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MONODROMIES AT INFINITY OF NON-TAME POLYNOMIALS

by Kiyoshi Takeuchi & Mihai Tibăr

Abstract. — Polynomials that we usually encounter in mathematics are non-convenient and hence non-tame at infinity. We consider the monodromy at infinity and the monodromies around the bifurcation points of polynomial functions $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ which are non-tame at infinity and might have non-isolated singularities. Our description of their Jordan blocks in terms of the Newton polyhedra and the motivic Milnor fibers relies on two new issues: the non-atypical eigenvalues of the monodromies and the corresponding concentration results for their generalized eigenspaces.

Résumé (*Monodromies à l'infini des polynômes non-modérés*). — Les polynômes qu'on rencontre d'habitude en mathématiques sont généralement non-commodes et donc non-modérés à l'infini. On considère ici la monodromie à l'infini et les monodromies autour les valeurs de bifurcation des fonctions polynômiales $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ qui sont non-modérés à l'infini et peuvent avoir des singularités non-isolées. Notre description de leurs blocs de Jordan en termes des polyèdres de Newton et des fibres de Milnor motiviques s'appuie sur deux nouveaux concepts : les valeurs propres non-atypiques des monodromies et les résultats de concentration pour leurs espaces propres généralisés.

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1. Introduction

For a polynomial map $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$, it is well-known that there exists a finite subset $B \subset \mathbb{C}$ such that the restriction

$$\mathbb{C}^n \setminus f^{-1}(B) \longrightarrow \mathbb{C} \setminus B \tag{1.1}$$

of f is a locally trivial fibration. We denote by B_f the smallest subset $B \subset \mathbb{C}$ satisfying this condition. We call the elements of B_f *bifurcation points of* f. For $f(x) = \sum_{v\in\mathbb{Z}^n_+} a_v x^v$ $(a_v \in \mathbb{C})$ we call the convex hull of $\mathrm{supp} f = \{v \in \mathbb{R}^n_+ \mid a_v \neq 0\}$ in $\mathbb{R}^n$ the Newton polytope of f and denote it by $NP(f)$. After Kushnirenko [11], the convex hull $\Gamma_\infty(f) \subset \mathbb{R}^n_+$ of $\{0\} \cup NP(f)$ in $\mathbb{R}^n$ is called the Newton polyhedron at infinity of f.

DEFINITION 1.1. — We say that f is *convenient* if $\Gamma_\infty(f)$ intersects the positive part of the i-th axis of $\mathbb{R}^n$ for any $1 \leq i \leq n$.

If f is convenient and non-degenerate at infinity (see Definition 2.1), then by a result of Broughton [1] it is *tame at infinity.* In this tame case he proved that one has the concentration

$$H^j(f^{-1}(R);\mathbb{C}) = 0 \quad (j \neq 0, n-1) \tag{1.2}$$

for the generic fiber $f^{-1}(R)$ $(R \gg 0)$ of f. After this fundamental result many mathematicians studied tame polynomials. However, polynomials that we usually encounter in mathematics are non-convenient and hence non-tame at infinity. According to the fundamental result of Némethi and Zaharia [18], they have a lot of singularities at infinity in general. The study of non-tame polynomials is important for the Jacobian Conjecture since non-tame polynomials are the only interesting objects in the problem. Their study would be useful also in the mirror symmetry, where the Landau-Ginzburg potentials may be non-convenient. Moreover, in what concerns the evaluation of the bifurcation set B_f, non-tame polynomials were studied by many mathematicians and with different methods, in particular by Némethi and Zaharia [18], [31] by using Newton polyhedra. The main reason why non-tame polynomials could not be studied precisely before is that one cannot expect to have the concentration (1.2) for them.

In this paper we overcome this difficulty on non-tame polynomials by improving the above-mentioned result of Broughton [1]. Let $C_R = \{x \in \mathbb{C} \mid |x| = R\}$ $(R \gg 0)$ be a sufficiently large circle in $\mathbb{C}$ such that $B_f \subset \{x \in \mathbb{C} \mid |x| < R\}$. Then by restricting the locally trivial fibration $\mathbb{C}^n \setminus f^{-1}(B_f) \longrightarrow \mathbb{C} \setminus B_f$ to C_R we obtain a geometric monodromy automorphism $\Phi^\infty_f\colon f^{-1}(R) \xrightarrow{\sim} f^{-1}(R)$ and the linear maps

$$\Phi^\infty_j\colon H^j(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R);\mathbb{C}) \quad (j = 0, 1, \ldots) \tag{1.3}$$

associated to it, where the orientation of C_R is taken to be counter-clockwise as usual. We call Φ_j^∞'s the (cohomological) *monodromies at infinity of f*. In the last few decades many mathematicians studied Φ_j^∞'s from various points of view. In the tame case, Libgober-Sperber [12] obtained a beautiful formula which expresses the semisimple part (i.e., the eigenvalues) of Φ_{n-1}^∞ in terms of the Newton polyhedron at infinity of f (see [13] for its generalizations). Recently in [14] (see also [6]) the first author proved formulae for its nilpotent part, i.e., its Jordan normal form, by using the motivic Milnor fiber at infinity of f. However, the methods of [12], [13] and [14] etc. do not apply beyond the tame case by the absense of the concentration (1.2) for non-tame polynomials (see [16] for a partial result). In this paper, even for non-tame polynomials we show that the desired cohomological concentration holds for the generalized eigenspaces of Φ_j^∞ for "good" eigenvalues associated to f. Then by avoiding the remaining "bad" eigenvalues, we can successfully generalize the results in [12], [13] and [14] etc. to non-tame polynomials and completely determine the Jordan normal forms of Φ_{n-1}^∞. More precisely, in Definition 2.10 by using the Newton polyhedron at infinity $\Gamma_\infty(f)$ we define a finite subset $A_f \subset \mathbb{C}$ of "bad" eigenvalues which we call *atypical engenvalues* of f. Then we have the following refinement of the main result of Broughton [1]. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ let $H^j(f^{-1}(R);\mathbb{C})_\lambda \subset H^j(f^{-1}(R);\mathbb{C})$ be the generalized eigenspace for the eigenvalue λ of the monodromy at infinity Φ_j^∞.

THEOREM 1.2. — *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-convenient polynomial such that* $\dim\Gamma_\infty(f) = n$*. Assume that f is non-degenerate at infinity. Then for any non-atypical eigenvalue $\lambda \notin A_f$ of f we have the concentration*

$$H^j(f^{-1}(R);\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1) \tag{1.4}$$

for the generic fiber $f^{-1}(R) \subset \mathbb{C}^n$ $(R \gg 0)$ of f.

This theorem allows non-isolated singularities of f and also the situation where the fibers may have cohomological perturbation "at infinity". Indeed, some of its atypical fibers $f^{-1}(b)$ $(b \in B_f)$ e.g., $f^{-1}(0)$ have non-isolated singularities in general. In the "tame" case one has only isolated singularities in $\mathbb{C}^n$ and either vanishing cycles at infinity do not occur at all or they occur at isolated points only (in the sense of [25], [29]), and then the concentration of cohomology (1.2) follows.

Theorem 1.2 will be proved by refining the proof of Sabbah's theorem [24, Theorem 13.1] in our situation. More precisely we construct a new compactification $\widetilde{X_\Sigma}$ of $\mathbb{C}^n$ and study the "horizontal" divisors at infinity for f in $\widetilde{X_\Sigma} \setminus \mathbb{C}^n$ very precisely to prove the concentration. With this main result at hand, by using the results in [14, Section 2] we can prove the generalizations of [12], [13] and [14, Theorems 5.9, 5.14 and 5.16] etc. to non-tame polynomials and completely determine the λ-part of the Jordan normal form of Φ_{n-1}^{∞} for any $\lambda \notin A_f$. Let us explain one of our results, which generalizes [14, Theorem 5.9]. Denote by $\mathrm{Cone}_{\infty}(f)$ the closed cone $\mathbb{R}_+\Gamma_{\infty}(f) \subset \mathbb{R}_+^n$ generated by $\Gamma_{\infty}(f)$. Let $q_1, \dots, q_l$ (resp. $\gamma_1, \dots, \gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_{\infty}(f)$ such that $q_i \in \mathrm{Int}(\mathrm{Cone}_{\infty}(f))$ (resp. the relative interior $\mathrm{rel.int}(\gamma_i)$ of γ_i is contained in $\mathrm{Int}(\mathrm{Cone}_{\infty}(f))$). For each q_i (resp. γ_i), denote by $d_i > 0$ (resp. $e_i > 0$) its "lattice distance" from the origin $0 \in \mathbb{R}^n$ (see Section 2.1 for the precise definition). For $1 \leq i \leq l'$, let Δ_i be the convex hull of $\{0\} \sqcup \gamma_i$ in $\mathbb{R}^n$. Then for $\lambda \neq 1$ and $1 \leq i \leq l'$ such that $\lambda^{e_i} = 1$ we set

$$(1.5) \quad n(\lambda)_i = \sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{ht}(v, \gamma_i) = k\} + \sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{ht}(v, \gamma_i) = e_i - k\},$$

where k is the minimal positive integer satisfying $\lambda = \zeta_{e_i}^k$ (we set $\zeta_d := \exp(2\pi\sqrt{-1}/d) \in \mathbb{C}$) and for $v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i)$ we denote by $\mathrm{ht}(v, \gamma_i)$ the lattice height of v from the base γ_i of Δ_i. Then we have the following extension of [14, Theorem 5.9] from tame to non-tame polynomials.

THEOREM 1.3. — *Assume that* $\dim\Gamma_{\infty}(f) = n$ *and* f *is non-degenerate at infinity. Then for any* $\lambda \notin A_f$ *we have*

1. *The number of the Jordan blocks for the eigenvalue* λ *with the maximal possible size* n *in* $\Phi_{n-1}^{\infty}\colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ *is equal to* $\sharp\{q_i \mid \lambda^{d_i} = 1\}$.
2. *The number of the Jordan blocks for the eigenvalue* λ *with the second maximal possible size* $n-1$ *in* Φ_{n-1}^{∞} *is equal to* $\sum_{i\colon \lambda^{e_i}=1} n(\lambda)_i$.

We can treat in a similar manner the following important monodromies at bifurcation points of f. Let $b \in B_f$ be such a bifurcation point. Choose sufficiently small $\varepsilon > 0$ such that

$$(1.6) \qquad B_f \cap \{x \in \mathbb{C} \mid |x - b| \leq \varepsilon\} = \{b\}$$

and set $C_{\varepsilon}(b) = \{x \in \mathbb{C} \mid |x - b| = \varepsilon\} \subset \mathbb{C}$. Then we obtain a locally trivial fibration $f^{-1}(C_{\varepsilon}(b)) \longrightarrow C_{\varepsilon}(b)$ over the small circle $C_{\varepsilon}(b) \subset \mathbb{C}$ and the monodromy automorphisms

$$(1.7) \qquad \Phi_j^b\colon H^j(f^{-1}(b+\varepsilon);\mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(b+\varepsilon);\mathbb{C}) \quad (j = 0, 1, \dots)$$

around the atypical fiber $f^{-1}(b) \subset \mathbb{C}^n$ associated to it. In Section 4 we apply our methods to the Jordan normal forms of Φ_j^b's. If f is non-convenient, then for some $b \in B_f$ the atypical fiber $f^{-1}(b) \subset \mathbb{C}^n$ may have "singularities at infinity". Even in such cases, we can define a finite subset $A_{f,b}^{\circ} \subset \mathbb{C}$ of "bad" eigenvalues for $b \in B_f$ and completely determine the λ-part of the Jordan normal form of Φ_{n-1}^b for any $\lambda \notin A_{f,b}^{\circ}$. In fact we obtain these results more generally, for polynomial maps $f : U \longrightarrow \mathbb{C}$ of affine algebraic varieties U. See Section 4 for the details.

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2. Preliminary notions and results

In this section, we introduce some preliminary notions and results which will be used in the proofs of our main theorems in Section 3.

2.1. Newton polyhedra and atypical eigenvalues. — Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial function. In the case of *tame* polynomials ([1], [11]) or *cohomologically tame* polynomials ([17]), the monodromies at infinity Φ_j^∞ were studied by several authors, basically since these conditions imply that there are no vanishing cycles at infinity. In turn, this fact implies that the cohomology of the general fibre of f is concentrated in the top dimension.

