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ON THE IWAHORI WEYL GROUP

BY TIMO RICHARZ

ABSTRACT. — Let F be a discretely valued complete field with valuation ring $\mathcal{O}_F$ and perfect residue field k of cohomological dimension ≤ 1. In this paper, we generalize the Bruhat decomposition in Bruhat and Tits [3] from the case of simply connected F-groups to the case of arbitrary connected reductive F-groups. If k is algebraically closed, Haines and Rapoport [4] define the Iwahori-Weyl group, and use it to solve this problem. Here we define the Iwahori-Weyl group in general, and relate our definition of the Iwahori-Weyl group to that of [4].

Let F be a discretely valued complete field with valuation ring $\mathcal{O}_F$ and perfect residue field k of cohomological dimension ≤ 1. In this paper, we generalize the Bruhat decomposition in Bruhat and Tits [3] from the case of simply connected F-groups to the case of arbitrary connected reductive F-groups. If k is algebraically closed, Haines and Rapoport [4] define the Iwahori-Weyl group, and use it to solve this problem. Here we define the Iwahori-Weyl group in general, and relate our definition of the Iwahori-Weyl group to that of [4]. Furthermore, we study the length function on the Iwahori-Weyl group, and use it to determine the number of points in a Bruhat cell, when k is a finite field. Except for Lemma 1.3 below, the results are independent of [4], and are directly based on the work of Bruhat and Tits [2], [3].

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Let $\bar{F}$ be the completion of a separable closure of F. Let $\breve{F}$ be the completion of the maximal unramified subextension with valuation ring $\mathcal{O}_{\breve{F}}$ and residue field $\bar{k}$. Let $I = \mathrm{Gal}(\bar{F}/\breve{F})$ be the inertia group of $\breve{F}$, and let $\Sigma = \mathrm{Gal}(\breve{F}/F)$.

Let G be a connected reductive group over F, and denote by $\mathscr{B} = \mathscr{B}(G, F)$ the enlarged Bruhat-Tits building. Fix a maximal F-split torus A. Let $\mathscr{A} = \mathscr{A}(G, A, F)$ be the apartment of $\mathscr{B}$ corresponding to A.

1.1. Definition of the Iwahori-Weyl group. — Let $M = Z_G(A)$ be the centralizer of A, an anisotropic group, and let $N = N_G(A)$ be the normalizer of A. Denote by $W_0 = N(F)/M(F)$ the relative Weyl group.

DEFINITION 1.1. — i) The *Iwahori-Weyl group* $W = W(G, A, F)$ is the group

$$W \stackrel{\text{def}}{=} N(F)/M_1,$$

where M_1 is the unique parahoric subgroup of $M(F)$.

ii) Let $\mathfrak{a} \subset \mathscr{A}$ be a facet and $P_{\mathfrak{a}}$ the associated parahoric subgroup. The subgroup $W_{\mathfrak{a}}$ of the Iwahori-Weyl group corresponding to $\mathfrak{a}$ is the group

$$W_{\mathfrak{a}} \stackrel{\text{def}}{=} P_{\mathfrak{a}} \cap N(F)/M_1.$$

The group $N(F)$ operates on $\mathscr{A}$ by affine transformations

$$\nu : N(F) \longrightarrow \mathrm{Aff}(\mathscr{A}). \tag{1.1}$$

The kernel $\ker(\nu)$ is the unique maximal compact subgroup of $M(F)$ and contains the compact group M_1. Hence, the morphism (1.1) induces an action of W on $\mathscr{A}$.

Let G_1 be the subgroup of $G(F)$ generated by all parahoric subgroups, and define $N_1 = G_1 \cap N(F)$. Fix an alcove $\mathfrak{a}_C \subset \mathscr{A}$, and denote by B the associated Iwahori subgroup. Let $\mathbb{S}$ be the set of simple reflections at the walls of $\mathfrak{a}_C$. By Bruhat and Tits [3, Prop. 5.2.12], the quadruple

$$(G_1, B, N_1, \mathbb{S}) \tag{1.2}$$

is a (double) Tits system with affine Weyl group $W_{\mathrm{af}} = N_1/N_1 \cap B$, and the inclusion $G_1 \subset G(K)$ is B-N-adapted of connected type.

