TWO DIMENSIONAL WATER WAVES IN HOLOMORPHIC COORDINATES II: GLOBAL SOLUTIONS

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Abstract. — This article is concerned with the infinite depth water wave equation in two space dimensions. We consider this problem expressed in position-velocity potential holomorphic coordinates, and prove that small localized data leads to global solutions. This article is a continuation of authors' earlier paper [8].

1. Introduction

We consider the two dimensional infinite depth water wave equations with gravity but without surface tension. This is governed by the incompressible Euler's equations with boundary conditions on the water surface. Under the additional assumption that the flow is irrotational the fluid dynamics can be expressed in terms of a one-dimensional evolution of the water surface coupled with the trace of the velocity potential on the surface.

Mihaela Ifrim, Department of Mathematics, University of California at Berkeley •
E-mail : ifrim@math.berkeley.edu

Daniel Tataru, Department of Mathematics, University of California at Berkeley •
E-mail : tataru@math.berkeley.edu

This problem was previously considered by many other authors. The local in time existence and uniqueness of solutions was proved in [14, 21, 17], both for finite and infinite depth. Later, Wu [19] proved almost global existence for small localized data. Very recently, global results for small localized data were independently obtained by Alazard-Delort [3] and by Ionescu-Pusateri [11]. Extensive work was also done on the same problem in three or higher space dimensions, and also on related problems with surface tension, vorticity, finite bottom, etc. Without being exhaustive, we list some of the more recent references [2, 1, 5, 6, 7, 12, 4, 13, 15, 16, 18, 20, 22].

An essential choice in any approach to this problem is that of the coordinates used. The citations above largely rely on either Eulerian or Lagrangian coordinates. Instead, the present article relies on holomorphic coordinates, which were originally introduced by Nalimov [14]; these are briefly described below. In the earlier article [8], using holomorphic coordinates, we revisited this problem in order to provide a new, self-contained approach, which considerably simplified and improved on many of the results mentioned above. Our results included:

(i) local well-posedness in Sobolev spaces, improving on previous regularity thresholds, e.g., those in [2].

(ii) cubic lifespan bounds for small data. These are proved using a *modified energy method*, first introduced in the authors' previous article [9]. The idea there is that instead of trying to transform the equation using the normal form method, which does not work well in quasilinear settings, one can produce quasilinear energy functionals, which are conserved to cubic order.

(iii) almost global well-posedness for small localized data, providing a shorter, simpler alternative to Wu's approach in [19], with lower regularity thresholds.

Here we improve the result in (iii) to a global statement, improving and simplifying the earlier results of Alazard-Delort [3] and by Ionescu-Pusateri [11] in several ways; see the discussion following our main theorem.

We first recall the set-up and the equations. We denote the water domain at time t by $\Omega(t)$, and the water surface at time t by $\Gamma(t)$. We think of $\Gamma(t)$ as being asymptotically flat at infinity. Rather than working in cartesian coordinates and the Eulerian setting, we use time dependent coordinates defined via a conformal map $\mathscr{T} : \mathbb{H} \to \Omega(t)$, where $\mathbb{H}$ is the lower half plane, $\mathbb{H} := \{\alpha + i\beta \; : \; \beta < 0\}$. We also have $\mathscr{T}(\mathbf{R}) = \Gamma(t)$. We call these the holomorphic coordinates.

The real variable α is then used to parametrize the free surface $\Gamma(t)$. We say that a function of α is holomorphic if its Fourier transform is supported in $(-\infty, 0]$. They can be described by the relation $Pf = f$, where the projector operator P to negative frequencies can be defined using the Hilbert transform

H as

$$P := \frac{1}{2}(I - iH).$$

Our variables (Z, Q) are functions of t and α which represent the position of the water surface $\Gamma(t)$, respectively the holomorphic extension of the velocity potential restricted to $\Gamma(t)$, expressed in the holomorphic coordinates. In view of our choice of coordinates, it is natural to consider the evolution of (Z, Q) within the closed subspace of holomorphic functions within various Sobolev spaces.

In position-velocity potential holomorphic coordinates the equations have the form

$$\begin{cases} Z_t + F Z_\alpha = 0 \\ Q_t + F Q_\alpha - i(Z - \alpha) + P\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0, \end{cases}$$

where

$$F := P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J := |Z_\alpha|^2.$$

For the derivation of the above equations we refer the reader to [8], Appendix A. With the substitution $W := Z - \alpha$ they become

(1.1)
$$\begin{cases} W_t + F(1 + W_\alpha) = 0 \\ Q_t + F Q_\alpha - iW + P\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0, \end{cases}$$

where

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |1 + W_\alpha|^2.$$

We can also differentiate and rewrite the system in terms of the diagonal variables

$$(\mathbf{W}, R) := \left(W_\alpha, \frac{Q_\alpha}{1 + W_\alpha}\right).$$

This yields the self-contained system

(1.2)
$$\begin{cases} \mathbf{W}_t + b\mathbf{W}_\alpha + \dfrac{(1 + \mathbf{W})R_\alpha}{1 + \bar{\mathbf{W}}} = (1 + \mathbf{W})M \\ R_t + bR_\alpha = i\left(\dfrac{\mathbf{W} - a}{1 + \mathbf{W}}\right), \end{cases}$$

where the real *advection velocity* b is given by

$$b := P\left[\frac{Q_\alpha}{J}\right] + \bar{P}\left[\frac{\bar{Q}_\alpha}{J}\right],$$

and the real *frequency-shift* a is

$$a := i\left(\bar{P}\left[\bar{R}R_\alpha\right] - P\left[R\bar{R}_\alpha\right]\right).$$

The auxiliary function M has the expression

$$M := \frac{R_\alpha}{1 + \bar{\mathbf{W}}} + \frac{\bar{R}_\alpha}{1 + \mathbf{W}} - b_\alpha = \bar{P}[\bar{R}Y_\alpha - R_\alpha\bar{Y}] + P[R\bar{Y}_\alpha - \bar{R}_\alpha Y],$$

written in terms of Y given by

$$Y := \frac{\mathbf{W}}{1 + \mathbf{W}}.$$

In particular, we remark that the linearization of the system (1.1) around the zero solution is a dispersive partial differential equation of the form

(1.3)
$$\begin{cases} w_t + q_\alpha = 0 \\ q_t - iw = 0. \end{cases}$$

Now we recall the function spaces introduced in [8]. The system (1.3) is a well-posed linear evolution in the space $\dot{\mathcal{H}}_0$ of holomorphic functions endowed with the $L^2 \times \dot{H}^{\frac{1}{2}}$ norm. A conserved energy for this system is

$$E_0(w, r) = \int \frac{1}{2}|w|^2 + \frac{1}{2i}(r\bar{r}_\alpha - \bar{r}r_\alpha)\, d\alpha.$$

The nonlinear system (1.1) also admits a conserved energy, which has the form

$$E(W, Q) = \int \frac{1}{2}|W|^2 + \frac{1}{2i}(Q\bar{Q}_\alpha - \bar{Q}Q_\alpha) - \frac{1}{4}(\bar{W}^2 W_\alpha + W^2\bar{W}_\alpha)\, d\alpha.$$

As suggested by the above energy, the function spaces for the differentiated water wave system (1.2) are the spaces $\dot{\mathcal{H}}_n$ endowed with the norm

$$\|(\mathbf{W}, R)\|^2_{\dot{\mathcal{H}}_n} := \sum_{k=0}^{n} \|\partial_\alpha^k(\mathbf{W}, R)\|^2_{L^2 \times \dot{H}^{\frac{1}{2}}},$$

where $n \geq 1$.

To describe the lifespan of the solutions we define the control norms

$$A := \|\mathbf{W}\|_{L^\infty} + \|Y\|_{L^\infty} + \|D^{\frac{1}{2}}R\|_{L^\infty \cap B_2^{0,\infty}},$$

respectively

$$B := \|D^{\frac{1}{2}}\mathbf{W}\|_{BMO} + \|R_\alpha\|_{BMO}.$$

Here A is a scale invariant quantity related to the critical homogeneous $\dot{\mathcal{H}}_{\frac{1}{2}}$ norm of $(\mathbf{W}, R)$, while B corresponds to the homogeneous $\dot{\mathcal{H}}_1$ norm of $(\mathbf{W}, R)$. We note that B and all but the Y component of A are directly controlled by the $\dot{\mathcal{H}}_1$ norm of the solution.

The main local well-posedness result in [8] is

THEOREM 1. — *Let $n \geq 1$. The system (1.2) is locally well-posed for data in $\dot{\mathcal{H}}_n(\mathbb{R})$ so that $|\mathbf{W} + 1| > c > 0$. Further, the solution can be continued for as long as A and B remain bounded.*

To state the global result we need to return to the original set of variables (W, Q). We also take advantage of the scale invariance of the water wave equations. Precisely, it is invariant with respect to the scaling law

$$(W(t, \alpha), Q(t, \alpha)) \to (\lambda^{-2} W(\lambda t, \lambda^2 \alpha), \lambda^{-3} Q(\lambda t, \lambda^2 \alpha)).$$

This suggests that we should use the scaling vector field

$$S = t\partial_t + 2\alpha\partial_\alpha,$$

and its action on the pair (W, Q), namely

$$\mathbf{S}(W, Q) := ((S - 2)W, (S - 3)Q),$$

which solve the linearized equations, see [8]. However, these are not the correct diagonal variables; instead the diagonal variables are $\mathbf{AS}(W, Q)$, where the diagonalization operator $\mathbf{A}$ is given by

$$\mathbf{A}(w, q) := (w, q - Rw).$$

Then we define the weighted energy

$$\|(W, Q)(t)\|_{\mathcal{WH}}^2 := \|(W, Q)(t)\|_{\mathcal{H}_0}^2 + \|(\mathbf{W}, R)(t)\|_{\mathcal{H}_6}^2 + \|\mathbf{AS}(W, Q)(t)\|_{\mathcal{H}_0}^2.$$

