Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

RIGIDITY OF REDUCIBILITY OF GEVREY QUASI-PERIODIC COCYCLES ON $U(n)$

Xuanji Hou & Georgi Popov

Tome 144
Fascicule 1

2016

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 1-52

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 1, tome 144, janvier 2016

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement Europe : 178 €, hors Europe : 194 € ($ 291)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

ISSN 0037-9484

Directeur de la publication : Marc PEIGNÉ

Bull. Soc. math. France
144 (1), 2016, p. 1–52

RIGIDITY OF REDUCIBILITY OF GEVREY QUASI-PERIODIC COCYCLES ON $U(n)$

BY XUANJI HOU & GEORGI POPOV

ABSTRACT. — We consider the reducibility problem of cocycles (α, A) on $\mathbb{T}^d \times U(n)$ in Gevrey classes, where α is a Diophantine vector. We prove that, if a Gevrey cocycle is conjugated to a constant cocycle (α, C) by a suitable measurable conjugacy $(0, B)$, then for almost all C it can be conjugated to (α, C) in the same Gevrey class, provided that A is sufficiently close to a constant. If B is continuous we obtain that it is Gevrey smooth. We consider as well the global problem of reducibility in Gevrey classes when $d = 1$.

RÉSUMÉ (*Rigidité de réductibilité des cocycles quasi-périodiques de Gevrey sur* $U(n)$)
On considère le problème de la réductibilité de cocycles (α, A) sur $\mathbb{T}^d \times U(n)$ dans les classes de Gevrey, où α est Diophantien. Si A est proche d'une constante et le Gevrey cocycle (α, A) est conjuqué au cocycle constant (α, C) par une conjugaison mesurable $(0, B)$, on montre que pour presque tous C le cocycle peut êtrte conjuguer à (α, C) dans la même classe de Gevrey . Si B est continue on obtient qu'elle est Gevrey. On considère aussi le problème de la réductibilité globale dans les classes de Gevrey dans le cas où $d = 1$.

Texte reçu le 5 juillet 2013, accepté le 21 janvier 2014.

XUANJI HOU, School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, P. R. China • *E-mail :* hxj@mail.ccnu.edu.cn.
GEORGI POPOV, Université de Nantes, Laboratoire de mathématiques Jean Leray, CNRS: UMR 6629, France • *E-mail :* georgi.popov@univ-nantes.fr

Key words and phrases. — reducibility of quasi-periodic cocycles, Gevrey classes.

X.H. Supported by National Natural Science Foundation of China (Grant 11371019) and Program for New Century Excellent Talents in University (Grant NCET-12-0869).
G.P. Partially supported by Agence Nationale de la Recherche, Projet DYNPDE: ANR-2010-BLAN-102-03.

1. Introduction

This article is concerned with the reducibility of cocycles in Gevrey classes on the unitary group $U(n)$. A cocycle on $U(n)$ is a diffeomorphism of $\mathbb{T}^d \times U(n)$, $\mathbb{T}^d$ being the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, given by the skew-product

$$(\alpha, A): \ \mathbb{T}^d \times \mathbb{C}^n \to \mathbb{T}^d \times \mathbb{C}^n$$
$$(\theta, v) \mapsto (\theta + \alpha, A(\theta)v),$$

where $\alpha \in \mathbb{T}^d$ and $A : \mathbb{T}^d \to U(n)$ is a map. The corresponding dynamics is defined by the iterates of the cocycle by composition $(\alpha, A)^n$, $n \in \mathbb{Z}$. We denote by $C^r(\mathbb{T}^d, U(n))$ $(r = 0, 1, \ldots, \infty, \omega)$ the set of all C^r functions A. For any $\rho \geq 1$ and $L > 0$ we denote by $\mathcal{G}^\rho_L(\mathbb{T}^d, U(n))$ the class of Gevrey$-\mathcal{G}^\rho$ functions with an exponent ρ and Gevrey constant L. A map $A \in C^\infty(\mathbb{T}^d, U(n))$ belongs to that class if it satisfies (2.10) (see Section 2.2). Denote by $SW^{\mathcal{G}}_\rho(\mathbb{T}^d, U(n))$ $(SW^r(\mathbb{T}^d, U(n)))$, the set of all Gevrey$-\mathcal{G}^\rho$ (C^r) quasi-periodic cocycles on $U(n)$.

The dynamics is particularly simple if (α, A) is a constant cocycle. The cocycle (α, A) is said to be constant if A is a constant matrix. Two cocycles $(\alpha, A), (\alpha, \widetilde{A}) \in SW^r(\mathbb{T}^d, U(n))$ are said to be conjugated if there exists $B : \mathbb{T}^d \to U(n)$ such that

$$Ad(B).(\alpha, A) := (\alpha, B(\cdot + \alpha)^{-1} A B) = (\alpha, \widetilde{A}),$$

which means that $B(\theta + \alpha)^{-1} A(\theta) B(\theta) = \widetilde{A}(\theta)$ for any $\theta \in \mathbb{T}^d$. The cocycle (α, A) is said to be reducible if it is conjugated to a constant one. We say also that the conjugation or the reducibility is Gevrey$-\mathcal{G}^\rho$, C^r, or measurable, if B belongs to the corresponding class of functions.

Reducibility problem of cocycles has been investigated for a long time. The local reducibility problem (the cocycle is close to a constant one) is usually studied using KAM-type iterations. In particular, Eliasson's KAM method developed in [3] gives full-measure reducibility for generic one-parameter families of cocycles [2, 4, 10, 9, 5, 6]. The global reducibility problem (cocycles are no longer close to a constant one) has been studied by Avila, Krikorian and others. By means of a renormalization scheme Krikorian obtained a global density result for C^∞ cocycles on $SU(2)$ [11] and also results for cocycles on $SL(2, \mathbb{R})$ [1, 12]. Almost reducibility for Gevrey cocycles has been studied by Chavaudret in [2].

The *rigidity problem* we are interested in, can be formulated as follows. Suppose that a Gevrey$-\mathcal{G}^\rho$ cocycle is measurably reducible. Is it also Gevrey$-\mathcal{G}^\rho$ reducible? In the case of C^∞ or C^ω cocycles the rigidity problem has been investigated in [1, 12, 7, 6].

In this paper, we will focus our attention on the Gevrey case. We will prove a local rigidity result of reducibility in Gevrey classes which can be viewed as a Gevrey analogue of the main result in [7]. To this end we use techniques developed in [17]. When $d = 1$, the local result together with Krikorian's renormalization scheme imply as in [11, 1] a global rigidity result for Gevrey quasi-periodic cocycles on $\mathbb{T}^1 \times U(n)$.

Why are we interested in Gevrey classes? Gevrey classes appear naturally in the KAM theory when dealing with Diophantine frequencies [16, 17]. They provide a natural framework for studying KAM systems, Birkhoff normal forms with an exponentially small reminder terms and the Nekhoroshev theory, and give an inside relation between these theories [14, 15, 16, 17]. One can consider as well the more general Roumieu classes of non-quasi-analytic functions. In the case of Bruno-Rüssmann arithmetic conditions we suggest that similar results hold in appropriate Roumieu spaces.

To formulate the main results we recall certain arithmetic conditions. Given $\gamma > 0$ and $\tau > d-1$, we say that $\alpha \in \mathbb{R}^d$ is (γ, τ)-Diophantine if

$$|e^{2\pi i\langle k,\alpha\rangle} - 1| > \frac{\gamma^{-1}}{|k|^{\tau}}, \quad 0 \neq k \in \mathbb{Z}^d, \tag{1.1}$$

and we denote by $\mathrm{DC}\,(\gamma, \tau)$ the set of all such Diophantine vectors. Hereafter, $i := \sqrt{-1}$ stands for the imaginary unit. It is well known that $\mathrm{DC}\,(\tau) := \bigcup_{\gamma>0} \mathrm{DC}\,(\gamma, \tau)$ is a set of full Lebesgue measure. For any given $\alpha \in \mathbb{R}^d$, we denote by $\Upsilon(\alpha; \chi, \nu)$ the set of all vectors $(\phi_1, \ldots, \phi_n) \in \mathbb{R}^n$, satisfying

$$|\langle k, \alpha\rangle + \phi_p - \phi_q - j| \geq \frac{\chi}{(1+|k|)^{\nu}} \tag{1.2}$$

for any $p \neq q \in \{1, 2, \ldots, n\}$, $k \in \mathbb{Z}^d$ and $j \in \mathbb{Z}$. The set

$$\Upsilon(\alpha) := \bigcup_{\chi,\nu>0} \Upsilon(\alpha; \chi, \nu)$$

has full Lebesgue measure in $\mathbb{R}^n$. Recall that the Lie group $U(n)$ consists of all $A \in GL(n, \mathbb{C})$ satisfying $A^*A = I$. Hereafter, I stands for the identity matrix and A^* is the adjoint matrix to A in $M_n = M_n(\mathbb{C})$. The corresponding Lie algebra $u(n)$ is the set of $X \in gl(n, \mathbb{C})$ satisfying $X^* + X = 0$. Any $A \in U(n)$ is diagonalizable, and the set of eigenvalues of A, denoted by $\mathrm{Spec}\,(A)$, is a subset of $\{z \in \mathbb{C} : |z| = 1\}$. Denote by $\Sigma(\alpha; \chi, \nu)$ the set of $A \in U(n)$ with spectrum $\mathrm{Spec}\,(A) := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ satisfying

$$|\lambda_p - \lambda_q e^{2\pi i\langle k,\alpha\rangle}| \geq \frac{\chi}{(1+|k|)^{\nu}} \tag{1.3}$$

for any $p \neq q \in \{1, 2, \ldots, n\}$ and $k \in \mathbb{Z}^d$. Let $\Sigma(\alpha) = \bigcup_{\chi,\nu>0} \Sigma(\alpha; \chi, \nu)$. It is obvious that $A \in \Sigma(\alpha)$ if and only if

$$\mathrm{Spec}\,(A) = \{e^{2\pi i\varrho_1}, e^{2\pi i\varrho_2}, \ldots, e^{2\pi i\varrho_n}\}$$

with $(\varrho_1, \varrho_2, \ldots, \varrho_n) \in \Upsilon(\alpha)$.

In Section 2.3 we assign to any measurable map $B : \mathbb{T}^d \to M_n$ a number $\lceil B \rfloor$ which evaluates the distance from B to the set of "totally degenerate maps". A measurable maps $C : \mathbb{T}^d \to M_n$ will be called *totally degenerate* if there exist constant matrices $S, T \in U(n)$ such that the first row of the matrix $SC(\theta)T$ is zero for a.e. $\theta \in \mathbb{T}^d$. We say that $B : \mathbb{T}^d \to M_n$ is ϵ-non-degenerate if $\lceil B \rfloor \geq \epsilon$.

We are going to state the main results of the article.

THEOREM 1.1. — *Let $\rho > 1$ and $(\alpha, Ae^G) \in SW_\rho^{\mathscr{G}}(\mathbb{T}^d, U(n))$, where $\alpha \in DC(\gamma, \tau)$, $A \in U(n)$ is a constant matrix and $G \in \mathscr{G}_L^\rho(\mathbb{T}^d, u(n))$. Then for any $\ell > 1$ there is a positive constant $\delta = \delta(d, n, \rho, L, \gamma, \tau, \ell)$ such that for any $\epsilon \in (0, 1]$ the following holds. If the cocycle (α, Ae^G) is conjugated to a constant cocycle (α, C) with $C \in \Sigma(\alpha)$ by a measurable map $B : \mathbb{T}^d \to U(n)$ where*

$$\lceil B^* \rfloor \geq \epsilon > 0 \quad and \quad \|G\|_L < \delta\epsilon^\ell, \tag{1.4}$$

then (α, Ae^G) can be conjugated to (α, C) by a Gevrey map $\widetilde{B} \in \mathscr{G}^\rho(\mathbb{T}^d, U(n))$ in the same Gevrey class. Moreover, $\widetilde{B}(\theta) = B(\theta)$ for a.e. $\theta \in \mathbb{T}^d$, which implies that B is a $\mathscr{G}^\rho$ map if it is continuous.

Making use of the above local result and of the renormalization we obtain a global rigidity result. The renormalization scheme we apply in this paper has been developed by Krikorian and it is often used when studying the global properties of 1-dimensional quasi-periodic cocycles [1, 6, 11, 12]. To formulate the global result in the case $d = 1$ we need the following arithmetic condition on α involving the Gauss map $G : (0, 1) \to (0, 1)$, where $G(x) = \{x\}^{-1}$ and $\{x\}$ stands for the fractional part of x. We denote by $\mathrm{RDC}\,(\gamma, \tau)$ the set of all irrational $\alpha \in (0, 1)$ such that $G^m(\alpha)$ belongs to $\mathrm{DC}\,(\gamma, \tau)$ for infinitely many $m \in \mathbb{N}$. It can be shown that $\mathrm{RDC}\,(\gamma, \tau)$ is of full Lebesgue measure in $(0, 1)$ as long as $\mathrm{DC}\,(\gamma, \tau)$ is of positive measure [1]. We set as well $\mathrm{RDC} = \bigcup_{\gamma,\tau>0} \mathrm{RDC}\,(\gamma, \tau)$.

The global result is stated as follows.

THEOREM 1.2. — *For any $\alpha \in \mathrm{RDC}$, if $(\alpha, A) \in SW_\rho^{\mathscr{G}}(\mathbb{T}^1, U(n))$ is conjugated to a constant cocycle (α, C) with $C \in \Sigma(\alpha)$ by a measurable $B : \mathbb{T}^1 \to U(n)$, then it can be conjugated to (α, C) by a Gevrey map $\widetilde{B} \in \mathscr{G}^\rho(\mathbb{T}^d, U(n))$ of the same class. Moreover, $\widetilde{B}(\theta) = B(\theta)$ for a.e. $\theta \in \mathbb{T}^1$, which implies that B is Gevrey–$\mathscr{G}^\rho$ if it is continuous.*

Remark 1.1. — *We remark that the proofs in this paper can also be generalized to obtain similar local and global results for Gevrey cocycles on compact semisimle Lie groups.*

The article is organized as follows. In Sect. 2 we give certain facts about analytic, Gevrey and measurable functions which are needed in the sequel. In particular we prove the Approximation Lemma and the Inverse Approximation Lemma for Gevrey functions $P : \mathbb{T}^d \to u(n)$ of Gevrey index $\rho > 1$ which gives the optimal approximation of P with analytic functions P_j in the complex strips $\mathbb{T}^d_{h_j}$, where $h_j = h_0\delta^j$, $j \in \mathbb{N}$, and $0 < \delta < 1$. By optimal we mean that $P_j - P_{j-1}$ is $O(\exp(-Ch_m^{-1/(\rho-1)})$ in $\mathbb{T}^d_{h_j}$, where $C > 0$ is a constant. We point out that the approximation with the truncated Fourier series is not optimal. In Sect. 2.3 we introduce the important quantity $\lceil B \rfloor$ giving a sort of a "distance" between a measurable map $B : \mathbb{T}^d \to M_n$ and the set of "totally degenerate maps". The definition of $\lceil B \rfloor$ is invariant with respect to the choice of the unitary bases in $\mathbb{C}^n$. We introduce the sets $\Gamma(N, \epsilon)$ and $\Pi(\widetilde{N}, \xi, \varepsilon)$ in order to keep track on the evolution of the quantity $\lceil \cdot \rfloor$ when performing certain operations on B such as truncation of the Fourier series of B up to order N and multiplication. The set Π obeys a simple rule under multiplication which allows one to use it successfully in the Iterative Lemma.

In Sect. 3 we prove the local rigidity result. First we establish the KAM Step—Proposition 3.1. It provides a conjugation of an analytic cocycle (α, Ae^F) in $\mathbb{T}^d_h$ with sup-norm $|F|_h \le \epsilon \ll 1$ to another one $(\alpha, A_+e^{F_+})$ with sup-norm $|F_+|_{(1-\kappa)h} \le \epsilon^{1+\sigma}$, where $0 < \kappa < 1$ and $0 < \sigma \ll 1$ are constants and $A, A_+ \in U(n)$ are constant matrices. The sup-norm of the conjugating operator R, however, can be estimated only by ϵ^{-K_*}, where the constant $K_* \ge 1$ may be large due to the presence of resonances. On the other hand, it belongs to a certain $\Pi(N, 1/n, \epsilon^{1-4\sigma})$, N being the order of the truncated Fourier series of F, which gives control on $\lceil R \rfloor$. To solve the corresponding homological equations for the non-resonant terms we use a variant of the inverse function theorem—Lemma 3.1. Iterating the KAM step we obtain almost reducibility with optimal estimates in Lemma 3.5. By optimal we mean again that the small constants ε_m in Lemma 3.5 are of the size of $\exp(-Ch_m^{-\frac{1}{\rho-1}})$, $C > 0$. In Sect. 3.3 we prove reducibility in the Gevrey class $SW^{\mathcal{G}}_\rho(\mathbb{T}^d, U(n))$ provided that the cocycle is reducible by a measurable conjugation B satisfying (1.4) (see Lemma 3.9). The idea (see Lemma 3.6) is first to consider the conjugation with B^*R_m for $m \gg 1$, where R_m gives the the conjugation to the cocycle $(\alpha, A_m e^{F_m})$ in the Iterative Lemma. Using the ϵ-non-degeneracy of B^* given by (1.4) and the relation $R^{(m)} \in \Pi(L_m, n^{-m}, \epsilon/4n^m)$ in (3.84) with some $L_m \in \mathbb{N}$, we obtain that the eigenvalues of A_m satisfy a suitable non-resonant condition. This allows us to estimate R_m by $\varepsilon_m^{1/2}$ using the KAM Step (Proposition 3.1, (ii)). Then the Inverse Approximation Lemma gives a conjugation in the class $\mathscr{G}^\rho(\mathbb{T}^d, U(n))$. We point out that there is no loss of Gevrey regularity.

In Sect. 4 we prove Theorem 1.2 adapting the renormalization scheme to the case of Gevrey classes.

2. Preliminaries

In this section we introduce the necessary tools to prove the KAM Step and the Iterative Lemma. In Sect. 2.1 we recall well-known facts on the Fourier series of analytic functions $P : \mathbb{T}^d_{h_j} \to u(n)$. In Sect. 2.2 we prove the Approximation Lemma and the Inverse Approximation Lemma for Gevrey functions $P : \mathbb{T}^d \to u(n)$ of Gevrey index $\rho > 1$ which gives the best approximation of P with analytic functions P_j in the complex strips $\mathbb{T}^d_{h_j}$, where $h_j = h_0\delta^j$, $j \in \mathbb{N}$, and $0 < \delta < 1$. By "best approximation" we mean that the sup-norm of $P_j - P_{j+1}$ is of the size of $\exp(-Ch_j^{-\frac{1}{\rho-1}})$ in $\mathbb{T}^d_{h_{j+1}}$, where $C > 0$. We point out the usual approximation with the truncated Fourier series is not optimal, it gives an estimate with $\exp(-Ch_j^{-\frac{1}{\rho}})$. The usual approximation with entire functions due to Moser is not optimal either. In Sect. 2.3 we introduce the important invariant $\lceil B \rfloor$ for measurable functions $B : \mathbb{T}^d \to M_n$ and the sets Γ and Π, which we need in the KAM Step and in the Iterative Lemma.

2.1. Analytic functions. — Denote by the $M_n = M_n(\mathbb{C})$ the linear space of all $n \times n$ matrices with norm $|A| = \sup\{\|Au\| : \|u\| = 1\}$, where $\|\cdot\|$ is the norm on $\mathbb{C}^n$ associated with the Hermitian inner product on it. Given $h > 0$ we set

$$\mathbb{R}^d_h := \{\theta \in \mathbb{C}^d : |\mathrm{Im}\, \theta_j| < h, 1 \le j \le d\},$$

$$\mathbb{T}^d_h := \mathbb{R}^d_h/\mathbb{Z}^d = \{\theta \in \mathbb{C}^d/\mathbb{Z}^d : |\mathrm{Im}\, \theta_j| < h, 1 \le j \le d\},$$

and for any analytic function $F : \mathbb{R}^d_h \to M_n$ ($F : \mathbb{T}^d_h \to M_n$) we define

$$|F|_h = \sup_{|\mathrm{Im}\,\theta| < h} |F(\theta)|.$$

Denote by $C^\omega_h(\mathbb{T}^d, M_n)$ the Banach space of all analytic functions $F : \mathbb{T}^d_h \to M_n$, equipped with the sup-norm $|\cdot|_h$. The Fourier expansion of F is given by

$$F(\theta) = \sum_{k \in \mathbb{Z}^d} \widehat{F}(k) e^{2\pi i \langle k, \theta \rangle},$$

and the Fourier coefficients satisfy the estimate

$$|\widehat{F}(k)| \le |F|_h e^{-2\pi|k|h}. \tag{2.5}$$

We introduce as well the Wiener norm

$$|F|_{1,h} := \sum_{k\in\mathbb{Z}^n} |\widehat{F}(k)| e^{2\pi|k|h} \tag{2.6}$$

and we denote by $\mathfrak{B}_h$ space of all $F \in C_h^\omega(\mathbb{T}^d, M_n)$ with bonded norm $|F|_{1,h} < \infty$. One can easily see that $\mathfrak{B}_h$ is a Banach space and even a Banach algebra—for any $F, G \in \mathfrak{B}_h$ one has

$$|FG|_{1,h} \le |F|_{1,h}|G|_{1,h}. \tag{2.7}$$

Taking into account (2.5) we get the following relation between the two norms

$$|F|_h \le |F|_{1,h},\ |F|_{1,h_+} \le |F|_h \sum_{k\in\mathbb{Z}^d} e^{-2\pi|k|(h-h_+)} \le \frac{c_*}{(h-h_+)^d}|F|_h \tag{2.8}$$

for any $0 < h_+ < h$, where $c_* = c_*(d)$ is a positive constant.

