}^{+}(p_2^{+})^{-1} \mu_\chi(F) \otimes \omega_{M/X}^{\otimes -1} \xrightarrow{+1}. \tag{4.5}$$

*Proof.* — By the definition of multi-microlocalization and Theorem 4.1 we have (see also Remark 2.1)

$$\nu_\chi(F) \to \tau^! R\tau_! \nu_\chi(F) \to Rp_{1*}^{+} p_2^{+!} \mu_\chi(F) \xrightarrow{+1}.$$

Since $\tau$ and $p_2^{+}$ are projections, we have

$$\tau^! \simeq \tau^{-1} \otimes \omega_{S_\chi/M} \simeq \tau^{-1} \otimes \omega_{M/X}^{\otimes -1},$$
$$p_2^{+!} \simeq (p_2^{+})^{-1} \otimes \omega_{P^+/M} \simeq (p_2^{+})^{-1} \otimes \omega_{M/X}^{\otimes -1}.$$

Hence we prove the theorem. □

By Theorem 3.6, under the identifications $T^*S_\chi^* = T^*S_\chi = S_{\chi^*}$, we have

$$\mathrm{SS}(\mu_\chi(F)) = \mathrm{SS}(\nu_\chi(F)) \subset C_{\chi^*}(\mathrm{SS}(F)).$$

In particular we obtain

$$\operatorname{supp} \mu_\chi(F) \subset S_\chi^* \cap C_{\chi^*}(\mathrm{SS}(F)). \tag{4.6}$$

Thus we obtain:

Corollary 4.3. — *If $\dot{S}_\chi^* \cap C_{\chi^*}(\mathrm{SS}(F)) = \varnothing$, then*

$$\nu_\chi(F) \xrightarrow{\sim} \tau^{-1} R\Gamma_M(F) \otimes \omega_{M/X}^{\otimes -1}.$$

**4.2. Solutions of $\mathscr{D}$-modules in complex domains.** — Let $\chi = \{Y_i\}_{i=1}^{\ell}$, and assume that each $Y_i$ and $Y := \bigcap_{i=1}^{\ell} Y_i$ are complex submanifolds of $X$. As usual, let $\mathscr{D}_X$ be the sheaf of holomorphic differential operators on $X$. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, and $\mathrm{Ch}\,\mathscr{M}$ the characteristic variety of $\mathscr{M}$. Then, for $F = R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$, it is known that $\mathrm{SS}(F) = \mathrm{Ch}\,\mathscr{M}$. From (4.5), we have

$$\begin{aligned} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_\chi(\mathscr{O}_X)) &\to \tau^{-1} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, R\Gamma_Y(\mathscr{O}_X)) \otimes \omega_{Y/X}^{\otimes -1} \\ &\to Rp_{1*}^{+}(p_2^{+})^{-1} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_\chi(\mathscr{O}_X)) \otimes \omega_{Y/X}^{\otimes -1} \xrightarrow{+1}. \end{aligned}$$

Let $f : Y \hookrightarrow X$ be the canonical embedding. We can define the following natural mappings associated with $f$:

$$T^*Y \xleftarrow{f_d} Y \underset{X}{\times} T^*X \xrightarrow{f_\pi} T^*X.$$

We define the *inverse image* of $\mathscr{M}$ by

$$\boldsymbol{D}f^*\mathscr{M} := \mathscr{O}_Y \underset{f^{-1}\mathscr{O}_X}{\overset{L}{\otimes}} f^{-1}\mathscr{M}.$$

Assume that $Y$ is non-characteristic for $\mathscr{M}$; that is, $T^*_YX \cap \mathrm{Ch}\,\mathscr{M} \subset T^*_XX$. Then, it is known that $\boldsymbol{D}f^*\mathscr{M}$ is identified with $Df^*\mathscr{M} := H^0\boldsymbol{D}f^*\mathscr{M}$, and $Df^*\mathscr{M}$ is a coherent $\mathscr{D}_Y$-module.

THEOREM 4.4. — *Assume that $Y$ is non-characteristic for $\mathscr{M}$. Then*

$$\begin{aligned} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_\chi(\mathscr{O}_X)) &\xrightarrow{\sim} \tau^{-1}R\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M}, \mathscr{O}_Y) \\ &\simeq \tau^{-1}f^{-1}R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X). \end{aligned}$$

*Proof.* — By the non-characteristic condition and Cauchy-Kovalevskaya-Kashiwara theorem, we obtain the following isomorphisms:

$$\begin{aligned} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, R\Gamma_Y(\mathscr{O}_X)) \otimes \omega_{Y/X}^{\otimes -1} &\simeq R\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M}, \mathscr{O}_Y) \\ &\simeq f^{-1}R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X). \end{aligned}$$