It appears that there are classes of non-tame polynomials which have the latter property of concentratedness and therefore Φ_{n-1}^∞ is the unique non-trivial monodromy at infinity: polynomials which have *isolated singularities at infinity* in a certain sense, like *isolated $\mathcal{W}$-singularities* (studied in [25], [26], [27]) or *isolated t-singularities* (studied in [26], [28], [29]). However in this setting there are no results on Jordan normal forms, except for the case of two variables, e.g., [4].

DEFINITION 2.1 ([11]). — We say that the polynomial $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v$ ($a_v \in \mathbb{C}$) is *non-degenerate at infinity* if for any face γ of $\Gamma_\infty(f)$ such that $0 \notin \gamma$ the complex hypersurface $\{x \in (\mathbb{C}^*)^n \mid f_\gamma(x) = 0\}$ in $(\mathbb{C}^*)^n$ is smooth and reduced, where we set $f_\gamma(x) = \sum_{v \in \gamma \cap \mathbb{Z}_+^n} a_v x^v$.

If f is convenient and non-degenerate at infinity, then by a result of Broughton [1] it is tame at infinity. However in this paper, we do not assume that f is convenient.

DEFINITION 2.2. — Assume that $\dim \Gamma_\infty(f) = n$. Then we say that a face $\gamma \prec \Gamma_\infty(f)$ is *atypical* if $0 \in \gamma$ and there exists a facet i.e., an $(n-1)$-dimensional face Γ of $\Gamma_\infty(f)$ containing γ whose non-zero inner conormal vectors are not contained in the first quadrant $\mathbb{R}_+^n$ of $\mathbb{R}^n$.

REMARK 2.3. — Our definition above is closely related to that of *bad faces* of $NP(f)$ in Némethi-Zaharia [18]. If $\gamma \prec NP(f)$ is a bad face of $NP(f)$, then the convex hull of $\{0\} \cup \gamma$ in $\mathbb{R}^n$ is an atypical one of $\Gamma_\infty(f)$. However, not all the atypical faces of $\Gamma_\infty(f)$ are obtained in this way.

EXAMPLE 2.4. — Let $n = 3$ and consider a non-convenient polynomial $f(x,y,z)$ on $\mathbb{C}^3$ whose Newton polyhedron at infinity $\Gamma_\infty(f)$ is the convex hull of the points $(2,0,0), (2,2,0), (2,2,3) \in \mathbb{R}^3_+$ and the origin $0 = (0,0,0) \in \mathbb{R}^3$. Then the line segment connecting the point $(2,2,0)$ and the origin $0 \in \mathbb{R}^3$ is an atypical face of $\Gamma_\infty(f)$. However the triangle whose vertices are the points $(2,0,0)$, $(2,2,0)$ and the origin $0 \in \mathbb{R}^3$ is not so.

EXAMPLE 2.5. — Let $n = 3$ and consider a non-convenient polynomial $f(x,y,z)$ on $\mathbb{C}^3$ whose Newton polyhedron at infinity $\Gamma_\infty(f)$ is the convex hull of the points $(2,0,0), (0,2,0), (1,1,2) \in \mathbb{R}^3_+$ and the origin $0 = (0,0,0) \in \mathbb{R}^3$. Then the line segment connecting the point $(2,0,0)$ and the origin $0 \in \mathbb{R}^3$ is an atypical face of $\Gamma_\infty(f)$.

If $\dim \Gamma_\infty(f) = n$, to the n-dimensional integral polytope $\Gamma_\infty(f)$ in $\mathbb{R}^n$ we can naturally associate a subdivision of (the dual vector space of) $\mathbb{R}^n$ into rational convex polyhedral cones as follows. For an element $u \in \mathbb{R}^n$ of (the dual vector space of) $\mathbb{R}^n$ define the supporting face $\gamma_u \prec \Gamma_\infty(f)$ of u in $\Gamma_\infty(f)$ by

$$\gamma_u = \left\{ v \in \Gamma_\infty(f) \mid \langle u, v \rangle = \min_{w \in \Gamma_\infty(f)} \langle u, w \rangle \right\}. \tag{2.1}$$

Then we introduce an equivalence relation $\sim$ on (the dual vector space of) $\mathbb{R}^n$ by $u \sim u' \iff \gamma_u = \gamma_{u'}$. We can easily see that for any face $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$ the closure of the equivalence class associated to γ in $\mathbb{R}^n$ is an $(n - \dim\gamma)$-dimensional rational convex polyhedral cone $\sigma(\gamma)$ in $\mathbb{R}^n$. Moreover the family $\{\sigma(\gamma) \mid \gamma \prec \Gamma_\infty(f)\}$ of cones in $\mathbb{R}^n$ thus obtained is a subdivision of $\mathbb{R}^n$ and satisfies the axiom of fans (see [7] and [19] etc.). We call it the dual fan of $\Gamma_\infty(f)$. Then we have the following characterization of atypical faces of $\Gamma_\infty(f)$.

LEMMA 2.6. — *Assume that* $\dim \Gamma_\infty(f) = n$ *and let* $\gamma \prec \Gamma_\infty(f)$ *be a face of* $\Gamma_\infty(f)$ *such that* $0 \in \gamma$. *Then* γ *is atypical if and only if the cone* $\sigma(\gamma)$ *which corresponds to it in the dual fan of* $\Gamma_\infty(f)$ *is not contained in* $\mathbb{R}^n_+$.

For a subset $S \subset \{1, 2, \ldots, n\}$ we define a coordinate subspace $\mathbb{R}^S \simeq \mathbb{R}^{|S|}$ of $\mathbb{R}^n$ by

$$\mathbb{R}^S = \{v = (v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_i = 0 \quad \text{for any} \quad i \notin S\}. \tag{2.2}$$

The following lemma should be obvious.

LEMMA 2.7. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and let* $\gamma \prec \Gamma_\infty(f)$ *be a face of* $\Gamma_\infty(f)$ *such that* $0 \in \gamma$. *Let* $\mathbb{R}^S \subset \mathbb{R}^n$ *be the minimal coordinate subspace of* $\mathbb{R}^n$ *containing* γ *and assume that* $\dim\gamma < \dim\mathbb{R}^S = |S|$. *Then* γ *is an atypical face of* $\Gamma_\infty(f)$.

By this lemma we can easily prove the following proposition.

PROPOSITION 2.8. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and a face* $\gamma \prec \Gamma_\infty(f)$ *of* $\Gamma_\infty(f)$ *such that* $0 \in \gamma$ *is non-atypical. Let* $\mathbb{R}^S \subset \mathbb{R}^n$ *be the minimal coordinate subspace of* $\mathbb{R}^n$ *containing* γ. *Then we have* $\dim\gamma = \dim\mathbb{R}^S = |S|$ *and there exist exactly* $n - |S|$ *facets i.e.,* $(n-1)$*-dimensional faces* $\Gamma_i \prec \Gamma_\infty(f)$ $(i \notin S)$ *of* $\Gamma_\infty(f)$ *containing* γ. *Moreover they are explicitly given by*

$$\Gamma_i = \Gamma_\infty(f) \cap \{v = (v_1, \dots, v_n) \in \mathbb{R}^n \mid v_i = 0\} \qquad (i \notin S). \tag{2.3}$$

DEFINITION 2.9. — We say that a face $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$ is *at infinity* if $0 \notin \gamma$. We say that such a face γ is moreover *admissible* if it is not contained in any atypical face of $\Gamma_\infty(f)$.

For a face at infinity $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$, let Δ_γ be the convex hull of $\{0\} \sqcup \gamma$ in $\mathbb{R}^n$. Denote by $\mathbb{L}(\Delta_\gamma)$ the $(\dim\gamma + 1)$-dimensional linear subspace of $\mathbb{R}^n$ spanned by Δ_γ and consider the lattice $M_\gamma = \mathbb{Z}^n \cap \mathbb{L}(\Delta_\gamma) \simeq \mathbb{Z}^{\dim\gamma+1}$ in it. Then there exists a unique non-zero primitive vector u_γ in its dual lattice which takes its maximum in Δ_γ exactly on $\gamma \prec \Delta_\gamma$:

$$\gamma = \left\{ v \in \Delta_\gamma \mid \langle u_\gamma, v\rangle = \max_{w \in \Delta_\gamma} \langle u_\gamma, w\rangle \right\}. \tag{2.4}$$

We set

$$d_\gamma = \max_{w \in \Delta_\gamma} \langle u_\gamma, w\rangle \in \mathbb{Z}_{>0} \tag{2.5}$$

and call it the *lattice distance* of γ from the origin $0 \in \mathbb{R}^n$. The following definition will be used in Sections 2.2 and 3.

DEFINITION 2.10. — Assume that $\dim\Gamma_\infty(f) = n$. Then we say that a complex number $\lambda \in \mathbb{C}$ is an *atypical eigenvalue of* f if either $\lambda = 1$ or there exists a non-admissible face at infinity $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$ such that $\lambda^{d_\gamma} = 1$. We denote by $A_f \subset \mathbb{C}$ the set of the atypical eigenvalues of f.

EXAMPLE 2.11. — Let $n = 2$ and consider a non-convenient polynomial $f(x, y)$ on $\mathbb{C}^2$ whose Newton polyhedron at infinity $\Gamma_\infty(f)$ is the convex hull of the points $(1, 3), (3, 0), (3, 2) \in \mathbb{R}^2_+$ and the origin $0 = (0, 0) \in \mathbb{R}^2$. Then the line segment connecting the point $(1, 3)$ and the origin is the unique atypical face of $\Gamma_\infty(f)$ and we have $A_f = \{1\}$.

Let $\overrightarrow{e_i} = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) \in \mathbb{R}^n$ $(i = 1, 2, \ldots, n)$ be the standard basis of $\mathbb{R}^n$. The following result will be used in Section 2.3.

PROPOSITION 2.12. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and let* $\gamma \prec \Gamma_\infty(f)$ *be a face at infinity of* $\Gamma_\infty(f)$. *Let* $\mathbb{R}^S \subset \mathbb{R}^n$ *be the minimal coordinate subspace of* $\mathbb{R}^n$ *containing* γ *and* $\sigma \subset \mathbb{R}^n$ *the cone which corresponds to* $\gamma \prec \Gamma_\infty(f)$ *in the dual fan of* $\Gamma_\infty(f)$. *Then* γ *is admissible if and only if there exist some integral vectors* $\overrightarrow{f_1}, \ldots, \overrightarrow{f_k} \in (\mathbb{R}^n \setminus \mathbb{R}^n_+) \cap \mathbb{Z}^n$ *such that*

$$\min_{v \in \Gamma_\infty(f)} \langle \overrightarrow{f_j}, v \rangle < 0 \qquad (1 \leq j \leq k) \tag{2.6}$$

and

$$\sigma = (\sum_{i \notin S} \mathbb{R}_+ \overrightarrow{e_i}) + (\sum_{j=1}^{k} \mathbb{R}_+ \overrightarrow{f_j}). \tag{2.7}$$

Proof. — Recall that γ is the intersection of the facets of $\Gamma_\infty(f)$ containing it and σ is generated by their inner conormal vectors. The intersection γ' of the ones containing the origin $0 \in \mathbb{R}^n$ is the minimal face of $\Gamma_\infty(f)$ such that $0 \in \gamma'$ and $\gamma \prec \gamma'$. If γ is admissible, then γ' is non-atypical and by Proposition 2.8 $\overrightarrow{e_i}$ $(i \notin S)$ are the inner conormal vectors of the facets of $\Gamma_\infty(f)$ containing γ'. This implies that the cone σ' which corresponds to γ' in the dual fan of $\Gamma_\infty(f)$ is explicitly given by

$$\sigma' = (\sum_{i \notin S} \mathbb{R}_+ \overrightarrow{e_i}). \tag{2.8}$$

Moreover the inner conormal vectors $\overrightarrow{f_1}, \ldots, \overrightarrow{f_k} \in (\mathbb{R}^n \setminus \mathbb{R}^n_+) \cap \mathbb{Z}^n$ of the other facets (at infinity) of $\Gamma_\infty(f)$ containing γ satisfy the condition

$$\min_{v \in \Gamma_\infty(f)} \langle \overrightarrow{f_j}, v \rangle < 0 \qquad (1 \leq j \leq k). \tag{2.9}$$