To control the evolution of the weighted energy we still use a pointwise type control norm, but one which is somewhat stronger than A and B. Precisely, we define

$$\|(W, R)\|_X := \|W\|_{L^\infty} + \|R\|_{L^\infty} + \||D|^{\frac{3}{2}} W\|_{L^\infty} + \|DR\|_{L^\infty}.$$

Now we can state our main result:

THEOREM 2. — a) *(Global solutions) Let $\epsilon \ll 1$. Then for each initial data $(W(0), Q(0))$ for the system* (1.1) *satisfying*

$$(1.4) \qquad \|(W, Q)(0)\|_{\mathcal{WH}}^2 \leq \epsilon,$$

the solution is global, and satisfies

$$(1.5) \qquad \|(W, Q)(t)\|_{\mathcal{WH}}^2 \lesssim \epsilon t^{C\epsilon^2},$$

as well as

$$(1.6) \qquad \|(W, R)\|_X \lesssim \frac{\epsilon}{\sqrt{t}}.$$

b) *(Asymptotic profile) There exists a function Ψ satisfying*

$$\|(1 + v^{-2})^{-5} \Psi\|_{L^2} + \|v\partial_v \Psi\|_{L^2} \lesssim \epsilon,$$

so that we have the asymptotic formulas

(1.7)

$$(W, Q)(t, \alpha) = \frac{1}{\sqrt{t}} e^{i \frac{t^2}{4\alpha}} \Psi(\alpha/t) e^{\frac{i}{2}(2\alpha/t)^{-5} \ln t |\Psi(\alpha/t)|^2} \left(1, \frac{t}{2\alpha}\right) + e_\alpha(t, \alpha/t),$$

$$(\hat{W}, \hat{Q})(t, \xi) = \sum_{-4v^2 = \xi} |v|^{-\frac{3}{2}} e^{\pm it \sqrt{|\xi|}} \left[\Psi(v) e^{\frac{i}{2}(2v)^{-5} \ln t |\Psi(v)|^2} (1, (2v)^{-1}) + e_\xi(t, \xi)\right],$$

where the last sum has two terms, depending on the sign of v, and the errors e_α and e_ξ satisfy bounds of the form

(1.8) $$e_\alpha(t, v) = O_{L^\infty}(\epsilon t^{-\frac{6}{11}}), \quad e_\xi(t, \xi) = O_{L^\infty}(\epsilon t^{-\frac{1}{22}}).$$

We recall that results of this type were recently proved in work of Alazard-Delort [3] and Ionescu-Pusateri [11]. Our result here, based on the setup in the previous article [8], provides a stronger statement in terms of the Sobolev regularity of the data, both at high frequency compared to [3] and [11], and at low frequency compared to [11]. We also give a much simpler proof, based on the idea of testing with wave packets. The main idea of the proof is described in the simpler setting of the one dimensional cubic NLS in the companion article [10]. However, this article does not rely directly on any of the results proved in [10]. While not needed for the proof of the global well-posedness result, the modified scattering type asymptotic profile in part (b) is easily obtained as a direct byproduct of our proof. A similar asymptotic profile, but in Eulerian coordinates, was obtained by Alazard and Delort in [3].

The organization of the paper is as follows. In the next section we set-up the bootstrap argument for the proof of part (a) of the theorem. The energy estimates were already proved in [8], and they are simply recalled here. In preparation for the proof of the pointwise bounds we also recall the normal form transformation associated to the two dimensional water wave equation, along with some related energy bounds. The core of the paper is Section 3, where we prove the global pointwise bounds and close the bootstrap argument. Our approach is based on the idea of testing with wave packets, which was first developed in [10]. The asymptotic profile is readily obtained at the end of the proof.

2. The bootstrap setup

The main difficulty in the proof of the theorem is in establishing the sharp pointwise decay rate in (1.6). Since in our context $\|(W, R)(t)\|_X$ is a continuous function of t, without any loss of generality , we can make the bootstrap

assumption

$$(2.1) \qquad \|(W,R)(t)\|_X \le \frac{C\epsilon}{\sqrt{t}}$$

with a fixed large universal constant C. Here C is chosen with the property that

$$1 \ll C \ll \epsilon^{-\frac{1}{2}}.$$

Then we need to show that for any solution satisfying (1.4) and (2.1) in a time interval $[0,T]$, we must also have (1.5) and (1.6).

The energy estimates are nontrivial, but they were already established in [8]. Precisely, from Proposition 6.1 in [8] we have

PROPOSITION 2.1. — *Assume that (1.4) and (2.1) hold in a time interval* $[0,T]$. *Then we also have the energy estimate*

$$(2.2) \qquad \|(W,Q)(t)\|_{\mathcal{WH}}^2 \lesssim \epsilon t^{C_*^2 \epsilon^2}, \qquad t \in [0,T], \qquad C_* \lesssim C.$$

Thus, the bound (1.5) follows directly from our bootstrap assumption. A pointwise bound is also established in [8], see Proposition 6.2:

PROPOSITION 2.2. — *Assume that (2.2) holds in a time interval* $[0,T]$. *Then we also have the pointwise bounds*

$$(2.3) \qquad |W| + |W_\alpha| + \||D|^{\frac{1}{2}} W_\alpha| + \||D|^{\frac{1}{2}} Q| + |R| + |R_\alpha| \lesssim \epsilon t^{C_*^2 \epsilon^2} \omega(\alpha,t),$$

where

$$\omega(\alpha,t) := \frac{1}{t^{\frac{1}{22}}} + \frac{1}{(|\alpha|/t + t/|\alpha|)^{\frac{1}{2}}}.$$

We remark that this proposition gives the bound

$$(2.4) \qquad \|(W,R)\|_X \lesssim \epsilon t^{C_*^2 \epsilon^2} t^{-\frac{1}{2}},$$

which is only sufficient in order to close the bootstrap up to an exponential time T, $T \lesssim e^{-c\epsilon^{-2}}$, and thus prove the almost global result. The bound (2.4) will not be so useful to us since we already have the bootstrap assumption (2.1). However, we can get more use out of (2.3); what (2.3) shows is that there is extra decay away from $|\alpha| \approx t$, so it suffices to improve the pointwise bound in (2.3) in a region of the form

$$(2.5) \qquad \Omega = \left\{ t^{-\frac{1}{11}} \lesssim \frac{|\alpha|}{t} \lesssim t^{\frac{1}{11}} \right\}.$$

Here the threshold $\frac{1}{11}$ was chosen somewhat arbitrarily; any smaller power would work as well.

The goal of the remainder of the paper is to establish the bound (1.6) in Ω. A key tool in this endeavor is the normal form transformation, which is discussed next.

2.1. The normal form transformation. — For the almost global result in [8], as well as for the global result here, a very useful observation is that the quadratically nonlinear terms may be removed from the water-wave equations by the near-identity, normal form transformation

$$(2.6) \qquad \tilde{W} = W - 2\mathfrak{M}_{\Re W} W_\alpha, \qquad \tilde{Q} = Q - 2\mathfrak{M}_{\Re W} R,$$

where the holomorphic multiplication operator $\mathfrak{M}_f$ is given by $\mathfrak{M}_f g = P[fg]$. For a more symmetric form of this transformation, one can replace R by Q_α. However, it is more convenient to use the diagonal variable R. The goal of normal form transformation is to remove the quadratic terms in the equation. Precisely, we have

PROPOSITION 2.3. — *The normal form variables solve a cubic equation:*

$$(2.7) \qquad \begin{cases} \tilde{W}_t + \tilde{Q}_\alpha = \tilde{G} \\ \tilde{Q}_t - i\tilde{W} = \tilde{K}, \end{cases}$$

where $\tilde{G}, \tilde{K}$ *are cubic (and higher order) functions of* $(W, \mathbf{W}, R, \mathbf{W}_\alpha, R_\alpha)$, *given by*

$$(2.8) \qquad \begin{cases} \tilde{G} = 2P[(F - R)_\alpha \Re W + \mathbf{W}_\alpha F \Re W + \mathbf{W} \Re(\mathbf{W} F) + F_\alpha \mathbf{W} \Re W] \\ \qquad - P[\bar{\mathbf{W}} R \bar{Y} - \mathbf{W}(P[\bar{R} Y] + \bar{P}[R\bar{Y}])] \\ \tilde{K} = P\left[((1 + \bar{\mathbf{W}})\bar{F} - \bar{R})R + 2P[bR_\alpha]\Re W + 2iP\left[\dfrac{\mathbf{W}^2 + a}{1 + \mathbf{W}}\right]\Re W\right]. \end{cases}$$

We remark here that the normal form transformation cannot be used directly to study well-posedness questions for the water wave equation as the cubic and higher order terms on the right are higher order than the leading linear part. However, in [8] we were able to use it in order to derive the pointwise bounds in (2.3), and here we will be able to further use it to get to (1.6).