We denote by $T_N F$ and $R_N F$ ($N \in \mathbb{N}$) the truncated trigonometric polynomial of F of order N and the corresponding remainder term respectively, i.e.,

$$T_N F = \sum_{|k|\le N} \widehat{F}(k) e^{2\pi i\langle k,\theta\rangle} \quad \text{and} \quad R_N F = \sum_{|k|>N} \widehat{F}(k) e^{2\pi i\langle k,\theta\rangle}.$$

One obtains as in (2.8) the well-known estimate

$$|R_N F|_{h_+} \le |R_N F|_{1,h_+} \le \frac{c_* N^d}{(h-h_+)^d} e^{-N(h-h_+)} |F|_h \tag{2.9}$$

where $0 < h_+ < h$.

For any subset $\Omega \subseteq M_n$, we denote by $C_h^\omega(\mathbb{T}^d, \Omega)$ the set of all $F \in C_h^\omega(\mathbb{T}^d, M_n)$ satisfying $F(\mathbb{T}^d) \subseteq \Omega$. In particular the space $C_h^\omega(\mathbb{T}^d, u(n))$ consists of all analytic functions $F : \mathbb{T}_h^d \to M_n$ such that $F(\theta)^* = -F(\theta)$ for each $\theta \in \mathbb{T}^d$. This is a Banach subspace of $C_h^\omega(\mathbb{T}^d, M_n)$ and $F \in C_h^\omega(\mathbb{T}^d, M_n)$ is in $C_h^\omega(\mathbb{T}^d, u(n))$ if and only if

$$\widehat{F}(k)^* = -\widehat{F}(-k).$$

2.2. Approximation and inverse approximation lemma for Gevrey functions. — Given $\rho \ge 1$, $L > 0$, and a subset $\Omega \subseteq M_n$, we denote by $\mathscr{G}_L^\rho(\mathbb{T}^d, \Omega)$ the set of all C^∞ functions $P : \mathbb{T}^d \to \Omega$ such that

$$\|P\|_L := \sup_{k\in\mathbb{N}^d} \sup_{\theta\in\mathbb{T}^d} (|\partial^k P(\theta)| L^{-|k|} k!^{-\rho}) < \infty \tag{2.10}$$

where $|k| = k_1 + \cdots + k_d$ and $k! = k_1! \cdots k_d!$ for $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$. Hereafter we suppose that Ω is closed in M_n. Then $\mathscr{G}_L^\rho(\mathbb{T}^d, \Omega)$ is complete. For $\rho = 1$ this space consists of analytic functions. When $\rho > 1$ the space $\mathscr{G}_L^\rho(\mathbb{T}^d, \Omega)$ is not quasi-analytic, i.e., the unique continuation rule does not hold any more and there exist functions with compact support. On the other hand, functions of that class can be nicely approximated by analytic functions as follows as we shall see below.

PROPOSITION 2.1 (Approximation Lemma). — *Fix $\rho > 1$, $L \geq 1$, $0 < \delta < 1$ and set $h_j = h_0\delta^j$, $j \in \mathbb{N}$, where $0 < h_0 \leq 1/(2L)$. Then for any $P \in \mathscr{G}^\rho_L(\mathbb{T}^d, u(n))$ there is a sequence $P_j \in C^\omega_{h_j}(\mathbb{T}^d, u(n))$, $j \geq 0$, such that*

$$\sup_{\theta\in\mathbb{T}^d} |P_j(\theta) - P(\theta)| \leq C_0 L^d \exp\left(-(cLh_j)^{-\frac{1}{\rho-1}}\right) \|P\|_L$$

and

$$|P_{j+1} - P_j|_{h_{j+1}} \leq C_0 L^d e^{-(cLh_j)^{-1/(\rho-1)}} \|P\|_L,$$

$$|P_0|_{h_0} \leq C_0(1 + L^d e^{-(cLh_0)^{-1/(\rho-1)}}) \|P\|_L,$$

where $c = c(\rho)$ and $C_0 = C_0(d, \rho)$ are positive constants depending only on ρ and on d and ρ respectively.

Proof. — Proposition 2.1 is a variant of Proposition 3.1 [17]. The proof given bellow is adapted to the case when P takes its values in $u(n)$ simplifying as well some arguments of [17].

1. Almost analytic extension of P. We recall the following estimates from [17].

LEMMA 2.1. — *There is a constant $C(\rho) \geq 1$, depending only on ρ, such that for any $t \in (0, 1]$ and $m \in \mathbb{N}$ satisfying*

$$1 \leq m \leq t^{-\frac{1}{\rho-1}} + 1\,, \tag{2.11}$$

the following inequality holds

$$t^m\, m!^{\,\rho-1} \leq C(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m}. \tag{2.12}$$

Proof. — Stirling's formula implies

$$\begin{aligned} t^m\, m!^{\,\rho-1} &\leq C_1(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m} \exp\left((\rho-1)m\ln m + m\ln t\right) \\ &= C_1(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m} \exp\left\{(\rho-1)m\ln\left(m\, t^{\frac{1}{\rho-1}}\right)\right\}. \end{aligned}$$

Moreover, (2.11) yields

$$m\ln\left(m\, t^{\frac{1}{\rho-1}}\right) \leq m\ln\left(1 + t^{\frac{1}{\rho-1}}\right) \leq m\, t^{\frac{1}{\rho-1}} \leq 1 + t^{\frac{1}{\rho-1}} \leq 2\,,$$

which proves (2.12). □

We define an almost analytic extensions F_j of P in $\mathbb{T}^d_{2h_j}$, $j \geq 0$, as follows

$$F_j(\theta + i\widetilde{\theta}) = \sum_{k \in \mathcal{M}_j} \partial_\theta^k P(\theta) \frac{(i\widetilde{\theta})^k}{k!}, \quad \theta \in \mathbb{T}^d, \ \widetilde{\theta} \in \mathbb{R}^d. \tag{2.13}$$

The index set $\mathcal{M}_j$ consists of all multi-indices $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ such that $k_1 \leq N_j, \dots, k_d \leq N_j$, where

$$N_j = \left[(2Lh_j)^{-\frac{1}{\rho-1}}\right] \tag{2.14}$$

and $[x] = \inf\{k \in \mathbb{Z} : x \geq k\}$ is the integer part of $x \in \mathbb{R}$. Estimating (2.13) term by term and using (2.10) one obtains

$$|F_j|_{2h_j} \leq \|P\|_L \sum_{k \in \mathcal{M}_j} (2Lh_j)^{|k|} k!^{\,\rho-1} .$$

Let $k = (k_1, \dots, k_s, \dots k_d) \in \mathcal{M}_j$ and $k_s > 0$. Then $0 < k_s \leq N_j = \left[(2Lh_j)^{-\frac{1}{\rho-1}}\right]$ and setting $t = 2Lh_j \leq 2Lh_0 < 1$ and $m = k_s$ in Lemma 2.1 one obtains

$$(2Lh_j)^{k_s} k_s!^{\,\rho-1} \leq C(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m}$$

which implies

$$|F_j|_{2h_j} \leq \left(1 + C(\rho) \sum_{m=1}^{\infty} m^{(1-\rho)/2} e^{-(\rho-1)m}\right)^d \|P\|_L = C_1(\rho, d)\|P\|_L.$$

On the other hand, applying $\bar{\partial}_s := \frac{1}{2}\left(\frac{\partial}{\partial \theta_s} + i \frac{\partial}{\partial \widetilde{\theta}_s}\right)$, $1 \leq s \leq d$, to F_j one gets

$$2\bar{\partial}_s F_j(\theta + i\widetilde{\theta}) = \sum_{k \in \mathcal{M}_j^s} \partial_\theta^k \partial_{\theta_s} P(\theta) \frac{(i\widetilde{\theta})^k}{k!} \tag{2.15}$$

where $\mathcal{M}_j^s$ consists of all multi-indices $k = (k_1, \dots, k_s, \dots k_d) \in \mathcal{M}_j$ such that $k_s = N_j$. Each term in the sum can be estimated in $\mathbb{T}^d_{2h_j}$ by

$$L(2Lh_j)^{|k|} k!^{\,\rho-1} (k_s + 1)^\rho \|P\|_L.$$

Since

$$(2Lh_j)^{-\frac{1}{\rho-1}} \leq k_s = N_j < (2Lh_j)^{-\frac{1}{\rho-1}} + 1,$$

one obtains from (2.12) (with $t = 2L_1 h_j$ and $m = k_s = N_j$) the estimate

$$(2Lh_j)^{k_s}\, k_s!^{\,\rho-1} (k_s + 1)^\rho \leq C (2Lh_j)^{-\frac{\rho}{\rho-1} - \frac{1}{2}} \exp\left(-(\rho-1)(2Lh_j)^{-\frac{1}{\rho-1}}\right).$$

Using the inequality $x^m e^{-x} \leq m!$ for $m \in \mathbb{N}$ and $x \geq 0$ this yields

$$
\begin{aligned}
|\bar{\partial}_s F_j|_{2h_j} &\leq C_1\, L(Lh_j)^{-\frac{\rho}{\rho-1}-\frac{1}{2}} \exp\left(-(\rho-1)(2Lh_j)^{-\frac{1}{\rho-1}}\right) \|P\|_L \\
&\leq C\, L \exp\left(-(cLh_j)^{-\frac{1}{\rho-1}}\right) \|P\|_L\,,
\end{aligned} \tag{2.16}
$$

where $c = 3(\rho-1)^{1-\rho}$ and C_1 and $C = C(\rho, d) > 0$ are positive constants depending only on d and ρ. In the same way, differentiating (2.15) one obtains the estimate

$$
|\bar{\partial}^l F_j|_{2h_j} \leq C(\rho, d)\, L^{|l|} \exp\left(-(cLh_j)^{-\frac{1}{\rho-1}}\right) \|P\|_L \tag{2.17}
$$

for any $l = (l_1, \dots, l_d) \in \mathbb{N}^d$ of length $1 \leq |l| = l_1 + \cdots + l_d \leq d$ and with components $0 \leq l_s \leq 1$, $s \in \{1, \dots, d\}$. Moreover, $F_j(\theta) = P(\theta)$ for any $\theta \in \mathbb{T}^d$ and (2.13) yields

$$
\forall\, z \in \mathbb{T}^d_{2h_j},\ F_j(z)^* = -F_j(\bar{z}). \tag{2.18}
$$

From now on we consider F_j as $\mathbb{Z}^d$-periodic functions on $\mathbb{R}^d_{2h_j}$ with values in M_n which means that $F_j(z+p) = F_j(z)$ for any $p \in \mathbb{Z}^d$.

2. Construction of P_j. We are going to approximate F_j by 1-periodic analytic in $\mathbb{R}^d_{h_j}$ functions using Green's formula

$$
\frac{1}{2\pi i}\int_{\partial D} \frac{f(\eta)}{\eta - \zeta}\, d\eta + \frac{1}{2\pi i}\iint_D \frac{\bar{\partial} f(\eta)}{\eta - \zeta}\, d\eta \wedge d\bar{\eta} = \begin{cases} f(\zeta) & \text{if } \zeta \in D \\ 0 & \text{if } \zeta \notin \bar{D} \end{cases} \tag{2.19}
$$

where $D \subset \mathbb{C}$ is a bounded domain symmetric with respect to the real axis and with a piecewise smooth boundary ∂D which is positively oriented with respect to D, $\bar{D} = D \cup \partial D$, and $f \in C^1(\bar{D}, M_n)$. Notice that

$$
F(\zeta) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\eta)}{\eta - \zeta}\, d\eta
$$

is analytic in D with values in M_n.

Lemma 2.2. — *Suppose that $f(z)^* = -f(\bar{z})$ for any $z \in \partial D$. Then $F(z)^* = -F(\bar{z})$ for any $z \in D$.*

The proof is immediate using the symmetry of ∂D with respect to the involution $z \to \bar{z}$.

Denote by $D_j \subset \mathbb{C}$ the open rectangle $\{z \in \mathbb{C} : |\mathrm{Re}\, z| < 1/2,\ |\mathrm{Im}\, z| < 2h_j\}$, by ∂D_j its boundary which is positively oriented with respect to D_j, and by Γ_j the union of the oriented segments

$$
\Gamma_j := [-1/2 - 2ih_j, 1/2 - 2ih_j] \cup [1/2 + 2ih_j, -1/2 + 2ih_j].
$$

Given $\eta \in \mathbb{C}$, we consider the 1-periodic meromorphic function

$$\zeta \mapsto K(\eta,\zeta) := \frac{1}{\eta-\zeta} + \sum_{k=1}^{\infty}\left(\frac{1}{\eta-\zeta+k} + \frac{1}{\eta-\zeta-k}\right).$$

Obviously, $K(\eta,\zeta) = -K(\zeta,\eta)$ and the meromorphic function $\eta \to K(\eta,\zeta)$ is 1-periodic for any ζ fixed. Set $D := \{z \in \mathbb{C},\ |\mathrm{Re}\, z| < 1/2,\ |\mathrm{Im}\, z| < 1/2\}$. Writing $K = K_0 + K_1$, where

$$K_0(\eta,\zeta) := \sum_{k=-2}^{2} \frac{1}{\eta-\zeta+k} \quad \text{and} \quad K_1(\eta,\zeta) := 2\sum_{k=3}^{\infty} \frac{\eta-\zeta}{(\eta-\zeta)^2-k^2}$$

one can find $C > 0$ such that

$$\forall\, z \in D, \quad i\int_D |K(\eta,z)|\, d\eta \wedge d\bar{\eta} \le C. \tag{2.20}$$

Consider the function

$$F_{j,1}(z) := \frac{1}{2\pi i}\int_{\Gamma_j} F_j(\eta_1, z_2, \dots, z_d) K(\eta_1, z_1)\, d\eta_1\ , \quad z \in \mathbb{R}^d_{2h_j}.$$

It is smooth and $\mathbb{Z}^d$-periodic in the strip $\mathbb{R}^d_{2h_j}$ and analytic with respect to z_1. Moreover, for any $z \in \mathbb{R}^d_{2h_j}$ such that $z_1 \in D_j$ we have

$$F_{j,1}(z) = \frac{1}{2\pi i}\int_{\partial D_j} F_j(\eta_1, z_2, \dots, z_d) K(\eta_1, z_1)\, d\eta_1$$

since the function under the integral is 1-periodic with respect to η_1. Using (2.18) we obtain by Lemma 2.2 that $F_{j,1}(z)^* = -F_{j,1}(\overline{z})$ for any $z \in \mathbb{R}^d_{2h_j}$ such that $z_1 \in D_j$ and by continuity and periodicity we get it for any $z \in \mathbb{R}^d_{2h_j}$. Moreover, (2.19) yields

$$F_{j,1}(z) = F_j(z) - \frac{1}{2\pi i}\int_{D_j} \bar{\partial}_{\eta_1} F_j(\eta_1, z_2, \dots, z_d) K(\eta_1, z_1)\, d\eta_1 \wedge d\bar{\eta}_1.$$

Set $F_{j,0}(z) := F_j(z)$ and denote by $\mathcal{U}_{j,1}$ the set of all $(z_1, \dots, z_d) \in \mathbb{R}^d_{2h_j}$ such that $|\mathrm{Im}\, z_1| \le h_j$. Using (2.16), (2.17) and (2.20) we obtain for any multi-index $l = (0, l_2, \dots, l_d) \in \mathbb{N}^d$ with $0 \le l_s \le 1$ for $2 \le s \le d$ the following estimate

$$\left|\bar{\partial}^l (F_{j,1} - F_{j,0})\right|_{\mathcal{U}_{j,1}} \ \le\ C\, L^d \exp\left(-(cLh_j)^{-\frac{1}{\rho-1}}\right) \|P\|_L, \tag{2.21}$$

where $C = C(\rho, d) > 0$. For $2 \le s \le d$ we define by recurrence $\mathcal{U}_{j,s}$ as the set of all $(z_1, \dots z_s, \dots z_d)$ in $\mathcal{U}_{j,s-1}$ such that $|\mathrm{Im}\, z_s| \le h_j$ and set

$$F_{j,s}(z) := \frac{1}{2\pi i}\int_{\Gamma_j} F_{j,s-1}(z_1, \dots, z_{s-1}, \eta_s, z_{s+1}, \dots, z_d) K(\eta_s, z_s)\, d\eta_s$$

for $z \in U_{j,s-1}$. By construction $F_{j,s}$ is a smooth $\mathbb{Z}^d$-periodic function with values in M_n and also analytic with respect to the variables $(z_1, \dots, z_s)$. It follows by induction that $F_{j,s} - F_{j,s-1}$ satisfies (2.21) in $\mathcal{U}_{j,s}$ for any $l = (0, \ldots, 0, l_{s+1}, \ldots, l_d)$ with $0 \le l_s \le 1$ and that $F_{j,s}(z)^* = -F_{j,s}(\overline{z})$ in $\mathcal{U}_{j,s}$.

Finally, the function $P_j := F_{j,d}$ is analytic and $\mathbb{Z}^d$-periodic in $\mathbb{R}^d_{h_j}$. Moreover, $P_j(z)^* = -P_j(\overline{z})$ in $\mathbb{R}^d_{h_j}$, hence, $P_j \in C^\omega_{h_j}(\mathbb{T}^d, u(n))$. Moreover,

$$|P_j - F_j|_{h_j} \ \le \ C\,L^d \exp\left(-(cLh_j)^{-\frac{1}{\rho-1}}\right) \|P\|_L.$$

In particular,

$$\begin{aligned}|P_{j+1} - P_j|_{h_{j+1}} &\le |P_{j+1} - F_{j+1}|_{h_{j+1}} + |P_j - F_j|_{h_{j+1}} + |F_{j+1} - F_j|_{h_{j+1}} \\ &\le C\,L^d \exp\left(-(cLh_j)^{-\frac{1}{\rho-1}}\right) \|P\|_L.\end{aligned}$$

Moreover,

$$|P_j(\theta) - P(\theta)| \ \le \ C\,L^d \exp\left(-(cLh_j)^{-\frac{1}{\rho-1}}\right) \|P\|_L$$

in $\mathbb{R}^d$, since $F_j(\theta) = P(\theta)$ for θ real. Finally,

$$|P_0|_{h_0} \le |F_0|_{h_0} + |P_0 - F_0|_{h_0} \le \ C\left(1 + L^d \exp\left(-(cLh_0)^{-\frac{1}{\rho-1}}\right)\right) \|P\|_L.$$

This completes the proof of the proposition. □

Conversely, there is also the following.

Proposition 2.2. — *(Inverse Approximation Lemma) Let $0 < \delta < 1$, $0 < h_0 \le 1$ and $h_j = h_0 \delta^j$, $j \in \mathbb{N}$. Let Ω be a closed subset of M_n and $P_j \in C^\omega_{h_j}(\mathbb{T}^d, \Omega)$, $j \ge 0$, satisfy*

$$|P_{j+1} - P_j|_{h_j} \le C_0 e^{-(Lh_j)^{-1/(\rho-1)}}$$

for any $j \ge 0$, where $C_0, L > 0$. Then there is $C = C(\rho, d) \ge 1$ and $c_0 = c_0(\rho, d) \ge 1$ and $P \in \mathcal{G}^\rho_{c_0 L}(\mathbb{T}^d, \Omega)$ such that $\lim P_j = P$ in $\mathcal{G}^\rho_{c_0 L}(\mathbb{T}^d, \Omega)$ and

$$\|P - P_j\|_{c_0 L} \le \frac{CC_0}{1-\delta} L^2 e^{-\frac{1}{2}(Lh_j)^{-1/(\rho-1)}}$$

for any $j \in \mathbb{N}$.