Then the assertion immediately follows. □

2.2. Motivic Milnor fibers at infinity. — Following Denef-Loeser [2], [3] and Guibert-Loeser-Merle [8], one defined the motivic reincarnations of the general fibers of polynomial functions $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ as follows (for the details, see Matsui-Takeuchi [14], Raibaut [21] and Esterov-Takeuchi [6]). In this paper, we follow the terminologies of [5], [9] and [10] etc. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a general polynomial map. First, take a smooth compactification X of $\mathbb{C}^n$. Next, by eliminating the points of indeterminacy of the meromorphic extension of f to X with the help of Hironaka's theorem we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\iota} & \widetilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{C} & \xrightarrow[j]{} & \mathbb{P}^1 \end{array} \tag{2.10}$$

such that horizontal arrows are open embeddings, g is a proper holomorphic map and $\widetilde{X} \setminus \mathbb{C}^n$, $Y := g^{-1}(\infty)$ are normal crossing divisors in $\widetilde{X}$. Take a local coordinate h of $\mathbb{P}^1$ in a neighborhood of $\infty \in \mathbb{P}^1$ such that $\infty = \{h = 0\}$ and set $\widetilde{g} = h \circ g$. Note that $\widetilde{g}$ is a holomorphic function defined on a neighborhood of the closed subvariety $Y = \widetilde{g}^{-1}(0) = g^{-1}(\infty) \subset \widetilde{X} \setminus \mathbb{C}^n$ of $\widetilde{X}$. Then for $R \gg 0$ we have

(2.11)

$$H^j_c(f^{-1}(R); \mathbb{C}) \simeq H^j\psi_h(j_!Rf_!\mathbb{C}_{\mathbb{C}^n}) \simeq H^j\psi_h(Rg_!\iota_!\mathbb{C}_{\mathbb{C}^n}) \simeq H^j(Y; \psi_{\widetilde{g}}(\iota_!\mathbb{C}_{\mathbb{C}^n})).$$

Here ψ_h and $\psi_{\widetilde{g}}$ are nearby cycle functors (for the definition, see [5] and [10] etc.) and we used the commutativity of the direct image $Rg_* = Rg_!$ for the proper map g with them (see e.g., [5, Proposition 4.2.11]) in the last isomorphism. Moreover $H^j(Y; \psi_{\widetilde{g}}(\iota_!\mathbb{C}_{\mathbb{C}^n}))$ stands for the j-th hypercohomology of the complex $\psi_{\widetilde{g}}(\iota_!\mathbb{C}_{\mathbb{C}^n})$ of sheaves on Y. Let us define an open subset Ω of $\widetilde{X}$ by

$$\Omega = \mathrm{Int}(\iota(\mathbb{C}^n) \sqcup Y) \tag{2.12}$$

and set $U = \Omega \cap Y$. Then U (resp. the complement of Ω in $\widetilde{X}$) is a normal crossing divisor in Ω (resp. $\widetilde{X}$). By using this very special geometric situation we can easily prove the isomorphisms

$$H^j(Y; \psi_{\widetilde{g}}(\iota_!\mathbb{C}_{\mathbb{C}^n})) \simeq H^j(Y; \psi_{\widetilde{g}}(\iota'_!\mathbb{C}_{\Omega})) \simeq H^j_c(U; \psi_{\widetilde{g}}(\mathbb{C}_{\widetilde{X}})), \tag{2.13}$$

where $\iota' \colon \Omega \hookrightarrow \widetilde{X}$ is the inclusion. Indeed, the isomorphism $\psi_{\widetilde{g}}(\iota_!\mathbb{C}_{\mathbb{C}^n}) \simeq \psi_{\widetilde{g}}(\iota'_!\mathbb{C}_{\Omega})$ follows from the fact that the intersections of the Milnor fiber of $\widetilde{g}$ at a point of Y with $\iota(\mathbb{C}^n)$ and Ω are the same. Moreover by showing that the stalk of $\psi_{\widetilde{g}}(\iota'_!\mathbb{C}_{\Omega})$ at each point of $Y \setminus U = Y \setminus \Omega$ is zero, we obtain $\psi_{\widetilde{g}}(\iota'_!\mathbb{C}_{\Omega})|_{Y \setminus U} \simeq 0$ and hence the isomorphism

$$(\psi_{\widetilde{g}}(\mathbb{C}_{\widetilde{X}}))_U \simeq (\psi_{\widetilde{g}}(\iota'_!\mathbb{C}_{\Omega}))_U \xrightarrow{\sim} \psi_{\widetilde{g}}(\iota'_!\mathbb{C}_{\Omega}). \tag{2.14}$$

Now let $E_1, E_2, \dots, E_k$ be the irreducible components of the normal crossing divisor $U = \Omega \cap Y$ in $\Omega \subset \widetilde{X}$. For each $1 \leq i \leq k$, let $b_i > 0$ be the order of the zero of $\widetilde{g}$ along E_i. For a non-empty subset $I \subset \{1, 2, \dots, k\}$, let us set

$$E_I = \bigcap_{i \in I} E_i, \qquad E_I^{\circ} = E_I \setminus \bigcup_{i \notin I} E_i \tag{2.15}$$

and $d_I = \gcd(b_i)_{i\in I} > 0$. Then, as in [3, Section 3.3], we can construct an unramified Galois covering $\widetilde{E_I^\circ} \longrightarrow E_I^\circ$ of E_I° as follows. First, for a point $p \in E_I^\circ$ we take an affine open neighborhood $W \subset \Omega \setminus (\cup_{i\notin I} E_i)$ of p on which there exist regular functions ξ_i $(i \in I)$ such that $E_i \cap W = \{\xi_i = 0\}$ for any $i \in I$. Then on W we have $\widetilde{g} = \widetilde{g_{1,W}}(\widetilde{g_{2,W}})^{d_I}$, where we set $\widetilde{g_{1,W}} = \widetilde{g}\prod_{i\in I}\xi_i^{-b_i}$ and $\widetilde{g_{2,W}} = \prod_{i\in I}\xi_i^{\frac{b_i}{d_I}}$. Note that $\widetilde{g_{1,W}}$ is a unit on W and $\widetilde{g_{2,W}}\colon W \longrightarrow \mathbb{C}$ is a regular function. It is easy to see that E_I° is covered by such affine open subsets W of $\Omega \setminus (\cup_{i\notin I} E_i)$. Then as in [3, Section 3.3] by gluing the varieties

$$\widetilde{E_{I,W}^\circ} = \{(t,z) \in \mathbb{C}^* \times (E_I^\circ \cap W) \mid t^{d_I} = (\widetilde{g_{1,W}})^{-1}(z)\} \tag{2.16}$$

together in an obvious way, we obtain the variety $\widetilde{E_I^\circ}$ over E_I°. Now for $d \in \mathbb{Z}_{>0}$, let $\mu_d \simeq \mathbb{Z}/\mathbb{Z}d$ be the multiplicative group consisting of the d-roots in $\mathbb{C}$. We denote by $\hat{\mu}$ the projective limit $\varprojlim_d \mu_d$ of the projective system $\{\mu_i\}_{i\geq 1}$ with morphisms $\mu_{id} \longrightarrow \mu_i$ given by $t \longmapsto t^d$. Then the unramified Galois covering $\widetilde{E_I^\circ}$ of E_I° admits a natural μ_{d_I}-action defined by assigning the automorphism $(t,z) \longmapsto (\zeta_{d_I}t, z)$ of $\widetilde{E_I^\circ}$ to the generator $\zeta_{d_I} := \exp(2\pi\sqrt{-1}/d_I) \in \mu_{d_I}$. Namely the variety $\widetilde{E_I^\circ}$ is equipped with a good $\hat{\mu}$-action in the sense of [3, Section 2.4]. Following the notations in [3], denote by $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ the ring obtained from the Grothendieck ring $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ of varieties over $\mathbb{C}$ with good $\hat{\mu}$-actions by inverting the Lefschetz motive $\mathbb{L} \simeq \mathbb{C} \in \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$. Recall that $\mathbb{L} \in \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ is endowed with the trivial action of $\hat{\mu}$.

Definition 2.13 ([14] and [21]). — We define the *motivic Milnor fiber at infinity* $\mathcal{S}_f^\infty$ of the polynomial map $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ by

$$\mathcal{S}_f^\infty = \sum_{I\neq\varnothing}(1-\mathbb{L})^{|I|-1}[\widetilde{E_I^\circ}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \tag{2.17}$$

Remark 2.14. — By Guibert-Loeser-Merle [8, Theorem 3.9], the motivic Milnor fiber at infinity $\mathcal{S}_f^\infty$ of f does not depend on the compactification X of $\mathbb{C}^n$. This fact was informed to us by Schürmann (private communication) and Raibaut [21]. For a proof of the independence on the compactification, see [22, Theorem 2.4].

As in [3, Section 3.1.2 and 3.1.3], we denote by $\mathrm{HS}^{\mathrm{mon}}$ the abelian category of Hodge structures with a quasi-unipotent endomorphism. Then, to the object $\psi_h(j_!Rf_!\mathbb{C}_{\mathbb{C}^n}) \in \mathbf{D}_c^b(\{\infty\})$ and the semisimple part of the monodromy automorphism acting on it, we can associate an element

$$[H_f^\infty] = \sum_{j\in\mathbb{Z}}(-1)^j[H^j\psi_h(j_!Rf_!\mathbb{C}_{\mathbb{C}^n})] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{2.18}$$

as in [2] and [3], where the weight filtration of the limit mixed Hodge structure $[H^j\psi_h(j_!Rf_!\mathbb{C}_{\mathbb{C}^n})] \in \mathrm{HS}^{\mathrm{mon}}$ is the "relative" monodromy filtration defined by the monodromy at infinity with respect to the variation of the mixed Hodge structure on $\mathbb{C}\setminus B_f$ whose underlying local system is $H^j(Rf_!\mathbb{C}_{\mathbb{C}^n})$ (see also [23] and [24] etc.). To describe the element $[H_f^\infty] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ in terms of $\mathcal{S}_f^\infty \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, let

$$\chi_h\colon \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \longrightarrow \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{2.19}$$

be the Hodge characteristic morphism defined in [3] which associates to a variety Z with a good μ_d-action the Hodge structure

$$\chi_h([Z]) = \sum_{j\in\mathbb{Z}}(-1)^j[H_c^j(Z;\mathbb{Q})] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{2.20}$$

with the actions induced by the one $z \longmapsto \exp(2\pi\sqrt{-1}/d)z$ $(z \in Z)$ on Z. Then as in [14, Theorem 4.4] and [22, Theorem 2.10], by applying [2, Theorem 4.2.1] and [8, Section 3.16] to our situation (2.11) and (2.13), we obtain the following result.

THEOREM 2.15. — *In the Grothendieck group* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, *we have the equality*

$$[H_f^\infty] = \chi_h(\mathcal{S}_f^\infty). \tag{2.21}$$

2.3. Motivic Milnor fibers at infinity via toric compactifications. — By using Newton polyhedrons at infinity, we can rewrite Theorem 2.15 more explicitly as follows. We shall adapt the constructions in [6] and [14] to our more general setting of non-convenient polynomials. For this purpose, we introduce a new toric compactification of $\mathbb{C}^n$. Let $f \in \mathbb{C}[x_1,\ldots,x_n]$ be a "non-convenient" polynomial such that $\dim\Gamma_\infty(f) = n$. Assume that f is non-degenerate at infinity. Now let us consider $\mathbb{C}^n$ as a toric variety associated with the fan Σ_0 in $\mathbb{R}^n$ formed by all the faces of the first quadrant $\mathbb{R}^n_+ \subset \mathbb{R}^n$. Denote by $T \simeq (\mathbb{C}^*)^n$ the open dense torus in it. Let Σ_1 be a subdivision of the dual fan of $\Gamma_\infty(f)$ which contains Σ_0 as its subfan. Then we can construct a smooth subdivision Σ of Σ_1 without subdividing the cones in Σ_0 (see e.g., [20, Lemma 2.6, Chapter II, page 99]). This implies that the toric variety X_Σ associated with Σ is a smooth compactification of $\mathbb{C}^n$. Our construction of X_Σ coincides with the one in Zaharia [31]. Recall that T acts on X_Σ and the T-orbits are parametrized by the cones in Σ. For a cone $\sigma \in \Sigma$ denote by $T_\sigma \simeq (\mathbb{C}^*)^{n-\dim\sigma}$ the corresponding T-orbit. We have also natural affine open subsets $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n$ of X_Σ associated to n-dimensional cones σ in Σ as follows. Let σ be an n-dimensional cone in Σ and $\{w_1,\ldots,w_n\} \subset \mathbb{Z}^n$ the set of the (non-zero) primitive vectors on the edges of σ. Then by the smoothness of Σ the semigroup ring $\mathbb{C}[\mathbb{Z}^n \cap \sigma]$ is isomorphic to the polynomial ring $\mathbb{C}[y_1,\dots,y_n]$. This implies that the affine open subset $\mathbb{C}^n(\sigma) := \mathrm{Spec}(\mathbb{C}[\mathbb{Z}^n \cap \sigma])$ of X_Σ is isomorphic to $\mathbb{C}^n_y$. Moreover, on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$ the function f has the following form:

$$f(y) = \sum_{v \in \mathbb{Z}^n_+} a_v y_1^{\langle w_1, v\rangle} \cdots y_n^{\langle w_n, v\rangle} = y_1^{b_1} \cdots y_n^{b_n} \times f_\sigma(y), \tag{2.22}$$

where we set $f = \sum_{v\in\mathbb{Z}^n_+} a_v x^v$,

$$b_i = \min_{v\in\Gamma_\infty(f)} \langle w_i, v\rangle \leq 0 \qquad (i = 1, 2, \dots, n) \tag{2.23}$$

and $f_\sigma(y)$ is a polynomial on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$. In $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$ the hypersurface $Z := \overline{f^{-1}(0)} \subset X_\Sigma$ is explicitly written as $\{y \in \mathbb{C}^n(\sigma) \mid f_\sigma(y) = 0\}$. By (2.22) we see that f is extended to a meromorphic function on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$. The variety X_Σ is covered by such affine open subsets. Let τ be a d-dimensional face of the n-dimensional cone $\sigma \in \Sigma$. For simplicity, assume that $w_1, \dots, w_d$ generate τ. Then in the affine chart $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$ the T-orbit T_τ associated to τ is explicitly defined by

$$T_\tau = \{(y_1,\dots,y_n) \in \mathbb{C}^n(\sigma) \mid y_1 = \cdots = y_d = 0,\ y_{d+1},\dots,y_n \neq 0\} \simeq (\mathbb{C}^*)^{n-d}.$$