In order to work with $(\tilde{W}, \tilde{Q})$ instead of (W, Q) we need to be able to transfer the energy information from (W, Q) to $(\tilde{W}, \tilde{Q})$, and the pointwise bounds in the opposite direction. This was also done in [8]:

PROPOSITION 2.4. — *Assume that the energy bound* (2.2) *holds in a time interval* $[0, T]$. *Then we have the following estimates for* $(\tilde{W}, \tilde{Q})$:
  (i) *Energy estimates:*

$$(2.9) \qquad \qquad \|(\tilde{W}, \tilde{Q})\|_{\mathcal{H}_5} \lesssim \epsilon t^{C_*^2 \epsilon^2},$$

$$(2.10) \qquad \|(2\alpha\partial_\alpha \tilde{W} + t\partial_\alpha \tilde{Q}, 2\alpha\partial_\alpha \tilde{Q} - it\tilde{W})\|_{\mathcal{H}_0 + \mathcal{H}_{-1}} \lesssim \epsilon t^{C_*^2 \epsilon^2}.$$

  (ii) *Pointwise comparison:*

$$(2.11) \qquad \qquad \|(W - \tilde{W}, R - \tilde{Q}_\alpha)\|_X \lesssim \epsilon^2 t^{-\frac{5}{8} + 2C_* \epsilon^2}.$$

For the proof of this result we refer the reader to the corresponding results in [8] as follows. The energy estimates in (2.9) are contained in Lemma 6.4. The bound (2.10) is based on the computation in (6.13), so it requires both Lemma 6.4 and Lemma 6.5. The estimate (2.11) is a consequence of Lemma 6.3, and is used in [8] to prove that the bound (2.3) for (W, R) is equivalent to its counterpart for $(\tilde{W}, \tilde{Q}_\alpha)$, namely

$$(2.12) \qquad |\tilde{W}| + ||D|^{\frac{1}{2}} \tilde{W}_\alpha| + ||D|^{\frac{1}{2}} \tilde{Q}| + |\tilde{Q}_{\alpha\alpha}| \lesssim \epsilon t^{C_*^2 \epsilon^2} \omega(\alpha, t).$$

We remark that in view of (2.11), for the proof of (1.6), it suffices to obtain the uniform bounds associated to $(\tilde{W}, \tilde{Q})$,

$$(2.13) \qquad |\tilde{W}| + ||D|^{\frac{1}{2}} \tilde{W}_\alpha| + ||D|^{\frac{1}{2}} \tilde{Q}| + |\tilde{Q}_{\alpha\alpha}| \lesssim \epsilon t^{-\frac{1}{2}}$$

in the region Ω.

Next we consider the right hand side terms $\tilde{G}$ and $\tilde{K}$ in the equations for $(\tilde{W}, \tilde{Q})$. It is convenient to decompose them into cubic and higher terms.

$$\tilde{G} = \tilde{G}^{(3)} + \tilde{G}^{(4+)}, \qquad \tilde{K} = \tilde{K}^{(3)} + \tilde{K}^{(4+)},$$

where

$$(2.14) \quad \begin{cases} \tilde{G}^{(3)} = 2P[-\partial_\alpha P\left[R\bar{\mathbf{W}} - \bar{R}\mathbf{W}\right]\Re W + (R\mathbf{W})_\alpha \Re W + \mathbf{W}\Re(\mathbf{W}R)] \\ \qquad - P[\bar{\mathbf{W}}^2 R - \mathbf{W}(P[\bar{R}\mathbf{W}] + \bar{P}[R\bar{\mathbf{W}}])] \\ \tilde{K}^{(3)} = 2P\left[P[(R+\bar{R})R_\alpha]\Re W + i\mathbf{W}^2\Re W + P[R\bar{R}_\alpha]\Re W\right] \\ \qquad + P\left[\bar{R}\bar{\mathbf{W}}R - R\bar{P}\left[\bar{R}\mathbf{W} - R\bar{\mathbf{W}}\right]\right]. \end{cases}$$

The higher order terms play a perturbative role in the long time behavior, because of the better decay. However, the cubic terms may drive the asymptotic dynamics, and need to be considered in greater detail. For this reason, it is also convenient to express the cubic terms as cubic in $(\tilde{W}, \tilde{Q}_\alpha)$ rather than (W, R). For the perturbative part of the terms above we have the following

PROPOSITION 2.5. — *Assume that the bound* (2.2) *holds. Then we have*

$$(2.15) \qquad \|(\tilde{G}^{4+}, \tilde{K}^{4+})\|_{\dot{\mathcal{H}}_0} \lesssim \epsilon^4 t^{-\frac{3}{2}+3C_*^2\epsilon^2},$$

respectively

$$(2.16) \quad \|(\tilde{G}^3(W,R), \tilde{K}^3(W,R)) - (\tilde{G}^3(\tilde{W}, \tilde{Q}_\alpha), \tilde{K}^3(\tilde{W}, \tilde{Q}_\alpha))\|_{\dot{\mathcal{H}}_0} \lesssim \epsilon^4 t^{-\frac{3}{2}+3C_*^2\epsilon^2}.$$

Proof. — The first estimate (2.15) follows from by interpolation and Hölder's inequality from the bounds

$$\|(W_\alpha, R)\|_{\dot{\mathcal{H}}_5} \lesssim \epsilon t^{C_*^2\epsilon^2},$$

$$|W| + |W_\alpha| + ||D|^{\frac{1}{2}} W_\alpha| + |R| + |R_\alpha| \lesssim \epsilon t^{-\frac{1}{2}+C_*^2\epsilon^2},$$

which in turn are consequences of (2.2) and (2.3). For the second bound (2.16) we also need to use once the estimate (2.11). The details are somewhat tedious but routine, and are left for the reader.   □

3. Pointwise decay

3.1. Testing by packets. — In order to establish the global pointwise decay estimates we use the method of testing by wave packets, first introduced in the companion paper [10] in the context of the one dimensional cubic NLS equation. The procedure we apply is very simple; we pick a ray $\{\alpha = vt\}$ and establish decay along this ray by testing with a wave packet moving along the ray. A wave packet is an approximate solution to the linear system (1.3), with $O(1/t)$ errors.

To motivate the definition of this packet we recall some useful facts. In view of the dispersion relation $\tau = \pm\sqrt{|\xi|}$, a ray with velocity v is associated with waves which have spatial frequency

$$\xi_v = -\frac{1}{4v^2}.$$

Secondly, for waves with initial data localized at the origin, the spatial frequency corresponding with a position (α, t) is

$$\xi(\alpha, t) = -\frac{t^2}{4\alpha^2}.$$

This is associated with the phase function

$$\phi(t, \alpha) = \frac{t^2}{4\alpha}.$$

Then our wave packets will be combinations of functions of the

$$\mathbf{u}(t, \alpha) = v^{-\frac{3}{2}} \chi\left(\frac{\alpha - vt}{t^{\frac{1}{2}} v^{\frac{3}{2}}}\right) e^{i\phi(t,\alpha)},$$

where χ is a smooth compactly supported bump function with integral one

$$(3.1) \qquad\qquad \int \chi(y)\, dy = 1.$$

Our packets are localized around the ray $\{\alpha = vt\}$ on the scale $\delta\alpha = t^{\frac{1}{2}} v^{\frac{3}{2}}$. This exact choice of scale is determined by the phase function ϕ. Precisely, the quadratic expansion of ϕ near $\alpha = vt$ reads

$$\phi(t, \alpha) = \phi(t, vt) + (\alpha - vt)\phi_\alpha(t, vt) + O(t^{-1}v^{-3}(\alpha - vt)^2),$$

and our scale $\delta\alpha$ represents exactly the scale on which ϕ is well approximated by its linearization. We further remark that there is a threshold $v \approx t$ above

which ϕ is essentially zero, and the above considerations are no longer relevant. We confine our analysis to the region where ϕ is strongly oscillatory,

$$|v| \ll t^{\frac{1}{2}}.$$

The power $\frac{1}{2}$ here is somewhat arbitrary, any choice less than 1 would do. Under this assumption, the function $\mathbf{u}$ is strongly localized at frequency ξ_v. For later use, we record here some ways to express this localization.

LEMMA 3.1. — *a) Let $\mathbf{u}$ be defined as above. Then its Fourier transform and that of $\partial_v \mathbf{u}$ have the form*

(3.2)
$$\hat{\mathbf{u}}(\xi) = t^{\frac{1}{2}} \chi_1 \left(\frac{\xi + (4v^2)^{-1}}{t^{-\frac{1}{2}} v^{-\frac{3}{2}}} \right) e^{-it\sqrt{|\xi|}},$$
$$\partial_v \hat{\mathbf{u}}(\xi) = tv^{-\frac{3}{2}} \chi_2 \left(\frac{\xi + (4v^2)^{-1}}{t^{-\frac{1}{2}} v^{-\frac{3}{2}}} \right) e^{-it\sqrt{|\xi|}},$$

where χ_1 and χ_2 are Schwartz functions so that in addition,

(3.3)
$$\int \chi_1(\xi)\, d\xi = 1 + O(v^{\frac{1}{2}} t^{-\frac{1}{2}}).$$

b) For $s \geq 0$, $\lambda_v = (4v^2)^{-1}$ and P_{λ_v} the associated dyadic frequency projector we have

(3.4)
$$P_{\lambda_v}(|D|^s - (4v^2)^{-s})\mathbf{u}(\alpha, t) = (4v^2)^{-s} t^{-\frac{1}{2}} v^{\frac{1}{2}} \chi_3 \left(\frac{\alpha - vt}{t^{\frac{1}{2}} v^{\frac{3}{2}}} \right) e^{i\phi(t,\alpha)},$$

where χ_3 is also a Schwartz function.

The proof of the lemma is straightforward, and left for the reader. In order to obtain (3.3), the key idea is to replace phases by their quadratic approximations; see also the similar computation in [10].