LEMMA 1.2. — *i) There is an equality $N_1 \cap B = M_1$.*

ii) The inclusion $N(F) \subset G(F)$ induces a group isomorphism $N(F)/N_1 \xrightarrow{\cong} G(F)/G_1$.

Proof. — The group $N_1 \cap B$ operates trivially on $\mathscr{A}$ and so is contained in $\ker(\nu) \subset M(F)$. In particular, $N_1 \cap B = M(F) \cap B$. But $M(F) \cap B$ is a parahoric subgroup of $M(F)$ and therefore equal to M_1.
The group morphism $N(F)/N_1 \to G(F)/G_1$ is injective by definition. We have to show that $G(F) = N(F) \cdot G_1$. This follows from the fact that the inclusion $G_1 \subset G(F)$ is B-N-adapted, cf. [2, 4.1.2]. □

Kottwitz defines in [5, §7] a surjective group morphism

$$\kappa_G : G(F) \longrightarrow X^*(Z(\hat{G})^I)^\Sigma. \tag{1.3}$$

Note that in [*loc. cit.*] the residue field k is assumed to be finite, but the arguments extend to the general case.

Lemma 1.3. — *There is an equality $G_1 = \ker(\kappa_G)$ as subgroups of $G(F)$.*

Proof. — For any facet $\mathfrak{a}$, let $\mathrm{Fix}(\mathfrak{a})$ be the subgroup of $G(F)$ which fixes $\mathfrak{a}$ pointwise. The intersection $\mathrm{Fix}(\mathfrak{a}) \cap \ker(\kappa_G)$ is the parahoric subgroup associated to $\mathfrak{a}$, cf. [4, Proposition 3]. This implies $G_1 \subset \ker(\kappa_G)$. For any facet $\mathfrak{a}$, let $\mathrm{Stab}(\mathfrak{a})$ be the subgroup of $G(F)$ which stabilizes $\mathfrak{a}$. Fix an alcove $\mathfrak{a}_C$. There is an equality

$$\mathrm{Fix}(\mathfrak{a}_C) \cap G_1 = \mathrm{Stab}(\mathfrak{a}_C) \cap G_1, \tag{1.4}$$

and (1.4) holds with G_1 replaced by $\ker(\kappa_G)$, cf. [4, Lemma 17]. Assume that the inclusion $G_1 \subset \ker(\kappa_G)$ is strict, and let $\tau \in \ker(\kappa_G)\backslash G_1$. By Lemma 1.2 ii), there exists $g \in G_1$ such that $\tau' = \tau \cdot g$ stabilizes $\mathfrak{a}_C$, and hence τ' is an element of the Iwahori subgroup $\mathrm{Stab}(\mathfrak{a}_C) \cap \ker(\kappa_G)$. This is a contradiction, and proves the lemma. □

By Lemma 1.2, there is an exact sequence

$$1 \longrightarrow W_{\mathrm{af}} \longrightarrow W \xrightarrow{\kappa_G} X^*(Z(\hat{G})^I)^\Sigma \longrightarrow 1. \tag{1.5}$$

The stabilizer of the alcove $\mathfrak{a}_C$ in W maps isomorphically onto $X^*(Z(\hat{G})^I)^\Sigma$ and presents W as a semidirect product

$$W = X^*(Z(\hat{G})^I)^\Sigma \ltimes W_{\mathrm{af}}. \tag{1.6}$$

For a facet $\mathfrak{a}$ contained in the closure of $\mathfrak{a}_C$, the group $W_\mathfrak{a}$ is the parabolic subgroup of W_{af} generated by the reflections at the walls of $\mathfrak{a}_C$ which contain $\mathfrak{a}$.

Theorem 1.4. — *Let $\mathfrak{a}$ (resp. $\mathfrak{a}'$) be a facet contained in the closure of $\mathfrak{a}_C$, and let $P_\mathfrak{a}$ (resp. $P_{\mathfrak{a}'}$) be the associated parahoric subgroup. There is a bijection*

$$\begin{aligned} W_\mathfrak{a}\backslash W/W_{\mathfrak{a}'} &\xrightarrow{\cong} P_\mathfrak{a}\backslash G(F)/P_{\mathfrak{a}'} \\ W_\mathfrak{a}\, w\, W_{\mathfrak{a}'} &\longmapsto P_\mathfrak{a}\, n_w\, P_{\mathfrak{a}'}, \end{aligned}$$

where n_w denotes a representative of w in $N(F)$.