Hence by Corollary 4.3, it is enough to show $\dot{S}^*_\chi \cap C_{\chi^*}(\mathrm{Ch}\,\mathscr{M}) = \varnothing$. Assume that there exists a point

$$(x^{(0)};\, \xi^{(1)}, \dots, \xi^{(\ell)}) \in \dot{S}^*_\chi \cap C_{\chi^*}(\mathrm{Ch}\,\mathscr{M}).$$

Then by Theorem 3.7 there exist sequences

$$\begin{aligned} \{(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}; \xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)})\}_{n=1}^\infty &\subset \mathrm{Ch}\,\mathscr{M}, \\ \{(c_n^{(1)}, \dots, c_n^{(\ell)})\}_{n=1}^\infty &\subset (\mathbb{R}^+)^\ell, \end{aligned}$$

such that:

$$\begin{cases} \lim\limits_{n\to\infty} c_n^{(j)} = \infty, \qquad (j = 1, \dots, \ell), \\ \lim\limits_{n\to\infty} (x_n^{(0)}, x_n^{(1)}c_n^{(\hat{J}_1)}, \dots, x_n^{(\ell)}c_n^{(\hat{J}_\ell)}; \xi_n^{(0)}c_n, \xi_n^{(1)}c_n^{(\hat{J}_1^c)}, \dots, \xi_n^{(\ell)}c_n^{(\hat{J}_\ell^c)}) \\ \qquad = (x^{(0)}, 0, \dots, 0; 0, \xi^{(1)}, \dots, \xi^{(\ell)}), \end{cases}$$

where $c_n := \prod_{j=1}^\ell c_n^{(j)}$, $\hat{J}_j^c := \{1, \dots, \ell\} \smallsetminus \hat{J}_j$, and $c_n^{(J)} := \prod_{j\in J} c_n^{(j)}$ for any $J \subseteq \{1, \dots, \ell\}$. In particular we have $\lim\limits_{n\to\infty}(x_n^{(1)}, \dots, x_n^{(\ell)}, \xi_n^{(0)}c_n) = (0, \dots, 0, 0)$. Since $(\xi^{(1)}, \dots, \xi^{(\ell)}) \neq 0$, we may assume that

$$t_n := |(\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)})| > 0.$$

We consider the sequence $\{(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}; t_n^{-1}(\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}))\}_{n=1}^\infty \subset \mathrm{Ch}\,\mathscr{M}$. By extracting a subsequence, we may assume that there exists $\zeta_0 \neq 0$ such that

$$\lim_{n\to\infty} t_n^{-1}(\xi^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}) = \zeta_0\,.$$

Choose $1 \leqslant k \leqslant \ell$ with $\lim_{n\to\infty} \xi_n^{(k)} c_n^{(\hat{J}_k^c)} = \xi^{(k)} \neq 0$. Since $\lim_{n\to\infty} c_n^{(j)} = \infty$, we have

$$\lim_{n\to\infty} \frac{\xi_n^{(0)}}{\xi_n^{(k)}} = \lim_{n\to\infty} \frac{\xi_n^{(0)} c_n}{\xi_n^{(k)} c_n^{(\hat{J}_k^c)} c_n^{(\hat{J}_k)}} = 0.$$

Therefore, we see that $\zeta_0 = (0, \zeta_{01}, \ldots, \zeta_{0\ell}) \neq 0$. Hence

$$\begin{aligned}\lim_{n\to\infty} &(x_n^{(0)}, x_n^{(1)}, \ldots, x_n^{(\ell)}; t_n^{-1}(\xi^{(0)}, \xi_n^{(1)}, \ldots, \xi_n^{(\ell)})) \\ &= (x^{(0)}, 0, \ldots, 0; \zeta_0) \in \dot{T}_Y^* X \cap \operatorname{Ch} \mathcal{M},\end{aligned}$$

which contradicts the non-characteristic condition. Thus $\dot{S}_\chi^* \cap C_{\chi^*}(\operatorname{Ch}\mathcal{M}) = \varnothing$. Hence we obtain the desired result. $\square$

Example 4.5. — Take a point $p = (q; \xi) \in \underset{X, 1\leqslant j\leqslant \ell}{\times} T_{Y_j}\iota(Y_j)$, and set $x_0 := \tau(p) \in Y$. Recall the notation in Remark 1.12. Then under the assumption of Theorem 4.4, every solution to $\mathcal{M}$ defined on $\operatorname{Cone}_\chi(p, \epsilon)$ for some $\epsilon > 0$ extends automatically a solution defined on a full neighborhood of $x_0$.