Applying the method of testing by wave packets for the water wave equation is slightly more complicated than in the case of the cubic NLS in [10] due the fact that we are dealing with a system, and we need to choose the two components to match. However, our system is simple enough, so is suffices to first choose the Q component and then use the second of the two linear equations in (1.3) to match W,

$$(\mathbf{w}, \mathbf{q}) = (-iv\partial_t \mathbf{u}, v\mathbf{u}).$$

Then we have

(3.5) $\mathbf{w} = \dfrac{1}{2}\mathbf{u} + \left(\dfrac{vt - \alpha}{2\alpha} \chi \left(\dfrac{\alpha - vt}{t^{\frac{1}{2}} v^{\frac{3}{2}}} \right) + \dfrac{i(vt + \alpha)}{2t^{\frac{3}{2}} v^{\frac{1}{2}}} \chi' \left(\dfrac{\alpha - vt}{t^{\frac{1}{2}} v^{\frac{3}{2}}} \right) \right) v^{-\frac{3}{2}} e^{i\phi(t,\alpha)}.$

The second term above is better by a $v^{\frac{1}{2}}t^{-\frac{1}{2}}$ factor, so it will play a negligible role in most of our analysis. However, it is crucial in improving the error in the first linear equation in (1.3), which is given by

$$(3.6) \qquad\qquad \mathbf{g} := \partial_t \mathbf{w} + \partial_\alpha \mathbf{q} = v(\partial_\alpha - i\partial_t^2)\mathbf{u}.$$

Indeed, computing the error in (3.6) we obtain

$$(3.7) \quad (\partial_\alpha - i\partial_t^2)\mathbf{u}$$
$$= \frac{e^{i\phi}}{v^{\frac{3}{2}}}\partial_\alpha \left[\frac{(\alpha - vt)}{2\alpha}\chi - i\frac{(\alpha + vt)^2}{4v^{\frac{3}{2}}t^{\frac{5}{2}}}\chi' \right] + \frac{e^{i\phi}}{v^{\frac{3}{2}}}\left[\frac{(\alpha - vt)}{2\alpha^2}\chi - i\frac{(\alpha - vt)}{4v^{\frac{3}{2}}t^{\frac{5}{2}}}\chi' \right].$$

The first term is the leading one, and, as expected, has size t^{-1} times the size of $\mathbf{w}$; further, it exhibits some additional structure, manifested in the presence of ∂_α, which we will take advantage of later on. The terms in the second bracket of the RHS are better by another $t^{\frac{1}{2}}$ factor, so no further structure information is needed.

This shows that the choice of a such wave packet is a reasonable approximate solution for the solution the linear system. Precisely, as in [10], our test packets $(\mathbf{w}, \mathbf{q})$ are good approximate solutions for the linear system associated to our problem only on the dyadic time scale $\delta t \leq t$.

The outcome of testing the normal form solutions to the water wave system with the wave packet $(\mathbf{w}, \mathbf{q})$ is the scalar complex valued function $\gamma(t, v)$, defined in the region $\{|v| \leq t^{\frac{1}{2}}\}$:

$$\gamma(t, v) = \langle (\tilde{W}, \tilde{Q}), (\mathbf{w}, \mathbf{q}) \rangle_{\mathcal{H}_0},$$

which we will use as a good measure of the size of $(\tilde{W}, \tilde{Q})$ along our chosen ray. Here it is important that we use the complex pairing in the inner product.

Now we have two tasks. Firstly, we need to show that γ is a good representation of the pointwise size of $(\tilde{W}, \tilde{Q})$ and their derivatives:

PROPOSITION 3.2. — *Assume that* (2.9) *and* (2.10) *hold. Then in* Ω *we have the following bounds for* γ:

$$(3.8) \ \ \|(1 + v^{-2})^5\gamma\|_{L_v^2} + \|v(1 + v^{-2})^{-1}\partial_v\gamma\|_{L_v^2} + \|v^{\frac{1}{2}}(1 + v^{-2})^2\gamma\|_{L^\infty} \lesssim \epsilon t^{C_*^2 \epsilon^2},$$

as well as the approximation bounds for $(\tilde{W}, \tilde{Q})$ *and their derivatives:*

$$(|D|^s\tilde{W}, |D|^{s+\frac{1}{2}}\tilde{Q})(t, vt) = |\xi_v|^s t^{-\frac{1}{2}}e^{i\phi(t,vt)}\gamma(t, v)(1, \operatorname{sgn} v) + \mathbf{err}_s,$$
$$(3.9) \quad (\hat{\tilde{W}}, |\xi|^{\frac{1}{2}}\hat{\tilde{Q}})(t, \xi) = \sum_{\xi = -(4v^2)^{-1}} |v|^{-\frac{3}{2}}(e^{it\operatorname{sgn} v|\xi|^{\frac{1}{2}}}\gamma(t, v)(1, \operatorname{sgn} v) + \hat{\mathbf{er}}),$$

where

(3.10)

$$\|(1+v^{-2})^{\frac{3}{2}-s}\mathbf{err}_{s+\frac{1}{2}}\|_{L_v^2} + \|(1+v^{-2})^{\frac{3}{2}-s}\mathbf{err}_s\|_{L^\infty} \lesssim \epsilon t^{-\frac{6}{11}+C_*^2\epsilon^2}, \quad 0 \le s \le \frac{3}{2},$$

$$\|v^{-1}(1+v^{-2})^{\frac{3}{2}}\hat{\mathbf{err}}\|_{L_v^2} + \|(1+v^{-2})^{\frac{3}{2}}\hat{\mathbf{err}}\|_{L^\infty} \lesssim \epsilon t^{-\frac{1}{22}+C_*^2\epsilon^2}.$$

Secondly, we need to show that γ stays bounded, which we do by establishing a differential equation for it:

PROPOSITION 3.3. — *Assume that* (2.2), (2.3), (2.9), (2.10) *and* (2.11) *hold. Then within the set* Ω *the function* γ *solves an asymptotic ordinary differential equation of the form*

(3.11)
$$\dot{\gamma} = \frac{i}{2t(2v)^5}\gamma|\gamma|^2 + \sigma,$$

where σ *satisfies the* L^2 *and* L^∞ *bounds*

(3.12)
$$\|(1+v^{-2})^2\sigma\|_{L^2} + \|(1+v^{-2})^2\sigma\|_{L^\infty} \lesssim \epsilon t^{-\frac{23}{22}+C_*^2\epsilon^2}.$$

We now use the two propositions to conclude the proof of (2.13). By virtue of (3.9) and (3.10), in order to prove (2.13) it suffices to establish its analogue for γ, namely

(3.13)
$$|\gamma(t,v)| \lesssim \epsilon(1+v^{-2})^{-\frac{3}{2}} \qquad \text{in } \Omega.$$

On the other hand, from (2.12) we directly obtain

(3.14)
$$|\gamma(t,v)| \lesssim \epsilon(1+v^{-2})^{-\frac{3}{2}}\omega(v,t)t^{C_*^2\epsilon} \text{ in } \Omega.$$

Our goal now is to use the ode (3.11) in order to transition from (3.14) to (3.13) along rays $\alpha = vt$. We consider three cases for v:

(i) Suppose first that $v \approx 1$, i.e., $|\alpha| \approx t$. Then we initially have

$$|\gamma(t)| \lesssim \epsilon, \qquad t \approx 1.$$

Integrating (3.11) we conclude that

$$|\gamma(t)| \lesssim \epsilon, \qquad t \ge 1,$$

and then (3.13) follows.

(ii) Assume now that $v \ll 1$, i.e., $|\alpha| \ll t$. Then, as t increases, the ray $\alpha = vt$ enters Ω at some point t_0 with $v \approx t_0^{-\frac{1}{11}}$. Then by (3.14) we obtain

$$|\gamma(t_0,v)| \lesssim \epsilon v^4 v^{\frac{1}{2}}t^{C_*^2\epsilon} \lesssim \epsilon v^2.$$

We use this to initialize γ. For larger t we use (3.11) to conclude that

$$|\gamma(t)| \lesssim \epsilon v^4 + \int_{t_0}^\infty \epsilon v^4 s^{-\frac{23}{22}+C_*^2\epsilon^2}ds \approx \epsilon v^4(1 + t_0^{-\frac{1}{22}+C_*^2\epsilon^2}) \lesssim \epsilon v^4, \qquad t > t_0.$$

Then (3.13) follows.

(iii) Finally, consider the case $v \gg 1$, i.e., $|\alpha| \gg t$. Again, as t increases, the ray $\alpha = vt$ enters Ω at some point t_0 $v \approx t_0^{\frac{1}{11}}$, therefore by (3.14) we obtain

$$|\gamma(t_0, v)| \lesssim \epsilon v^{-\frac{1}{2}} t_0^{C_*^2 \epsilon} \lesssim \epsilon.$$

We use this to initialize γ. For larger t we use (3.11) to conclude that

$$|\gamma(t)| \lesssim \epsilon + \int_{t_0}^{\infty} \epsilon s^{-\frac{23}{22} + C_*^2 \epsilon^2} \approx \epsilon(1 + t_0^{-\frac{1}{22} + C_*^2 \epsilon^2}) \lesssim \epsilon, \qquad t > t_0.$$

Then (3.13) again follows.

The remainder of the paper is devoted to the proof of the two propositions above.