Hence we have

$$X_\Sigma = \bigcup_{\dim\sigma = n} \mathbb{C}^n(\sigma) = \bigsqcup_{\tau\in\Sigma} T_\tau. \tag{2.24}$$

Now f extends to a meromorphic function on X_Σ, which may still have points of indeterminacy. For simplicity we denote this meromorphic extension also by f. From now on, we will eliminate its points of indeterminacy by blowing up X_Σ (see [13, Section 3] and [14, Section 3] for the details). For a cone σ in Σ by taking a non-zero vector u in the relative interior rel.int(σ) of σ we define a face $\gamma(\sigma)$ of $\Gamma_\infty(f)$ by

$$\gamma(\sigma) = \left\{ v \in \Gamma_\infty(f) \mid \langle u, v\rangle = \min_{w\in\Gamma_\infty(f)} \langle u, w\rangle \right\}. \tag{2.25}$$

This face $\gamma(\sigma)$ does not depend on the choice of $u \in$ rel.int(σ) and is called the supporting face of σ in $\Gamma_\infty(f)$. Following [12], we say that a T-orbit T_σ in X_Σ (or a cone $\sigma \in \Sigma$) is at infinity if its supporting face $\gamma(\sigma) \prec \Gamma_\infty(f)$ is at infinity i.e., $0 \notin \gamma(\sigma)$. We can easily see that f has poles on the union of T-orbits at infinity. Let $\rho_1, \rho_2, \dots, \rho_m$ be the 1-dimensional cones at infinity in Σ and set $T_i = T_{\rho_i}$. We call the cones ρ_i rays at infinity in Σ. Then $T_1, T_2, \dots, T_m$ are the $(n-1)$-dimensional T-orbits at infinity in X_Σ. For any $i = 1, 2, \dots, m$ the toric divisor $D_i := \overline{T_i}$ is a smooth hypersurface in X_Σ and the poles of f are contained in $D_1 \cup \cdots \cup D_m$. Let us denote the (unique non-zero) primitive vector in $\rho_i \cap \mathbb{Z}^n$ by u_i. Then the order $a_i > 0$ of the pole of f along D_i is given by

$$a_i = - \min_{v \in \Gamma_\infty(f)} \langle u_i, v \rangle. \tag{2.26}$$

Moreover by the non-convenience of f, there exist some cones $\sigma \in \Sigma$ such that $\sigma \notin \Sigma_0$ and $0 \in \gamma(\sigma)$ i.e., $\gamma(\sigma)$ is an atypical face of $\Gamma_\infty(f)$. For such σ the function f extends holomorphically to a neighborhood of $T_\sigma \subset X_\Sigma \setminus \mathbb{C}^n$. For this reason we call them *horizontal T-orbits* in X_Σ (in the tame case where f is convenient, they do not appear). Since f may have non-trivial monodromies at infinity on such T_σ, for the proof of our main theorem we need some detailed study on them in Proposition 3.4. Note also that by the non-degeneracy at infinity of f, for any non-empty subset $I \subset \{1, 2, \dots, m\}$ the hypersurface $Z = \overline{f^{-1}(0)}$ in X_Σ intersects $D_I := \bigcap_{i \in I} D_i$ transversally (or the intersection is empty). At such intersection points, f has indeterminacy. Now, in order to eliminate the indeterminacy of the meromorphic function f on X_Σ, we first consider the blow-up $\pi_1 \colon X_\Sigma^{(1)} \longrightarrow X_\Sigma$ of X_Σ along the $(n-2)$-dimensional smooth subvariety $D_1 \cap Z$. Then the indeterminacy of the pull-back $f \circ \pi_1$ of f to $X_\Sigma^{(1)}$ is improved. If $f \circ \pi_1$ still has points of indeterminacy on the intersection of the exceptional divisor E_1 of π_1 and the proper transform $Z^{(1)}$ of Z, we construct the blow-up $\pi_2 \colon X_\Sigma^{(2)} \longrightarrow X_\Sigma^{(1)}$ of $X_\Sigma^{(1)}$ along $E_1 \cap Z^{(1)}$. By repeating this procedure a_1 times, we obtain a tower of blow-ups

$$X_\Sigma^{(a_1)} \underset{\pi_{a_1}}{\longrightarrow} \cdots \underset{\pi_2}{\longrightarrow} X_\Sigma^{(1)} \underset{\pi_1}{\longrightarrow} X_\Sigma. \tag{2.27}$$

Then the pull-back of f to $X_\Sigma^{(a_1)}$ has no indeterminacy over T_1. It also extends to a holomorphic function on (an open dense subset of) the exceptional divisor of the last blow-up π_{a_1}. For this reason we call it and its proper transform in the variety $\widetilde{X_\Sigma}$ that we construct below *horizontal exceptional divisors*. For the details see the figures in [13, page 420]. Next we apply this construction to the proper transforms of D_2 and Z in $X_\Sigma^{(a_1)}$. Then we obtain also a tower of blow-ups

$$X_\Sigma^{(a_1)(a_2)} \longrightarrow \cdots \longrightarrow X_\Sigma^{(a_1)(1)} \longrightarrow X_\Sigma^{(a_1)} \tag{2.28}$$

and the indeterminacy of the pull-back of f to $X_\Sigma^{(a_1)(a_2)}$ is eliminated over $T_1 \sqcup T_2$. By applying the same construction to (the proper transforms of) $D_3, D_4, \dots, D_m$, we finally obtain a birational morphism $\pi \colon \widetilde{X_\Sigma} \longrightarrow X_\Sigma$ such that $g := f \circ \pi$ has no point of indeterminacy on the whole $\widetilde{X_\Sigma}$. Note that the smooth compactification $\widetilde{X_\Sigma}$ of $\mathbb{C}^n$ thus obtained is not a toric variety any more. On $\widetilde{X_\Sigma}$ we thus have constructed m (smooth) horizontal exceptional divisors $F_1, F_2, \dots, F_m$. By our construction of the blow-up $\pi \colon \widetilde{X_\Sigma} \longrightarrow X_\Sigma$,

$F_1 \cup F_2 \cup \cdots \cup F_m$ is a normal crossing divisor in $\widetilde{X_\Sigma}$ and for any non-empty subset $I \subset \{1, 2, \dots, m\}$ and $t \in \mathbb{C}$ the hypersurface $g^{-1}(t) \subset \widetilde{X_\Sigma}$ intersects $F_I := \cap_{i \in I} F_i$ transversally. Now let us consider the direct image sheaves

$$\mathscr{L}_{I,j} := H^j R(g|_{F_I \cap g^{-1}(\mathbb{C})})_* \mathbb{C}_{F_I \cap g^{-1}(\mathbb{C})} \qquad (j \in \mathbb{Z}) \tag{2.29}$$

on $\mathbb{C}$. Then as in the proof of [24, Lemma 8.5 (2)] by using the commutativity of nearby cycle functors and the direct image by the proper map $g|_{F_I \cap g^{-1}(\mathbb{C})}$, we see that the sheaves $\mathscr{L}_{I,j}$ are locally constant on $\mathbb{C}$. Since $\mathbb{C}$ is simply connected, they are in fact globally constant on $\mathbb{C}$. Namely the monodromy of the map

$$g|_{F_I \cap g^{-1}(\mathbb{C})} : F_I \cap g^{-1}(\mathbb{C}) \longrightarrow \mathbb{C} \tag{2.30}$$

is trivial. By eliminating the points of indeterminacy of the meromorphic extension of f to X_Σ we have constructed the commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\iota} & \widetilde{X_\Sigma} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathbb{C} & \xrightarrow[j]{} & \mathbb{P}^1. \end{array} \tag{2.31}$$

Take a local coordinate h of $\mathbb{P}^1$ in a neighborhood of $\infty \in \mathbb{P}^1$ such that $\infty = \{h = 0\}$ and set $\widetilde{g} = h \circ g$, $Y = \widetilde{g}^{-1}(0) = g^{-1}(\infty) \subset \widetilde{X_\Sigma}$ and $\Omega = \mathrm{Int}(\iota(\mathbb{C}^n) \sqcup Y)$ as before. Since in our explicit construction above of $\widetilde{X_\Sigma}$ we used only some blow-ups along codimension-two smooth subvarieties on the normal crossing divisor $D_1 \cup \cdots \cup D_m \subset X_\Sigma$, the resulting hypersurface $Y = \widetilde{g}^{-1}(0) \subset \widetilde{X_\Sigma}$ is again normal crossing. For simplicity, let us set $\widetilde{g} = \frac{1}{f}$. Then the divisor $U = Y \cap \Omega$ in Ω contains not only the proper transforms $D_1', \dots, D_m'$ of $D_1, \dots, D_m$ in $\widetilde{X_\Sigma}$ but also the exceptional divisors of the blow-up: $\widetilde{X_\Sigma} \longrightarrow X_\Sigma$. So the motivic Milnor fiber at infinity $\mathcal{S}_f^\infty$ of $f \colon \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by this compactification $\widetilde{X_\Sigma}$ of $\mathbb{C}^n$ contains also unramified Galois coverings of some subsets of these exceptional divisors. However they are not necessary to compute the Hodge realization of $\mathcal{S}_f^\infty$ as follows. For each non-empty subset $I \subset \{1, 2, \dots, m\}$, set $D_I = \bigcap_{i \in I} D_i$,

$$D_I^\circ = D_I \setminus \left\{ \left(\bigcup_{i \notin I} D_i \right) \cup \overline{f^{-1}(0)} \right\} \subset X_\Sigma \tag{2.32}$$

and $d_I = \gcd(a_i)_{i \in I} > 0$. Then the function $\widetilde{g} = \frac{1}{f}$ is regular on D_I° and we can decompose it as $\frac{1}{f} = \widetilde{g_1}(\widetilde{g_2})^{d_I}$ globally on a Zariski open neighborhood W of D_I° in X_Σ, where $\widetilde{g_1}$ is a unit on W and $\widetilde{g_2} \colon W \longrightarrow \mathbb{C}$ is regular. Therefore we can construct an unramified Galois covering $\widetilde{D_I^\circ}$ of D_I° with a natural μ_{d_I}-action as in (2.16). Let $[\widetilde{D_I^\circ}]$ be the element of the ring $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ which corresponds to $\widetilde{D_I^\circ}$.

Then as in [14, Theorem 4.7] we obtain the following result. We need only strata D_I° on which $\widetilde{g}$ vanishes for it.

THEOREM 2.16. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and* f *is non-degenerate at infinity. Then we have the equality*

$$\chi_h\left(\mathscr{S}_f^\infty\right) = \sum_{I\neq\varnothing} \chi_h\left((1-\mathbb{L})^{|I|-1}[\widetilde{D_I^\circ}]\right) \tag{2.33}$$

in the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$.