**4.3. Solutions of $\mathcal{D}$-modules in real domains. —** Next, we consider the real cases.

Example 4.6. — Let us take $\chi = \{N, M\}$ with $N \subset M \subset X$ (Takeuchi's case). Then $S_\chi = T_N M \underset{M}{\times} T_M X$ and $S_\chi^* = T_N^* M \underset{M}{\times} T_M^* X \simeq T^*_{N \underset{M}{\times} T_M^* X} T_M^* X$. Under the notation of [11], Takeuchi defines functors

$$\begin{aligned}\nu_{NM} &:= \nu_\chi \colon D^b(k_X) \to D^b(k_{T_N M \underset{M}{\times} T_M X}),\\ \nu\mu_{NM} &:= \nu_\chi^{\wedge_2} \colon D^b(k_X) \to D^b(k_{T_N M \underset{M}{\times} T_M^* X}),\\ \mu_{NM} &:= \mu_\chi \colon D^b(k_X) \to D^b(k_{T_N^* M \underset{M}{\times} T_M^* X}).\end{aligned}$$

Let $M$ be a real analytic manifold, and $\chi = \{N_i\}_{i=1}^{\ell} \subset M$. Assume that each $N_i$ and $N := \bigcap_{i=1}^{\ell} N_i$ are real analytic submanifolds of $M$. We consider the multi-normal deformation $\widetilde{M}_\chi$ along $\chi$. Let $X$ be the complexification of $M$, and $Y$ the complexification of $N$ in $X$. Let $\iota\colon M \hookrightarrow X$ the canonical embedding. Let $\mathcal{B}_M$ be the sheaf of hyperfunctions on $M$. Then by (4.5) we obtain

$$\begin{aligned}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{B}_M)) &\to \tau^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1}\\ &\to Rp_{1*}^{+}(p_2^{+})^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} \xrightarrow{+1}.\end{aligned}$$

For any conic subset $A \subset T^*X$ we can define $\iota^{\#}(A) := T^*M \cap C_{T^*_M X}(A)$ ([4, Definition 6.2.3]). Note that $(x_0;\xi_0) \in \iota^{\#}(A)$ if and only if there exists a sequence $\{(x_\nu + \sqrt{-1}\,y_\nu;\xi_\nu + \sqrt{-1}\,\eta_\nu)\}_{\nu=1}^{\infty} \subset A$ such that

$$\lim_{\nu\to\infty}(x_\nu + \sqrt{-1}\,y_\nu;\xi_\nu) = (x_0;\xi_0), \quad \lim_{\nu\to\infty}|y_\nu|\,|\eta_\nu| = 0.$$

THEOREM 4.7. — *Assume that $N \hookrightarrow M$ is hyperbolic for $\mathscr{M}$; that is,*

$$\dot{T}^*_N M \cap \iota^{\#}(\operatorname{Ch}\mathscr{M}) = \varnothing. \tag{4.7}$$

*Then*

$$R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_{\chi}(\mathscr{B}_M)) \xrightarrow{\sim} \tau^{-1}R\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M}, \mathscr{B}_N).$$

*Proof.* — By (4.7), we see that $Y$ is non-characteristic for $\mathscr{M}$ on a neighborhood of $N$. By the non-characteristic division theorem, we have

$$R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, R\Gamma_N(\mathscr{B}_M)) \otimes \omega_{N/M}^{\otimes -1} \simeq R\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M}, \mathscr{B}_N).$$

By Corollary 6.4.4 of [4], we have

$$\mathrm{SS}(R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)) \subset \iota^{\#}\,\mathrm{SS}(R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)) = \iota^{\#}(\operatorname{Ch}\mathscr{M}).$$

As in the proof of Theorem 4.4 we have

$$\operatorname{supp}(R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_{\chi}(\mathscr{B}_M))) \cap \dot{S}^*_{\chi} \subset \dot{S}^*_{\chi} \cap C_{\chi^*}(\iota^{\#}(\operatorname{Ch}\mathscr{M})) = \varnothing.$$

and we obtain the desired result. □

As usual, let $\mathscr{E}_X$ be the sheaf of *ring of microdifferential operators* on $T^*X$ and $\{\mathscr{E}_X(m)\}_{m\in\mathbb{Z}}$ the usual *order filtration* on $\mathscr{E}_X$. Let $U$ be a $\mathbb{C}^\times$-conic open subset of $\dot{T}^*X$, and $\Sigma$ a $\mathbb{C}^\times$-conic involutory closed analytic subset of $U$. Set $\mathscr{I}_\Sigma := \{P \in \mathscr{E}_X(1)|_U;\ \sigma_1(P)|_\Sigma \equiv 0\}$ and $\mathscr{E}_\Sigma := \bigcup_{m\in\mathbb{N}_0}\mathscr{I}_\Sigma^m$. Here $\sigma_m(P)$ denotes the *principal symbol* of $P \in \mathscr{E}_X(m)$, and we set $\mathscr{I}_\Sigma^0 := \mathscr{E}_X(0)|_U$. Namely, $\mathscr{E}_\Sigma \subset \mathscr{E}_X|_U$ is a subring generated by $\mathscr{I}_\Sigma$.