Proof. — For any $k \in \mathbb{N}^d$ and $j \ge 0$ one obtains by Cauchy

$$\sup_{\theta \in \mathbb{T}^d} \left|\partial^k (P_{j+1}(\theta) - P_j(\theta))\right| \le C_0 k!\, h_j^{-|k|-1} e^{-(Lh_j)^{-1/(\rho-1)}}.$$

Using the inequality $x^m e^{-x} \le m!$ for

$$x = \frac{1}{2}(Lh_j)^{-1/(\rho-1)} \text{ and } m = [(\rho-1)(|\alpha|+2)] + 1),$$

where $[x] = \inf\{m \in \mathbb{Z} : x \geq m\}$ stands for the integer part of $x \in \mathbb{R}$, one gets the following estimate

$$\begin{aligned}&\sup_{\theta\in\mathbb{T}^d} \left|\partial^k(P_{j+1}(\theta) - P_j(\theta))\right| \\ &\leq C_0(2^{\rho-1}L)^{|k|+2}k!\,([(\rho-1)(|k|+2)]+1)!\,h_j\, e^{-\frac{1}{2}(Lh_j)^{-1/(\rho-1)}}\end{aligned}$$

for $j \gg 1$. On the other hand, using the properties of the Gamma function, one obtains

$$\begin{aligned}([(\rho-1)(|k|+2)]+1)! &= \Gamma([(\rho-1)(|k|+2)]+2) \leq \Gamma((\rho-1)(|k|+2)+2) \\ \leq c_1^{|k|+1}\Gamma(|k|+1)^{\rho-1} &= c_1^{|k|+1}|k|!^{\,\rho-1} \leq c_2^{|k|+1}k!^{\,\rho-1}\end{aligned}$$

where $c_1 = c_1(\rho) \geq 1$ and $c_2 = c_2(\rho, d) \geq 1$. Setting $c_0 := 2^{\rho-1}c_2$ this implies

$$\|P_{j+1} - P_j\|_{c_0L} \leq C_0(c_0L)^2 h_j e^{-\frac{1}{2}(Lh_j)^{-1/(\rho-1)}},$$

and we get

$$\|P_m - P_j\|_{c_0L} \leq C_0(c_0L)^2 e^{-\frac{1}{2}(Lh_j)^{-1/(\rho-1)}} \frac{\delta^j}{1-\delta}$$

for $m > j \gg 1$, hence, the sequence $P_m = P_0 + \sum_{j=1}^m (P_j - P_{j-1})$ is Cauchy in $\mathcal{G}^\rho_{c_0L}(\mathbb{T}^d, \Omega)$, which is a complete space since Ω is closed. Taking the limit as $m \to \infty$ we get $P \in \mathcal{G}^\rho_{c_0L}(\mathbb{T}^d, \Omega)$ and the estimate of $P - P_j$. □

COROLLARY 2.1. — *The sequence P_j in Proposition 2.1 satisfies the estimate*

$$\|P - P_j\|_{c_0L} \leq \frac{cc_0}{1-\delta} L^{d+2} e^{-\frac{1}{2}(cLh_j)^{-1/(\rho-1)}} \|P\|_L.$$

for some $c = c(\rho, d) > 0$, $c = c(\rho) > 0$ and $c_0 = c_0(\rho, d) > 0$.

2.3. Measurable functions with values in $U(n)$.— In this section we introduce the important invariant $\lceil B \rfloor$ for measurable functions $B : \mathbb{T}^d \to M_n$ and the sets Γ and Π, which we need in the KAM step and in the Iterative Lemma. This sets give information on the quantity $\lceil\cdot\rfloor$ under truncation of the Fourier series of B and under multiplication.

Consider the Fourier expansion

$$B(\theta) \sim \sum_{k\in\mathbb{Z}^d} \widehat{B}(k)e^{2\pi i\langle k,\theta\rangle}$$

of a measurable function $B : \mathbb{T}^d \to M_n$ and denote by $\widehat{b}_{p,q}(k)$ the (p, q) entry of $\widehat{B}(k)$. Note that any measurable function $B : \mathbb{T}^d \to U(n)$ is always in $L^2(\mathbb{T}^d, M_n)$ since $|B(\theta)| = 1$ for any $\theta \in \mathbb{T}^d$ (the norm $|\cdot|$ on $M_n = M_n(\mathbb{C})$ is fixed in Sect. 2.1). To measure the minimal size of the rows of the $n \times n$ matrix $\widetilde{B}$ with entries

$$\widetilde{b}_{p,q} := \sup_{k\in\mathbb{Z}^d} |\widehat{b}_{p,q}(k)| \tag{2.22}$$

we define

$$\lceil B \rfloor_0 := \min_{1\le p\le n} \max_{1\le q\le n} |\widetilde{b}_{p,q}| = \min_{1\le p\le n} \sup\{|\widehat{b}_{p,q}(k)| : k\in\mathbb{Z}^d,\ 1\le q\le n\}.$$

The equality $\lceil B \rfloor_0 = 0$ means that there is a row of the matrix $\widetilde{B}$ equal to 0, or equivalently that there is a row of B which is zero for a.e. $\theta\in\mathbb{T}^d$, which implies that $\det B(\theta) = 0$ for a.e. $\theta\in\mathbb{T}^d$. In particular,

$$B : \mathbb{T}^d \to U(n) \text{ measurable } \Longrightarrow \lceil B \rfloor_0 > 0 \tag{2.23}$$

since $|\det B(\theta)| = 1$. On the other hand, $\lceil B \rfloor_0 \ge \epsilon$ if and only if for any $p\in\{1,\dots,n\}$ there is $q\in\{1,\dots,n\}$ and $k\in\mathbb{Z}^d$ such that

$$|\widehat{b}_{p,q}(k)| \ge \epsilon. \tag{2.24}$$

The quantity $\lceil\ \ \rfloor_0$ has the following properties.

LEMMA 2.3. — 1. *For any measurable $B : \mathbb{T}^d \to M_n$ and $T\in U(n)$,*

$$\lceil BT \rfloor_0 \ \ge\ \frac{1}{n}\lceil B \rfloor_0.$$

2. *Let $W(\theta) = \exp\big(2\pi i\,\mathrm{diag}\,(\langle k^{(1)},\theta\rangle,\dots,\langle k^{(n)},\theta\rangle)\big)$, where $k^{(1)},\dots,k^{(n)}\in\mathbb{Z}^d$. Then*

$$\forall\, S,T\in U(n),\quad \lceil SWT \rfloor_0 \ge n^{-3/2},$$

3. *For any constant function $B\in U(n)$, $\lceil B \rfloor_0 \ge 1/\sqrt{n}$.*

Proof. — Set $Q = BT$ and denote by $\hat{Q}_{p,q}(k)$, $1\le p,q\le n$, and $\hat{Q}_p(k)$, $1\le p\le n$, the corresponding entries and rows of $\hat{Q}(k)$, $k\in\mathbb{Z}^d$. The rows T_q^* of T^*, $1\le q\le n$, form an orthonormal basis of $\mathbb{C}^n$ and we get

$$\max_{1\le q\le n}\left|\hat{Q}_{p,q}(k)\right| = \max_{1\le q\le n}\left|\langle \hat{B}_p(k), T_q^*\rangle\right| \ge \frac{1}{n}\|\hat{B}_p(k)\| \ge \frac{1}{n}\max_{1\le q\le n}\left|\hat{B}_{p,q}(k)\right|$$

which proves the first part of the lemma. To prove the second one, it will be enough to show that for any $S = (S_{p,q})_{1\le p,q\le n}\in U(n)$

$$\lceil SW \rfloor_0 \ge n^{-1/2}.$$

The (p,q) entry of $SW(\theta)$ is $S_{p,q}e^{2\pi i\langle k^{(q)},\theta\rangle}$. For any given p, $\sum_{q=1}^n |S_{p,q}|^2 = 1$, so there exists q such that $|S_{p,q}| \ge n^{-1/2}$. Hence, for any given p there exists q such that $|S_{p,q}e^{2\pi i\langle k^{(q)},\theta\rangle}| \ge n^{-1/2}$, which implies that $\lceil SW \rfloor_0 \ge n^{-1/2}$. We have also shown in particular that $\lceil B \rfloor_0 \ge n^{-1/2}$ for any constant $B\equiv S\in U(n)$, which is the third conclusion. □

In general, the quantity $\lceil B \rfloor_0$ can not be controlled when multiplying B by a matrix $S \in U(n)$ from the left. To make it invariant with respect to the choice of the unitary bases in $\mathbb{C}^n$ or under multiplication with $S, T \in U(n)$ from both left and right, we define

$$\lceil B \rfloor := \inf_{S,T \in U(n)} \lceil SB(\cdot)T \rfloor_0.$$

Thus $\lceil B \rfloor = 0$ if and only if there are constant matrices $S, T \in U(n)$ such that the first row of the matrix $S\widetilde{B}T$ (the definition of $\widetilde{B}$ is given in (2.22)) is zero, or equivalently, the first row of $SB(\theta)T$ is zero for a.e. $\theta \in \mathbb{T}^d$. Such maps B will be called totally degenerate. We say that $B : \mathbb{T}^d \to M_n$ is ϵ-non-degenerate if $\lceil B \rfloor \geq \epsilon$.

Lemma 2.4. — 1. *For any constant function* $B \in U(n)$ *we have* $\lceil B \rfloor \geq 1/\sqrt{n}$.
2. $\lceil B \rfloor > 0$ *for any measurable* $B : \mathbb{T}^d \to U(n)$.
3. *Set* $W(\theta) = \exp\big(2\pi i \operatorname{diag}(\langle k^{(1)}, \theta\rangle, \ldots, \langle k^{(n)}, \theta\rangle)\big)$ *where* $k^{(1)}, \ldots, k^{(n)}$ *belong to* $\mathbb{Z}^d$. *Then* $\lceil W \rfloor \geq n^{-3/2}$.

Proof. — We are going to prove *2.* Suppose that there are sequences $\{S_j\}_{j\in\mathbb{N}}, \{T_j\}_{j\in\mathbb{N}} \subset U(n)$ such that

$$\lim \lceil S_j B T_j \rfloor_0 = 0.$$

Let $S, T \in U(n)$ be accumulation points of the sequences $\{S_j\}_{j\in\mathbb{N}}$ and $\{T_j\}_{j\in\mathbb{N}}$. Then $SBT : \mathbb{T}^d \to U(n)$ is again measurable and one can easily show that $\lceil SBT \rfloor_0 = 0$ which leads to a contradiction to (2.23). The first and the third parts of the lemma follow from Lemma 2.3. □

Given $N \in \mathbb{N}$ and $\epsilon > 0$, we say that $B \in L^2(\mathbb{T}^d, M_n)$ is (N, ϵ)-non-degenerate if the truncated Fourier series of B up order N is ϵ-non-degenerate, i.e.,

$$\lceil T_N B \rfloor \geq \epsilon,$$

where $T_N B$ is defined in Sect. 2.1. We denote by $\Gamma(N, \epsilon)$ the set of (N, ϵ)-non-degenerate maps $B \in L^2(\mathbb{T}^d, M_n)$. We point out that the definition of $\Gamma(N, \epsilon)$ here is different from that in [7]—in contrast to [7], the set $\Gamma(N, \epsilon)$ is invariant under the action of $U(n)$ from both left and right on the target space M_n. This set has the following properties which can be easily checked as in [7], Lemma 3.1.

Lemma 2.5. — 1. $S\Gamma(N, \epsilon)T = \Gamma(N, \epsilon)$ *for any* $S, T \in U(n)$
2. $B \in \Gamma(N, \lceil B \rfloor / 2)$ *for* N *large enough since* $\lim_{N\to\infty} \lceil T_N B \rfloor = \lceil B \rfloor$

3. $\Gamma(N,\epsilon)W \subseteq \Gamma(N+\widetilde{N},\epsilon/n)$, *where*

$$W(\theta) := \exp\left(2\pi i \operatorname{diag}\left(\langle k^{(1)},\theta\rangle,\dots,\langle k^{(n)},\theta\rangle\right)\right),$$

$$k^{(1)},\dots,k^{(n)} \in \mathbb{Z}^d \quad \text{and} \quad \max\{|k^{(1)}|,\dots,|k^{(n)}|\} \le \widetilde{N}$$

4. $\Gamma(N,\epsilon)P \subseteq \Gamma(N,\epsilon-\varepsilon)$ *for any measurable* $P:\mathbb{T}^d \to U(n)$ *with*

$$\varepsilon = \sup_{\theta\in\mathbb{T}^d} |P(\theta)-I|.$$

Proof. — We shall sketch the proof of *3*, the other items follow immediately from the proof of Lemma 3.1 [7]. Take $B\in\Gamma(N,\epsilon)$, $S\in U(n)$ and set $E := SB$. The Fourier coefficients of EW and E are related by the identity

$$\widehat{EW}_{p,q}(k) = \widehat{E}_{p,q}(k+k^{(q)}), \quad 1\le p,q\le n,\ k\in\mathbb{Z}^n.$$

In particular, for any $p\in\{1,\dots,n\}$ fixed and $k\in\mathbb{Z}^n$ with $|k|\le N$ there is $q\in\{1,\dots,n\}$ and $l\in\mathbb{Z}^n$ with $|l|\le N+\widetilde{N}$ such that $\widehat{E}_{p,q}(k) = \widehat{EW}_{p,q}(l-k^{(q)})$, hence, $\lceil T_{N+\widetilde{N}}EW\rfloor_0 \ge \lceil T_N E\rfloor_0$. Now Lemma 2.3 implies that $\forall S,T\in U(n)$

$$\begin{aligned}\lceil ST_{N+\widetilde{N}}BWT\rfloor_0 &\ge \frac{1}{n}\lceil ST_{N+\widetilde{N}}BW\rfloor_0\\ &\ge \frac{1}{n}\lceil T_N SB\rfloor_0 = \frac{1}{n}\lceil ST_N B\rfloor_0 \ge \lceil T_N B\rfloor \ge \frac{\epsilon}{n}\end{aligned}$$

and we get $\lceil T_{N+\widetilde{N}}BW\rfloor \ge \epsilon/n$. □

The set $\Gamma(N,\epsilon)$ provides information of the quantity $\lceil\cdot\rfloor$ after truncating the Fourier series of a function up to order N, which is needed in KAM step. In order to evaluate $\lceil\cdot\rfloor$ for the product of two functions PB where P is L^2 and B in $\Gamma(N,\delta)$ (this occurs in the Iterative Lemma below), it is convenient to introduce the following notation. For any $\widetilde{N}\in\mathbb{N}$ and ξ, $\varepsilon\in\mathbb{R}$ we denote by $\Pi(\widetilde{N},\xi,\varepsilon)$ the set of all $P\in L^2(\mathbb{T}^d,M_n)$ such that the operator of multiplication from the left by P maps $\Gamma(N,\delta)$ into $\Gamma(N+\widetilde{N},\xi\delta-\varepsilon)$, i.e.,

$$B\in\Gamma(N,\delta) \quad\Rightarrow\quad BP\in\Gamma(N+\widetilde{N},\xi\delta-\varepsilon). \tag{2.25}$$

The above relation means that PB is $(N+\widetilde{N},\xi\delta-\varepsilon)$-non-degenerate if B is (N,δ)-non-degenerate. The definition of the sets $\Gamma(N,\epsilon)$ and $\Pi(\widetilde{N},\xi,\varepsilon)$ seems technical but it turns out to be quite helpful in Sections 3.2 and 3.3. Using the definition of Π and Lemma 2.5 we obtain

LEMMA 2.6. — 1. $S\in\Pi(0,1,0)$ *for any* $S\in U(n)$,

2. $S\Pi(\widetilde{N},\xi,\varepsilon)T = \Pi(\widetilde{N},\xi,\varepsilon)$ *for any* $S,T\in U(n)$,
3. *The map* $\theta\to\exp\left(2\pi i\operatorname{diag}\left(\langle k^{(1)},\theta\rangle,\dots,\langle k^{(n)},\theta\rangle\right)\right)$ *belongs to* $\Pi(\widetilde{N},1/n,0)$ *provided that*

$$k^{(1)},\dots,k^{(n)}\in\mathbb{Z}^d \quad \text{and} \quad \max\{|k^{(1)}|,\dots,|k^{(n)}|\}\le\widetilde{N},$$

4. $P \in \Pi(0,1,\varepsilon)$ *for any measurable* $P : \mathbb{T}^d \to U(n)$ *with* $\varepsilon = \sup_{\theta\in\mathbb{T}^d} |P(\theta) - I|$,
5. $\Pi(\widetilde{N},\xi,\varepsilon_1) \subseteq \Pi(\widetilde{N},\xi,\varepsilon_2)$ *if* $\varepsilon_1 \geq \varepsilon_2$.

The set Π behaves nicely under multiplication. It obeys the following simple rule which allows us to keep control on the quantity $\lceil\cdot\rfloor$ in the Iterative Lemma.

Lemma 2.7. — *If* $P_1 \in \Pi(\widetilde{N}_1,\xi_1,\varepsilon_1)$ *and* $P_2 \in \Pi(\widetilde{N}_2,\xi_2,\varepsilon_2)$, *then*

$$P_1P_2 \in \Pi(\widetilde{N}_1+\widetilde{N}_2,\xi_1\xi_2,\xi_2\varepsilon_1+\varepsilon_2).$$

Proof. — For any $B \in \Gamma(N,\delta)$, we have

$$BP_1 \in \Gamma(N+\widetilde{N}_1,\xi_1\delta-\varepsilon_1).$$

Now for P_1P_2, we have

$$\begin{aligned} B(P_1P_2) = (BP_1)P_2 &\in \Gamma(N+\widetilde{N}_1+\widetilde{N}_2,\xi_2(\xi_1\delta-\varepsilon_1)-\varepsilon_2) \\ &= \Gamma(N+(\widetilde{N}_1+\widetilde{N}_2),\xi_1\xi_2\delta-(\xi_2\varepsilon_1+\varepsilon_2)), \end{aligned}$$

which implies that

$$P_1P_2 \in \Pi(\widetilde{N}_1+\widetilde{N}_2,\xi_1\xi_2,\xi_2\varepsilon_1+\varepsilon_2).$$

□

The following assertion gives information on the quantity $\lceil\cdot\rfloor$ for sequences of measurable functions with values in $U(n)$ when passing to a limit.

Lemma 2.8. — *Let* $B_m : \mathbb{T}^d \to U(n)$ *and* $D_m : \mathbb{T}^d \to U(n)$, $m \in \mathbb{N}$, *be two sequences of measurable functions such that* $\lceil D_m\rfloor \geq \delta > 0$ *and*

$$\lim_{m\to\infty}\int_{\mathbb{T}^d}|B_m(\theta)-D_m(\theta)|d\theta = 0,$$

Then

$$\underline{\lim}_{m\to\infty}\lceil B_m\rfloor \geq \delta.$$

Proof. — Fix $S,T \in U(n)$ and $\epsilon > 0$. There exists $m_0 > 0$ such that

$$\int_{\mathbb{T}^d}|S(B_m(\theta)-D_m(\theta))T|d\theta < \epsilon$$

as $m \geq m_0$. Now for any $k \in \mathbb{Z}^d$

$$|S(\widehat{B_m - D_m})T(k)| \leq \int_{\mathbb{T}}|\{S(B_m(\theta)-D_m(\theta))T\}e^{-2\pi i\langle k,\theta\rangle}|d\theta < \epsilon,$$

hence, by the definition of $\lceil\ \ \rfloor_0$,

$$\lceil SB_mT\rfloor_0 = \lceil SD_m + S(B_m-D_m)T\rfloor_0 > \lceil SD_mT\rfloor_0 - \epsilon.$$

By the definition of $\lceil\ \ \rfloor$ we obtain $\lceil B_m\rfloor > \lceil D_m\rfloor - \epsilon$. We then get the desired conclusion. □

3. Local Setting

In this section we prove Theorem 1.1 which provides a local rigidity result of the reducibility problem in Gevrey classes. Firstly, we describe the KAM step in the case of analytic cocycles. Next we approximate a Gevrey cocycle by a sequence of analytic cocycles and apply the KAM step. In this way we get a sequence of analytic cocycles tending to a constant. Then we use a convergence argument to obtain Gevrey reducibility under a suitable smallness assumption.