3.2. Approximation errors.

— Here we prove Proposition 3.2, using the estimates (2.9) and (2.10). In order to symmetrize the problem it is useful to introduce the normalized variables $(w, r) = (\tilde{W}, |D|^{\frac{1}{2}} \tilde{Q})$, which satisfy the bounds

$$\|(w, r)\|_{H^5} \leq \epsilon t^{C_*^2 \epsilon^2}, \qquad \|(2\alpha\partial_\alpha w - it|D|^{\frac{1}{2}} r, 2\alpha\partial_\alpha r - it|D|^{\frac{1}{2}} w)\|_{H^{-1}} \lesssim \epsilon t^{C_*^2 \epsilon^2}.$$

Then we rewrite γ in terms of these variables as

$$\gamma = \int w\bar{\mathbf{w}} + rD^{\frac{1}{2}}\bar{\mathbf{q}} \, d\alpha.$$

For the purpose of proving (3.10) we can simplify somewhat the expression of γ. The lower order terms in $\mathbf{w}$ in (3.5) are better by a factor of $v^{\frac{1}{2}} t^{-\frac{1}{2}}$, therefore we can readily replace $\mathbf{w}$ by $\frac{1}{2}\mathbf{u}$, modulo errors which satisfy (3.10). Also, in view of Lemma 3.1, we can also substitute $D^{\frac{1}{2}}\bar{\mathbf{q}}$ by $\frac{t}{2|\alpha|}\bar{\mathbf{q}}$ and further by $\pm\frac{1}{2}\mathbf{u}$, with errors that are also $v^{\frac{1}{2}} t^{-\frac{1}{2}}$ better. In view of these considerations, it suffices to prove Proposition 3.2 with γ redefined as

$$(3.15) \qquad\qquad \gamma(t, v) = \frac{1}{2} \int (w \pm r)\bar{\mathbf{q}} \, d\alpha.$$

Then Proposition 3.2 is a consequence of the following Lemma:

LEMMA 3.4. — *Let γ be defined as in (3.15) in the region Ω, where (w, r) are holomorphic functions which satisfy*

$$(3.16) \quad \|(w, r)\|_{H^5} \leq 1, \qquad \|(2\alpha\partial_\alpha w - it|D|^{\frac{1}{2}} r, 2\alpha\partial_\alpha r - it|D|^{\frac{1}{2}} w)\|_{H^{-1}} \lesssim 1.$$

Then γ satisfies the bounds

$$(3.17) \quad \|(1 + v^{-2})^5 \gamma\|_{L_v^2} + \|v(1 + v^{-2})^{-1}\partial_v \gamma\|_{L_v^2} \lesssim 1, \qquad |\gamma| \lesssim v^{-\frac{1}{2}}(1 + v^{-2})^{-2}.$$

Moreover, the following error bounds for γ also hold:

$$|D|^s(w,r)(t,vt) = |\xi_v|^s t^{-\frac{1}{2}} e^{i\phi(t,vt)} \gamma(t,v)(1,\operatorname{sgn} v) + \mathbf{err}_s,$$

(3.18)

$$(\hat{w},\hat{r})(t,\xi) = \sum_{\xi = -(4v^2)^{-1}} |v|^{-\frac{3}{2}} (e^{it \operatorname{sgn} v|\xi_v|^{\frac{1}{2}}} \gamma(t,v)(1,\operatorname{sgn} v) + \mathbf{e\hat{r}r}),$$

where

(3.19)

$$\|(1+v^{-2})^{\frac{3}{2}-s} \mathbf{err}_{s+\frac{1}{2}}\|_{L_v^2} \lesssim t^{-\frac{5}{9}}, \; \|(1+v^{-2})^{\frac{3}{2}-s}\mathbf{err}_s\|_{L^\infty} \lesssim t^{-\frac{6}{11}}, \; 0 \le s \le 2,$$

$$\|v^{-1}(1+v^{-2})^{\frac{3}{2}}\mathbf{e\hat{r}r}\|_{L_v^2} \lesssim t^{-\frac{1}{22}}, \qquad \|(1+v^{-2})^{\frac{3}{2}}\mathbf{e\hat{r}r}\|_{L^\infty} \lesssim t^{-\frac{1}{22}}, \quad 0 \le s \le 2.$$

Proof. — We first note that our Hypothesis (3.16) on (w,r) is stable with respect to dyadic frequency localizations. At a fixed dyadic frequency λ, the operator

$$(w,r) \to (2\alpha\partial_\alpha w - it|D|^{\frac{1}{2}}r, 2\alpha\partial_\alpha r - it|D|^{\frac{1}{2}}w)$$

is elliptic outside the region $\lambda \approx t^2/\alpha^2$. Equivalently, the spatial region $\{\alpha \approx vt\}$ is matched to the dyadic frequency $\lambda_v = v^{-2}$. Thus, using elliptic estimates, we can decompose (w,r) into a leading part and an elliptic component,

$$(w,r) = (w_{\mathrm{ell}}, r_{\mathrm{ell}}) + \sum_v \chi_{\alpha \approx vt} P_{\lambda_v}(w,r).$$

We consider the two parts separately.

(i) *The elliptic part.* — For $(w_{\mathrm{ell}}, r_{\mathrm{ell}})$, the bound (3.16) translates into

$$\|(w_{\mathrm{ell}}, r_{\mathrm{ell}})\|_{H^5} \le 1, \qquad \|\alpha\partial_\alpha(w_{\mathrm{ell}}, r_{\mathrm{ell}})\|_{L^2} + \||t|D|^{\frac{1}{2}}(w_{\mathrm{ell}}, r_{\mathrm{ell}})\|_{H^{-1}} \lesssim 1.$$

By interpolation and Bernstein's inequality this leads to the bounds

$$\||D|^{s+\frac{1}{2}}(w_{\mathrm{ell}}, r_{\mathrm{ell}})\|_{L_\alpha^2} + \||D|^s(w_{\mathrm{ell}}, r_{\mathrm{ell}})\|_{L^\infty} \lesssim t^{-\frac{6}{11} - \frac{2}{11}(\frac{3}{2}-s)}, \qquad 0 \le s \le \frac{3}{2}.$$

One can also switch to an L_v^2 norm, using the fact that the norms L_v^2 and L_α^2 are related by

$$\|f\|_{L_\alpha^2} = t^{\frac{1}{2}} \|f\|_{L_v^2}.$$

On the Fourier side, we similarly obtain the pointwise bound

$$|(\hat{w}_{\mathrm{ell}}, \hat{r}_{\mathrm{ell}})| \lesssim (\xi^{\frac{1}{2}}\langle\xi\rangle^2 + t^{\frac{1}{2}}\xi\langle\xi\rangle^{-1})^{-1}.$$

These estimates allow us to place $(w_{\mathrm{ell}}, r_{\mathrm{ell}})$ into the error term in (3.18). Further, in view of (3.2), the contribution of $(w_{\mathrm{ell}}, r_{\mathrm{ell}})$ to γ has size t^{-N}, and can be also placed in the error.

(ii) *The hyperbolic part.* — Here spatial dyadic regions are diagonally matched with dyadic frequency regions, and dyadic l^2 summability is inherited from the latter. Hence it suffices to consider a fixed dyadic velocity range $\{v \approx v_0\}$, associated to the spatial region $\{|\alpha| \approx v_0 t\}$, and (w, r) localized at frequency $\lambda_{v_0} \approx v_0^{-2}$.

In order to fix signs, we first need to differentiate between the two symmetric cases $v_0 > 0$ and $v_0 < 0$. Without any restriction in generality we take $v_0 > 0$. Then subtracting the two components in the second term in (3.16) we obtain

$$\|(2\alpha|D| + t|D|^{\frac{1}{2}})(w - r)\|_{L^2} \lesssim 1 + v_0^{-2}.$$

The operator above is elliptic in $\{\alpha \approx v_0 t\}$, therefore we obtain

$$\|\chi_{\alpha \approx v_0 t}(w - r)\|_{L^2} \lesssim \frac{v_0(1 + v_0^{-2})}{t},$$

which is comparable to the estimates obtained in the elliptic case. Thus, as there, we can bound $|D|^s(w - r)$ in L^2 and in L^∞ and place it into the error term of (3.18).

Hence, we can freely replace w and r by $y = \frac{w+r}{2}$ in (3.18). We note that γ is already expressed in terms of y. To reduce the problem completely to an estimate for y we need one last step. Combining again the two components in the second term in (3.16) we obtain

$$\||D|^{\frac{1}{2}}\chi_{\alpha \approx v_0 t}(4\alpha^2\partial_\alpha + it^2)(w, r)\|_{L^2_\alpha} \lesssim t(1 + v_0^{-2}),$$

which yields the same bound for y. Choosing $\chi_{\alpha \approx vt}$ to be supported at spatial frequency $\ll v^{-2}$, we can cancel the $|D|^{\frac{1}{2}}$ and conclude that

$$(3.20) \qquad \|\chi_{\alpha \approx v_0 t}Ly\|_{L^2_\alpha} \lesssim \frac{1 + v_0^{-2}}{v_0 t}, \qquad L = \partial_\alpha + \frac{it^2}{4\alpha^2}.$$

On the other hand, from the first relation in (3.16) we obtain

$$(3.21) \qquad \|y\|_{L^2_\alpha} \lesssim (1 + v_0^{-2})^{-5}.$$

From here on we will work only with the function y.

It is convenient to rewrite the bounds on y in terms of the auxiliary function $u := e^{-i\phi}y$, which satisfies $\partial_\alpha u = e^{-i\phi}(\partial_\alpha + \frac{it^2}{4\alpha^2})y$. Then for u we have

$$(3.22) \qquad \|\chi_{\alpha \approx v_0 t}\partial_\alpha u\|_{L^2_\alpha} \lesssim \frac{1 + v_0^{-2}}{v_0 t}, \qquad \|u\|_{L^2_\alpha} \lesssim (1 + v_0^{-2})^{-5}.$$

Combining these bounds we get by interpolation

$$\|\chi_{\alpha \approx v_0 t}u\|_{L^\infty} \lesssim t^{-\frac{1}{2}}v_0^{-\frac{1}{2}}(1 + v_0^{-2})^{-2},$$

which also is transferred back to y,

$$(3.23) \qquad \|\chi_{\alpha \approx v_0 t}y\|_{L^\infty} \lesssim t^{-\frac{1}{2}}v_0^{-\frac{1}{2}}(1 + v_0^{-2})^{-2}.$$

The bounds (3.21) and (3.23) lead directly to L^2 and L^∞ bounds for γ,

$$(3.24) \qquad \|\gamma\|_{L_v^2(v\approx v_0)} \lesssim (1 + v_0^{-2})^{-5}, \qquad \|\gamma\|_{L^\infty(v\approx v_0)} \lesssim v_0^{-\frac{1}{2}}(1 + v_0^{-2})^{-2}.$$

To estimate $\partial_v\gamma = \langle y, \partial_v\mathbf{u}\rangle_{L^2}$ we write $\partial_v\mathbf{u}$ in the form

$$\partial_v\mathbf{u} = -v^{-\frac{3}{2}}e^{i\phi}\left(t\partial_\alpha\chi\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right) + \frac{3}{2}\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{5}{2}}}\chi'\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right)\right),$$

and compute using integration by parts

$$\partial_v\gamma = \int v^{-\frac{3}{2}}t\partial_\alpha u(t,\alpha)\chi\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right)d\alpha - \int v^{-\frac{3}{2}}u(t,\alpha)\frac{3}{2}\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{5}{2}}}\chi'\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right)d\alpha.$$

Now we can bound the two integrals using (3.22) to obtain

$$\|\partial_v\gamma\|_{L_v^2(v\approx v_0)} \lesssim \frac{1 + v_0^{-2}}{v_0},$$

which, together to (3.24), concludes the proof of (3.8).