Proof. — Conjugating with elements of $N(F)$ which stabilize the alcove $\mathfrak{a}_C$, we are reduced to proving that

$$W_{\mathfrak{a}}\backslash W_{\mathrm{af}}/W_{\mathfrak{a}'} \longrightarrow P_{\mathfrak{a}}\backslash G_1/P_{\mathfrak{a}'} \tag{1.7}$$

is a bijection. But (1.7) is a consequence of the fact that the quadruple (1.2) is a Tits system, cf. [1, Chap. IV, §2, n^o 5, Remark. 2)]. □

Remark 1.5. — Let $G_{\mathrm{sc}} \to G_{\mathrm{der}}$ be the simply connected cover of the derived group G_{der} of G, and denote by A_{sc} the preimage of the connected component $(A \cap G_{\mathrm{der}})^0$ in G_{sc}. Then A_{sc} is a maximal F-split torus of G_{sc}. Let $W_{\mathrm{sc}} = W(G_{\mathrm{sc}}, A_{\mathrm{sc}})$ be the associated Iwahori-Weyl group. Consider the group morphism $\varphi : G_{\mathrm{sc}}(F) \to G_{\mathrm{der}}(F) \subset G(F)$. Then $G_1 = \varphi(G_{\mathrm{sc}}(F)) \cdot M_1$ by the discussion above [3, Proposition 5.2.12], and this yields an injective morphism of groups

$$W_{\mathrm{sc}} \longrightarrow W$$

which identifies W_{sc} with W_{af}.

1.2. Passage to $\breve{F}$. — Let S be a maximal $\breve{F}$-split torus which is defined over F and contains A, cf. [3]. Denote by $\mathscr{A}^{\mathrm{nr}} = \mathscr{A}(G, S, \breve{F})$ the apartment corresponding to S over $\breve{F}$. The group Σ acts on $\mathscr{A}^{\mathrm{nr}}$, and there is a natural Σ-equivariant embedding

$$\mathscr{A} \longrightarrow \mathscr{A}^{\mathrm{nr}}, \tag{1.8}$$

which identifies $\mathscr{A}$ with the Σ-fixpoints $(\mathscr{A}^{\mathrm{nr}})^{\Sigma}$, cf. [3, 5.1.20]. The facets of $\mathscr{A}$ correspond to the Σ-invariant facets of $\mathscr{A}^{\mathrm{nr}}$.

Let $T = Z_G(S)$ (a maximal torus) be the centralizer of S, and let $N_S = N_G(S)$ be the normalizer of S. Let T_1^{nr} be the unique parahoric subgroup of $T(\breve{F})$. Denote by $W^{\mathrm{nr}} = W(G, S, \breve{F})$ the Iwahori-Weyl group

$$W^{\mathrm{nr}} = N_S(\breve{F})/T_1^{\mathrm{nr}}$$

over $\breve{F}$. The group Σ acts on W^{nr}, and the group of fixed points $(W^{\mathrm{nr}})^{\Sigma}$ acts on $\mathscr{A}$ by (1.8). We have

$$(W^{\mathrm{nr}})^{\Sigma} = N_S(F)/T_1,$$

since $H^1(\Sigma, T_1^{\mathrm{nr}})$ is trivial. For an element $n \in N_S(F)$ the tori A and nAn^{-1} are both maximal F-split tori of S and hence are equal. This shows $N_S(F) \subset N(F)$, and we obtain a group morphism

$$(W^{\mathrm{nr}})^{\Sigma} = N_S(F)/T_1 \longrightarrow N(F)/M_1 = W, \tag{1.9}$$

which is compatible with the actions on $\mathscr{A}$.

Lemma 1.6. — *The morphism* (1.9) *is an isomorphism, i.e.,* $(W^{\mathrm{nr}})^{\Sigma} \xrightarrow{\cong} W$.