3.1. The KAM step. — The KAM scheme we are using here is close to that in [8]. We want to conjugate a cocycle (α, Ae^F) with small F to a constant one. In other words, we are looking for a constant matrix $\widetilde{A} \in U(n)$ and a $u(n)$-valued function Y with a small norm, such that

$$Ad(e^Y).(\alpha, Ae^F) = (\alpha, \widetilde{A})$$

which means that

$$e^{-Y(\cdot+\alpha)} Ae^F e^Y = \widetilde{A}. \tag{3.26}$$

The corresponding (affine) linearized equation reads

$$Y - A^{-1}Y(\cdot + \alpha)A = -A^{-1}F + A^{-1}\widetilde{A} - I. \tag{3.27}$$

If the inverse of the operator

$$\begin{aligned} \mathscr{A} :\ C^\omega_h(\mathbb{T}^d, u(n)) &\longrightarrow C^\omega_h(\mathbb{T}^d, u(n)) \\ Y \quad &\longmapsto Y - A^{-1}Y(\cdot + \alpha)A \end{aligned} \tag{3.28}$$

was bounded then the Equation (3.26) could have been solved by means of the implicit function theorem. The presence of small divisors, however, does not allow doing this. Indeed, expanding Y in Fourier series one immediately observes that there is a lot of resonant terms which makes it impossible to find bounded solutions of (3.27) in general. To overcome this obstruction, we follow the standard approach to normal forms—keep resonant terms and remove non-resonant ones at each step of the iteration. To this end we divide the initial space into two spaces, one of resonant modes and another one containing only non-resonant terms where a suitable lower bound of the operator (3.28) can be obtained. On the other hand, the space $C^\omega_h(\mathbb{T}^d, u(n))$ equipped with the sup-norm is not adapted for estimating the operator (3.28) below. For this reason we fix $0 < \tilde{h} < h$ and consider the operator (3.28) in the Banach space

$$\mathfrak{B}_{\tilde{h}} := \left\{ X \in C^\omega_{\tilde{h}}(\mathbb{T}^d, u(n)) :\ |X|_{1,\tilde{h}} < \infty \right\}$$

equipped with the norm $|\cdot|_{1,\tilde{h}}$ (see Sect. 2.1). The advantage of this norm is that it gives a lower bound of (3.28) if there is a lower bound of each of the

Fourier coefficients. More precisely, given $\eta \in (0,1)$ and $A \in U(n)$ we suppose that there is a decomposition

$$\mathfrak{B}_{\tilde{h}} = \mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})} \oplus \mathfrak{B}_{\tilde{h}}^{(\mathrm{re})}$$

on a direct sum of two closed sub-spaces $\mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$ and $\mathfrak{B}_{\tilde{h}}^{(\mathrm{re})}$ (the decomposition depends on A and η) in such a way that for any $Y \in \mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$ the following relations hold

$$\mathscr{A}(Y)(\cdot) = Y - A^{-1}Y(\cdot+\alpha)A \in \mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})} \quad \text{and} \quad |\mathscr{A}(Y)|_{1,\tilde{h}} \geq \eta |Y|_{1,\tilde{h}}\,. \tag{3.29}$$

Let Π_{nre} (Π_{re}) be the standard projection from $\mathfrak{B}_{\tilde{h}}$ onto $\mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$ ($\mathfrak{B}_{\tilde{h}}^{(\mathrm{re})}$). We call $\mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$ ($\mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$) the η-nonresonant (η-resonant) subspace. With all these assumptions, one can solve (3.26) partially, which is summarized in the following lemma.

LEMMA 3.1. — *There is a universal constant $\delta_* \in (0,1)$, such that for any $F \in \mathfrak{B}_{\tilde{h}}$ satisfying $|F|_{1,\tilde{h}} \leq \delta_* \eta^2$, there exist $Y \in \mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$ and $F^{(\mathrm{re})} \in \mathfrak{B}_{\tilde{h}}^{(\mathrm{re})}$ such that*

$$e^{-Y(\cdot+\alpha)}Ae^{F}e^{Y} = Ae^{F^{(\mathrm{re})}},$$

i.e.,

$$Ad(e^{Y}).(\alpha, Ae^{F}) = (\alpha, Ae^{F^{(\mathrm{re})}}), \tag{3.30}$$

with the estimates

$$|Y|_{1,\tilde{h}} \leq \frac{2}{\eta}|F|_{1,\tilde{h}}, \quad |F^{(\mathrm{re})}|_{1,\tilde{h}} \leq \mathrm{cst.}\ |F|_{1,\tilde{h}}.$$

Proof. — Lemma 3.1 is a counterpart of Lemma 3.1 [8] in the discrete case. The lemma follows from the implicit function theorem. Given F with $|F|_{1,\tilde{h}} \ll 1$ we are looking for a solution $Y \in \mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$ of the equation

$$\mathscr{H}(F,Y) := \Pi_{\mathrm{nre}} \log\{A^{-1}e^{-Y(\cdot+\alpha)}Ae^{F}e^{Y}\} = 0 \tag{3.31}$$

where

$$\log(X) = \sum_{r\geq 1} \frac{1}{r}(I-X)^r, \quad \text{for} \quad X \in M_n,\ |I-X| < 1.$$

We are going to solve (3.31) by means of a fixed point argument for contraction maps. To this end, we firstly compute the partial derivative of $\mathscr{H}$ with respect to Y. Taking the power series expansions of e^Y and e^F at $Y = F = 0$ we get

$$
\begin{aligned}
A^{-1}e^{-Y(\cdot+\alpha)}Ae^Fe^Y &= I - A^{-1}Y(\cdot+\alpha)A + Y \\
&\quad + A^{-1}\Big\{\sum_{r\geq 2}\frac{(-1)^r}{r!}Y(\cdot+\alpha)^r\Big\}Ae^Fe^Y \\
&\quad + A^{-1}e^{-Y(\cdot+\alpha)}Ae^F\Big\{\sum_{r\geq 2}\frac{1}{r!}Y^r\Big\} \\
&\quad + A^{-1}Y(\cdot+\alpha)A\Big\{\sum_{r\geq 1}\frac{1}{r!}F^r\Big\} + \Big\{\sum_{r\geq 1}\frac{1}{r!}F^r\Big\}Y. \qquad (3.32)
\end{aligned}
$$

Using the above formula we obtain

$$
\begin{aligned}
\mathscr{L}_{(F,Y)}Z &:= \lim_{t\to 0}\frac{1}{t}\{\mathscr{H}(F, Y+tZ) - \mathscr{H}(F,Y)\} \\
&= \Pi_{\mathrm{nre}}\{Z - A^{-1}Z(\cdot+\alpha)A + \mathscr{E}(F,Y)Z\}, \qquad (3.33)
\end{aligned}
$$

for any $Z \in \mathfrak{B}_{\tilde h}^{(\mathrm{nre})}$, where $\mathscr{E} = \mathscr{E}(F,Y)$ is a linear operator in $\mathfrak{B}_{\tilde h}$ depending on F, Y. Moreover, applying (2.7) one gets a positive constant cst. such that

$$
|\mathscr{E}(F,Y)Z|_{1,\tilde h} \leq \mathrm{cst.}(|F|_{1,\tilde h} + |Y|_{1,\tilde h})\,|Z|_{1,\tilde h} \qquad (3.34)
$$

for any $F, Y \in \mathfrak{B}_{\tilde h}$ with $|Y|_{1,\tilde h} \leq 1$ and $|F|_{1,\tilde h} \leq 1$. In particular, using the definition of the space $\mathfrak{B}_{\tilde h}^{(\mathrm{nre})}$ we obtain

$$
\mathscr{L}_{(0,0)}Z = \Pi_{\mathrm{nre}}\{Z - A^{-1}Z(\cdot+\alpha)A\} = Z - A^{-1}Z(\cdot+\alpha)A \qquad (3.35)
$$

as well as the estimate

$$
|\mathscr{L}_{(0,0)}Z|_{1,\tilde h} = |Z - A^{-1}Z(\cdot+\alpha)A|_{1,\tilde h} \geq \eta|Z|_{1,\tilde h}. \qquad (3.36)
$$

Using (3.33)–(3.36) we prove that there exists a constant $\delta_* \in (0,1)$ such that

$$
|(\mathscr{L}_{(F,Y)} - \mathscr{L}_{(0,0)})Z|_{1,\tilde h} \leq \frac{\eta}{2}|Y|_{1,\tilde h} \qquad (3.37)
$$

provided that $|F|_{1,\tilde h} < \delta_*\eta^2$ and $|Y|_{1,\tilde h} < 2\delta_*\eta$. Hence,

$$
\|\mathscr{L}_{(0,0)}^{-1}\|_{1,\tilde h} \leq \frac{1}{\eta} \quad \text{and} \quad \|\mathscr{L}_{(F,Y)} - \mathscr{L}_{(0,0)}\|_{1,\tilde h} \leq \frac{\eta}{2} \qquad (3.38)
$$

for $|F|_{1,\tilde h} < \delta_*\eta^2$ and $|Y|_{1,\tilde h} < 2\delta_*\eta$, where $\|\cdot\|_{1,\tilde h}$ is the operator norm corresponding to the norm $|\cdot|_{1,\tilde h}$.

Denote by $\mathscr{W}_{\tilde h}$ the ball

$$
\mathscr{W}_{\tilde h} := \{Y \in \mathfrak{B}_{\tilde h}^{(\mathrm{nre})} : |Y|_{1,\tilde h} \leq 2\delta_*\eta\}
$$

which is complete with respect to the norm $|\cdot|_{1,\tilde{h}}$. For any fixed $F \in \mathfrak{B}_{\tilde{h}}$ satisfying $|F|_{1,\tilde{h}} \leq \delta_* \eta^2$, we consider the map

$$\mathcal{F}(Y) := Y - \mathcal{L}_{(0,0)}^{-1} \Pi_{\mathrm{nre}} \log\{A^{-1} e^{-Y(\cdot+\alpha)} A e^F e^Y\}$$

from $\mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$ to itself. It follows from (3.38) that

$$\|\mathrm{Id} - \mathcal{L}_{(0,0)}^{-1} \mathcal{L}_{(F,Y)}\|_{1,\tilde{h}} \leq \|\mathcal{L}_{(0,0)}^{-1}\|_{1,\tilde{h}} \|\mathcal{L}_{(0,0)} - \mathcal{L}_{(F,Y)}\|_{1,\tilde{h}} \leq \frac{1}{\eta} \cdot \frac{\eta}{2} = \frac{1}{2}$$

as long as $Y \in \mathcal{W}_{\tilde{h}}$. On the other hand, for any $Y_1, Y_2 \in \mathcal{W}_{\tilde{h}}$ there is $\xi \in [0,1]$ such that

$$\mathcal{F}(Y_2) - \mathcal{F}(Y_1) = \left(\mathrm{Id} - \mathcal{L}_{(0,0)}^{-1} \mathcal{L}_{(F, Y_1 + \xi(Y_2 - Y_1))}\right)(Y_2 - Y_1).$$

Then for any $Y_1, Y_2 \in \mathcal{W}_{\tilde{h}}$ we obtain

$$|\mathcal{F}(Y_2) - \mathcal{F}(Y_1)|_{1,\tilde{h}} \leq \frac{1}{2} |Y_2 - Y_1|_{1,\tilde{h}}. \tag{3.39}$$

Set $Y_0 = 0$ and define Y_j inductively by $Y_j = \mathcal{F}(Y_{j-1})$ for $j \geq 1$. Note that

$$Y_1 = \mathcal{F}(0) = -\mathcal{L}_{(0,0)}^{-1} \Pi_{\mathrm{nre}} F$$

which implies

$$|Y_1 - Y_0|_{1,\tilde{h}} = |\mathcal{F}(0)|_{1,\tilde{h}} \leq \frac{1}{\eta} |F|_{1,\tilde{h}} \leq \delta_* \eta \tag{3.40}$$

in view of (3.38). This means that $Y_1 \in \mathcal{W}_{\tilde{h}}$. One can prove inductively that Y_j belongs to $\mathcal{W}_{\tilde{h}}$ for each $j \geq 1$, and that

$$|Y_j - Y_{j-1}|_{1,\tilde{h}} \leq \frac{1}{2^{j-1}} |Y_1 - Y_0|_{1,\tilde{h}}. \tag{3.41}$$

In fact, if $Y_1, \ldots, Y_{j-1}$ are all in $\mathcal{W}_{\tilde{h}}$ and for all $s \in \{1, \ldots, j-1\}$

$$|Y_s - Y_{s-1}|_{1,\tilde{h}} \leq \frac{1}{2^{s-1}} |Y_1 - Y_0|_{1,\tilde{h}},$$

then (3.39) implies

$$|Y_j - Y_{j-1}|_{1,\tilde{h}} = |\mathcal{F}(Y_{j-1}) - \mathcal{F}(Y_{j-1})| \leq \frac{1}{2} |Y_{j-1} - Y_{j-2}|_{1,\tilde{h}} \leq \frac{1}{2^{j-1}} |Y_1 - Y_0|_{1,\tilde{h}},$$

and using (3.40) one obtains

$$\begin{aligned} |Y_j|_{1,\tilde{h}} &= |(Y_j - Y_{j-1}) + (Y_{j-1} - Y_{j-2}) + \cdots + (Y_1 - Y_0)|_{1,\tilde{h}} \\ &\leq 2|Y_1 - Y_0|_{1,\tilde{h}} \leq \frac{2}{\eta} |F|_{1,\tilde{h}} \leq 2\delta_* \eta. \end{aligned}$$

Since $\mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$ is closed, the limit $Y = \lim_{j \to \infty} Y_j$ exists in $\mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})}$. Moreover,

$$|Y|_{1,\tilde{h}} \leq \frac{2}{\eta} |F|_{1,\tilde{h}} \leq 2\delta_* \eta, \tag{3.42}$$

which means that $Y \in \mathcal{W}_{\tilde{h}}$. In particular, Y satisfies the equation $\mathcal{F}(Y) = Y$ and in fact Y is unique in $\mathcal{W}_{\tilde{h}}$ in view of (3.39). Hence,

$$\Pi_{\mathrm{nre}} \log\{A^{-1}e^{-Y(\cdot+\alpha)}Ae^F e^Y\} = 0$$

and setting

$$F^{(\mathrm{re})} := \Pi_{\mathrm{re}} \log\{A^{-1}e^{-Y(\cdot+\alpha)}Ae^F e^Y\} \tag{3.43}$$

we obtain (3.30). It remains to estimate $F^{(\mathrm{re})}$. It follows from (3.32), (3.42) , and from the implication

$$Y \in \mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})} \Rightarrow \mathcal{A}(Y) = Y - A^{-1}Y(\cdot + \alpha)A \in \mathfrak{B}_{\tilde{h}}^{(\mathrm{nre})},$$

that

$$|F^{(\mathrm{re})}|_{1,\tilde{h}} \leq \mathrm{cst.}(|Y|^2_{1,\tilde{h}} + |F|_{1,\tilde{h}}) \tag{3.44}$$

Then using (3.42) and the assumption $|F|_{1,\tilde{h}} \leq \delta_* \eta^2$, we obtain

$$\begin{aligned} |F^{(\mathrm{re})}|_{1,\tilde{h}} &\leq \mathrm{cst.}\left(\frac{4}{\eta^2}|F|^2_{1,\tilde{h}} + |F|_{1,\tilde{h}}\right) \\ &\leq \mathrm{cst.}\left(\frac{4}{\eta^2}(\delta_*\eta^2)|F|_{1,\tilde{h}} + |F|_{1,\tilde{h}}\right) \leq \mathrm{cst.}|F|_{1,\tilde{h}}. \end{aligned} \tag{3.45}$$

This completes the proof of the lemma. □

The previous lemma will be used in the K.A.M. step. Before formulating it we recall the notion of resonant (non-resonant) pairs. A pair $(\lambda, \widetilde{\lambda})$ of complex numbers is said to be (N, δ)-non-resonant (with respect to α) if

$$\forall\, k \in \mathbb{Z}^d, \quad 0 \neq |k| \leq N \ : \quad |e^{i\langle k,\alpha\rangle}\widetilde{\lambda} - \lambda| \geq \delta, \tag{3.46}$$

otherwise it is said to be (N, δ)-resonant. Given $A \in U(n)$, we write $A \in \mathrm{NR}\,(N, \delta)$ if any pair of eigenvalues of A is (N, δ) non-resonant, otherwise we write $A \in \mathrm{RS}\,(N, \delta)$. We fix a small constant $0 < \sigma = \sigma(\ell) < 1$ by

$$\sigma = \min(1/100, (\ell - 1)(5\ell)^{-1}) \tag{3.47}$$

where $\ell > 1$ appears in (1.4).

Proposition 3.1. — *Let $\alpha \in \mathrm{DC}\,(\gamma, \tau)$. For any given $\kappa \in (0, 1)$, there exist*

$$\delta_0 = \delta_0(\kappa, \gamma, \tau, d) \in (0, 1), \quad \chi = \chi(\kappa, \gamma, \tau, d) \geq 1 \quad \textit{and} \quad K_* = K_*(\kappa) \geq 1$$

such that the following holds.

(i) *For any* $h, \varepsilon \in (0,1)$, $A \in U(n)$, $N \geq 1$ *and* $F \in C^\omega_h(\mathbb{T}^d, U(n))$ *satisfying*

$$0 < \varepsilon < \delta_0 h^\chi, \quad N = \left(\frac{2n+1}{\kappa}\right)^n \left(\frac{2}{h}\log\frac{1}{\varepsilon} + 1\right), \quad \text{and} \quad |F|_h \leq \varepsilon$$

there is $R \in C^\omega_{(1-\kappa)h}(\mathbb{T}^d, U(n))$, $A_+ \in U(n)$ *and* $F_+ \in C^\omega_{(1-\kappa)h}(\mathbb{T}^d, U(n))$ *such that*

$$\mathrm{Ad}\,(R).(\alpha, Ae^F) = (\alpha, A_+ e^{F_+})$$

where

$$|F_+|_{(1-\kappa)h} \leq \varepsilon_+ := \varepsilon^{1+\sigma}, \quad |R|_{(1-\kappa)h} \leq \varepsilon^{-K_*}, \quad \textit{and} \quad R \in \Pi(N, 1/n, \varepsilon^{1-4\sigma}).$$

(ii) *If* $A \in \mathrm{NR}\,(N, \varepsilon^\sigma)$, *then* $R = e^Y$ *with* $Y \in C^\omega_h(\mathbb{T}^d, U(n))$ *and*

$$|Y|_h < \varepsilon^{1-2\sigma}, \quad |A_+ - A| \leq \varepsilon^{1/2}.$$

Proof. — The proof of the proposition is long and we divide it in several steps.