It remains to estimate the L^2 and L^∞ norms of the error in (3.10). We begin with the physical space error bounds. The idea is to bound the difference

$$\mathtt{err} = y(t, vt) - t^{-\frac{1}{2}}e^{i\phi(t,vt)}\langle y, \mathbf{u}\rangle_{L^2}$$

in both L_v^2 and L^∞ in terms of $\|y\|_{L_\alpha^2}$ and $\|Ly\|_{L_\alpha^2}$. Precisely, we claim that
(3.25)
$$\|\mathtt{err}\|_{L^\infty(v\approx v_0)} \lesssim v_0^{\frac{3}{4}}t^{\frac{1}{4}}\|\chi_{\alpha\approx v_0 t}Ly\|_{L_\alpha^2}, \qquad \|\mathtt{err}\|_{L_v^2(v\approx v_0)} \lesssim v_0^{\frac{3}{2}}\|\chi_{\alpha\approx v_0 t}Ly\|_{L_\alpha^2}.$$

Restated in terms of u we have $e^{-i\phi}\mathtt{err} = y(t, vt) - t^{-\frac{1}{2}}\langle u, e^{-i\phi}\mathbf{u}\rangle_{L^2}$. Hence, using (3.1), we can write

$$(3.26) \qquad e^{-i\phi}\mathtt{err} = \int (u(t, vt) - u(t, (v - z)t))\chi\left(t^{\frac{1}{2}}v^{-\frac{3}{2}}z\right)v^{-\frac{3}{2}}t^{\frac{1}{2}}\,dz.$$

By Hölder's inequality

$$|u(t, vt) - u(t, (v - z)t)| \leq |z|^{\frac{1}{2}}\|\partial_v u(t, vt)\|_{L_v^2} = |z|^{\frac{1}{2}}t^{\frac{1}{2}}\|\partial_\alpha u\|_{L_\alpha^2}.$$

As in (3.26) z is restricted to $|z| \lesssim v_0^{\frac{3}{2}}t^{-\frac{1}{2}}$; by (3.22) we obtain

$$|\mathtt{err}| \lesssim v_0^{-\frac{1}{4}}t^{-\frac{3}{4}}.$$

Hence the pointwise part of (3.25) follows.

To prove the L_v^2 bound in (3.25), we estimate the RHS of (3.26) in terms of $\partial_v u$,

$$|\mathtt{err}| \lesssim \int_0^1\int |z||\partial_v u(t, (v - hz)t)|\chi\left(t^{\frac{1}{2}}v^{-\frac{3}{2}}z\right)v^{-\frac{3}{2}}t^{\frac{1}{2}}\,dzdh.$$

Thus, we can now evaluate the L_v^2 of the LHS of (3.26) as follows,

$$\|\mathbf{err}\|_{L_v^2(v \approx v_0)} \lesssim \|\partial_v u(t,vt)\|_{L_v^2(v \approx v_0)} \int |z|\chi\left(t^{\frac{1}{2}}v^{-\frac{3}{2}}v_1\right)v^{-\frac{3}{2}}t^{\frac{1}{2}}\,dz$$

$$\approx t^{-\frac{1}{2}}v_0^{\frac{3}{2}}\|\partial_v u(t,vt)\|_{L_v^2(v \approx v_0)} = v_0^{\frac{3}{2}}\|\chi_{\alpha \approx v_0 t}\partial_\alpha u\|_{L_\alpha^2}.$$

This completes the proof of (3.25). In turn, (3.25) together with (3.20) applies directly to the case $s = 0$ of (3.10) to give

$$\|\mathbf{err}_0\|_{L^\infty(v \approx v_0)} \lesssim v_0^{-\frac{1}{4}}(1 + v_0^{-2})t^{-\frac{3}{4}}, \qquad \|\mathbf{err}_0\|_{L_v^2(v \approx v_0)} \lesssim v_0^{\frac{1}{2}}(1 + v_0^{-2})t^{-1},$$

which suffices for (3.19).

In order to consider also the case $s > 0$, we write

$$\mathbf{err}_s = |D|^s y(t,vt) - (4v^2)^{-s}t^{-\frac{1}{2}}e^{i\phi}\langle y, \mathbf{u}\rangle_{L_\alpha^2}$$

$$= |D|^s y(t,vt) - t^{-\frac{1}{2}}e^{i\phi}\langle |D|^s y, \mathbf{u}\rangle_{L_\alpha^2} - t^{-\frac{1}{2}}e^{i\phi}\langle y, (|D|^s - (4v^2)^{-s})\mathbf{u}\rangle_{L_\alpha^2}$$

$$:= \mathbf{err}_s^1 + \mathbf{err}_s^2.$$

For $\mathbf{err}_s^1$ we apply (3.25) with y replaced by $|D|^s y$, to obtain the same bound as before but with an added v_0^{-2s} factor,

$$\|\mathbf{err}_s^1\|_{L^\infty(v \approx v_0)} \lesssim v_0^{-\frac{1}{4}}(1 + v^{-2})^{1-s}t^{-\frac{3}{4}}, \|\mathbf{err}_{s+\frac{1}{2}}^1\|_{L_v^2(v \approx v_0)} \lesssim v_0^{-\frac{1}{2}}(1 + v^{-2})^{1-s}t^{-1},$$

which is unfavorable if $v_0 < 1$. But then we can still interpolate with the $t^{-\frac{1}{2}}$ bound from (3.23) and (3.24) to remove the negative power of v_0.

For $\mathbf{err}_s^2$ instead we use the cancellation in (3.4) to conclude that $P_\lambda(|D|^s - (4v^2)^{-s})\mathbf{u}$ is a bump function comparable to $(4v^2)^{-s}v^{\frac{1}{2}}t^{-\frac{1}{2}}\mathbf{u}$. Hence we obtain a direct bound as in (3.24), but with the same additional factor,

$$\|\mathbf{err}_s^2\|_{L^\infty(v \approx v_0)} \lesssim v_0^{-2s}(1 + v_0^{-2})^{-2}t^{-1}, \quad \|\mathbf{err}_s^1\|_{L_v^2(v \approx v_0)} \lesssim v_0^{\frac{1}{2}-2s}(1 + v_0^{-2})^{-5}t^{-1},$$

which again suffices.

Finally, we consider the Fourier space error estimates. For this we first need to switch the bounds for y to the Fourier space. For a fixed frequency λ we have two dyadic regions in v where the hyperbolic components of (w,r) are supported, namely those for which $v_0^{-2} \approx \lambda$. Thus we will get two contributions in the approximation to $(\hat{w}, \hat{r})(t, \xi)$. As above, let us restrict ourselves to the contribution corresponding to $v > 0$. Adding the two components in the second term in (3.16) we obtain

$$\|(2\alpha|D| - t|D|^{\frac{1}{2}})y\|_{L^2} \lesssim 1 + v_0^{-2}.$$

Since y is localized at frequency v_0^{-2}, taking a Fourier transform and estimating commutation errors via the first part of (3.16), we obtain the main bounds for $\hat{y}$,

$$\|\hat{y}\|_{L^2} \lesssim (1 + v^{-2})^{-5}, \qquad \|(\partial_\xi + it|\xi|^{-\frac{1}{2}})\hat{y}\|_{L^2} \lesssim v_0^2(1 + v_0^{-2}).$$

On the other hand, the quantity to bound in L_v^2 and L^∞ is

$$\mathbf{e\hat{r}r}_s = v^{\frac{3}{2}}\hat{y}(t,\xi_v) - e^{-it\xi_v}\langle \hat{y}, \hat{\mathbf{u}}\rangle, \qquad \xi_v = -(4v^2)^{-1}.$$

Given the expression of $\hat{\mathbf{u}}$ in (3.2), the error $\mathbf{e\hat{r}r}$ is estimated in the same fashion as in the proof of (3.25).

$\square$

3.3. The evolution of γ. — Here we track the evolution of $\gamma(t,v)$ and prove Proposition 3.3. In view of the energy conservation relation for the linear system (1.3), we directly obtain the relation

$$\dot{\gamma}(t) = \int \tilde{G}\bar{\mathbf{w}} + \tilde{W}\bar{\mathbf{g}} + i\tilde{K}_\alpha \bar{\mathbf{q}}\,d\alpha.$$

We successively consider all terms on the right. With the exception of a single term, namely the resonant part of G, see below, all contributions will be placed into the error term σ.

A. The contribution of $\bar{\mathbf{g}}$. — This is

$$I_1 = v^{-\frac{3}{2}}\int \tilde{W}e^{-i\phi}\left(\partial_\alpha\left[\frac{(\alpha-vt)}{2\alpha}\chi - i\frac{(\alpha+vt)^2}{4v^{\frac{3}{2}}t^{\frac{5}{2}}}\chi'\right] + \left[\frac{(\alpha-vt)}{2\alpha^2}\chi - i\frac{(\alpha-vt)}{4v^{\frac{3}{2}}t^{\frac{5}{2}}}\chi'\right]\right)\,d\alpha.$$

We use (3.9) to replace W in terms of γ, and the contribution of the error $\tilde{W} - t^{-\frac{1}{2}}e^{i\phi}\gamma(t,v)$ is directly estimated in both L^2 and L^∞ via (3.10).