Proof. — Let $\mathfrak{a}_C$ be a Σ-invariant alcove of $\mathscr{A}^{\mathrm{nr}}$. The morphism (1.9) is compatible with the semidirect product decomposition (1.6) given by $\mathfrak{a}_C$. We are reduced to proving that the morphism

$$(W_{\mathrm{af}}^{\mathrm{nr}})^{\Sigma} \longrightarrow W_{\mathrm{af}}$$

is an isomorphism. It is enough to show that $(W_{\mathrm{af}}^{\mathrm{nr}})^{\Sigma}$ acts simply transitively on the set of alcoves of $\mathscr{A}$. Let $\mathfrak{a}_{C'}$ another Σ-invariant alcove of $\mathscr{A}^{\mathrm{nr}}$. Then there is a unique $w \in W_{\mathrm{af}}^{\mathrm{nr}}$ such that $w \cdot \mathfrak{a}_C = \mathfrak{a}_{C'}$. The uniqueness implies $w \in (W_{\mathrm{af}}^{\mathrm{nr}})^{\Sigma}$. □

COROLLARY 1.7. — *Let $\mathfrak{a} \subset \mathscr{A}$ be a facet, and denote by $\mathfrak{a}^{\mathrm{nr}} \subset \mathscr{A}^{\mathrm{nr}}$ the unique facet containing $\mathfrak{a}$. Then $W_{\mathfrak{a}} = (W_{\mathfrak{a}^{\mathrm{nr}}})^{\Sigma}$ under the inclusion $W \hookrightarrow W^{\mathrm{nr}}$.* □

1.3. The length function on W. — Let $\mathscr{R} = \mathscr{R}(G, A, F)$ be the set of affine roots. We regard $\mathscr{R}$ as a subset of the affine functions on $\mathscr{A}$. The Iwahori-Weyl group W acts on $\mathscr{R}$ by the formula

$$(w \cdot \alpha)(x) = \alpha(w^{-1} \cdot x) \tag{1.10}$$

for $w \in W$, $\alpha \in \mathscr{R}$ and $x \in \mathscr{A}$. This action preserves non-divisible [(1)] roots.

Fix an alcove $\mathfrak{a}_C$ in $\mathscr{A}$. By (1.6), W is the semidirect product of W_{af} with the stabilizer of the alcove $\mathfrak{a}_C$ in W. Hence, W is a quasi-Coxeter system and is thus equipped with a Bruhat-Chevalley partial order $\leq$ and a length function l.

For $\alpha \in \mathscr{R}$, we write $\alpha > 0$ (resp. $\alpha < 0$), if α takes positive (resp. negative) values on $\mathfrak{a}_C$. For $w \in W$, define

$$\mathscr{R}(w) \overset{\mathrm{def}}{=} \{\alpha \in \mathscr{R} \mid \alpha > 0 \quad \text{and} \quad w\alpha < 0\}. \tag{1.11}$$

We have $\mathscr{R}(w) = \mathscr{R}(\tau w)$ for any τ in the stabilizer of $\mathfrak{a}_C$. Let $\mathbb{S}$ be the reflections at the walls of $\mathfrak{a}_C$. These are exactly the elements in W_{af} of length 1. For any $s \in \mathbb{S}$, there exists a unique non-divisible root $\alpha_s \in \mathscr{R}(s)$. In particular, $\mathscr{R}(s)$ has cardinality ≤ 2.

LEMMA 1.8. — *Let $w \in W$ and $s \in \mathbb{S}$. If $\alpha \in \mathscr{R}(s)$, then $w\alpha > 0$ if and only if $w \leq ws$.*

Proof. — We may assume that $w \in W_{\mathrm{af}}$ and that $\alpha_s = \alpha$ is non-divisible. We show that $w\alpha_s < 0$ if and only if $ws \leq w$. If $w\alpha_s < 0$, then fix a reduced decomposition $w = s_1 \cdots s_n$ with $s_i \in \mathbb{S}$. There exists an index i such that

$$s_{i+1} \cdots s_n \alpha_s > 0 \qquad \text{and} \qquad s_i \cdot s_{i+1} \cdots s_n \alpha_s < 0,$$

(1) An element $\alpha \in \mathscr{R}$ is called non-divisible, if $\frac{1}{2}\alpha \notin \mathscr{R}$.

i.e., $s_{i+1}\cdots s_n\alpha_s = \alpha_{s_i}$ is the unique non-divisible root in $\mathscr{R}(s_i)$. Hence,

$$s_{i+1}\cdots s_n \cdot s \cdot s_n \cdots s_{i+1} = s_i,$$

and $ws \leq w$ holds true. Conversely, if $ws \leq w$, then $ws \leq (ws)s$. This implies that $(ws)\alpha_s > 0$ by what we have already shown. But $s\alpha_s = -\alpha_s$, and $w\alpha_s < 0$ holds true. □

Lemma 1.9. — *Let w, $v \in W$. Then*

$$\mathscr{R}(wv) \subset \mathscr{R}(v) \sqcup v^{-1}\mathscr{R}(w),$$

and equality holds if and only if $l(wv) = l(w) + l(v)$.