Step 1. Choosing the constants. — There exists $S \in U(n)$ such that $SAS^* = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$. Hence, without loss of generality, we may assume that A is diagonal, $A = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$. Set

$$N_j = \left(\frac{\kappa}{2n+1}\right)^{n-j} N, \quad j = 0, 1, \ldots, n. \tag{3.48}$$

Choosing properly the constants $\delta_0 = \delta_0(\kappa) > 0$ and $\chi > 1$ and taking $\varepsilon \in (0, \delta_0 h^\chi)$ we can assume that $\varepsilon^{\sigma/2}$ is smaller than any of the finitely many universal constants arising below, and that the following estimates hold

$$\begin{aligned} & e^{-N_n h} \leq \cdots \leq e^{-N_1 h} \leq e^{-N_0 h} \\ & \quad = \exp\left\{-h\left(\frac{2}{h}\log\frac{1}{\varepsilon} + 1\right)\right\} \leq \varepsilon^2, \end{aligned} \tag{3.49}$$

$$\begin{aligned} \frac{N^d}{(\kappa h)^d} &= \frac{1}{(\kappa h)^d}\left(\frac{2n+1}{\kappa}\right)^{nd}\left(\frac{2}{h}\log\frac{1}{\varepsilon} + 1\right)^d \\ &\leq \frac{3^d(2n+1)^{nd}}{\kappa^{(n+1)d} h^{2d}}\left(\log\frac{1}{\varepsilon}\right)^d \leq \left(\frac{1}{\varepsilon}\right)^\sigma, \end{aligned} \tag{3.50}$$

$$\begin{aligned} & 2\gamma(4n)^{\tau+1} N^\tau \\ & \quad = 2\gamma(4n)^{\tau+1}\left(\frac{2n+1}{\kappa}\right)^{\tau n}\left(\frac{2}{h}\log\frac{1}{\varepsilon} + 1\right)^\tau < \left(\frac{1}{\varepsilon}\right)^\sigma. \end{aligned} \tag{3.51}$$

For technical reasons we assume as well that the inequality

$$|X|, |Y| \leq \varepsilon^{1/2}, \; X, Y \in M_n$$

implies

$$\left|e^{-X}e^{X+Y} - I\right| < 1, \quad |\log\{e^{-X}e^{X+Y}\}| \leq 2(|X|^2 + |Y|). \tag{3.52}$$

Step 2. The operator $\mathscr{C}$. — Expanding Y in Fourier series we write the operator $\mathscr{C}$ in (3.28) as follows

$$\sum_{k\in\mathbb{Z}^d}\widehat{Y}(k)e^{2\pi i\langle k,\theta\rangle}\overset{\mathscr{C}}{\longmapsto}\sum_{k\in\mathbb{Z}^d}\{\widehat{Y}(k)-A^{-1}e^{2\pi i\langle k,\alpha\rangle}\widehat{Y}(k)A\}e^{2\pi i\langle k,\theta\rangle}. \tag{3.53}$$

Denote by $E(p,q)$ the elementary matrix with entries $E(p,q)_{s,t}=1$ if $(s,t)=(p,q)$ and 0 otherwise. Then (3.53) becomes

$$\sum_{k\in\mathbb{Z}^d}e^{2\pi i\langle k,\theta\rangle}\left\{\sum_{1\le p,q\le n}\widehat{y}_{p,q}(k)E(p,q)\right\}$$
$$\overset{\mathscr{C}}{\longmapsto}\sum_{k\in\mathbb{Z}^d}e^{2\pi i\langle k,\theta\rangle}\left\{\sum_{1\le p,q\le n}\overline{\lambda_p}(\lambda_p-\lambda_q e^{2\pi i\langle k,\alpha\rangle})\widehat{y}_{p,q}(k)E(p,q)\right\} \tag{3.54}$$

where $\widehat{y}_{p,q}(k)$ denotes the (p,q) entry of $\widehat{Y}(k)$. In this way the corresponding homological Equation (3.27) splits into a system of equations

$$\overline{\lambda_p}(\lambda_p-\lambda_q e^{2\pi i\langle k,\alpha\rangle})\widehat{y}_{p,q}(k)=f_{p,q},\quad |k|\le N,\ 1\le p,q\le n.$$

We would like to solve it and to get "good" estimates for the solutions, or equivalently to invert the operator $\mathscr{C}$ in a suitable space. To do this we have to deal with the divisor $|\lambda_p-e^{2\pi i\langle k,\alpha\rangle}\lambda_q|$ which could be arbitrary small if the pair (λ_p,λ_q) is (N,ε^σ)-resonant and only ε^σ-small if it is (N,ε^σ)-non-resonant (note that all $\lambda_1,\dots,\lambda_n$ are on the unit circle of $\mathbb{C}$). When $p=q$, the divisor takes the form $|1-e^{2\pi i\langle k,\alpha\rangle}|$ since $|\lambda_p|=1$, which is only ε^σ-small for $|k|\le N$. Indeed (1.1) and (3.51)) imply that it is larger than $2(4n)^{\tau+1}\varepsilon^\sigma$. To deal with the case $p\neq q$, we recall a simple fact known as "the uniqueness of the $(2nN,2n\varepsilon^\sigma$-resonance" which says that for any λ_p and λ_q the following relation holds

$$\left\{\begin{array}{l}|\lambda_p-e^{2\pi i\langle k,\alpha\rangle}\lambda_q|<2n\varepsilon^\sigma\\ |\lambda_p-e^{2\pi i\langle l,\alpha\rangle}\lambda_q|<2n\varepsilon^\sigma\\ |k|,|l|\le 2nN\end{array}\right\}\quad\Rightarrow\quad k=l. \tag{3.55}$$

In fact, taking into account (1.1), the violation of (3.55) would imply

$$4n\varepsilon^\sigma\ge|e^{2\pi i\langle k,\alpha\rangle}-e^{2\pi i\langle l,\alpha\rangle}|\ge\frac{\gamma^{-1}}{|k-l|^\tau}\ge\frac{1}{\gamma(4nN)^\tau},$$

which contradicts (3.51).

Step 3. Structure of the resonances. — We are going to describe the structure of the spectrum of A dividing it into blocks of resonant pairs of eigenvalues.

Lemma 3.2. — *There exist* $0 \leq j \leq n-1$ *and* $1 \leq m \leq n$*, such that* $\mathrm{Spec}\,(A) = \{\lambda_1, \ldots, \lambda_n\}$ *can be divided into* m *subsets* $\Lambda_1, \ldots, \Lambda_m$*, with the properties*

a) *If* λ_p *and* λ_q *belong to one and the same* Λ_r *then they are* $(nN_j, n\varepsilon^\sigma)$*-resonant;*

b) *If* λ_p, λ_q *belong to different subsets then they are* $(N_{j+1}, \varepsilon^\sigma)$*-nonresonant.*

Proof. — We will say that λ_p, λ_q are (L, a)-connected if there exists a (L, a)-*resonant path of length* r

$$\lambda_{p_0}, \lambda_{p_1}, \ldots, \lambda_{p_{r-1}}, \lambda_{p_r} \in \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \quad \text{with} \quad p_0 = p,\ p_r = q,$$

such that $(\lambda_{p_0}, \lambda_{p_1})$, $(\lambda_{p_1}, \lambda_{p_2})$,$\cdots$, $(\lambda_{p_{r-1}}, \lambda_{p_r})$ are all (L, a)-resonant. One can easily check that such a (L, a)-connected pair (λ_p, λ_q) is (rL, ra)-resonant.

Note that any λ_p, λ_q in a $(N_j, \varepsilon^\sigma)$-connected component are $(nN_j, n\varepsilon^\sigma)$-resonant. Indeed, suppose that the pair λ_p, λ_q can be connected by a $(N_j, \varepsilon^\sigma)$-resonant path of length r. Without loss of generality, eliminating the "closed loops," we can assume that $r \leq n$, hence, the pair is $(nN_j, n\varepsilon^\sigma)$-resonant as well.

To prove the assertion, let us firstly divide $\{\lambda_1, \ldots, \lambda_n\}$ into $(N_0, \varepsilon^\sigma)$-connected components. If any λ_p and λ_q belonging to two different $(N_0, \varepsilon^\sigma)$-connected components are $(N_1, \varepsilon^\sigma)$-nonresonant, we finish the proof choosing $\Lambda_1, \ldots, \Lambda_m$ to be the $(N_0, \varepsilon^\sigma)$-connected components. Otherwise, we consider the $(N_1, \varepsilon^\sigma)$-connected components and repeat the procedure. More precisely, if there is j such that any λ_p and λ_q belonging to different $(N_j, \varepsilon^\sigma)$-connected components are $(N_{j+1}, \varepsilon^\sigma)$-nonresonant, we denote by $\Lambda_1, \ldots, \Lambda_m$ the corresponding $(N_j, \varepsilon^\sigma)$-connected components and finish the proof. Otherwise, we consider the $(N_{j+1}, \varepsilon^\sigma)$-connected components. Thus there are two possibilities: a) either we stop at the j-th step and set $m = j$; b) or the number of the $(N_{j+1}, \varepsilon^\sigma)$-connected components is strictly less than the number of the $(N_j, \varepsilon^\sigma)$-connected components. In the latter case, there is $j \leq n-1$ such that at the j-th step there is only one $(N_j, \varepsilon^\sigma)$-connected component $\{\lambda_1, \ldots, \lambda_n\}$ and we take $m = 1$ and $\Lambda_1 = \{\lambda_1, \ldots, \lambda_n\}$. □

From now on we fix $j \leq n-1$ as in Lemma 3.2. Taking account of the structure of the resonances we are going to define the spaces $\mathfrak{B}_h^{(\mathrm{nre})}$ and $\mathfrak{B}_h^{(\mathrm{re})}$ and verify the hypothesis of Lemma 3.1. To this end we assign to any $p \in \{1, \ldots, n\}$ an integer vector $k^{(p)} \in \mathbb{Z}^d$ as follows. First for any $1 \leq t \leq m$ we choose a representative $\lambda_{p_t} \in \Lambda_t$ and set $k^{(p_t)} = 0$. Let $p \in \{1, \ldots, n\}$ and $p \notin \{p_1, \ldots, p_m\}$. There exits $t \in \{1, \ldots, m\}$ such that $\lambda_p \in \Lambda_t$ and by Lemma 3.2, *a)*, and (3.55) there is a *unique* $k^{(p)} \in \mathbb{Z}^d$ such that

$$|k^{(p)}| \leq nN_j \quad \text{and} \quad |\lambda_p - \lambda_{p_t} e^{2\pi i \langle k^{(p)}, \alpha \rangle}| < n\varepsilon^\sigma.$$

Let λ_p and λ_q belong to one and the same component Λ_t. Then

$$\max\left\{|\lambda_p - \lambda_{p_t} e^{2\pi i\langle k^{(p)},\alpha\rangle}|,\ |\lambda_q - \lambda_{p_t} e^{2\pi i\langle k^{(q)},\alpha\rangle}|\right\} < n\varepsilon^\sigma$$

which implies

$$|\lambda_p - \lambda_q e^{2\pi i\langle k^{(p)}-k^{(q)},\omega\rangle}| < 2n\varepsilon^\sigma.$$

By the uniqueness (3.55) of the $(2nN, 2n\varepsilon^\sigma)$-resonance, the integer vector $k^{(p)} - k^{(q)}$ can be characterized as the unique $k \in \mathbb{Z}^d$ satisfying the inequalities

$$|k| \le 2nN \quad \text{and} \quad |\lambda_p - \lambda_q e^{2\pi i\langle k,\alpha\rangle}| < 2n\varepsilon^\sigma. \tag{3.56}$$

This implies the relation

$$\lambda_p, \lambda_q \in \Lambda_t,\ k \neq k^{(p)} - k^{(q)},\ |k| \le 2nN \implies |\lambda_p - \lambda_q e^{2\pi i\langle k,\alpha\rangle}| \ge 2n\varepsilon^\sigma. \tag{3.57}$$

The existence of $k \in \mathbb{Z}^d$ satisfying (3.56) follows from lemma 3.2, *a)*. Indeed, the eigenvalues $\lambda_p, \lambda_q \in \Lambda_t$ are $(nN_j, n\varepsilon^\sigma)$-resonant, which means there is $k^{(p,q)} \in \mathbb{Z}^d$ satisfying

$$|k^{(p,q)}| \le nN_j \le nN \quad \text{and} \quad |\lambda_p - \lambda_q e^{2\pi i\langle k,\alpha\rangle}| < n\varepsilon^\sigma$$

and by (3.57) we get $k^{(p,q)} = k^{(p)} - k^{(q)}$. In particular, $|k^{(p)} - k^{(q)}| = |k^{(p,q)}| \le nN_j$, and we obtain the relation

$$\lambda_p, \lambda_q \in \Lambda_t,\ N_j < |k| \le 2nN \implies |\lambda_p - \lambda_q e^{2\pi i\langle k,\alpha\rangle}| \ge 2n\varepsilon^\sigma. \tag{3.58}$$

Moreover, (3.58) and lemma 3.2, *b)* imply

$$\lambda_p, \lambda_q \in \operatorname{Spec}(A),\quad nN_j < |k| \le N_{j+1} \implies |\lambda_p - \lambda_q e^{2\pi i\langle k,\alpha\rangle}| \ge \varepsilon^\sigma. \tag{3.59}$$

Step 4. The spaces $\mathfrak{B}_{\tilde h}^{(\mathrm{nre})}$ and $\mathfrak{B}_{\tilde h}^{(\mathrm{re})}$. — Let us go back to the expansion (3.54). Denote by $\mathcal{Z}_k$, $k \in \mathbb{Z}^d$, the following subset of $\{1, \dots, n\}^2$

$$\mathcal{Z}_k := \{(p,q) :\ \lambda_p, \lambda_q \in \Lambda_t \quad \text{for some } 1 \le t \le m \text{ and } k = k^{(p)} - k^{(q)}\} \tag{3.60}$$

and by $\mathcal{Z}_k^{\mathrm{c}} := \{1, \dots, n\}^2 - \mathcal{Z}_k$ the complement of $\mathcal{Z}_k$. Note that

$$(p,q) \in \mathcal{Z}_k \Leftrightarrow (q,p) \in \mathcal{Z}_{-k} \quad \text{and} \quad (p,q) \in \mathcal{Z}_0 \Rightarrow p = q.$$

Recall that for any (p,q) there is at most one k such that $(p,q) \in \mathcal{Z}_k$, which implies that there is no more than n^2 non-empty $\mathcal{Z}_k$.

Set $\tilde h = (1 - \kappa/2)h$ and consider the space $\mathfrak{B}_{\tilde h}$ consisting of all $X \in C^\omega_{\tilde h}(\mathbb{T}^d, u(n))$ of finite norm $|X|_{1,\tilde h} < \infty$. We define $\mathfrak{B}_{\tilde h}^{(\mathrm{re})}$ as the space of all $X \in \mathfrak{B}_{\tilde h}$ such that

$$\left(T_{N_{j+1}} X\right)(\theta) = \sum_{|k| \le nN_j} e^{2\pi i\langle k,\theta\rangle} \left\{ \sum_{(p,q)\in\mathcal{Z}_k} \widehat{x}_{p,q}(k) E(p,q) \right\} \tag{3.61}$$

and denote by $\mathfrak{B}_{\tilde h}^{(\mathrm{nre})}$ the space of all $X \in \mathfrak{B}_{\tilde h}$ of the form

$$X(\theta) = \sum_{|k|\le N_{j+1}} e^{2\pi i\langle k,\theta\rangle}\{\sum_{(p,q)\in \mathcal{Z}_k^c} \widehat{x}_{p,q}(k)E(p,q)\} \tag{3.62}$$

$$= \sum_{|k|\le nN_j} e^{2\pi i\langle k,\theta\rangle} \sum_{(p,q)\in \mathcal{Z}_k^c} \widehat{x}_{p,q}(k)E(p,q) + \sum_{nN_j<|k|\le N_{j+1}} \widehat{X}(k)e^{2\pi i\langle k,\theta\rangle}$$

(recall that the truncation $T_N F$ of the Fourier series of F is defined in Sect. 2.1 and that $\widehat{f}_{p,q}(k)$ is the (p,q) entry of $\widehat{F}(k)$). Both spaces are closed in $\mathfrak{B}_{\tilde h}$ and obviously

$$\mathfrak{B}_{\tilde h}^{(\mathrm{re})} = \mathfrak{B}_{\tilde h}^{(\mathrm{nre})} \oplus \mathfrak{B}_{\tilde h}^{(\mathrm{re})}.$$

Moreover, the following assertion holds true.

Lemma 3.3. — *Let $X \in \mathfrak{B}_{\tilde h}^{(\mathrm{nre})}$. Then*

$$X - A^{-1}X(\cdot+\alpha)A \in \mathfrak{B}_{1,\tilde h}^{(\mathrm{nre})} \quad \textit{and} \quad |X - A^{-1}X(\cdot+\alpha)A|_{1,\tilde h} \ge \varepsilon^{\sigma}|X|_{1,\tilde h}.$$

Proof. — The first relation is evident. To estimate below the first sum in (3.62) we use (3.57) and for the second sum in (3.62) we make use of (3.59). □

Step 5. Applying Lemma 3.1. — The previous lemma says that (3.29) holds with $\eta = \varepsilon^{\sigma}$. By assumption and by (2.8) and (3.50) we get as well

$$|F|_{1,\tilde h} \le c_*(\kappa h)^{-d}|F|_h \le c_*(\kappa h)^{-d}\varepsilon < \varepsilon^{1-3\sigma/2}$$

and applying Lemma 3.1 we find $Y \in \mathfrak{B}_{\tilde h}^{(\mathrm{nre})}$ and $\widetilde F \in \mathfrak{B}_{\tilde h}^{(\mathrm{re})}$ such that

$$\begin{cases} |Y|_{\tilde h} \le |Y|_{1,\tilde h} \le 2\varepsilon^{-\sigma}|F|_{1,\tilde h} \le 2c_*(\kappa h)^{-d}\varepsilon^{1-\sigma} \le \varepsilon^{1-2\sigma}, \\ |\widetilde F|_{\tilde h} \le |\widetilde F|_{1,\tilde h} \le \mathrm{cst.}|F|_{1,\tilde h} \le \varepsilon^{1-2\sigma} \end{cases} \tag{3.63}$$

and

$$\mathrm{Ad}\,(e^{Y}).(\alpha, Ae^{F}) = (\alpha, Ae^{\widetilde F}).$$

Setting $Q(\theta) = \mathrm{diag}\,(e^{2\pi i\langle k^{(1)},\theta\rangle}, \dots, e^{2\pi i\langle k^{(n)},\theta\rangle})$ we consider

$$Ad(Q).(\alpha, Ae^{\widetilde F}) = (\alpha, Q(\cdot+\alpha)^{-1}AQ\exp\{Q^{-1}\widetilde F Q\}).$$

Since $\widetilde F \in \mathfrak{B}_{\tilde h}^{(\mathrm{re})}$ is of the form (3.61) with f replaced by $\widetilde f$ we get

$$Q(\theta)^{-1}\widetilde F(\theta)Q(\theta) = \sum_{|k|\le nN_j}\sum_{(p,q)\in\mathcal{Z}_k} e^{2\pi i\langle k+k^{(q)}-k^{(p)},\theta\rangle}\widehat{\widetilde f}_{p,q}(k)E(p,q)$$

$$+ Q(\theta)^{-1}(R_{N_{j+1}}\widetilde F)(\theta)Q(\theta) = (I) + (II).$$

It follows from the definition of $\mathcal{Z}_k$ that (I) is a constant and by (2.8) and (3.50) one gets

$$\begin{aligned}|(I)| &\leq \sum_{|k|\leq nN_j} \Big|\sum_{(p,q)\in\mathcal{Z}_k} \widehat{\widetilde{f}}_{p,q}(k)E(p,q)\Big| \\ &\leq n^2|\widetilde{F}|_{1,\tilde{h}} \leq \varepsilon^{1-3\sigma}.\end{aligned}$$

On the other hand,

$$|Q|_h \leq e^{nN_jh} \leq e^{\kappa Nh} \leq \frac{1}{2}\varepsilon^{-K_*}, \tag{3.64}$$

where $K_* = K_*(\kappa) > 0$ is a constant depending only on κ. In view of (2.9) and (3.48)-(3.50) this implies

$$\begin{aligned}|(II)|_{(1-\kappa)h} &\leq e^{2nN_jh}|R_{N_{j+1}}\widetilde{F}|_{(1-\kappa)h} \\ &\leq e^{2nN_jh}\left\{c_*\frac{N_{j+1}^d}{(\kappa h)^d}e^{-\kappa N_{j+1}h}\varepsilon^{1-2\sigma}\right\} \\ &= e^{-N_jh}\left\{c_*\frac{N_{j+1}^d}{(\kappa h)^d}\varepsilon^{1-2\sigma}\right\} \\ &\leq \varepsilon^2\left\{c_*\frac{N_{j+1}^d}{(\kappa h)^d}\varepsilon^{1-2\sigma}\right\} \leq \varepsilon^{1+2\sigma}.\end{aligned}$$

Finally, we obtain the conjugation

$$Ad(Qe^Y).(\alpha, Ae^F) = (\alpha, \widetilde{A}e^{(I)+(II)}), \tag{3.65}$$

where $\widetilde{A} := Q(\cdot+\alpha)^{-1}AQ$ is a constant. Moreover, there exists $F_+ \in C^\omega_{(1-\kappa)h}(\mathbb{T}^d, U(n))$ (one can take $F_+ = \log\{e^{-(I)}e^{(I)+(II)}\}$) such that

$$\widetilde{A}e^{(I)+(II)} = \widetilde{A}e^{(I)}e^{F_+}$$

and using (3.52) we get

$$|F_+|_{(1-\kappa)h} \leq 2\left(|(I)|^2 + |(II)|_{(1-\kappa)h}\right) \leq 4\varepsilon^{1+2\sigma} \leq \varepsilon^{1+\sigma}.$$

Recall that (I) is a constant. Then $A_+ = \widetilde{A}e^{(I)} \in U(n)$ is a constant matrix which satisfies

$$|A_+ - \widetilde{A}| \leq 2|(I)| \leq 2\varepsilon^{1-3\sigma} \leq \varepsilon^{1/2} \tag{3.66}$$

in view of (3.52). Setting $R = e^YQ$, one obtains from (3.65) the equality

$$Ad(R).(\alpha, Ae^F) = (\alpha, A_+e^{F_+}). \tag{3.67}$$

One can check easily the desired estimates.

The relation

$$R = e^YQ \in \Pi(N, 1/n, \varepsilon^{1-4\sigma})$$

follows from Lemma 2.7, since $e^Y \in \Pi(0,1,\varepsilon^{1-4\sigma})$ by Lemma 2.6, *4)*, and $Q \in \Pi(N, 1/n, 0)$ by Lemma 2.6, *3)*.

If $A \in \mathrm{NR}\,(N, \varepsilon^\sigma)$ using (3.63) and (3.66)) we get $Q = I$, $R = e^Y$ and $A = \widetilde{A}$ with $|Y|_h < \varepsilon^{1-2\sigma}$ and $|A_+ - A| \le \varepsilon^{1/2}$. This completes the proof of the proposition. □

REMARK 3.1. — *This proof can extended for cocycles on other groups, for example, $GL(n,\mathbb{C})$, $GL(n,\mathbb{R})$, $SO(n,\mathbb{R})$, as well as for general compact semi-simple Lie groups.*

For any $\kappa > 0$, repeating the KAM step infinitely many times and choosing $h_m = (1 - \frac{\kappa}{2^m})h$ and $\varepsilon_m = \varepsilon^{(1+\sigma)^m}$ and the corresponding N_m as above, we are going to obtain almost reducibility with analytic radius of conjugations decreasing to $(1-2\kappa)h$ in the next section. The sequence ε_m will be well-adapted to the corresponding Gevrey class choosing $\varepsilon = \varepsilon_0$ as in (3.70). Almost reducibility for Gevrey cocycles has been proved by Chavaudret in [2]. We point out that the KAM Step and the KAM iteration scheme below are somewhat different from that in [2].