For a face at infinity $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$, by using the lattice $M_\gamma = \mathbb{Z}^n \cap \mathbb{L}(\Delta_\gamma) \simeq \mathbb{Z}^{\dim\gamma+1}$ in $\mathbb{L}(\Delta_\gamma) \simeq \mathbb{R}^{\dim\gamma+1}$ we set $T_{\Delta_\gamma} := \mathrm{Spec}(\mathbb{C}[M_\gamma]) \simeq (\mathbb{C}^*)^{\dim\gamma+1}$. Moreover let $\mathbb{L}(\gamma)$ be the smallest affine linear subspace of $\mathbb{R}^n$ containing γ and for $v \in M_\gamma$ define their lattice heights $\mathrm{ht}(v,\gamma) \in \mathbb{Z}$ from $\mathbb{L}(\gamma)$ in $\mathbb{L}(\Delta_\gamma)$ so that we have $\mathrm{ht}(0,\gamma) = d_\gamma > 0$. Then to the group homomorphism $M_\gamma \longrightarrow \mathbb{C}^*$ defined by $v \longmapsto \zeta_{d_\gamma}^{\mathrm{ht}(v,\gamma)}$ we can naturally associate an element $\tau_\gamma \in T_{\Delta_\gamma}$. We define a Laurent polynomial $g_\gamma = \sum_{v\in M_\gamma} b_v x^v$ on T_{Δ_γ} by

$$b_v = \begin{cases} a_v & (v \in \gamma), \\ -1 & (v = 0), \\ 0 & (\text{otherwise}), \end{cases} \tag{2.34}$$

where $f = \sum_{v\in\mathbb{Z}_+^n} a_v x^v$. Then the Newton polytope $NP(g_\gamma)$ of g_γ is Δ_γ, $\mathrm{supp} g_\gamma \subset \{0\} \sqcup \gamma$ and the hypersurface $Z_{\Delta_\gamma}^* = \{x \in T_{\Delta_\gamma} \mid g_\gamma(x) = 0\}$ is non-degenerate (see [14, Section 4]). Since $Z_{\Delta_\gamma}^* \subset T_{\Delta_\gamma}$ is invariant by the multiplication $l_{\tau_\gamma} : T_{\Delta_\gamma} \xrightarrow{\sim} T_{\Delta_\gamma}$ by τ_γ, $Z_{\Delta_\gamma}^*$ admits an action of μ_{d_γ}. We thus obtain an element $[Z_{\Delta_\gamma}^*]$ of $\mathscr{M}_{\mathbb{C}}^{\hat{\mu}}$. For a face at infinity $\gamma \prec \Gamma_\infty(f)$ let $s_\gamma > 0$ be the dimension of the minimal coordinate subspace of $\mathbb{R}^n$ containing γ and set $m_\gamma = s_\gamma - \dim\gamma - 1 \geq 0$. Finally, for $\lambda \in \mathbb{C}$ and an element $H \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ denote by $H_\lambda \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ the eigenvalue λ-part of H. Then by applying the proof of [14, Theorem 5.7 (i)] to the geometric situation in Proposition 2.12, we obtain the following result.

THEOREM 2.17. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and* f *is non-degenerate at infinity. Then for any* $\lambda \notin A_f$ *we have the equality*

$$[H_f^\infty]_\lambda = \chi_h(\mathscr{S}_f^\infty)_\lambda = \sum_\gamma \chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z_{\Delta_\gamma}^*])_\lambda \tag{2.35}$$

in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, *where in the sum* $\sum_\gamma$ *the face* γ *of* $\Gamma_\infty(f)$ *ranges through the admissible ones at infinity.*

Proof. — By Proposition 2.12 the argument at the end of the proof of [14, Theorem 5.7 (i)] holds for admissible faces at infinity of $\Gamma_\infty(f)$. But it does not hold for non-admissible ones by the presence of horizontal T-orbits in X_Σ. Hence it suffices to avoid atypical eigenvalues $\lambda \in A_f$. □

Remark 2.18. — The referee pointed out to us that by applying the Hodge realization and the localization to generalized eigenspaces to the formula for motivic Milnor fibers in [22, Theorem 3.24] one can obtain (2.35) for any $\lambda \in \mathbb{C}$ and Theorem 2.17 holds true without any restriction on $\lambda \in \mathbb{C}$.

3. Main results

In this section, we consider non-convenient polynomials $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ such that $\dim \Gamma_\infty(f) = n$. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ let $H^j(f^{-1}(R);\mathbb{C})_\lambda \subset H^j(f^{-1}(R);\mathbb{C})$ be the generalized eigenspace for the eigenvalue λ of the monodromy at infinity $\Phi_j^\infty \colon H^j(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$. Denote by $\Phi_{j,\lambda}^\infty$ the restriction of Φ_j^∞ to $H^j(f^{-1}(R);\mathbb{C})_\lambda$. Assuming also that f is non-degenerate at infinity, for non-atypical eigenvalues $\lambda \notin A_f$ of f we will prove the concentration

$$H^j(f^{-1}(R);\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1) \tag{3.1}$$

for the λ-parts $H^j(f^{-1}(R);\mathbb{C})_\lambda$ of the cohomology groups of the generic fiber $f^{-1}(R)$ $(R \gg 0)$ of f. This implies that the Jordan normal forms of the λ-parts $\Phi_{j,\lambda}^\infty$ of the monodromies at infinity of f can be completely determined by $\Gamma_\infty(f)$ as in [14, Section 5]. For this purpose we first consider Laurent polynomials on $T = (\mathbb{C}^*)^n$. Let $f' \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial on $T = (\mathbb{C}^*)^n$. We define its Newton polytope $NP(f') \subset \mathbb{R}^n$ as usual and let $\Gamma_\infty(f') \subset \mathbb{R}^n$ be the convex hull of $\{0\} \cup NP(f')$ in $\mathbb{R}^n$. We say that a face $\gamma \prec \Gamma_\infty(f')$ is at infinity if $0 \notin \gamma$. By using faces at infinity of $\Gamma_\infty(f')$ we define also the non-degeneracy at infinity of f' as in Definition 2.1.

Definition 3.1. — Assume that $\dim \Gamma_\infty(f') = n$. Then we say that a face $\gamma \prec \Gamma_\infty(f')$ is atypical if $0 \in \gamma$. Moreover a face at infinity $\gamma \prec \Gamma_\infty(f')$ is called admissible if it is not contained in any atypical one.

As in Definition 2.10, by using non-admissible faces at infinity of $\Gamma_\infty(f')$ we define the subset $A_{f'} \subset \mathbb{C}$ of the atypical eigenvalues of f' such that $1 \in A_{f'}$. Finally let us recall the following result of Libgober-Sperber [12] on the monodromies at infinity $\Psi_j^\infty : H^j((f')^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^j((f')^{-1}(R);\mathbb{C})$ $(R \gg 0)$

of $f' : T = (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$. We define the monodromy zeta function at infinity $\zeta_{f'}^{\infty}(t) \in \mathbb{C}((t))$ of f' by

$$\zeta_{f'}^{\infty}(t) = \prod_{j=0}^{n-1} \det(\mathrm{id} - t\Psi_j^{\infty})^{(-1)^j} \in \mathbb{C}((t)). \tag{3.2}$$

For a face at infinity $\gamma \prec \Gamma_\infty(f')$ let $\mathbb{L}(\gamma) \simeq \mathbb{R}^{\dim\gamma}$ be the minimal affine subspace of $\mathbb{R}^n$ containing γ.

Proposition 3.2 (Libgober-Sperber [12]). — *Assume that* $\dim\Gamma_\infty(f') = n$ *and f' is non-degenerate at infinity. Then we have*

$$\zeta_{f'}^{\infty}(t) = \prod_{\gamma} (1 - t^{d_\gamma})^{(-1)^{n-1}\mathrm{Vol}_{\mathbb{Z}}(\gamma)} \in \mathbb{C}((t)), \tag{3.3}$$

where in the product $\prod_\gamma$ the face $\gamma \prec \Gamma_\infty(f')$ ranges through those at infinity such that $\dim\gamma = n-1$, *$d_\gamma > 0$ is the lattice distance of γ from the origin* $0 \in \mathbb{R}^n$, *and* $\mathrm{Vol}_{\mathbb{Z}}(\gamma) \in \mathbb{Z}_{>0}$ *is the normalized* $(n-1)$*-dimensional volume of γ with respect to the lattice* $\mathbb{L}(\gamma) \cap \mathbb{Z}^n \simeq \mathbb{Z}^{n-1}$.

Remark 3.3. — Proposition 3.2 holds true even if $\dim\Gamma_\infty(f') < n$. In such a case, the fiber $(f')^{-1}(R) \subset T$ $(R \gg 0)$ has a product decomposition $(f')^{-1}(R) = A \times T'$ for a smaller torus $T' = (\mathbb{C}^*)^{n'}$ $(n' = n - \dim\Gamma_\infty(f') > 0)$ such that its monodromy automorphism at infinity acts non-trivially only on the first factor A. Since the topological Euler characteristic of T' is zero, by the Künneth formula we get $\zeta_{f'}^{\infty}(t) \equiv 1$ in this case.

Proposition 3.4. — *Let $f' \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial on $T = (\mathbb{C}^*)^n$ such that* $\dim\Gamma_\infty(f') = n$. *Assume that f' is non-degenerate at infinity. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ denote the generalized eigenspace for the eigenvalue λ of its monodromy at infinity $\Psi_j^{\infty} : H^j((f')^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^j((f')^{-1}(R);\mathbb{C})$ $(R \gg 0)$ by $H^j((f')^{-1}(R);\mathbb{C})_\lambda \subset H^j((f')^{-1}(R);\mathbb{C})$. Then for any $\lambda \notin A_{f'}$ we have the concentration*

$$H^j((f')^{-1}(R);\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1) \tag{3.4}$$

for the generic fiber $(f')^{-1}(R) \subset T$ $(R \gg 0)$ of f'. If $\lambda \notin A_{f'}$ satisfies the condition $H^{n-1}((f')^{-1}(R);\mathbb{C})_\lambda \neq 0$, then there exists a facet at infinity $\gamma \prec \Gamma_\infty(f')$ such that $\lambda^{d_\gamma} = 1$. Moreover for such λ the relative monodromy filtration of $H^{n-1}((f')^{-1}(R);\mathbb{C})_\lambda$ $(R \gg 0)$ coincides with the absolute one (up to some shift).

Proof. — We will prove the proposition by induction on n. If $n = 1$ the assertion is obvious. Assume that we already proved it for the lower dimensions $1, 2, \dots, n-1$. Let Σ_1' be the dual fan of $\Gamma_\infty(f')$ in $\mathbb{R}^n$ and Σ' its smooth subdivision. Then the toric variety $X_{\Sigma'}$ associated to Σ' is a smooth compactification of T. By eliminating the points of indeterminacy of the meromorphic extension of f' to $X_{\Sigma'}$ as in Section 2.3 we obtain a commutative diagram:

$$\begin{tikzcd} T \arrow[r, "\iota'"] \arrow[d, "f'"'] & \widetilde{X_{\Sigma'}} \arrow[d, "g'"] \\ \mathbb{C} \arrow[r, "j"'] & \mathbb{P}^1 \end{tikzcd} \tag{3.5}$$

of holomorphic maps, where j and ι' are open embeddings and g' is proper. Now restricting the map $g' : \widetilde{X_{\Sigma'}} \longrightarrow \mathbb{P}^1$ to $\mathbb{C} \subset \mathbb{P}^1$ we set $K' = (g')^{-1}(\mathbb{C}) = \widetilde{X_{\Sigma'}} \setminus (g')^{-1}(\infty)$. Let $\kappa' : K' \longrightarrow \mathbb{C}$ be the restriction of g' to K'. Denote the normal crossing divisor $K' \setminus T$ in K' by D' and let $i_{D'} : D' \longrightarrow K'$ and $i_T : T \longrightarrow K'$ be the inclusions. Then we obtain also a commutative diagram:

$$\begin{tikzcd} T \arrow[r, "i_T"] \arrow[d, "f'"'] & K' \arrow[d, "\kappa'"] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C}. \end{tikzcd} \tag{3.6}$$