The contribution of γ, on the other hand, is written using integration by parts as

$$\tilde{I}_1 = v^{-\frac{3}{2}}t^{-\frac{1}{2}}\int -\gamma_\alpha\left[\frac{(\alpha-vt)}{2\alpha}\chi - i\frac{(\alpha+vt)^2}{4v^{\frac{3}{2}}t^{\frac{5}{2}}}\chi'\right] + \gamma\left[\frac{(\alpha-vt)}{2\alpha^2}\chi - i\frac{(\alpha-vt)}{4v^{\frac{3}{2}}t^{\frac{5}{2}}}\chi'\right]\,d\alpha.$$

Now we can easily bound the two terms using (3.8).

B. The contribution of $\tilde{G}$ and $\tilde{K}$. — For this we consider in more detail the structure of $\tilde{G}$ and $\tilde{K}$. We will successively peel off favorable terms until we are left only with the leading resonant part.

B1. Quartic and higher order terms. — We decompose them into cubic and higher terms,

$$\tilde{G} = \tilde{G}^{(3)} + \tilde{G}^{(4+)}, \qquad \tilde{K} = \tilde{K}^{(3)} + \tilde{K}^{(4+)}.$$

In view of (2.15), we can estimate the contribution of the quartic and higher terms in L^∞,

$$\left|\int \tilde{G}^{4+}\bar{\mathbf{w}} + i\tilde{K}_\alpha^{4+}\bar{\mathbf{q}}\,d\alpha\right| \lesssim \epsilon^4 t^{-\frac{3}{2}+3C_*^2\epsilon^2}\|(\mathbf{w},\mathbf{q})\|_{\mathcal{H}_0} \lesssim \epsilon^4 v^{-\frac{1}{2}}t^{-\frac{5}{4}+3C_*^2\epsilon^2},$$

which suffices in Ω. The L^2 bound is similar, using again (2.15).

In the same way, by using (2.16), we can estimate the contribution of the difference

$$(\tilde{G}^3(W,R), \tilde{K}^3(W,R)) - (\tilde{G}^3(\tilde{W}, \tilde{Q}_\alpha), \tilde{K}^3(\tilde{W}, \tilde{Q}_\alpha)),$$

which also contains only quartic and higher terms. Thus we have substituted $(\tilde{G}, \tilde{K})$ with the cubic expressions $(\tilde{G}^3(\tilde{W}, \tilde{Q}_\alpha), \tilde{K}^3(\tilde{W}, \tilde{Q}_\alpha))$.

B2. Cubic terms. — To better understand the cubic interactions we need the following heuristic analysis:

(i) in the physical space, waves at frequency ξ move with velocity $\pm\frac{1}{2}|\xi|^{-\frac{1}{2}}$. Since our data is localized near the origin, it follows that the bulk of the solution at (α, t) is at space-time frequency $(\xi, \tau) = \left(-\dfrac{t^2}{4\alpha^2}, \dfrac{t}{2\alpha}\right)$. Thus the worst cubic interactions are those of waves with equal frequency.

(ii) In the frequency space, trilinear interactions of equal frequency waves $(\xi, \pm\sqrt{|\xi|})$ (with $\xi < 0$) can only lead back to the characteristic set if exactly one complex conjugation is present.

This leads to the following classification of the terms in $(\tilde{G}^3(\tilde{W}, \tilde{Q}_\alpha), \tilde{K}^3(\tilde{W}, \tilde{Q}_\alpha))$:

A. Nonresonant trilinear terms: these are either (A1) terms with no complex conjugates, or (A2) terms with two complex conjugates.

B. Resonant trilinear terms: terms with exactly one conjugation. For such terms one may further define a notion of principal symbol, which is the leading coefficient in the expression obtained by substituting the factors in the trilinear form by the expressions in (3.9) [1] Thus one can isolate a linear subspace of resonant terms for which this symbol vanishes, which we call *null terms*. Hence on the full class of resonant trilinear terms we can further define an equivalence relation, modulo null terms.

Now we turn our attention to the situation at hand, where we recall that

$$\begin{cases} \tilde{G}^{(3)}(W,R) = 2P[-\partial_\alpha P\left[R\bar{W} - \bar{R}W\right]\Re W + (RW)_\alpha \Re W + \mathbf{W}\Re(\mathbf{W}R)] \\ \qquad - P[\bar{\mathbf{W}}^2 R - \mathbf{W}(P[\bar{R}\mathbf{W}] + \bar{P}[R\bar{\mathbf{W}}])] \\ \tilde{K}^{(3)}(W,R) = 2P\left[P[(R+\bar{R})R_\alpha]\Re W + i\mathbf{W}^2\Re W + P[R\bar{R}_\alpha]\Re W\right] \\ \qquad + P\left[\bar{R}\bar{\mathbf{W}}R - R\bar{P}\left[\bar{R}\mathbf{W} - R\bar{\mathbf{W}}\right]\right]. \end{cases}$$

Based on the previous heuristics, we decompose

$$\begin{cases} \tilde{G}^{(3)} = \tilde{G}_r^{(3)} + \tilde{G}_{nr}^{(3)} + \tilde{G}_{\text{null}}^{(3)} \\ \tilde{K}^{(3)} = \tilde{K}_r^{(3)} + \tilde{K}_{nr}^{(3)} + \tilde{K}_{\text{null}}^{(3)}, \end{cases}$$

[1] Which corresponds to all three frequencies being equal.

where

$$
\begin{cases}
\tilde{G}_r^{(3)} = P[(R\mathbf{W})_\alpha \bar{W} + \mathbf{W}R\bar{\mathbf{W}}] \\[4pt]
\tilde{G}_{nr}^{(3)} = P[(R\mathbf{W})_\alpha W + \mathbf{W}\mathbf{W}R] \\[4pt]
\tilde{G}_{\mathrm{null}}^{(3)} = P[-2\partial_\alpha P[R\bar{\mathbf{W}} - \bar{R}\mathbf{W}]\Re W + \bar{\mathbf{W}}(\mathbf{W}\bar{R} - \bar{\mathbf{W}}R) + \mathbf{W}P[\bar{R}\mathbf{W} - R\bar{\mathbf{W}}]] \\[4pt]
\tilde{K}_r^{(3)} = 0 \\[4pt]
\tilde{K}_{nr}^{(3)} = P\left[\bar{R}\bar{\mathbf{W}}R\right] \\[4pt]
\tilde{K}_{\mathrm{null}}^{(3)} = P[-R\bar{P}\left[\bar{R}\mathbf{W} - R\bar{\mathbf{W}}\right] + 2P[|R|^2]_\alpha \Re W + 2(RR_\alpha + i\mathbf{W}^2)\Re W].
\end{cases}
$$

We will place all cubic contributions into the error term σ, except for the contribution of the resonant part $\tilde{G}_r$.

We note that for the most part the exact form of the expressions above is irrelevant. The only significant matter is the coefficient of the terms in $\tilde{G}_r^{(3)}$, which needs to be real [2].

We also remark that the leading projection in all terms can be harmlessly discarded, since it can be moved onto the wave packets, which decay rapidly at positive frequencies,

$$
\|(\mathbf{w}, \mathbf{q}) - P(\mathbf{w}, \mathbf{q})\|_{H^N} \lesssim t^{-N}.
$$

B2(a). *Substitution by the asymptotic expansion.* — The first step in our estimates for $(\tilde{G}^3(\tilde{W}, \tilde{Q}_\alpha), \tilde{K}^3(\tilde{W}, \tilde{Q}_\alpha))$ is to show that we can harmlessly replace $(\tilde{W}, \tilde{Q}_\alpha)$ by their leading asymptotic expression in (3.9). Denoting the resulting expressions by $(\tilde{G}^3(\gamma), \tilde{K}^3(\gamma))$, we claim that we can place the expression

$$
\langle (\tilde{G}^3(\tilde{W}, \tilde{Q}_\alpha) - \tilde{G}^3(\gamma), \tilde{K}^3(\tilde{W}, \tilde{Q}_\alpha)) - \tilde{K}^3(\gamma)), (\mathbf{w}, \mathbf{q}) \rangle_{\dot{H}_0}
$$

into the error term σ.

We first consider terms without inner projections. To fix the notations, consider the expression $\tilde{G}_r^{(3)}$, with the projection P removed. Expanding the derivatives, we have

$$
\tilde{G}_r^3(\tilde{W}, \tilde{Q}_\alpha) = \tilde{Q}_{\alpha\alpha}\tilde{W}_\alpha \bar{\tilde{W}} + \tilde{Q}_\alpha \tilde{W}_{\alpha\alpha}\bar{\tilde{W}} + \tilde{Q}_{\alpha\alpha}\tilde{W}_\alpha \bar{\tilde{W}}_\alpha.
$$

On the other hand, a direct computation, using (3.9), shows that

$$
(3.27) \qquad \tilde{G}_r^3(\gamma) = \frac{i}{t^{\frac{3}{2}}}\left(\frac{1}{2v}\right)^5 e^{i\phi}\gamma|\gamma|^2.
$$

Within the region of interest Ω, the difference $\tilde{G}_r^3(\tilde{W}, \tilde{Q}_\alpha) - \tilde{G}_r^3(\gamma)$ is a sum of cubic expressions with either one γ factor and two error factors, or viceversa.

---

[2] A similar constraint would be required of the coefficients in $\tilde{K}_r^{(3)}$, if they were nonzero.

Using (3.8) for γ and (3.10) for the errors, we immediately obtain the bound

$$\|(1 + v^{-2})^2 \tilde{G}_r^3(\tilde{W}, \tilde{Q}_\alpha) - \tilde{G}_r^3(\gamma)\|_{L^2 \cap L^\infty} \lesssim \epsilon^3 t^{-\frac{3}{2} - \frac{1}{18} + C_*^2 \epsilon^2},$$

where both norms are estimated in Ω. Matching this against $\mathbf{w}$ we can directly place the output in the error σ.