3.2. The iterative Lemma. — We assume that $A \in U(n)$ and $G \in \mathscr{G}^\rho_L(\mathbb{T}^d, u(n))$, where $\rho > 1$. Let σ , χ, δ_0 and K_* be the same as in the assumptions of Proposition 3.1. Let $0 < h_0 \le 1/2L$ (h_0 will be specified later) and set

$$\kappa = 1 - \left(1 + \frac{\sigma}{2}\right)^{1-\rho}, \text{ and}$$

$$h_m := (1-\kappa)h_{m-1} = (1-\kappa)^m h_0 = h_0\left(1+\frac{\sigma}{2}\right)^{(1-\rho)m}, \ m \in \mathbb{N}. \tag{3.68}$$

By Proposition 2.1 one can find a sequence of $G_m \in C^\omega_{h_m}(\mathbb{T}^d, u(n))$ satisfying

$$\begin{cases} |G_0|_{h_0} \le c_0 L^d (1 + e^{-(cLh_0)^{-1/(\rho-1)}})\|G\|_L \\ |G_m - G_{m-1}|_{h_m} \le c_0 L^d e^{-(cLh_{m-1})^{-1/(\rho-1)}}\|G\|_L, \ m \ge 1 \\ \sup_{\theta\in\mathbb{T}^d} |G(\theta) - G_{m-1}(\theta)| \le c_0 L^d e^{-(cLh_m)^{-1/(\rho-1)}}\|G\|_L, \ m \ge 1, \end{cases} \tag{3.69}$$

where $c = c(\rho) > 0$ and $c_0 = c_0(\rho, d) \ge 1$. We set

$$\begin{cases} \varepsilon_0 := \exp\left(-\frac{1}{16(1+K_*/\sigma)}(cLh_0)^{-\frac{1}{\rho-1}}\right), \text{ and} \\ \varepsilon_m := \varepsilon_0^{(1+\frac{\sigma}{2})^m} = \exp\left(-\frac{1}{16(1+K_*/\sigma)}(cLh_m)^{-\frac{1}{\rho-1}}\right), \quad m \in \mathbb{N}. \end{cases} \tag{3.70}$$

Choose $0 < h_0 \ll 1$ small enough depending only on σ, χ, L, ρ, and δ_0 such that

$$cLh_0 \le 1, \quad \varepsilon_0^{\sigma/2} \le \frac{1}{8}, \tag{3.71}$$

and

$$\varepsilon_m = \varepsilon_0^{(1+\frac{\sigma}{2})^m} = \exp\left(-\frac{1}{16(1+K_*/\sigma)}(cLh_m)^{-\frac{1}{\rho-1}}\right) \leq \delta_0 h_m^{\chi} \tag{3.72}$$

for any , $m \in \mathbb{N}$. We suppose as well that for any matrix P satisfying $|P-I| \leq \varepsilon_m \leq \varepsilon_0$ the following inequality holds true

$$|\log P| \leq 2|P-I| \leq 2\varepsilon_m. \tag{3.73}$$

It is enough to get the above inequality for $m = 0$ then they follow automatically for each $m \in \mathbb{N}$.

We are going to impose a smallness condition on G.

Lemma 3.4. — *Assume that*

$$\|G\|_L \leq \frac{\varepsilon_0}{2c_0 L^d}\,. \tag{3.74}$$

Then

$$\begin{cases} |G_0|_{h_0} \leq \varepsilon_0 \\ |G_m - G_{m-1}|_{h_m} \leq \varepsilon_{m+1}^{4(1+K_*/\sigma)}, \quad m \geq 1, \\ \sup_{\theta\in\mathbb{T}^d} |G(\theta) - G_{m-1}(\theta)|_{h_m} \leq \varepsilon_{m+1}^{4(1+K_*/\sigma)}. \end{cases} \tag{3.75}$$

Proof. — The claim follows directly from (3.69), (3.70) and (3.74) since

$$\begin{aligned} &e^{-(cLh_{m-1})^{-1/(\rho-1)}} = e^{-(1+\frac{\sigma}{2})^{-2}(cLh_{m+1})^{-\frac{1}{\rho-1}}} \\ &< e^{-\frac{1}{4}(cLh_{m+1})^{-\frac{1}{\rho-1}}} = \varepsilon_{m+1}^{4(1+K_*/\sigma)} \end{aligned} \tag{3.76}$$

for $m \geq 1$. □

From now on we denote by $(N_m)_{m\in\mathbb{N}}$ the increasing sequence

$$\begin{aligned} N_m &:= \left(\frac{2n+1}{\kappa}\right)^n \left(\frac{2}{h_m}\log\frac{1}{\varepsilon_m} + 1\right) \\ &= \left(\frac{2n+1}{1-(1+\frac{\sigma}{2})^{1-\rho}}\right)^n \left(\frac{2}{h_m}\log\frac{1}{\varepsilon_m} + 1\right). \end{aligned} \tag{3.77}$$

In view of (3.72), one can apply Proposition 3.1 for $\varepsilon = \varepsilon_m$, $h = h_m$ and $N = N_m$, $m \in \mathbb{N}$. Set

$$L_m = N_0 + \cdots + N_{m-1}. \tag{3.78}$$

We can state now the iterative lemma.

LEMMA 3.5. — *For each $m \geq 0$ there is $A_m \in U(n)$, $F_m \in C^{\omega}_{h_m}(\mathbb{T}^d, u(n))$, and $R^{(m)} \in C^{\omega}_{h_m}(\mathbb{T}^d, U(n))$, such that*

$$
\begin{cases}
\mathrm{Ad}\,(R^{(m)}).(\alpha, Ae^{G_{m-1}}) = (\alpha, A_m e^{F_m}) \\
|F_m|_{h_m} \leq \varepsilon_m \\
|R^{(m)}|_{h_m} \leq \varepsilon_m^{-2K_*/\sigma} \\
R^{(m)} \in \Pi(L_m, \frac{1}{n^m}, \frac{1}{n^{m-1}}\varepsilon_0^{1-4\sigma} + \cdots + \frac{1}{n}\varepsilon_{m-2}^{1-4\sigma} + \varepsilon_{m-1}^{1-4\sigma})
\end{cases}
\tag{3.79}
$$

(recall that the definition and property of Π *has been given in Section 2.3).*

Proof. — Applying Proposition 3.1 one can find $R^{(1)}$ such that

$$
\begin{cases}
\mathrm{Ad}\,(R^{(1)}).(\alpha, Ae^{G_0}) = (\alpha, A_1 e^{F_1}) \\
|F_1|_{h_1} \leq \varepsilon_0^{1+\sigma} < \frac{1}{8}\varepsilon_1 \\
|R^{(1)}|_{h_1} \leq \varepsilon_0^{-K_*} = \varepsilon_1^{-K_*/(1+\frac{\sigma}{2})} < \varepsilon_1^{-2K_*/\sigma} \\
R^{(1)} \in \Pi(N_0, 1/n, \varepsilon_0^{1-4\sigma}).
\end{cases}
$$

Arguing by induction assume that for given $m \geq 2$ we have

$$
\begin{cases}
\mathrm{Ad}\,(R^{(m)}).(\alpha, Ae^{G_{m-1}}) = (\alpha, A_m e^{F_m}) \\
|F_m|_{h_m} \leq \varepsilon_m \\
|R^{(m)}|_{h_m} \leq \varepsilon_m^{-2K_*/\sigma} \\
R^{(m)} \in \Pi(L_m, \frac{1}{n^m}, \frac{1}{n^{m-1}}\varepsilon_0^{1-4\sigma} + \cdots + \frac{1}{n}\varepsilon_{m-2}^{1-4\sigma} + \varepsilon_{m-1}^{1-4\sigma}).
\end{cases}
$$

Using Proposition 3.1 and (3.71) one can find R_m and $F^{(0)}_{m+1} \in C^{\omega}_{h_{m+1}}(\mathbb{T}^d, u(n))$ such that

$$
\begin{cases}
\mathrm{Ad}\,(R_m).(\alpha, A_m e^{F_m}) = (\alpha, A_{m+1} e^{F^{(0)}_{m+1}}) \\
|F^{(0)}_{m+1}|_{h_{m+1}} \leq \varepsilon_m^{1+\sigma} = \varepsilon_m^{\sigma/2}\varepsilon_m^{1+\sigma/2} \leq \frac{1}{8}\varepsilon_{m+1} \\
|R_m|_{h_{m+1}} \leq \varepsilon_m^{-K_*} \\
R_m \in \Pi(N_m, \frac{1}{n}, \varepsilon_m^{1-4\sigma}).
\end{cases}
$$

Setting $R^{(m+1)} = R^{(m)} R_m$ we obtain

$$
\begin{aligned}
&R^{(m+1)}(\cdot + \alpha)^{-1} A e^{G_m} R^{(m+1)} \\
&= R_m(\cdot + \alpha)^{-1} A_m e^{F_m} R_m + R^{(m+1)}(\cdot + \alpha)^{-1} A(e^{G_m} - e^{G_{m-1}}) R^{(m+1)} \\
&= A_{m+1} e^{F^{(0)}_{m+1}} + R^{(m+1)}(\cdot + \alpha)^{-1} A(e^{G_m} - e^{G_{m-1}}) R^{(m+1)}.
\end{aligned}
$$

Moreover,

$$
|R^{(m+1)}|_{h_{m+1}} \leq |R^{(m)}|_{h_m} |R_m|_{h_{m+1}} \leq \varepsilon_m^{-K_*(1+2/\sigma)} = \varepsilon_m^{-\frac{2(1+\sigma/2)K_*}{\sigma}} = \varepsilon_{m+1}^{-2K_*/\sigma}
$$

while Lemma 2.7 implies

$$\begin{aligned}R^{(m+1)} &\in \Pi(L_m + N_m, \frac{1}{n^{m+1}}, \frac{1}{n}(\frac{1}{n^{m-1}}\varepsilon_0^{1-4\sigma} + \cdots + \varepsilon_{m-1}^{1-4\sigma}) + \varepsilon_m^{1-4\sigma})\\ &= \Pi(L_{m+1}, \frac{1}{n^{m+1}}, \frac{1}{n^m}\varepsilon_0^{1-4\sigma} + \cdots + \frac{1}{n}\varepsilon_{m-1}^{1-4\sigma} + \varepsilon_m^{1-4\sigma}).\end{aligned}$$

Taking into account (3.75) one obtains

$$\begin{aligned}&|R^{(m+1)}(\cdot+\alpha)^{-1}A(e^{G_m} - e^{G_{m-1}})R^{(m+1)}|_{h_{m+1}}\\ &\le 2\varepsilon_{m+1}^{-4K_*/\sigma}\varepsilon_{m+1}^{4(1+K_*/\sigma)} = 2\varepsilon_{m+1}^4 \le \frac{1}{4}\varepsilon_{m+1},\end{aligned}$$

Moreover,

$$|A_{m+1}e^{F_{m+1}^{(0)}} - A_{m+1}|_{h_{m+1}} \le 2\times\frac{1}{8}\varepsilon_{m+1} = \frac{1}{4}\varepsilon_{m+1}$$

which implies

$$|R^{(m+1)}(\cdot+\alpha)^{-1}Ae^{G_m}R^{(m+1)} - A_{m+1}|_{h_{m+1}} \le \frac{1}{4}\varepsilon_{m+1} + \frac{1}{4}\varepsilon_{m+1} \le \frac{1}{2}\varepsilon_{m+1}.$$

Setting

$$F_{m+1} := \log\left(A_{m+1}^{-1}R^{(m+1)}(\cdot+\alpha)^{-1}Ae^{G_m}R^{(m+1)} - I\right)$$

and using (3.73) one obtains

$$\begin{cases}\mathrm{Ad}\,(R^{(m+1)}).(\alpha, Ae^{G_m}) = (\alpha, A_{m+1}e^{F_{m+1}})\\ |F_{m+1}|_{h_{m+1}} \le \varepsilon_{m+1}\\ |R^{(m+1)}|_{h_{m+1}} \le \varepsilon_{m+1}^{-2K_*/\sigma}\\ R^{(m+1)} \in \Pi(L_{m+1}, \frac{1}{n^{m+1}}, \frac{1}{n^m}\varepsilon_0^{1-4\sigma} + \cdots + \frac{1}{n}\varepsilon_{m-1}^{1-4\sigma} + \varepsilon_m^{1-4\sigma}).\end{cases}$$

This completes the induction argument and proves the iterative lemma. □

3.3. Gevrey reducibility. — In the previous section we have established almost reducibility of the Gevrey$-\mathscr{G}^\rho$ cocycle (α, Ae^G). More precisely, for each $m \in \mathbb{N}$ we have obtained in Lemma 3.5 a map $R^{(m)}$ which conjugates $(\alpha, Ae^{G_{m-1}})$ to $(\alpha, A_m e^{F_m})$, where F_m is of the size of ε_m. In general, the sequence $R^{(m)}$ diverges. Our aim in this section is to prove that the sequence $R^{(m)}$ is convergent in $\mathscr{G}^\rho$, provided that there exits a measurable function $B : \mathbb{T}^d \to U(n)$ with $\lceil B^{-1}\rfloor = \lceil B^*\rfloor \ge \epsilon$ and $C \in \Sigma(\alpha)$ such that

$$\mathrm{Ad}\,(B).(\alpha, Ae^G) = (\alpha, C) \tag{3.80}$$

where $0 < \epsilon \le 1$. To do this we impose condition (1.4). More precisely, choosing $0 < h_0 < 1$ sufficiently small we assume as well that

$$\varepsilon_0 := \exp\left(-\frac{1}{16(1+K_*/\sigma)}(cLh_0)^{-\frac{1}{\rho-1}}\right) = \tilde{\varepsilon}_0\epsilon^\ell \tag{3.81}$$

where $\tilde{\varepsilon}_0 := \tilde{\varepsilon}_0(n, \sigma, \kappa, \rho, L) \ll 1$ is sufficiently small so that all previous assumptions on ε_0 hold when $\epsilon = 1$. To do this we choose appropriately h_0 in a function of ϵ as well. Setting $\delta := 2c_0 L^d \tilde{\varepsilon}_0$ and using (3.74) we obtain the small constant in (1.4). We assume that

$$\tilde{\varepsilon}_0^{\sigma(1-4\sigma)/2} \leq \frac{1}{8n}. \tag{3.82}$$

We have $1 - 5\sigma \geq 1/\ell$ by (3.47) which implies

$$\varepsilon_0^{1-4\sigma} < \tilde{\varepsilon}_0^{\sigma} \varepsilon_0^{1/\ell} \leq \frac{1}{8n} \epsilon^{\ell} \tag{3.83}$$

and we obtain

$$\begin{aligned} \frac{1}{n^{m-1}} \varepsilon_0^{1-4\sigma} + \cdots + \frac{1}{n} \varepsilon_{m-2}^{1-4\sigma} + \varepsilon_{m-1}^{1-4\sigma} &< \frac{1}{n^{m-1}} \varepsilon_0^{1-4\sigma} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-1}}\right) \\ &< \frac{2}{n^{m-1}} \varepsilon_0^{1-4\sigma} \leq \frac{\epsilon}{4n^m}. \end{aligned}$$

This inequality combined with Lemma 2.6, 5) and Lemma 3.5, implies

$$\begin{aligned} R^{(m)} &\in \Pi(L_m, \frac{1}{n^m}, \frac{1}{n^{m-1}} \varepsilon_0^{1-4\sigma} + \cdots + \frac{1}{n} \varepsilon_{m-2}^{1-4\sigma} + \varepsilon_{m-1}^{1-4\sigma}) \\ &\subseteq \Pi(L_m, \frac{1}{n^m}, \frac{\epsilon}{4n^m}). \end{aligned} \tag{3.84}$$

Now we can get a good control on the size of R_m.

Lemma 3.6. — *For any m sufficiently large there is $Y_m \in C^{\omega}_{h_m}(\mathbb{T}^d, u(n))$ such that*

$$\begin{cases} R_m = e^{Y_m} \\ |Y_m|_{h_m} \leq \varepsilon_m^{1-2\sigma} \\ |A_m - A_{m+1}| \leq \varepsilon_m^{1/2}. \end{cases}$$

Proof. — The idea is to prove that $A_m \in \mathrm{NR}\,(N_m, \varepsilon_m^{\sigma})$ for large m which allows us to use Proposition 3.1, (ii). In the following we will need only the L^{∞} norm $\|\cdot\|_{\infty}$ on $\mathbb{T}^n$.

By Lemma 3.5 and (3.80)) we have the equalities

$$\begin{cases} \mathrm{Ad}\,(R^{(m)}).(\alpha, Ae^{G_{m-1}}) = (\alpha, A_m e^{F_m}), \\ \mathrm{Ad}\,(B).(\alpha, Ae^{G}) = (\alpha, C). \end{cases}$$

Setting $B_m = B^{-1} R^{(m)}$ and using (3.75) we obtain

$$\mathrm{Ad}\,(B_m).(\alpha, C) = \left(\alpha, A_m e^{F_m} + O(\varepsilon_{m+1}^{4(1+K_*/\sigma)})\right).$$

Hereafter, the symbol $O(\varepsilon_m^\alpha)$, $\alpha \in \mathbb{R}$, stands for a map $W : \mathbb{T}^d \to M_n(\mathbb{R})$ such that $\|W\|_\infty \le c\varepsilon_m^\alpha$, where $c > 0$ does not depend on m and θ. Since the L^∞-norm of any measurable unitary matrix function is 1 this yields

$$B_m(\cdot + \alpha)A_m(I + O(\varepsilon_m)) = CB_m,$$

which implies

$$B_m(\cdot + \alpha)A_m = CB_m + O(\varepsilon_m). \tag{3.85}$$

By Lemma 2.5 there exists N_*, such that $B^{-1} = B^* \in \Gamma(N_*, \epsilon/2)$, and using (3.84) and (2.25) applied to B^* we arrive at

$$\begin{aligned} B_m = B^* R^{(m)} &\in \Gamma(N_* + L_m, \frac{\epsilon}{2n^m} - \frac{\epsilon}{4n^m}) \\ &= \Gamma(N_* + L_m, \frac{\epsilon}{4n^m}). \end{aligned} \tag{3.86}$$

Recall that $C \in \Sigma(\alpha)$, which means that there exist $\chi, \nu > 0$, such that

$$|\mu_p e^{2\pi i \langle k,\alpha\rangle} - \mu_q| \ge \frac{\chi}{(1 + |k|)^\nu}, \quad \text{for any} \quad p \ne q, \quad k \in \mathbb{Z}^d, \tag{3.87}$$

where $\{\mu_1, \ldots, \mu_n\} = \mathrm{Spec}\,(C)$. Let $T \in U(n)$ be such that

$$TCT^* = \mathrm{diag}\,(\mu_1, \ldots, \mu_n).$$

Choose $S_m \in U(n)$ such that

$$S_m^* A_m S_m = \mathrm{diag}\,(\lambda_1^{(m)}, \ldots, \lambda_n^{(m)})$$

and set $D_m = T^* B_m S_m$. By Lemma 2.5

$$D_m \in \Gamma\left(N_* + L_m, \frac{\epsilon}{4n^m}\right) \tag{3.88}$$

and using (3.85) we obtain

$$D_m(\cdot + \alpha)\mathrm{diag}\,(\lambda_1^{(m)}, \ldots, \lambda_n^{(m)}) = \mathrm{diag}\,(\mu_1, \ldots, \mu_n) D_m + O(\varepsilon_m).$$

which means that

$$D_m(\cdot + \alpha)\mathrm{diag}\,(\lambda_1^{(m)}, \ldots, \lambda_n^{(m)}) + W_m = \mathrm{diag}\,(\mu_1, \ldots, \mu_n) D_m$$

with $\|W_m\|_\infty \le \mathrm{cst.}\,\varepsilon_m$. Thus for any $1 \le p, q \le n$ we have

$$e^{2\pi i\langle k,\alpha\rangle} \lambda_q^{(m)} \widehat{d}_{p,q}^{(m)}(k) + \widehat{w}_{p,q}^{(m)}(k) = \mu_p \widehat{d}_{p,q}^{(m)}(k)$$

where $\widehat{d}_{p,q}^{(m)}(k)$ and $\widehat{w}_{p,q}^{(m)}(k)$ stand fot the (p, q) entries of the matrices $\widehat{D}_m(k)$ and of $\widehat{W_m}(k)$ respectively. We have

$$|\widehat{w}_{p,q}^{(m)}(k)| \le \|W_m\|_\infty \le \mathrm{cst.}\varepsilon_m$$

which implies

$$|\widehat{d}_{p,q}^{(m)}(k)||\lambda_q^{(m)} e^{2\pi i\langle k,\alpha\rangle} - \mu_p| \le \mathrm{cst.}\varepsilon_m \tag{3.89}$$

for all $p, q \in \{1, \dots, n\}$ and $k \in \mathbb{Z}^d$. It follows from (3.88) and the definition of Γ in Section 2.3 that for any p there exist $q \in \{1, \dots, n\}$ and $k \in \mathbb{Z}^d$ with $|k| \le N_* + L_m$ such that

$$|\widehat{d}^{(m)}_{p,q}(k)| \ge \frac{\epsilon}{4n^m}$$

(q and k depend on m as well). Then using (3.89) and choosing $m_1 = m_1(\epsilon) \gg 1$ we obtain

$$\begin{aligned} |\lambda_q^{(m)} e^{2\pi i \langle k, \alpha\rangle} - \mu_p| &\le \text{cst.}\, \epsilon^{-1} n^m \varepsilon_m \\ &\le \text{cst.}\, \varepsilon_m^{1/2} \end{aligned} \tag{3.90}$$

for any $m \ge m_1$. Consider the map

$$f_m : \{1, \dots, n\} \to \{1, \dots, n\}$$

assigning to each $p \in \{1, \dots, n\}$ an integer $q \in \{1, \dots, n\}$ such that (3.90) holds with some $k \in \mathbb{Z}^d$ of length $|k| \le N_* + L_m$.