Note that the normal crossing divisor D' in K' is a union of horizontal T-orbits (which correspond to atypical faces $\gamma \prec \Gamma_\infty(f')$) and the horizontal exceptional divisors on the blow-up $\widetilde{X_{\Sigma'}}$ of $X_{\Sigma'}$ (see Section 2.3 for the details). By our induction hypothesis and Proposition 3.2, for $\lambda \notin A_{f'}$ the monodromies at infinity of the restrictions of κ' to these horizontal T-orbits have no λ-part. Moreover the corresponding monodromies at infinity over the horizontal exceptional divisors have only the eigenvalue $1 \in A_{f'}$ (see Section 2.3). On the other hand, by applying the functor $R\kappa'_* = R\kappa'_!$ to the distinguished triangle

$$(i_T)_!\mathbb{C}_T \longrightarrow R(i_T)_*\mathbb{C}_T \longrightarrow (i_{D'})_* i_{D'}^{-1}(R(i_T)_*\mathbb{C}_T) \longrightarrow +1 \tag{3.7}$$

we obtain a distinguished triangle

$$R(f')_!\mathbb{C}_T \longrightarrow R(f')_*\mathbb{C}_T \longrightarrow R(\kappa'|_{D'})_* i_{D'}^{-1}(R(i_T)_*\mathbb{C}_T) \longrightarrow +1. \tag{3.8}$$

Then by using the above description of $\kappa'|_{D'} : D' \longrightarrow \mathbb{C}$, for $\lambda \notin A_{f'}$ we can easily show the vanishing

$$\psi_{h,\lambda}(j_! R(\kappa'|_{D'})_* i_{D'}^{-1}(R(i_T)_*\mathbb{C}_T)) \simeq 0, \tag{3.9}$$

where $\psi_{h,\lambda}$ is the λ-part of the nearby cycle functor ψ_h. Indeed, by our construction of the normal crossing divisor D' there exists a Whitney stratification $D' = \sqcup_\alpha S_\alpha$ of D' such that the restriction of $H^j i_{D'}^{-1}(R(i_T)_*\mathbb{C}_T)$ to S_α is constant for any α and $j \in \mathbb{Z}$, and the monodromy at infinity of the restriction $\kappa'|_{S_\alpha} : S_\alpha \longrightarrow \mathbb{C}$ of κ' to S_α has only eigenvalues in the set $A_{f'} \subset \mathbb{C}$. Let $i_{S_\alpha} : S_\alpha \hookrightarrow D'$ be the inclusion map. Then by truncation functors in the derived category of constructible sheaves on D' and some standard distinguished triangles associated to the stratification $D' = \sqcup_\alpha S_\alpha$ we can decompose the constructible sheaf $i_{D'}^{-1}(R(i_T)_*\mathbb{C}_T)$ to the simpler ones $(i_{S_\alpha})_!\mathbb{C}_{S_\alpha}$ and reduce the proof of (3.9) to that of the vanishing

$$\psi_{h,\lambda}(j_!R(\kappa'|_{S_\alpha})_!\mathbb{C}_{S_\alpha}) \simeq 0 \qquad (\lambda \notin A_{f'}). \tag{3.10}$$

But this immediately follows from the fact that the monodromy at infinity of $\kappa'|_{S_\alpha} : S_\alpha \longrightarrow \mathbb{C}$ has only eigenvalues in $A_{f'}$. By (3.9) we thus obtain an isomorphism

$$\psi_{h,\lambda}(j_!R(f')_!\mathbb{C}_T) \simeq \psi_{h,\lambda}(j_!R(f')_*\mathbb{C}_T) \tag{3.11}$$

for $\lambda \notin A_{f'}$. Namely, for any $\lambda \notin A_{f'}$ and $j \in \mathbb{Z}$ we have an isomorphism

$$H_c^j((f')^{-1}(R);\mathbb{C})_\lambda \simeq H^j((f')^{-1}(R);\mathbb{C})_\lambda \qquad (R \gg 0). \tag{3.12}$$

Since the generic fiber $(f')^{-1}(R) \subset T$ $(R \gg 0)$ of f' is affine, the left (resp. right) hand side is zero for $j < n-1$ (resp. $j > n-1$). Hence we obtain the desired concentration

$$H^j((f')^{-1}(R);\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1) \tag{3.13}$$

for $R \gg 0$. Now the second assertion follows immediately from Proposition 3.2. Also the last assertion follows from the proof of Sabbah [24, Theorem 13.1] by using the isomorphism (3.11). This completes the proof. □

REMARK 3.5. — For a non-degenerate polynomial $f' \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the condition $0 \in \mathrm{Int}(NP(f'))$ implies that $f' : T = (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$ is "cohomologically tame at infinity" in the sense of Némethi-Sabbah [17] and Sabbah [24], and in this particular case the first assertion of Proposition 3.4 is due to [17]. Note also that Raibaut [22, page 67] defines the notion of "convenient" for Laurent polynomials precisely by the condition $0 \in \mathrm{Int}(NP(f'))$.

THEOREM 3.6. — *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-convenient polynomial such that* $\dim\Gamma_\infty(f) = n$. *Assume that f is non-degenerate at infinity. Then for any non-atypical eigenvalue $\lambda \notin A_f$ of f we have the concentration*

$$H^j(f^{-1}(R);\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1) \tag{3.14}$$

for the generic fiber $f^{-1}(R) \subset \mathbb{C}^n$ $(R \gg 0)$ of f. Moreover for such λ the relative monodromy filtration of $H^{n-1}(f^{-1}(R);\mathbb{C})_\lambda$ $(R \gg 0)$ coincides with the absolute one (up to some shift).

Proof. — We will freely use the notations in Section 2.3. For example, we consider the commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\ \iota\ } & \widetilde{X_\Sigma} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathbb{C} & \xrightarrow[\ j\ ]{} & \mathbb{P}^1. \end{array} \tag{3.15}$$

By restricting the map $g : \widetilde{X_\Sigma} \longrightarrow \mathbb{P}^1$ to $\mathbb{C} \subset \mathbb{P}^1$ we set $K = g^{-1}(\mathbb{C}) = \widetilde{X_\Sigma} \setminus g^{-1}(\infty)$ and $\kappa = g|_K : K \longrightarrow \mathbb{C}$. Set $D = K \setminus \mathbb{C}^n$ and let $i_D : D \longrightarrow K$ and $i : \mathbb{C}^n \longrightarrow K$ be the inclusions. Then we obtain also a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\ i\ } & K \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \kappa} \\ \mathbb{C} & = & \mathbb{C}. \end{array} \tag{3.16}$$

Note that the normal crossing divisor D in K is a union of horizontal T-orbits (which correspond to atypical faces $\gamma \prec \Gamma_\infty(f)$) and the horizontal exceptional divisors on $\widetilde{X_\Sigma}$. By Proposition 3.4, for $\lambda \notin A_f$ the monodromies at infinity of the restrictions of κ to these horizontal T-orbits have no λ-part. Moreover the corresponding monodromies at infinity over the horizontal exceptional divisors have only the eigenvalue $1 \in A_f$. On the other hand, by applying the functor $R\kappa_* = R\kappa_!$ to the distinguished triangle

$$i_!\mathbb{C}_{\mathbb{C}^n} \longrightarrow Ri_*\mathbb{C}_{\mathbb{C}^n} \longrightarrow (i_D)_* i_D^{-1}(Ri_*\mathbb{C}_{\mathbb{C}^n}) \longrightarrow +1 \tag{3.17}$$

we obtain a distinguished triangle

$$Rf_!\mathbb{C}_{\mathbb{C}^n} \longrightarrow Rf_*\mathbb{C}_{\mathbb{C}^n} \longrightarrow R(\kappa|_D)_* i_D^{-1}(Ri_*\mathbb{C}_{\mathbb{C}^n}) \longrightarrow +1. \tag{3.18}$$

Then by using the above description of $\kappa|_D : D \longrightarrow \mathbb{C}$, for $\lambda \notin A_f$ we can easily show

$$\psi_{h,\lambda}(j_! R(\kappa|_D)_* i_D^{-1}(Ri_*\mathbb{C}_{\mathbb{C}^n})) \simeq 0. \tag{3.19}$$

This implies that there exists an isomorphism

$$\psi_{h,\lambda}(j_! Rf_!\mathbb{C}_{\mathbb{C}^n}) \simeq \psi_{h,\lambda}(j_! Rf_*\mathbb{C}_{\mathbb{C}^n}). \tag{3.20}$$

Namely, for any $\lambda \notin A_f$ and $j \in \mathbb{Z}$ we have an isomorphism

$$H_c^j(f^{-1}(R);\mathbb{C})_\lambda \simeq H^j(f^{-1}(R);\mathbb{C})_\lambda \qquad (R \gg 0). \tag{3.21}$$

Since the generic fiber $f^{-1}(R) \subset \mathbb{C}^n$ ($R \gg 0$) of f is affine, the left (resp. right) hand side is zero for $j < n-1$ (resp. $j > n-1$). Hence we obtain the desired concentration

$$H^j(f^{-1}(R);\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1) \tag{3.22}$$

for $R \gg 0$. Moreover the last assertion follows from the proof of Sabbah [24, Theorem 13.1] by using the isomorphism (3.20). This completes the proof. □

REMARK 3.7. — As is clear from the proof above, Theorem 3.6 can be easily generalized to arbitrary polynomial maps $f : U \longrightarrow \mathbb{C}$ of affine algebraic varieties U. We leave the precise formulation to the reader.

If $n = 2$ the first assertion of Theorem 3.6 can be improved as follows.

LEMMA 3.8. — *Assume that a polynomial $f(x, y) \in \mathbb{C}[x, y]$ of two variables is non-degenerate at infinity and satisfies the condition* $\dim\Gamma_\infty(f) = 2$. *Then the generic fiber of $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is connected and hence $H^0(f^{-1}(R);\mathbb{C}) \simeq \mathbb{C}$ for $R \gg 0$. In particular, for any $\lambda \neq 1$ we have the concentration*

$$H^j(f^{-1}(R);\mathbb{C})_\lambda \simeq 0 \qquad (j \neq 1), \tag{3.23}$$

where $R \gg 0$.

Proof. — By the classification of open connected Riemann surfaces, it suffices to show that there is no decomposition of f of the form

$$f(x, y) = \hat{f}(\tilde{f}(x, y)) \tag{3.24}$$

by polynomials $\hat{f}(t)$ and $\tilde{f}(x, y)$ such that $\deg\hat{f}(t) \geq 2$. Assume that there exists such a decomposition $f = \hat{f} \circ \tilde{f}$ and set $m = \deg\hat{f} \geq 2$. Then we have $\Gamma_\infty(f) = m\Gamma_\infty(\tilde{f})$. Take a face at infinity $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$ satisfying $\dim\gamma = 1$ and let $\tilde{\gamma} \prec \Gamma_\infty(\tilde{f})$ be the corresponding one of $\Gamma_\infty(\tilde{f})$ such that $\gamma = m\tilde{\gamma}$. Denote by f_γ (resp. $\tilde{f}_{\tilde{\gamma}}$) the γ-part of f (resp. the $\tilde{\gamma}$-part of $\tilde{f}$). Then we have $f_\gamma = (\tilde{f}_{\tilde{\gamma}})^m$ (up to some non-zero constant multiple) for $m \geq 2$. This contradicts the non-degeneracy at infinity of f. □

For an element $[V] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, $V \in \mathrm{HS}^{\mathrm{mon}}$ with a quasi-unipotent endomorphism $\Theta\colon V \xrightarrow{\sim} V$, $p, q \geq 0$ and $\lambda \in \mathbb{C}$ denote by $e^{p,q}([V])_\lambda$ the dimension of the λ-eigenspace of the morphism $V^{p,q} \xrightarrow{\sim} V^{p,q}$ induced by Θ on the (p, q)-part $V^{p,q}$ of V. Then by Theorem 3.6 we obtain the following result.