DEFINITION 4.8. — (1) Let $U$ be a $\mathbb{C}^\times$-conic open subset of $\dot{T}^*X$, and $\Sigma$ a $\mathbb{C}^\times$-conic involutory closed analytic subset of $U$. Let $\mathfrak{M}$ be a coherent $\mathscr{E}_X$-module defined on $U$.

(a) An $\mathscr{E}_\Sigma$ submodule $\mathfrak{L}$ of $\mathfrak{M}$ is called an *$\mathscr{E}_\Sigma$-lattice* of $\mathfrak{M}$ if $\mathfrak{L}$ is $\mathscr{E}_X(0)$-coherent and $\mathscr{E}_X\mathfrak{L} = \mathfrak{M}$.

(b) We say that $\mathfrak{M}$ *has regular singularities along* $\Sigma$ if for any $p^* \in U$, there exist an open neighborhood $V$ of $p^*$ and an $\mathscr{E}_\Sigma$-lattice $\mathfrak{L} \subset \mathfrak{M}|_V$.

(2) Let $\Sigma$ be a $\mathbb{C}^\times$-conic involutory closed analytic subset of $\dot{T}^*X$, and $\mathscr{M}$ a coherent $\mathscr{D}_X$-module. Then we say that *$\mathscr{M}$ has regular singularities along $\Sigma$* if so does $\mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \pi^{-1}\mathscr{M}$.

We impose the

Condition 4.9. — (1) $\Lambda \subset \dot{T}^*X$ is a $\mathbb{C}^\times$-conic closed regular involutory complex submanifold,

(2) $\mathcal{M}$ has regular singularities along $\Lambda$,

(3) $\dot{T}^*_N M \cap \iota^\#(\Lambda) = \varnothing$.

By Condition 4.9 (2), we have $\operatorname{Ch}\mathcal{M} \subset \Lambda \sqcup \operatorname{supp}\mathcal{M}$. Hence by virtue of Condition 4.9 (3), we see that $Y$ is non-characteristic for $\mathcal{M}$ on neighborhood of $N$. Let $\mathcal{D}b_M$ be the sheaf of distributions on $M$.

Theorem 4.10. — *Assume Condition 4.9. Then*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{D}b_M)) \xrightarrow{\sim} \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{D}b_N).$$

*Proof.* — Consider

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{D}b_M)) &\to \tau^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} \\ &\to Rp^+_{1*}(p^+_2)^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} \xrightarrow{+1}. \end{aligned}$$

Under Condition 4.9, we have

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} \\ &\simeq R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{D}b_N), \end{aligned}$$

and

$$\mathrm{SS}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)) \subset \iota^\#(\Lambda \sqcup \operatorname{supp}\mathcal{M})$$

(see [2]). As in the proof of Theorem 4.4 we have

$$\operatorname{supp}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{D}b_M))) \cap \dot{S}^*_\chi \subset \dot{S}^*_\chi \cap C_{\chi*}(\iota^\#(\Lambda \sqcup \operatorname{supp}\mathcal{M})) = \varnothing,$$

and this entails that $Rp^+_{1*}(p^+_2)^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{D}b_M)) = 0$, and we obtain the desired result. □

Theorem 4.11. — *Assume Condition 4.9. Then*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{C}^\infty_M)) \xrightarrow{\sim} \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{C}^\infty_N).$$

*Proof.* — Consider

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{C}^\infty_M)) &\to \tau^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{C}^\infty_M)) \otimes \omega_{N/M}^{\otimes -1} \\ &\to Rp^+_{1*}(p^+_2)^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{C}^\infty_M)) \otimes \omega_{N/M}^{\otimes -1} \xrightarrow{+1}. \end{aligned}$$

Under Condition 4.9, we have

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{C}^\infty_M)) \otimes \omega_{N/M}^{\otimes -1} &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{W}^\infty_{M,N}) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{C}^\infty_N), \end{aligned}$$

and

$$\mathrm{SS}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^\infty_M)) \subset \iota^\#(\Lambda \sqcup \operatorname{supp}\mathcal{M})$$

Here $\mathscr{W}^{\infty}_{M,N}$ is the sheaf of Whitney functions on $N$ (see [13]). Thus the proof is the same as in Theorem 4.4. $\square$

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