LEMMA 3.7. — *There is $m_0 \in \mathbb{N}$ such that the map f_m is bijective for any $m \ge m_0$.*

Proof. — It suffices to prove that f_m is injective. Suppose that there are $p_1 \neq p_2$ such that $f_m(p_1) = f_m(p_2) = q$. Then there exist k_1, k_2 such that

$$\begin{cases} |\lambda_q^{(m)} e^{2\pi i \langle k_1, \alpha\rangle} - \mu_{p_1}| \le \text{cst.}\varepsilon_m^{1/2}, \\ |\lambda_q^{(m)} e^{2\pi i \langle k_2, \alpha\rangle} - \mu_{p_2}| \le \text{cst.}\varepsilon_m^{1/2}. \end{cases}$$

This implies

$$\begin{aligned} 0 &= |\lambda_q^{(m)} - \lambda_q^{(m)}| \\ &\ge |\mu_{p_1} e^{2\pi i \langle -k_1 + k_2, \alpha\rangle} - \mu_{p_2}| - \text{cst.}\varepsilon_m^{1/2} \\ &\ge \frac{\chi}{(2L_m + 2N_* + 1)^\nu} - \text{cst.}\varepsilon_m^{1/2} \end{aligned}$$

which can not be true for $m \gg 1$ in view of (3.77), (3.78) and (3.51). □

LEMMA 3.8. — *There is $m_0 \in \mathbb{N}$ such that $A_m \in \mathrm{NR}\,(N_m, \varepsilon_m^\sigma)$ for $m \ge m_0$.*

Proof. — Suppose that A_m is $(N_m, \varepsilon_m^\sigma)$-resonant. Then there exist $k \in \mathbb{Z}^d$ with $0 < |k| \le N_m$ and $p, q \in \{1, \dots n\}$ such that

$$|\lambda_p e^{2\pi i \langle k, \alpha\rangle} - \lambda_q| \le \varepsilon_m^\sigma.$$

If $p = q$, we get

$$\varepsilon_m^\sigma \ge |e^{2\pi i \langle k, \alpha\rangle} - 1| \ge \frac{2\pi\gamma^{-1}}{N_m^\tau}$$

which can not be true for $m \gg 1$. If $p \neq q$ and m is large enough, it follows from Lemma 3.7 that there exist $k(p), k(q) \in \mathbb{Z}^d$ satisfying $|k(p)|, |k(q)| \leq L_m + N_*$, such that

$$\max\{|\lambda_p e^{2\pi i\langle k(p),\alpha\rangle} - \widetilde{\mu}_p|,\ |\lambda_q e^{2\pi i\langle k(q),\alpha\rangle} - \widetilde{\mu}_q|\} \leq \text{cst.}\, \varepsilon_m^{1/2},$$

where

$$\widetilde{\mu}_p = \mu_{f_m^{-1}(p)} \neq \mu_{f_m^{-1}(q)} = \widetilde{\mu}_q.$$

Then setting $l = k - k(p) + k(q)$ we get

$$\begin{aligned} |\widetilde{\mu}_p e^{2\pi i\langle l,\alpha\rangle} - \widetilde{\mu}_q| &= |\widetilde{\mu}_p e^{2\pi i\langle k-k(p),\alpha\rangle} - \widetilde{\mu}_q e^{2\pi i\langle -k(q),\alpha\rangle}| \\ &\leq |\lambda_p e^{2\pi i\langle k,\alpha\rangle} - \lambda_q| + \text{cst.}\, \varepsilon_m^{1/2} \\ &\leq \varepsilon_m^{\sigma} + \text{cst.}\varepsilon_m^{1/2}. \end{aligned}$$

On the other hand, the assumption (3.87) yields

$$\begin{aligned} |\widetilde{\mu}_p e^{2\pi i\langle l,\alpha\rangle} - \widetilde{\mu}_q| &\geq \frac{\chi}{(|l|+1)^{\nu}} \\ &\geq \frac{\chi}{(N_m + 2L_m + 2N_* + 1)^{\nu}}, \end{aligned}$$

hence,

$$\frac{\chi}{(N_m + 2L_m + N_* + 1)^{\nu}} \leq \varepsilon_m^{\sigma} + \text{cst.}\, \varepsilon_m^{1/2}$$

which is not true for $m \gg 1$. □

Lemma 3.8 allows one to apply Proposition 3.1(ii), which completes the proof of Lemma 3.6. □

Now we can prove reducibility in Gevrey classes.

LEMMA 3.9 (Gevrey$-\mathscr{G}^{\rho}$ reducibility). — *There exist $R \in \mathscr{G}^{\rho}$ and $\widetilde{A} \in U(n)$ such that*

$$\text{Ad}\,(R).(\alpha, Ae^{G}) = (\alpha, \widetilde{A}). \tag{3.91}$$

Proof. — Recall from Lemma 3.74 and Lemma 3.5 that

$$\begin{aligned} &\text{Ad}\,(R^{(m)}).(\alpha, Ae^{G_{m-1}}) = (\alpha, A_m e^{F_m}), \\ &|G_m - G_{m-1}|_{h_m} \leq \varepsilon_{m+1}^{4(1+K_*/\sigma)} \quad \text{and} \quad |F_m - F_{m+1}|_{h_{m+1}} \leq 2\varepsilon_m. \end{aligned}$$

Moreover, by Lemma 3.6, there exists m_*, such that for any $m \geq m_*$ the following estimate holds true

$$\begin{aligned} |R^{(m+1)} - R^{(m)}|_{h_{m+1}} &= |R^{(m)}R_m - R^{(m)}|_{h_{m+1}} \\ &= |R^{(m)}(R_m - I)|_{h_{m+1}} \\ &= |R^{(m)}(e^{Y_m} - I)|_{h_{m+1}} \\ &\leq 2\varepsilon_m^{1-2\sigma}|R^{(m)}|_{h_{m+1}}. \end{aligned}$$

This implies

$$|R^{(m)}|_{h_{m+1}} \leq 2|R^{(m_*)}|_{h_{m_*+1}},$$

and

$$|R^{(m+1)} - R^{(m)}|_{h_{m+1}} \leq 4\varepsilon_m^{1-2\sigma}|R^{(m_*)}|_{h_{m_*+1}}.$$

Taking $m_* \gg 1$ we get for any $m \geq m_*$ the estimate

$$|R^{(m+1)} - R^{(m)}|_{h_{m+1}} \leq \varepsilon_m^{1/2}.$$

hence, for any $m \geq m_*$ we have

$$\zeta_m := \max\{|G_m - G_{m-1}|_{h_{m+1}},\ |F_m - F_{m+1}|_{h_{m+1}},\ |R^{(m+1)} - R^{(m)}|_{h_{m+1}}\} \leq \varepsilon_m^{1/2}.$$

Recall that

$$h_m = h_0\left(1 + \frac{\sigma}{2}\right)^{-m(\rho-1)} \quad \text{and} \quad \varepsilon_m = \exp\left\{-\frac{1}{16(K_*/\sigma+1)}(cLh_m)^{-\frac{1}{\rho-1}}\right\}.$$

Then for any m sufficiently large we get

$$\begin{aligned} &\exp\left\{-\left((32(K_*/\sigma+1))^{\rho-1}cLh_m\right)^{-\frac{1}{\rho-1}}\right\} \qquad (3.92) \\ &= \exp\left\{-\frac{1}{32(K_*/\sigma+1)}(cLh_m)^{-\frac{1}{\rho-1}}\right\} \\ &= \varepsilon_m^{1/2} \geq \zeta_m. \qquad (3.93) \end{aligned}$$

By the inverse approximation lemma (Proposition 2.2) and (3.92) choosing

$$\widetilde{L} \geq c_0(\rho, d)(32(K_*/\sigma+1))^{\rho-1}cL$$

we obtain that

- $R^{(m)}$ converges in $\mathscr{G}^\rho_{\widetilde{L}}$ to some $R \in \mathscr{G}^\rho_{\widetilde{L}}(\mathbb{T}^d, U(n))$,
- G_m converges in $\mathscr{G}_{\widetilde{L}}$ and it converges to G in C^0, so G_m converges to G in $\mathscr{G}^\rho_{\widetilde{L}}$,
- F_m converges in $\mathscr{G}^\rho_{\widetilde{L}}$, and F_m converges to 0 in C^0, hence, F_m converges to 0 in $\mathscr{G}^\rho_{\widetilde{L}}$.

To finish the proof, one has just to let $m \to \infty$ in

$$\mathrm{Ad}\,(R^{(m)}).(\alpha, Ae^{G_{m-1}}) = (\alpha, A_m e^{F_m})$$

with A_m converging to some $\widetilde{A} \in U(n)$ by Lemma 3.6. □

3.4. Proof of Theorem 1.1. — We are going to complete the proof of Theorem 1.1. By Lemma 3.9, there exist $R \in \mathcal{G}^{\rho}_{\widetilde{L}}$ and $\widetilde{A} \in U(n)$ such that

$$\mathrm{Ad}\,(R).(\alpha, Ae^{G}) = (\alpha, \widetilde{A}).$$

Moreover, there exist by assumption a measurable $B : \mathbb{T}^d \to U(n)$, $A \in U(n)$ and $C \in \Sigma(\alpha)$ such that

$$\mathrm{Ad}\,(B).(\alpha, Ae^{G}) = (\alpha, C).$$

Setting $V = RB^{-1}$, then $B = V^{-1}R$. R is analytic, to prove that B is almost surely Gevrey$-\mathcal{G}^{\rho}$, we just need to prove that V is so. In fact, we can prove that there exist $U_1, U_2, U_3 \in U(n)$ and $k^{(1)}, \ldots, k^{(n)} \in \mathbb{Z}^d$ such that

$$V(\theta) = U_1 U_3^* \exp\{2\pi i \mathrm{diag}\,(\langle k^{(1)}, \theta\rangle, \ldots, \langle k^{(n)}, \theta\rangle)\} U_2 \tag{3.94}$$

for a.e. $\theta \in \mathbb{T}^d$. Then for a.e. $\theta \in \mathbb{T}^d$ we have

$$\begin{aligned}\widetilde{B}(\theta) :&= V(\theta)^{-1} R(\theta)\\ &= U_1 U_3^* \exp\{2\pi i \mathrm{diag}\,(\langle k^{(1)}, \theta\rangle, \ldots, \langle k^{(n)}, \theta\rangle)\} U_2^* R(\theta) = B(\theta).\end{aligned}$$

Obviously $\widetilde{B}$ is in $\mathcal{G}^{\rho}_{\widetilde{L}}$ and we obtain the desired result.

To prove (3.94), let us consider the conjugation

$$\mathrm{Ad}\,(V).(\alpha, C) = (\alpha, \widetilde{A}).$$

We write

$$C = U_1 \mathrm{diag}\,(\mu_1, \ldots, \mu_n) U_1^* \quad \text{and} \quad \widetilde{A} = U_2 \mathrm{diag}\,(\widetilde{\mu}_1 \cdots, \widetilde{\mu}_n) U_2^*$$

where $U_1, U_2 \in U(n)$ and we get

$$\widetilde{V}(\cdot + \alpha) \mathrm{diag}\,(\mu_1, \ldots, \mu_n) = \mathrm{diag}\,(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_n) \widetilde{V},$$

where $\widetilde{V} := U_2^* V^{-1} U_1 : \mathbb{T}^d \to U(n)$ is measurable. Then for any $p, q \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}^d$ we obtain

$$(e^{2\pi i \langle k, \alpha\rangle} \mu_q - \widetilde{\mu}_p) \widehat{\widetilde{v}}_{p,q}(k) = 0.$$

where $\widehat{\widetilde{v}}_{p,q}(k)$ denotes the (p, q) entry of $\widehat{\widetilde{V}}(k)$. Since $\lceil \widetilde{V} \rfloor > 0$ there exist $k^{(1)}, \ldots, k^{(n)} \in \mathbb{Z}^d$, such that

$$\widetilde{\mu}_j = \mu_j e^{2\pi i \langle k^{(j)}, \alpha\rangle}, \quad j = 1, \ldots, n.$$

Setting $W(\theta) = \text{diag}\,(e^{2\pi i\langle k^{(1)},\theta\rangle}, \ldots, e^{2\pi i\langle k^{(n)},\theta\rangle})\widetilde{V}(\theta)$ we get

$$W(\cdot + \alpha)\text{diag}\,(\mu_1, \ldots, \mu_n) = \text{diag}\,(\mu_1, \ldots, \mu_n)W.$$

Then for all $p, q \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}^d$ we have

$$(e^{2\pi i\langle k,\alpha\rangle}\mu_q - \mu_p)\widehat{w}_{p,q}(k) = 0.$$

where $\widehat{w}_{p,q}(k)$ denotes the (p, q) entry of $\widehat{W}(k)$. The relation $C \in \Sigma(\alpha)$ implies that

$$\forall\, k \neq 0, \quad e^{2\pi i\langle k,\alpha\rangle}\mu_q - \mu_p \neq 0$$

and we obtain that $\widehat{W}(k) = 0$ for all $0 \neq k \in \mathbb{Z}^d$. Thus there exists $U_3 \in U(n)$, such that

$$W(\theta) = U_3$$

for a.e. $\theta \in \mathbb{T}^d$, which implies (3.94).

4. Global Setting

In this section we will prove the global rigidity result (Theorem 1.2) using Theorem 1.1 and adapting the renormalization scheme of a $\mathbb{Z}^2$-action developed by Krikorian [1, 11, 6] to the case of Gevrey classes.

4.1. $\mathbb{Z}^2$-Action in Gevrey classes. — Given $\rho > 1$ and $K > 0$ we denote by $\mathcal{G}^\rho_L([-K, K], U(n))$ the Banach space of all C^∞ functions $A : [-K, K] \to U(n)$ with finite norm

$$\|A\|^{\mathcal{G}}_{L,K} := \sup_{r\in\mathbb{N}} \sup_{|\theta|\leq K} (|\partial^r A(\theta)|L^{-|r|}r!^{-\rho}) < \infty.$$

Denote by $\mathcal{G}^\rho_L(\mathbb{R}, U(n))$ the functional space $\bigcap_{K>0} \mathcal{G}^\rho_L([-K, K], U(n))$ equipped by the projective limit topology (a sequence $A_m \in \mathcal{G}^\rho_L(\mathbb{R}, U(n))$ converges to $A \in \mathcal{G}^\rho_L(\mathbb{R}, U(n))$ if it converges in $\mathcal{G}^\rho_L([-K, K], U(n))$ for any $K > 0$) and set $\mathcal{G}^\rho(\mathbb{R}, U(n)) := \bigcup_{L>0} \mathcal{G}^\rho_L(\mathbb{R}, U(n))$ equipped by the corresponding inductive limit topology (a sequence $A_m \in \mathcal{G}^\rho(\mathbb{R}, U(n))$ converges to $A \in \mathcal{G}^\rho(\mathbb{R}, U(n))$ if there is $L > 0$ such that the sequence A_m converges to in $\mathcal{G}^\rho_L(\mathbb{R}, U(n))$).

Denote by $SW^{\mathcal{G}}_{L,\rho}(\mathbb{R}, U(n))$ the composition group of all $(\gamma, A) \in \mathbb{R} \times \mathcal{G}^\rho_L(\mathbb{R}, U(n))$ acting on $\mathbb{R} \times U(n)$ by

$$\begin{aligned}(\gamma, A) : \mathbb{R} \times U(n) &\to \mathbb{R} \times U(n)\\ (\theta, v) &\mapsto (\theta + \gamma, A(\theta)v).\end{aligned}$$

Denote as well by $\Lambda^{L,\rho}$ the set of all Gevrey$-\mathscr{G}^\rho$ fibered $\mathbb{Z}^2$-actions. By definition $\Phi \in \Lambda^{L,\rho}$ if it is a homomorphism from the additive group $\mathbb{Z}^2$ to the composition group $SW^{\mathscr{G}}_{L,\rho}(\mathbb{R}, U(n))$,

$$\begin{aligned} \Phi : \mathbb{Z}^2 &\to SW^{\mathscr{G}}_{L,\rho}(\mathbb{R}, U(n)) \\ (k_1,k_2) &\mapsto \Phi(k_1,k_2) = (\gamma^{\Phi}_{k_1,k_2}, A^{\Phi}_{k_1,k_2}). \end{aligned}$$

Any $\mathbb{Z}^2$-action Φ is completely determined by its values on $(1,0)$ and $(0,1)$ and we write it as follows

$$\Phi = \{\Phi(1,0), \Phi(0,1)\} = \{(\gamma^{\Phi}_{1,0}, A^{\Phi}_{1,0}), (\gamma^{\Phi}_{0,1}, A^{\Phi}_{0,1})\}.$$

We denote by Λ^ρ the union $\bigcup_{L>0} \Lambda^{L,\rho}$. To shorten the notations we sometimes skip the index ρ which is fixed.

A $\mathbb{Z}^2$-action Φ is said to be *normalized* if $\Phi(1,0) = (1, I)$. If Φ is normalized then $\Phi(0,1) = (\alpha, A)$ can be viewed as a cocycle in $SW^{\mathscr{G}}(\mathbb{T}^1, U(n))$, since A is automatically $\mathbb{Z}$-periodic. Conversely, to any $(\alpha, A) \in SW^{\mathscr{G}}(\mathbb{T}^1, U(n))$ one can associate a normalized $\mathbb{Z}^2$-action $\Phi = \{(1, I), (\alpha, A)\}$.

A $\mathbb{Z}^2$-action $\Phi \in \Lambda^{L,\rho}$ is said to be conjugated to a $\mathbb{Z}^2$-action $\widetilde{\Phi}$ by $B \in \mathscr{G}^\rho_L(\mathbb{R}, U(n))$ if for any $(k_1,k_2) \in \mathbb{Z}^2$

$$\begin{aligned} \mathrm{Ad}\,(B).\Phi(k_1,k_2) &:= (0,B)^{-1} \circ \Phi(k_1,k_2) \circ (0,B) \\ &= (0, B^{-1}) \circ \Phi(k_1,k_2) \circ (0,B) \\ &= \widetilde{\Phi}(k_1,k_2). \end{aligned}$$

The following lemma states that any Gevrey $\mathbb{Z}^2$-actions can be conjugated to a normalized one in the same Gevrey class.

Lemma 4.1 (Normalization). — *Any $\mathbb{Z}^2$-action of the form*

$$\Phi = \{(1, C), (\gamma, D)\} \in \Lambda^{L,\rho}$$

can be conjugated to a normalized one by some $P \in \mathscr{G}^\rho_{cL}(\mathbb{R}, U(n))$, where $c \geq 1$. Moreover, if

$$\sup_{|\theta| \leq 3} \|C(\theta) - I\| < 1/3 \tag{4.95}$$

then one can choose P so that

$$\|P - I\|^{\mathscr{G}}_{cL,2} \leq \mathrm{cst.} \|C - I\|^{\mathscr{G}}_{L,3}. \tag{4.96}$$

Proof. — Choose $X_0 \in u(n)$ such that $C(0) = e^{X_0}$, fix $0 < \delta < 3/5$ so that

$$\|e^{-X_0} C(\theta - 1) - I\| < 2/3 \quad \text{for} \quad \theta \in [1-\delta, 1+\delta] \tag{4.97}$$

and define $Y \in \mathscr{G}^\rho_L([1-\delta, 1+\delta], u(n))$ by

$$Y(\theta) := \log\left\{e^{-X_0} C(\theta - 1)\right\} \quad \text{for} \quad \theta \in [1-\delta, 1+\delta]. \tag{4.98}$$

We have

$$e^{X_0}e^{Y(\theta)} = C(\theta-1) \quad \text{for} \quad \theta \in [1-\delta, 1+\delta]. \tag{4.99}$$

Since $\log t = (1-t)h(t)$, where h is analytic in the unit ball $\{|t|<1\}$, we get the following inequality by estimating the composition of Gevrey functions (see [17], Proposition A.3)

$$\begin{aligned}\sup_{r\in\mathbb{N}}\sup_{|\theta-1|\le\delta}(|\partial^r Y(\theta)|L^{-|r|}r!^{-\rho}) &\le \text{cst.}\,\|C(0)^{-1}C(\cdot)-I\|^{\mathscr{G}}_{cL,\delta}\\ &\le \text{cst.}\,\|C-I\|^{\mathscr{G}}_{cL,\delta}\end{aligned} \tag{4.100}$$

where $c\ge 1$ is a constant independent of the constant $L\ge 1$.