COROLLARY 3.9. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and f is non-degenerate at infinity. Let $\lambda \notin A_f$. Then we have $e^{p,q}([H_f^\infty])_\lambda = 0$ for $(p, q) \notin [0, n-1] \times [0, n-1]$. Moreover for any $(p, q) \in [0, n-1] \times [0, n-1]$ we have the Hodge symmetry*

$$e^{p,q}([H_f^\infty])_\lambda = e^{n-1-q,n-1-p}([H_f^\infty])_\lambda. \tag{3.25}$$

Proof. — By Theorem 3.6, for $\lambda \notin A_f$ the weight filtration on the limit mixed Hodge structure $H^{n-1}(f^{-1}(R);\mathbb{C})_\lambda$ $(R \gg 0)$ which is originally defined to be the relative monodromy filtration coincides with the absolute one centered at $n-1$. This implies the Hodge symmetry

$$e^{p,q}([H_f^\infty])_\lambda = e^{n-1-q,n-1-p}([H_f^\infty])_\lambda. \tag{3.26}$$

for any $p, q \in \mathbb{Z}$. Then the remaining assertion immediately follows from the standard vanishing $e^{p,q}([H_f^\infty])_\lambda = 0$ for $(p,q) \notin [0,+\infty) \times [0,+\infty)$. □

Now by Theorems 2.17 and 3.6, Corollary 3.9 and the results in Section 2.2 (see the proof of [14, Theorem 5.7 (ii)]), we obtain the following theorem. In contrast to the tame case in [14, Theorem 5.7 (ii)] we need to take only the part of the motivic Milnor fiber at infinity of f which corresponds to the admissible faces at infinity of $\Gamma_\infty(f)$ (the same remark is applied also to Theorem 3.11).

Theorem 3.10. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and* f *is non-degenerate at infinity. Let* $\lambda \notin A_f$ *and* $k \geq 1$*. Then the number of the Jordan blocks for the eigenvalue* λ *with sizes* $\geq k$ *in* $\Phi_{n-1}^\infty \colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ *is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k,n-1+k} \left\{ \sum_\gamma e^{p,q}(\chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]))_\lambda \right\}, \tag{3.27}$$

where in the sum $\sum_\gamma$ *the face* γ *of* $\Gamma_\infty(f)$ *ranges through the admissible ones at infinity.*

By this theorem and the results of [14, Section 2] we can extend [14, Theorems 5.9, 5.14 and 5.16] to non-tame polynomials. Here we introduce only the extension of [14, Theorem 5.9]. Denote by $\mathrm{Cone}_\infty(f)$ the closed cone $\mathbb{R}_+\Gamma_\infty(f) \subset \mathbb{R}^n_+$ generated by $\Gamma_\infty(f)$. Let $q_1, \dots, q_l$ (resp. $\gamma_1, \dots, \gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces at infinity of $\Gamma_\infty(f)$ such that $q_i \in \mathrm{Int}(\mathrm{Cone}_\infty(f))$ (resp. the relative interior $\mathrm{rel.int}(\gamma_i)$ of γ_i is contained in $\mathrm{Int}(\mathrm{Cone}_\infty(f))$). For each q_i (resp. γ_i), denote by $d_i > 0$ (resp. $e_i > 0$) its lattice distance from the origin $0 \in \mathbb{R}^n$. For $1 \leq i \leq l'$, let Δ_i be the convex hull of $\{0\} \sqcup \gamma_i$ in $\mathbb{R}^n$. Then for $\lambda \neq 1$ and $1 \leq i \leq l'$ such that $\lambda^{e_i} = 1$ we set

$$\begin{aligned} n(\lambda)_i = \sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{ht}(v,\gamma_i) = k\} \\ + \sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{ht}(v,\gamma_i) = e_i - k\}, \end{aligned} \tag{3.28}$$

where k is the minimal positive integer satisfying $\lambda = \zeta_{e_i}^k$ and for $v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i)$ we denote by $\mathrm{ht}(v,\gamma_i)$ the lattice height of v from the base γ_i of Δ_i. Then we have the following extension of [14, Theorem 5.9] to non-tame polynomials.

THEOREM 3.11. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and* f *is non-degenerate at infinity. Let* $\lambda \notin A_f$. *Then we have*

1. *The number of the Jordan blocks for the eigenvalue* λ *with the maximal possible size* n *in* $\Phi_{n-1}^\infty\colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ *is equal to* $\sharp\{q_i \mid \lambda^{d_i} = 1\}$.
2. *The number of the Jordan blocks for the eigenvalue* λ *with the second maximal possible size* $n-1$ *in* Φ_{n-1}^∞ *is equal to* $\sum_{i\colon \lambda^{e_i}=1} n(\lambda)_i$.

REMARK 3.12. — By Proposition 3.4 we can similarly obtain the analogues of [14, Theorems 5.9, 5.14 and 5.16] for Laurent polynomials $f' \in \mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$. The results on the Jordan normal forms of their monodromies at infinity for the eigenvalues $\lambda \notin A_{f'}$ are explicitly described by the admissible faces at infinity of $\Gamma_\infty(f')$. We omit the details.

Moreover in the situation above, we can obtain also a closed formula for the multiplicities of the non-atypical eigenvalues $\lambda \notin A_f$ in the monodromy at infinity $\Phi_{n-1}^\infty\colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ as follows. We define the monodromy zeta function at infinity $\zeta_f^\infty(t) \in \mathbb{C}((t))$ of f by

$$\zeta_f^\infty(t) = \prod_{j=0}^{n-1} \det(\mathrm{id} - t\Phi_j^\infty)^{(-1)^j} \in \mathbb{C}((t)). \tag{3.29}$$

Then by our compactification $\widetilde{X_\Sigma}$ of $\mathbb{C}^n$ we obtain the following refinement of the previous results in [12] and [13]. In particular here we can remove the condition $(*)$ in [13, Theorem 3.1]. It can be removed also by applying Remark 3.3 to the tori in the standard decomposition (3.15) of $\mathbb{C}^n$ in the proof of [13, Theorem 3.1].

THEOREM 3.13. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and* f *is non-degenerate at infinity. Then we have*

$$\zeta_f^\infty(t) = \prod_\gamma (1 - t^{d_\gamma})^{(-1)^{s_\gamma - 1}\mathrm{Vol}_\mathbb{Z}(\gamma)} \in \mathbb{C}((t)), \tag{3.30}$$

where in the product $\prod_\gamma$ *the face* $\gamma \prec \Gamma_\infty(f)$ *ranges through those at infinity satisfying the condition* $m_\gamma = s_\gamma - \dim\gamma - 1 = 0$, *and* $\mathrm{Vol}_\mathbb{Z}(\gamma) \in \mathbb{Z}_{>0}$ *is the normalized* $(\dim\gamma)$*-dimensional volume of* γ *with respect to the lattice* $\mathbb{L}(\gamma) \cap \mathbb{Z}^n \simeq \mathbb{Z}^{\dim\gamma}$.

EXAMPLE 3.14. — Let $n = 3$ and consider a non-convenient polynomial $f(x,y,z)$ on $\mathbb{C}^3$ whose Newton polyhedron at infinity $\Gamma_\infty(f)$ is the convex hull of the points $(2,0,0), (0,2,0), (1,1,1) \in \mathbb{R}_+^3$ and the origin $0 = (0,0,0) \in \mathbb{R}^3$. Then the line segment connecting the point $(2,0,0)$ and the origin $0 \in \mathbb{R}^3$ is an atypical face of $\Gamma_\infty(f)$. Hence the 0-dimensional face at infinity

$\gamma = \{(2,0,0)\} \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$ contained in it is non-admissible. However it satisfies the condition $m_\gamma = s_\gamma - \dim\gamma - 1 = 1 - 0 - 1 = 0$. For the proof of Theorem 3.13 we have to consider also the contribution from such non-admissible faces at infinity of $\Gamma_\infty(f)$.

If we restrict ourselves to the non-atypical eigenvalues $\lambda \notin A_f$ for which we have the concentration

$$H^j(f^{-1}(R);\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1) \tag{3.31}$$

$(R \gg 0)$ in Theorem 3.6, we have the following result.

COROLLARY 3.15. — *Assume that* $\dim\Gamma_\infty(f) = n$ *and* f *is non-degenerate at infinity. Then for any* $\lambda \notin A_f$ *the multiplicity of the eigenvalue* λ *in the monodromy at infinity* Φ^∞_{n-1} *of* f *is equal to that of the factor* $(1-\lambda t) = \lambda \cdot (1/\lambda - t)$ *in the rational function*

$$\prod_\gamma (1-t^{d_\gamma})^{(-1)^{n-s_\gamma}\mathrm{Vol}_\mathbb{Z}(\gamma)} \in \mathbb{C}((t)), \tag{3.32}$$

where in the product $\prod_\gamma$ *the face* $\gamma \prec \Gamma_\infty(f)$ *of* $\Gamma_\infty(f)$ *ranges through the admissible ones at infinity satisfying the condition* $m_\gamma = s_\gamma - \dim\gamma - 1 = 0$.

4. Monodromies around atypical fibers

Let $f : U \longrightarrow \mathbb{C}$ be a polynomial map of an affine algebraic variety U and $B_f \subset \mathbb{C}$ the set of its bifurcation points. For a point $b \in B_f$ we choose sufficiently small $\varepsilon > 0$ such that

$$B_f \cap \{x \in \mathbb{C} \mid |x-b| \leq \varepsilon\} = \{b\} \tag{4.1}$$

and set $C_\varepsilon(b) = \{x \in \mathbb{C} \mid |x-b| = \varepsilon\} \subset \mathbb{C}$. Then we obtain a locally trivial fibration $f^{-1}(C_\varepsilon(b)) \longrightarrow C_\varepsilon(b)$ over the small circle $C_\varepsilon(b) \subset \mathbb{C}$ and the monodromy automorphisms

$$\Phi^b_j \colon H^j(f^{-1}(b+\varepsilon);\mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(b+\varepsilon);\mathbb{C}) \quad (j=0,1,\dots) \tag{4.2}$$

around the atypical fiber $f^{-1}(b) \subset U$ associated to it. Let h^b be a holomorphic local coordinate of $\mathbb{C}$ on a neighborhood of $b \in B_f$ such that $b = \{h^b(x) = 0\}$. Then for the object $\psi_{h^b}(Rf_!\mathbb{C}_U) \in \mathbf{D}^b_c(\{b\})$ we have isomorphisms

$$H^j\psi_{h^b}(Rf_!\mathbb{C}_U) \simeq H^j_c(f^{-1}(b+\varepsilon);\mathbb{C}) \qquad (j=0,1,\dots) \tag{4.3}$$

and the monodromy automorphisms on $H^j_c(f^{-1}(b+\varepsilon);\mathbb{C})$ are induced by the one on $\psi_{h^b}(Rf_!\mathbb{C}_U)$. To $H^j\psi_{h^b}(Rf_!\mathbb{C}_U)$ and the semisimple parts of the monodromy automorphisms acting on them, we can associate an element

$$[H^b_f] = \sum_{j\in\mathbb{Z}} (-1)^j [H^j\psi_{h^b}(Rf_!\mathbb{C}_U)] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}). \tag{4.4}$$

Recall that the weight filtration of $[H_f^b]$ is a relative one. In this situation, we can apply our methods in previous sections to the Jordan normal forms of Φ_j^b. For the sake of simplicity, let us assume here that the central fiber $f^{-1}(b) \subset U$ is reduced and has only isolated singular points $p_1, p_2, \dots, p_l \in f^{-1}(b) \subset U$. When f is non-tame at infinity, we have to consider also the singularities at infinity of f. For this purpose, let X be a smooth compactification of U for which there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\iota} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathbb{C} & \xrightarrow[j]{} & \mathbb{P}^1 \end{array} \tag{4.5}$$

of holomorphic maps. Here ι and j are inclusion maps and g is proper. We may assume also that the divisor at infinity $D = X \setminus U \subset X$ is normal crossing and all its irreducible components are smooth. We call the irreducible components of D contained in $g^{-1}(\infty) \subset D$ (resp. in $\overline{D \setminus g^{-1}(\infty)} \subset D$) "vertical" (resp. "horizontal") divisors at infinity of f in X. For the normal crossing divisor D let us consider the standard (minimal) stratification. Then for simplicity we assume also that the restriction $g|_{D\setminus g^{-1}(\infty)} : D \setminus g^{-1}(\infty) \longrightarrow \mathbb{C}$ of g to the horizontal part $D \setminus g^{-1}(\infty)$ of D has only stratified isolated singular points $p_{l+1}, \dots, p_{l+r}$ in $g^{-1}(b) \subset X$ and all of them are contained in the smooth part of $D \setminus g^{-1}(\infty)$. By our assumption on $f^{-1}(b) \subset U$ this implies that the hypersurface $\overline{f^{-1}(b)} = g^{-1}(b) \subset X$ in X has also an isolated singular point at each p_i $(l+1 \leq i \leq l+r)$.