The same strategy works for terms with an inner projection P, with two additional observations:

(i) Since P is nonlocal, one also needs to consider contributions from outside Ω. But this is straightforward due to the better decay estimate in (2.12) for v away from 1.

(ii) Since P is not bounded in L^∞, there is an additional logarithmic loss in the L^∞ bound for the error σ.

B2(b). *The contribution of the null terms* $\tilde{G}_{\mathrm{null}}^{(3)}(\gamma)$ *and* $\tilde{K}_{\mathrm{null}}^{(3)}(\gamma)$. — Here we simply note that $\tilde{G}_{\mathrm{null}}^{(3)}(\gamma) = 0$ and $\tilde{K}_{\mathrm{null}}^{(3)}(\gamma) = 0$, so after the previous step there is nothing left to do. We remark that cancellation actually occurs at the bilinear level for the "null expressions" of type

$$\tilde{W}_\alpha \bar{\tilde{Q}}_\alpha - \bar{\tilde{W}}_\alpha \tilde{Q}_\alpha, \qquad (|\bar{\tilde{Q}}|^2)_\alpha, \qquad \tilde{Q}_\alpha \tilde{Q}_{\alpha\alpha} + i\tilde{W}_\alpha^2.$$

B2(c). *The contribution of the nonresonant terms* $\tilde{G}_{nr}^{(3)}$ *and* $\tilde{K}_{nr}^{(3)}$. — Here it is important that we integrate against $\mathbf{w}$ and $\mathbf{q}$, as that fixes the frequency of the output at $\xi = -\frac{1}{4v^2}$. On the other hand the nonresonant trilinear expression will be concentrated at frequency 3ξ if no complex conjugate occur, respectively at frequency $-\xi$ if two conjugates occur. The easiest way to take advantage of this mismatch is via an integration by parts argument. Consider for instance the expression in $\tilde{G}_{nr}^{(3)}(\gamma)$, which has the form

$$\tilde{G}_{nr}^{(3)}(\gamma) = -\frac{3i}{t^{\frac{3}{2}}} \left(\frac{1}{2v}\right)^5 e^{3i\phi}\gamma^3.$$

Testing this against $\mathbf{w}$, we can write

$$\int \tilde{G}_{nr}^{(3)}(\gamma)\bar{\mathbf{w}}\, d\alpha = \int -\frac{3i}{t^{\frac{3}{2}}} \left(\frac{t}{2\alpha}\right)^5 e^{3i\phi}\gamma^3\mathbf{w}\, d\alpha$$

$$= \int -\frac{3i}{t^{\frac{3}{2}}} \left(\frac{t}{2\alpha}\right)^5 e^{2i\phi}\gamma^3\tilde{\chi}\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right)v^{-\frac{3}{2}}\, d\alpha.$$

Here the phase 2ϕ is nonstationary, so we can integrate by parts to place a derivative on either $\tilde{\chi}$ or on γ. In the first case we gain a $t^{-\frac{1}{2}}$ factor directly, while in the second case a similar gain comes from (3.8).

B2 (d). *The contribution of the resonant term* $\tilde{G}_r^{(3)}(\gamma)$. — Given the expression (3.27), all we need is to consider the integral

$$I = \int \frac{i}{t^{\frac{3}{2}}} \left(\frac{t}{2\alpha}\right)^5 \gamma(t.\alpha/t)|\gamma(t,\alpha/t)|^2 e^{-i\phi}\bar{\mathbf{w}}\,d\alpha.$$

Here $e^{-i\phi}\bar{\mathbf{w}}$ has the form

$$e^{-i\phi}\bar{\mathbf{w}} = \frac{1}{2}v^{-\frac{3}{2}}\chi\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right) + v^{-1}t^{-\frac{1}{2}}\tilde{\chi}\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right)$$

with Schwartz functions χ and $\tilde{\chi}$ so that $\int \chi = 1$. We can freeze the coefficient $\frac{t}{2\alpha}$ at $\alpha = vt$ at the expense of modifying $\tilde{\chi}$ to write

$$I = \frac{i}{t^{\frac{3}{2}}}(2v)^{-5}\int \gamma(t.\alpha/t)|\gamma(t,\alpha/t)|^2 \left[\frac{1}{2}v^{-\frac{3}{2}}\chi\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right) + v^{-1}t^{-\frac{1}{2}}\tilde{\chi}\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right)\right]\,d\alpha$$
$$:= J + \tilde{J}.$$

The contribution $\tilde{J}$ of the lower order term containing $\tilde{\chi}$ is part of the error σ, and is estimated directly in L^2 and L^∞ using the L^2 and L^∞ bounds for $|\gamma|^3$ derived from (3.8).

For the contribution J of the main term containing χ we freeze γ at the packet center to obtain the leading contribution:

$$J = \frac{i}{2t}(2v)^{-5}\gamma(t,v)|\gamma(t,v)|^2 + J_1,$$

where

$$J_1 = \frac{i}{2t^{\frac{3}{2}}}(2v)^{-5}\int \left[\gamma(t.\alpha/t)|\gamma(t,\alpha/t)|^2 - \gamma(t.v)|\gamma(t,v)|^2\right]v^{-\frac{3}{2}}\chi\left(\frac{\alpha - vt}{t^{\frac{1}{2}}v^{\frac{3}{2}}}\right)\,d\alpha.$$

We still need to bound the last integral in L^∞ and in L_v^2. For this we first use the estimates in (3.8) to conclude that

$$\|v^2(1 + v^{-2})^5\partial_v[\gamma(t,v)|\gamma(t,v)|^2]\|_{L_v^2} \lesssim t^{C_*^2\epsilon^2}.$$

Then, by the same argument as in the proof of (3.25), we obtain

$$\|(1 + v^{-2})J_1\|_{L^\infty} \lesssim t^{-\frac{5}{4}+C_*^2\epsilon^2}, \qquad \|J_1\|_{L_v^2} \lesssim t^{-\frac{3}{2}+C_*^2\epsilon^2},$$

which are more than sufficient in order to include J_1 in the error term σ. Thus the proof of Proposition 3.3 is concluded.

3.4. The asymptotic expansion of the solution. — To construct the asymptotic profile Ψ we use the differential equation in Proposition 3.3 for $\gamma(t, v)$. The inhomogeneous term $\sigma(t, v)$ is estimated in L^∞ and L_v^2 as showed in (3.12). The differential equation for γ in (3.11) can be explicitly solved in polar coordinates. Since $\sigma(t, v)$ in uniformly integrable in time, it follows that for each v, $\gamma(t, v)$ is well approximated at infinity by a solution to the unperturbed differential equation, in the sense that

$$(3.28) \qquad \gamma(t, v) = \Psi(v)\, e^{\frac{i}{2}(2v)^{-5}\ln t |\Psi(v)|^2} + O_{L^\infty}(\epsilon t^{-\frac{1}{18}+C_*^2\epsilon^2}).$$

Integrating the L_v^2 part of (3.12) leads to a similar L_v^2 bound

$$(3.29) \qquad \gamma(t, v) = \Psi(v)\, e^{\frac{i}{2}(2v)^{-5}\ln t |\Psi(v)|^2} + O_{L_v^2}(\epsilon t^{-\frac{1}{18}+C_*^2\epsilon^2}).$$

Both of these relations are valid in Ω, where we know that (3.8) holds. Then from the two relations in (3.18) we obtain the asymptotic expansions in (1.7) within Ω but for the normal form variables $(\tilde{W}, \tilde{Q})$. The transition to the original variables (W, Q) is straightforward in view of the expressions (2.6), and the pointwise decay bounds for (W, Q).

The next step is to establish the regularity of Ψ. On one hand, from (3.28) and (3.29) we get (within Ω)

$$\left\| \Psi(v) - \gamma(t, v) e^{-\frac{i}{2}(2v)^{-5}|\gamma(t,v)|^2 \log t} \right\|_{L_v^2 \cap L^\infty} \lesssim \epsilon t^{-\frac{1}{18}+C_*^2\epsilon^2} \log t,$$

while by (3.8) we have the L_v^2 bound

$$\left\| (1 + v^{-2})^5 \gamma(t, v) e^{-\frac{i}{2}(2v)^{-5}|\gamma(t,v)|^2 \log t} \right\|_{L_v^2}$$
$$+ \left\| v \partial_v [\gamma(t, v) e^{-\frac{i}{2}(2v)^{-5}|\gamma(t,v)|^2 \log t}] \right\|_{L_v^2} \lesssim \epsilon t^{C_*^2\epsilon^2} \log t,$$

and the L^∞ bound

$$\left\| v^{\frac{1}{2}}(1 + v^{-2})^{\frac{5}{2}} \gamma(t, v) e^{-\frac{i}{2}(2v)^{-5}|\gamma(t,v)|^2 \log t} \right\|_{L^\infty} \lesssim \epsilon t^{C_*^2\epsilon^2}.$$

As t increases, the range $[t^{-\frac{1}{9}}, t^{\frac{1}{9}}]$ of $|v|$ within Ω increases to cover the entire $\mathbb{R}^+$. Hence, by interpolation we obtain that for large enough C_*, Ψ has the regularity

$$(3.30) \qquad \begin{aligned} \left\| |v|^{\frac{1}{2}-C_*^2\epsilon^2}(1 + v^{-2})^{\frac{5}{2}-C_*^2\epsilon^2} \Psi \right\|_{L^\infty} &\lesssim \epsilon, \\ \left\| (1 + v^{-2})^{5-C_*^2\epsilon^2} \Psi \right\|_{L_v^2} &\lesssim \epsilon, \\ \left\| |v|^{1-C_*^2\epsilon^2} \Psi \right\|_{H_v^{1-C_*^2\epsilon^2}} &\lesssim \epsilon. \end{aligned}$$

Finally, we can combine the pointwise bounds for Ψ with the pointwise bounds for (W, Q) to extend the error estimates in (1.7) to the exterior of Ω.

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