Proof. — We may assume that w, $v \in W_{\mathrm{af}}$. Assume that $s = v \in \mathbb{S}$, which will imply the general case by induction on $l(v)$. The inclusion

$$\mathscr{R}(ws) \subset \mathscr{R}(s) \sqcup s\mathscr{R}(w) \tag{1.12}$$

is easy to see. If in (1.12) equality holds, then we have to show that $l(ws) = l(w)+1$, i.e., $w \leq ws$. In view of Lemma 1.8, it is enough to show that $w\alpha > 0$ for $\alpha \in \mathscr{R}(s)$. But this is equivalent to $\mathscr{R}(s) \subset \mathscr{R}(ws)$, and we are done. Conversely, if $w \leq ws$, then equality in (1.12) also follows from Lemma 1.8. □

Corollary 1.10. — *If every root in $\mathscr{R}$ is non-divisible, then $l(w) = |\mathscr{R}(w)|$ for every $w \in W$.*

1.4. The length function on W^{nr}. — In this section, the residue field k is finite of cardinality q. Let $\mathscr{R}^{\mathrm{nr}} = \mathscr{R}(G, S, \breve{F})$ be the set of affine roots over $\breve{F}$. Note that every root of $\mathscr{R}^{\mathrm{nr}}$ is non-divisible, since $G \otimes \breve{F}$ is residually split. Let W^{nr} be the Iwahori-Weyl group over F^{nr}. Denote by $\mathfrak{a}_C^{\mathrm{nr}}$ the unique Σ-invariant facet of $\mathscr{A}^{\mathrm{nr}}$ containing $\mathfrak{a}_C$. Let $\leq^{\mathrm{nr}}$ be the corresponding Bruhat order and l^{nr} the corresponding length function on W^{nr}. By Lemma 1.6, we may regard W as the subgroup of W^{nr} whose elements are fixed by Σ.

Let $w \in W$. If $\alpha \in \mathscr{R}^{\mathrm{nr}}(w)$, then its restriction to $\mathscr{A}$ is non-constant, and hence $\alpha \in \mathscr{R}$ by [7, 1.10.1]. We obtain a restriction map

$$\begin{aligned} \mathscr{R}^{\mathrm{nr}}(w) &\longrightarrow \mathscr{R}(w) \\ \alpha &\longmapsto \alpha|_{\mathscr{A}}. \end{aligned} \tag{1.13}$$

Proposition 1.11. — *The inclusion $W \subset W^{\mathrm{nr}}$ is compatible with the Bruhat orders in the sense that for $w, w' \in W$ we have $w \leq w'$ if and only if $w \leq^{\mathrm{nr}} w'$, and $l(w) = 0$ if and only if $l^{\mathrm{nr}}(w) = 0$. For $w \in W$, there is the equality*

$$|BwB/B| = q^{l^{\mathrm{nr}}(w)},$$

where B is the Iwahori subgroup in $G(F)$ attached to $\mathfrak{a}_C$.

Proof. — We need some preparation. Let $w \in W$, $s \in \mathbb{S}$ with $w \leq ws$.

Sublemma 1.12. — *There is an equality*

$$\mathscr{R}^{\mathrm{nr}}(ws) = \mathscr{R}^{\mathrm{nr}}(s) \sqcup s\mathscr{R}^{\mathrm{nr}}(w).$$

In particular, $l^{\mathrm{nr}}(ws) = l^{\mathrm{nr}}(w) + l^{\mathrm{nr}}(s)$.

Proof. — By Lemma 1.9 applied to $\mathscr{R}^{\mathrm{nr}}$, the inclusion '$\subset$' holds for general $w, s \in W^{\mathrm{nr}}$.