Remark 4.1. — If $U = \mathbb{C}^n$, $b \neq f(0)$ and in addition to the conditions in Theorems 3.6 and 3.10 (i.e., $\dim \Gamma_\infty(f) = n$ and f is non-degenerate at infinity) we assume that for any atypical face $\gamma \prec \Gamma_\infty(f)$ such that $\dim \gamma < n-1$ the γ-part $f_\gamma : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$ of f does not have the critical value b, then the meromorphic extension g of f to the compactification $X = \widetilde{X_\Sigma}$ satisfies the above-mentioned property in general (see also Némethi-Zaharia [18], Zaharia [31]). In this case the stratified isolated singular points $p_{l+1}, \dots, p_{l+r}$ are on the $(n-1)$-dimensional horizontal T-orbits which correspond to the atypical facets of $\Gamma_\infty(f)$.

For $l+1 \leq i \leq l+r$, in a neighborhood of p_i the divisor D is smooth and the function $g|_D : D \simeq \mathbb{C}^{n-1} \longrightarrow \mathbb{C}$ has an isolated singular point at $p_i \in D$. Therefore we may consider the (local) Milnor monodromies of $g|_D : D \simeq \mathbb{C}^{n-1} \longrightarrow \mathbb{C}$ at $p_i \in D$. Denote by $A_{f,b} \subset \mathbb{C}$ the union of their eigenvalues and $1 \in \mathbb{C}$. Then by applying the proof of Theorem 3.6 to this situation, we obtain the following result. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ let $H^j(f^{-1}(b+\varepsilon); \mathbb{C})_\lambda \subset$

$H^j(f^{-1}(b+\varepsilon);\mathbb{C})$ be the generalized eigenspace for the eigenvalue λ of the monodromy Φ_j^b around $f^{-1}(b)$.

THEOREM 4.2. — *In the above situation of the "isolated singularities at infinity" of f, for any $\lambda \notin A_{f,b}$ we have the concentration*

$$H^j(f^{-1}(b+\varepsilon);\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1). \tag{4.6}$$

Moreover for such λ the relative monodromy filtration of $H^{n-1}(f^{-1}(b+\varepsilon);\mathbb{C})_\lambda$ coincides with the absolute one (up to some shift).

Proof. — The proof is similar to those of Proposition 3.4 and Theorem 3.6, and it suffices to prove the isomorphism

$$\psi_{h^b,\lambda}(j_!Rf_!\mathbb{C}_U) \simeq \psi_{h^b,\lambda}(j_!Rf_*\mathbb{C}_U) \tag{4.7}$$

for any $\lambda \notin A_{f,b}$. As in the proof of (3.9) this follows from the vanishing

$$\psi_{h^b,\lambda}(R(g|_D)_*\mathbb{C}_D) \simeq R\Gamma((g|_D)^{-1}(b);\psi_{h^b\circ(g|_D),\lambda}(\mathbb{C}_D)) \simeq 0 \tag{4.8}$$

for $\lambda \notin A_{f,b}$. Since $g|_{D\setminus g^{-1}(\infty)} : D\setminus g^{-1}(\infty) \longrightarrow \mathbb{C}$ has only stratified isolated singular points $p_{l+1},\dots,p_{l+r}$ in $(g|_D)^{-1}(b) \subset D$ and the eigenvalues of the Milnor monodromies of $g|_D$ there are contained in the set $A_{f,b}$, we obtain

$$\psi_{h^b\circ(g|_D),\lambda}(\mathbb{C}_D) \simeq 0 \qquad (\lambda \notin A_{f,b}) \tag{4.9}$$

and hence (4.8). □

COROLLARY 4.3. — *Let $\lambda \notin A_{f,b}$. Then we have $e^{p,q}([H_f^b])_\lambda = 0$ for $(p,q) \notin [0,n-1]\times[0,n-1]$. Moreover for any $(p,q) \in [0,n-1]\times[0,n-1]$ we have the Hodge symmetry*

$$e^{p,q}([H_f^b])_\lambda = e^{n-1-q,n-1-p}([H_f^b])_\lambda. \tag{4.10}$$

From now on we shall use Theorem 4.2 and Corollary 4.3 to describe explicitly the Jordan normal form of Φ_{n-1}^b in terms of some Newton polyhedra associated to f. For this purpose, assume moreover that for any $1 \leq i \leq l+r$ there exists a local coordinate $y=(y_1,y_2,\dots,y_n)$ of X on a neighborhood W_i of p_i such that $p_i = \{y=0\}$ and the local defining polynomial $f_i(y) \in \mathbb{C}[y_1,\dots,y_n]$ of the hypersurface $\overline{f^{-1}(b)} = g^{-1}(b)$ (for which we have $\overline{f^{-1}(b)} = \{f_i(y)=0\}$) is convenient and non-degenerate at $y=0$ (see [30] etc.). We assume also that for $l+1 \leq i \leq l+r$ we have $D=\{y_n=0\}$ in W_i. For $1 \leq i \leq l+r$ let $\Gamma_+(f_i) \subset \mathbb{R}^n_+$ be the Newton polyhedron of f_i at $y=0$. Moreover for $l+1 \leq i \leq l+r$ we set

$$\Gamma_+^\circ(f_i) = \Gamma_+(f_i) \cap \{v=(v_1,\dots,v_n) \in \mathbb{R}^n \mid v_n = 0\}. \tag{4.11}$$

Note that $\Gamma_+^\circ(f_i)$ is nothing but the Newton polyhedron of the restriction $f_i|_D$ of f_i to $D=\{y_n=0\}$.

DEFINITION 4.4. — In the above situation of the "isolated singularities at infinity" of f, we say that a complex number $\lambda \in \mathbb{C}$ is an atypical eigenvalue for $b \in B_f$ if either $\lambda = 1$ or there exists a compact face $\gamma \prec \Gamma_+^\circ(f_i)$ of $\Gamma_+^\circ(f_i)$ for some $l+1 \leq i \leq l+r$ such that $\lambda^{d_\gamma} = 1$. We denote by $A_{f,b}^\circ \subset \mathbb{C}$ the set of the atypical eigenvalues for $b \in B_f$.

By the main theorem of Varchenko [30] we have $A_{f,b} \subset A_{f,b}^\circ$. On the other hand, as in [3], [14] and [15], for $1 \leq i \leq l+r$ by a toric modification $\pi_i : Y_i \longrightarrow W_i$ of W_i we can explicitly construct the motivic Milnor fiber $\mathcal{S}_{f_i,p_i} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ of f_i at p_i. See [15] for the details. For $l+1 \leq i \leq l+r$ let $(W_i \cap D)' \subset Y_i$ be the proper transform of $W_i \cap D = \{y_n = 0\}$ by π_i and $\mathcal{S}_{f_i,p_i}^\circ \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ the base change of $\mathcal{S}_{f_i,p_i}$ by the inclusion map $Y_i \setminus (W_i \cap D)' \hookrightarrow Y_i$. Let $[Z_{f,b}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ be the class of the variety $Z_{f,b} = f^{-1}(b) \setminus \{p_1, p_2, \dots, p_l\}$ with the trivial action of $\hat{\mu}$ and set

$$\mathcal{S}_f^b = [Z_{f,b}] + \sum_{i=1}^{l} \mathcal{S}_{f_i,p_i} + \sum_{i=l+1}^{l+r} \mathcal{S}_{f_i,p_i}^\circ \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \tag{4.12}$$

Then as in [14, Theorem 4.4] and [22, Theorem 2.10], by applying [2, Theorem 4.2.1] and [8, Section 3.16] to our situation, we obtain the following result.

THEOREM 4.5. — *In* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ *we have the equality*

$$[H_f^b] = \chi_h(\mathcal{S}_f^b). \tag{4.13}$$

By Theorems 4.2 and 4.5 and Corollary 4.3, for any $\lambda \notin A_{f,b}^\circ$ we can describe explicitly the λ-part of the Jordan normal form of Φ_{n-1}^b as follows. For $1 \leq i \leq l+r$ let $\gamma \prec \Gamma_+(f_i)$ be a compact face of $\Gamma_+(f_i)$. Denote by Δ_γ the convex hull of $\{0\} \sqcup \gamma$ in $\mathbb{R}^n$. Let $\mathbb{L}(\Delta_\gamma)$ be the $(\dim\gamma+1)$-dimensional linear subspace of $\mathbb{R}^n$ spanned by Δ_γ and consider the lattice $M_\gamma = \mathbb{Z}^n \cap \mathbb{L}(\Delta_\gamma) \simeq \mathbb{Z}^{\dim\gamma+1}$ in it. Then we set $T_{\Delta_\gamma} := \mathrm{Spec}(\mathbb{C}[M_\gamma]) \simeq (\mathbb{C}^*)^{\dim\gamma+1}$. Moreover for the points $v \in M_\gamma$ we define their lattice heights $\mathrm{ht}(v,\gamma) \in \mathbb{Z}$ from the affine hyperplane $\mathbb{L}(\gamma)$ in $\mathbb{L}(\Delta_\gamma)$ so that we have $\mathrm{ht}(0,\gamma) = d_\gamma > 0$. Then to the group homomorphism $M_\gamma \longrightarrow \mathbb{C}^*$ defined by $v \longmapsto \zeta_{d_\gamma}^{-\mathrm{ht}(v,\gamma)}$ we can naturally associate an element $\tau_\gamma \in T_{\Delta_\gamma}$. We define a Laurent polynomial $g_\gamma = \sum_{v \in M_\gamma} b_v y^v$ on T_{Δ_γ} by

$$b_v = \begin{cases} a_v & (v \in \gamma), \\ -1 & (v = 0), \\ 0 & (\text{otherwise}), \end{cases} \tag{4.14}$$

where $f_i = \sum_{v \in \mathbb{Z}_+^n} a_v y^v$. Then we have $NP(g_\gamma) = \Delta_\gamma$, $\mathrm{supp} g_\gamma \subset \{0\} \sqcup \gamma$ and the hypersurface $Z_{\Delta_\gamma}^* = \{y \in T_{\Delta_\gamma} \mid g_\gamma(y) = 0\}$ is non-degenerate by [14, Proposition 5.3]. Moreover $Z_{\Delta_\gamma}^* \subset T_{\Delta_\gamma}$ is invariant by the multiplication

$l_{\tau_\gamma}: T_{\Delta_\gamma} \xrightarrow{\sim} T_{\Delta_\gamma}$ by τ_γ, and hence we obtain an element $[Z^*_{\Delta_\gamma}]$ of $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$. Finally we define $m_\gamma \in \mathbb{Z}_+$ as in Section 2.2. Then (as in [14, Theorem 5.7] and [15, Theorem 4.3]) by using Theorems 4.2 and 4.5 and Corollary 4.3 we obtain the following results. In contrast to the tame case in [14, Theorem 5.7] and [15, Theorem 4.3], we need to take only the part of the motivic Milnor fiber $\mathcal{S}^b_f$ which corresponds to compact faces of $\Gamma_+(f_i)$ $(1 \leq i \leq l+r)$ not contained in $\Gamma^\circ_+(f_i)$ $(l+1 \leq i \leq l+r)$.

Theorem 4.6. — *In the above situation of the "isolated singularities at infinity" of f, for any $\lambda \notin A^\circ_{f,b}$ we have the equality*

$$[H^b_f]_\lambda = \chi_h(\mathcal{S}^b_f)_\lambda = \sum_{i=1}^{l+r} \sum_{\gamma \prec \Gamma_+(f_i)} \chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}])_\lambda \tag{4.15}$$

in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, where in the sum $\sum_{\gamma \prec \Gamma_+(f_i)}$ for $l+1 \leq i \leq l+r$ the face γ of $\Gamma_+(f_i)$ ranges through compact ones not contained in $\Gamma^\circ_+(f_i)$.

Theorem 4.7. — *In the above situation of the "isolated singularities at infinity" of f, let $\lambda \notin A^\circ_{f,b}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in Φ^b_{n-1} is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k,n-1+k} \left\{ \sum_{i=1}^{l+r} \sum_{\gamma \prec \Gamma_+(f_i)} e^{p,q}(\chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]))_\lambda \right\}, \tag{4.16}$$

where in the sum $\sum_{\gamma \prec \Gamma_+(f_i)}$ for $l+1 \leq i \leq l+r$ the face γ of $\Gamma_+(f_i)$ ranges through compact ones not contained in $\Gamma^\circ_+(f_i)$.

By this theorem and the results in [14, Section 2], for $\lambda \notin A^\circ_{f,b}$ we immediately obtain the analogues of [14, Theorems 5.9, 5.14 and 5.16] for the λ-part of the Jordan normal form of Φ^b_{n-1}. More precisely it suffices to neglect the compact faces of $\Gamma^\circ_+(f_i)$ for $l+1 \leq i \leq l+r$. We omit the details.

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