There is the inclusion $\mathscr{R}^{\mathrm{nr}}(s) \subset \mathscr{R}^{\mathrm{nr}}(ws)$: If $\alpha \in \mathscr{R}^{\mathrm{nr}}(s)$, then $\alpha|_{\mathscr{A}} \in \mathscr{R}(s)$ by (1.13). Since $w \leq ws$, we have $w \cdot \alpha|_{\mathscr{A}} > 0$ by Lemma 1.8. So $ws \cdot \alpha|_{\mathscr{A}} < 0$ which shows that $ws \cdot \alpha < 0$.

The inclusion $s\mathscr{R}^{\mathrm{nr}}(w) \subset \mathscr{R}^{\mathrm{nr}}(ws)$ follows similarly. □

SubLemma 1.12 implies that the inclusion $W \subset W^{\mathrm{nr}}$ is compatible with the Bruhat orders. To show the rest of the proposition, we may assume that $w \in W_{\mathrm{af}}$. Fix a reduced decomposition $w = s_1 \cdots s_n$ with $s_i \in \mathbb{S}$. By standard facts on Tits systems, the multiplication map

$$Bs_1B \times^B \cdots \times^B Bs_nB/B \longrightarrow BwB/B \tag{1.14}$$

is bijective. In view of SubLemma 1.12, we reduce to the case that $n = 1$, i.e., $s = w \in \mathbb{S}$ is a simple reflection. Let $\mathscr{B}$ be the Iwahori group scheme over $\mathscr{O}_F$ corresponding to the Iwahori subgroup B, and denote by $\mathscr{P}$ the parahoric group scheme corresponding to the parahoric subgroup $B \cup BsB$. Let $\bar{\mathscr{P}}_{\mathrm{red}}$ be the maximal reductive quotient of $\mathscr{P} \otimes k$. This is a connected reductive group over k of semisimple k-rank 1. The image of the natural morphism

$$\mathscr{B} \otimes k \longrightarrow \mathscr{P} \otimes k \longrightarrow \bar{\mathscr{P}}_{\mathrm{red}},$$

is a Borel subgroup $\bar{\mathscr{B}}$ of $\bar{\mathscr{P}}_{\mathrm{red}}$. This induces a bijection

$$P/B \longrightarrow \bar{\mathscr{P}}_{\mathrm{red}}(k)/\bar{\mathscr{B}}(k). \tag{1.15}$$

By Lang's Lemma, we have $\bar{\mathscr{P}}_{\mathrm{red}}(k)/\bar{\mathscr{B}}(k) = (\bar{\mathscr{P}}_{\mathrm{red}}/\bar{\mathscr{B}})(k)$. Let $\bar{s}$ be the image of s under (1.15), and denote by $C_{\bar{s}}$ the $\bar{\mathscr{B}}$-orbit of $\bar{s}$ in the flag variety $\bar{\mathscr{P}}_{\mathrm{red}}/\bar{\mathscr{B}}$. It follows that the image of BsB/B under (1.15) identifies with the k-points $C_{\bar{s}}(k)$. Note that $\langle \bar{s} \rangle$ is the relative Weyl group of $\bar{\mathscr{P}}_{\mathrm{red}}$ with respect to the reduction to k of the natural $\mathscr{O}_F$-structure on A. Then $C_{\bar{s}} \simeq \bar{U}$ where $\bar{U}$ denotes the unipotent radical of $\bar{\mathscr{B}}$. But $\bar{U}$ is an affine space and hence $|C_{\bar{s}}(k)| = q^{\dim(\bar{U})}$. On the other hand,

$$l^{\mathrm{nr}}(s) = |\mathscr{R}^{\mathrm{nr}}(s)| = \dim(\bar{U}),$$

where the last equality holds because $\mathscr{R}^{\mathrm{nr}}(s)$ may be identified with the positive roots of $\bar{\mathscr{P}}_{\mathrm{red}} \otimes \bar{k}$ with respect to $\overline{\bar{\mathscr{B}}} \otimes \bar{k}$. □

Remark 1.13. — i) If G is residually split, then $l(w) = l^{\mathrm{nr}}(w)$ for all $w \in W$.

ii) Tits attaches in [7, 1.8] to every vertex v of the local Dynkin diagram a positive integer $d(v)$. To the vertex v, there corresponds a non-divisible affine root $\alpha_v \in \mathscr{R}$, and a simple reflection $s_v \in \mathbb{S}$. Then Proposition 1.11 shows that $d(v) = l^{\mathrm{nr}}(s_v)$, cf. [7, 3.3.1].

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