Choosing $c \gg 1$ there is b in $\mathscr{G}^\rho_c \subset \mathscr{G}^\rho_{cL}$ such that $0\le b\le 1$, $b(\theta)=0$ for $\theta\in(-\infty,1/3]$, $b(\theta)=1$ for $\theta\in[2/3,+\infty]$ and b is strictly increasing on $[1/3,2/3]$. Set

$$b_\delta(\theta) = b\left(\frac{\theta-(1-\delta)}{\delta}\right) \quad \text{and} \quad f_\delta(\theta) = (1-\delta)+(\theta-(1-\delta))b_\delta(\theta)$$

and define

$$P(\theta) = e^{b_\delta(\theta)X_0}e^{b_\delta(\theta)Y(f_\delta(\theta))}, \quad \theta\in[-\delta,1+\delta]. \tag{4.101}$$

The functions b_δ and f_δ have the following properties

- $b_\delta(\theta)=1$ and $f_\delta(\theta)=\theta$ for $\theta\in[1-\delta/3,1+\delta]$,
- $1-\delta\le f_\delta(\theta)\le\theta$ and f_δ is increasing on $[1-2\delta/3,1-\delta/3]$,
- $b_\delta(\theta)=0$ and $f_\delta(\theta)=1-\delta$ for $\theta\in[-\delta,1-2\delta/3]$.

The theorem about the composition of Gevrey functions implies that $P\in\mathscr{G}^\rho_{cL}([1-\delta,1+\delta],U(n))$, where $c\ge 1$. Moreover, $P(\theta)=e^{X_0}e^{Y(\theta)}=C(\theta-1)$ for $\theta\in[1-\delta/3,1+\delta]$ and $P(\theta)=I$ for $\theta\in[-\delta,\delta]\subset[-\delta,1-2\delta/3]$ since $0<\delta<3/5$, and we obtain

$$P(\theta+1)^{-1}C(\theta)P(\theta)=I \quad \text{for} \quad |\theta|<\delta. \tag{4.102}$$

Now we extend P in $\mathbb{R}$ by

$$P(\theta)=\begin{cases}C(\theta-1)P(\theta-1), \ \theta\in[1,+\infty),\\ C(\theta)^{-1}P(\theta+1), \quad \theta\in(-\infty,1].\end{cases} \tag{4.103}$$

By (4.102) the function P is well defined. Moreover, $P\in\mathscr{G}^\rho_{cL}(\mathbb{R},U(n))$ with some $c\ge1$ independent of $L\ge 1$ and it satisfies the relation

$$\forall\,\theta\in\mathbb{R},\quad P(\theta+1)^{-1}C(\theta)P(\theta)=I.$$

It remains to prove (4.96) if (4.95) holds. We can choose now $X_0=\log C(0)$. Moreover, the inequality (4.97) holds for $\theta\in[-2,2]$, Y is well defined by (4.99) in $[-2,2]$ and as in (4.100) we get

$$\|X_0\|+\|Y\|^{\mathscr{G}}_{dL,2}\le \text{cst.}\|C-I\|^{\mathscr{G}}_{L,3}$$

where $d \geq 1$. Choose $\delta = 1/2$ in (4.101). Writing $e^X = I + Xg(X)$, where g is an entire function and using (4.101) and the theorems about the multiplication and the composition of Gevrey functions ([17], Proposition A.3) we obtain the following estimate in $\mathcal{G}^\rho_{cL}([-1/2, 3/2], M_n)$

$$\begin{aligned}\|P - I\|^{\mathcal{G}}_{cL,2} &= \|e^{-b_\delta X_0} - e^{b_\delta Y \circ f_\delta}\|^{\mathcal{G}}_{cL,2} \\ &\leq \|b_\delta X_0 g(b_\delta X_0)\|^{\mathcal{G}}_{cL,2} + \|(b_\delta Y \circ f_\delta) g(b_\delta Y \circ f_\delta)\|^{\mathcal{G}}_{cL,2} \\ &\leq \text{cst.}(\|X_0\| + \|Y\|^{\mathcal{G}}_{dL,2}) \leq \text{cst.}\|C - I\|^{\mathcal{G}}_{L,3}\end{aligned}$$

with $\delta = 1/2$ and a suitable $c > d$. Using (4.103) we obtain the estimate (4.96) in $\mathcal{G}^\rho_{cL}([-2, 2], M_n)$. □

When $A^\Phi_{k_1,k_2}((k_1, k_2) \in \mathbb{Z}^2)$ are all constants, we say that Φ is *constant*. The following simple lemma provides a normalization of constants.

LEMMA 4.2. — *Any constant* $\{(1, C), (\alpha, D)\}$ *can be conjugated to a normalized constant.*

Proof. — As C and D commute, they generate an abelian Lie subgroup $\mathcal{T}$ of $U(n)$, hence, one can choose X_0 in the Lie algebra of $\mathcal{T}$ satisfying $C = e^{X_0}$ and $X_0 D = D X_0$. Now $\{(1, e^{X_0}), (\alpha, D)\}$ can be conjugated to $\{(1, I), (\alpha, De^{-\alpha X_0})\}$ by $B(\theta) = e^{\theta X_0}$. □

A $\mathbb{Z}^2$-action $\Phi \in \Lambda$ is said to be *reducible* if it can be conjugated to a constant by some $B \in \mathcal{G}^\rho(\mathbb{R}, U(n))$. From Lemma 4.2 one obtains the following

LEMMA 4.3. — *Let* (α, A) *be a cocycle. Then the* $\mathbb{Z}^2$*-action* $\{(1, I), (\alpha, A)\}$ *is reducible if and only if* (α, A) *is reducible as a cocycle.*

4.2. Renormalization. — We recall from [1, 11] the following operations on Λ.

a) For any $\theta \in \mathbb{R}$ a translation T_θ is defined by

$$T_\theta(\Phi)(k_1, k_2) = (\gamma^\Phi_{k_1,k_2}, A^\Phi_{k_1,k_2}(\cdot + \theta)).$$

b) For any $\lambda \neq 0$ denote by M_λ the rescaling

$$M_\lambda(\Phi)(k_1, k_2) = (\lambda^{-1}\gamma^\Phi_{k_1,k_2}, A^\Phi_{k_1,k_2}(\lambda \cdot)).$$

c) For any $U \in GL(2, \mathbb{Z})$ we denote by N_U the base change

$$N_U(\Phi)(k_1, k_2) = \Phi((k_1, k_2)(U^T)^{-1}).$$

Reducibility is invariant under conjugation, translation, rescaling and base change.

Given an irrational $\alpha \in (0,1)$ we consider the continued fractional expansion

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

We set $\alpha_0 = \alpha$, and

$$\alpha_m = \cfrac{1}{a_{m+1} + \cfrac{1}{a_{m+2} + \cdots}}.$$

In fact, $\alpha_m = G^m(\alpha)$ where G is the Gauss map $x \mapsto \{1/x\}$ assigning to each $x \neq 0$ its fractional part $\{1/x\}$. The integers a_m are given by $a_m = [\alpha_{m-1}^{-1}]$, where $[\cdot]$ denotes the integer part. We also set $a_0 = 0$ for convenience.

Let $\beta_m = \prod_{j=0}^m \alpha_j$. Define

$$Q_0 = \begin{bmatrix} q_0 & p_0 \\ q_{-1} & p_{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$Q_m = \begin{bmatrix} q_m & p_m \\ q_{m-1} & p_{m-1} \end{bmatrix} = \begin{bmatrix} a_m & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_{m-1} & p_{m-1} \\ q_{m-2} & p_{m-2} \end{bmatrix}.$$

It is easy to see that $Q_m = U(\alpha_m) \cdots U(\alpha_1)$ where

$$U(x) = \begin{bmatrix} [x^{-1}] & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus $\det(Q_m) = q_m p_{m-1} - p_m q_{m-1} = (-1)^m$. Note that

$$\beta_m = (-1)^m (q_m \alpha - p_m) = \frac{1}{q_{m+1} + \alpha_{m+1} q_m},$$

$$\frac{1}{q_m + q_{m+1}} < \beta_m < \frac{1}{q_{m+1}}.$$

The renormalization operator R is defined by $R(\Phi) = M_{\alpha^\Phi} N_{U(\alpha)}(\Phi)$. Notice that if $\Phi \in \Lambda$ satisfies $\gamma_{1,0}^\Phi = 1$, then $\alpha^{R(\Phi)} = G(\alpha^\Phi)$ (recall that G is Gauss map) and

$$R^m(\Phi) = M_{\alpha_{m-1}} \circ N_{U(\alpha_{m-1})} \circ \cdots \circ M_{\alpha_0} \circ N_{U(\alpha_0)}(\Phi) = M_{\beta_{m-1}}(N_{Q_m}(\Phi)).$$

It is obvious that the reducibility is invariant under renormalization.

4.3. Proof of Theorem 1.2. — Let $\Phi = \{(1, I), (\alpha, A)\}$ be a normalized $\mathbb{Z}^2$-action such that $\alpha \in \mathrm{RDC}\,(\gamma, \tau)$ for some $\gamma, \tau > 0$. Suppose that there is a measurable function $B : \mathbb{T} \to U(n)$ satisfying

$$\mathrm{Ad}\,(B).(\alpha, A) = (\alpha, C) \tag{4.104}$$

where C is constant. Denote the spectrum of C by

$$\mathrm{spec}\,(C) = \{e^{2\pi i\phi_1}, \dots, e^{2\pi i\phi_n}\}.$$

By assumption $C \in \Sigma(\alpha)$ which means that $\phi := (\phi_1, \dots, \phi_n) \in \Upsilon(\alpha)$ (for the definition $\Upsilon(\alpha)$ we refer to (1.2)).

The following fact of B will be needed in the proof of Theorem 1.2.

LEMMA 4.4. — *For a.e. $\theta_0 \in \mathbb{T}$, we have*

$$\lim_{t\to 0+} \int_0^1 |B(\theta_0 + t\theta)B(\theta_0)^* - I| d\theta = 0.$$

Proof. — Denote by X the set of measurable continuity points of B and B^*. By the Lebesgue density theorem, X has full Lebesgue measure.

Fix $\theta_0 \in X$. For any $\epsilon > 0$, let

$$I_t(\epsilon) = \{\theta \in [\theta_0, \theta_0 + t] : |B(\theta)B(\theta_0)^* - I| < \epsilon\}$$

and $J_t(\epsilon) = [\theta_0, \theta_0 + t] - I_t(\epsilon)$. Note that

$$\lim_{t\to 0+} \frac{\mathrm{Leb.}(J_t(\epsilon))}{t} = 0,$$

where Leb. denotes Lebesgue measure. Now we have

$$\begin{aligned}
\int_{[0,1]} |B(\theta_0 + t\theta)B(\theta_0)^* - I| d\theta &= \frac{1}{t} \int_{[\theta_0, \theta_0 + t]} |B(\theta)B(\theta_0)^* - I| d\theta \\
&= \frac{1}{t} \left(\int_{I_t(\epsilon)} + \int_{J_t(\epsilon)} \right) |B(\theta)B(\theta_0)^* - I| d\theta \\
&\leq \epsilon + \frac{\mathrm{Leb.}(J_t(\epsilon))}{t} \to \epsilon
\end{aligned}$$

as $t \to 0+$, where Leb. stands for the Lebesgue measure. Since the inequality holds for any $\epsilon > 0$, the lemma is proved. □

Without loss of generality, we assume that the conclusion of Lemma 4.4 holds for $\theta_0 = 0$ (if not we make a translation). Consider

$$R^m(\Phi) = \{(1, A_{s_m}(\beta_{m-1}\cdot)), (\alpha_m, A_{\tilde{s}_m}(\beta_{m-1}\cdot))\}, \quad m = 1, 2, \dots$$

where

$$s_m := (-1)^{m-1} q_{m-1} \quad \text{and} \quad \tilde{s}_m := (-1)^m q_m.$$

We need the following lemma.

Lemma 4.5. — *There exists a constant $c > 0$ such that for any α irrational and that $A \in \mathcal{G}^{\rho}_{L}(\mathbb{T}, U(n))$ the sequences $A_{s_m}(\beta_{m-1}\cdot)$ and $A_{\tilde{s}_m}(\beta_{m-1}\cdot)$, $m \geq 1$, are uniformly bounded in $\mathcal{G}^{\rho}_{\widetilde{L}}(\mathbb{R}, U(n))$, where $\widetilde{L} = L/c$.*

Proof. — Recall from [1, 6] the following estimate

$$\forall\, m, r \in \{1, 2, \ldots\},\ \forall\, \theta \in \mathbb{T}: \quad \|\partial^r A_m(\theta)\| \leq m^r c^r \|\partial^r A\|_0,$$

where $\|\cdot\|_0$ is the C^0 norm and $c > 0$ is a constant. Applying it to both (α, A) and $(\alpha, A)^{-1}$ we obtain

$$||\partial^r A_k||_0 \leq |k|^r c^r ||\partial^r A||_0$$

for all $0 \neq k \in \mathbb{Z}$. It follows that for all $\theta \in \mathbb{R}$ and $m \geq 1$

$$||\partial^r A_{(-1)^{m-1}q_{m-1}}(\beta_{m-1}\theta)|| \leq |\beta_{m-1}q_{m-1}|^r c^r ||\partial^r A||_0 \leq c^r ||\partial^r A||_0;$$

$$||\partial^r A_{(-1)^m q_m}(\beta_{m-1}\theta)|| \leq |\beta_{m-1}q_m|^r c^r ||\partial^r A||_0 \leq c^r ||\partial^r A||_0.$$

By assumption $A \in \mathcal{G}^{\rho}_{L}(\mathbb{T}, U(n))$, thus there exists $M > 0$ such that

$$\sup_{r\in\mathbb{N}} \sup_{\theta\in\mathbb{R}} (\|\partial^r A(\theta)\| L^r r!^{-\rho}) \leq M.$$

This implies

$$\sup_{r\in\mathbb{N}} \sup_{\theta\in\mathbb{R}} (\|\partial^r A_{(-1)^{m-1}q_{m-1}}(\beta_{m-1}\theta)\| \left(\frac{L}{c}\right)^r r!^{-\rho}) \leq M,$$

$$\sup_{r\in\mathbb{N}} \sup_{\theta\in\mathbb{R}} (\|\partial^r A_{(-1)^m q_m}(\beta_{m-1}\theta)\| \left(\frac{L}{c}\right)^r r!^{-\rho}) \leq M$$

which completes the proof. □

Set $H(\cdot) := e^{h(\cdot)}$, where $h(\varphi_1, \ldots, \varphi_n) := 2\pi i \operatorname{diag}(\varphi_1, \ldots, \varphi_n)$. Choose $S \in U(n)$ so that

$$C = S^* H(\phi) S. \tag{4.105}$$

The following lemma gives us information about the limit points of the set of $\mathbb{Z}^2$-actions $\{R^m(\Phi) : m = 1, 2, \ldots\}$.

Lemma 4.6. — *There exist sequences $(m_j) \subset \mathbb{N}$ and (l_j), $(\widetilde{l}_j) \subset \mathbb{Z}^n$ and $\psi, \widetilde{\psi} \in [0, 1]^n$ such that $\alpha_{m_j} \in \mathrm{DC}(\gamma, \tau)$ and*

$$\alpha_\infty := \lim_{j\to\infty} \alpha_{m_j} \in \mathrm{DC}(\gamma, \tau);$$

$$\lim_{j\to\infty} (s_j \phi + l_j) = \psi;$$

$$\lim_{j\to\infty} (\widetilde{s}_j \phi + \widetilde{l}_j) = \widetilde{\psi},$$

where $s_j := (-1)^{m_j-1} q_{m_j-1}$ and $\widetilde{s}_j := (-1)^{m_j} q_{m_j}$, and the sequences of functions $A_{s_j}(\beta_{m_j-1}\cdot)$ and $A_{\widetilde{s}_j}(\beta_{m_j}\cdot)$ converge in $\mathcal{G}^{\rho}_{N}(\mathbb{R}, U(n))$ to

$B(0)^*S^*H(\psi)SB(0)$ *and* $B(0)^*S^*H(\widetilde{\psi})SB(0)$ *respectively, for some fixed* $N > \widetilde{L}$.

Proof. — By Lemma 4.5, there exist $\widetilde{L} > 0$, such that the sequences of functions $A_{s_m}(\beta_{m-1}\cdot)$ and $A_{\widetilde{s}_m}(\beta_m\cdot)$, $m \geq 1$, are uniformly bounded in $\mathscr{G}^\rho_{\widetilde{L}}(\mathbb{R}, U(n))$. Fix $N > \widetilde{L}$. Then for any $K > 0$ the inclusion

$$\mathscr{G}^\rho_{\widetilde{L}}([-K, K], U(n)) \hookrightarrow \mathscr{G}^\rho_N([-K, K], U(n))$$

is compact (see [13], Ch. 7) and by the diagonal procedure one can find a subsequence $\{m_j : j \in \mathbb{N}\}$ such that the sequences

$$U_j := A_{s_{m_j}}(\beta_{m_j-1}\cdot) \quad \text{and} \quad \widetilde{U}_j := A_{\widetilde{s}_{m_j}}(\beta_{m_j}\cdot),\ j \in \mathbb{N}, \tag{4.106}$$

converge in $\mathscr{G}^\rho_N(\mathbb{R}, U(n))$ to some U_∞ and $\widetilde{U}_\infty$, respectively, as $j \to \infty$. Recall that the topology of the space of Gevrey functions above is given in Sect. 4.1. Without loss of generality we assume that

$$H(s_j\phi) \to H(\psi), \quad H(\widetilde{s}_j\phi) \to H(\widetilde{\psi})$$

for some $\psi, \widetilde{\psi} \in [0, 1]^n$ and $\alpha_{m_j} \to \alpha_\infty$ for some $\alpha_\infty \in \mathrm{DC}\,(\gamma, \tau)$ (otherwise we choose subsequences).

Recall that 0 is a measurable continuity point of both B and B^*. Thus for any $\varepsilon > 0$ and $d > 1$ fixed

$$\lim_{j\to\infty} \frac{1}{2d\beta_{m_j}} \mathrm{Leb.}\big(I(0, \varepsilon) \cap [-\beta_{m_j}d, \beta_{m_j}d]\big) = 1,$$

where Leb. is the Lebesgue measure and

$$I(0, \varepsilon) := \{\vartheta \in \mathbb{R} : \|B(\vartheta) - B(0)\| < \varepsilon, \|B(\vartheta)^* - B(0)^*\| < \varepsilon\}.$$

By Lemma 4.5 the functions U_j defined in (4.106) are C^1-uniformly bounded on $\mathbb{R}$, hence, they are equicontinuous and there exists a positive $\widehat{\delta}$ such that

$$\|U_j(\theta) - U_j(\vartheta)\| < \varepsilon$$

for all j as long as $|\theta - \vartheta| < \widehat{\delta}$. Then for any $j \gg 1$ and $\theta \in [-d+1, d-1]$ there exists $\vartheta \in [-d+1, d-1] \cap (\theta - \widehat{\delta}, \theta + \widehat{\delta})$ such that

$$\beta_{m_j}\vartheta, \quad \beta_{m_j}(\vartheta + 1) \in I(0, \varepsilon)$$

and we obtain

$$\max\{\|U_j(\theta) - U_j(\vartheta)\|,\ \|B(\beta_{m_j}\vartheta) - B(0)\|,\ \|B(\beta_{m_j}(\vartheta + 1)) - B(0)\|\} < \varepsilon. \tag{4.107}$$

Using the conjugations (4.104) and (4.105) we get

$$B(\beta_{m_j}(\vartheta + 1))^{-1}U_j(\vartheta)B(\beta_{m_j}\vartheta) = S^*H(s_j\phi)S, \tag{4.108}$$

then by (4.107) and (4.108) we obtain that

$$\|B(0)^*\widetilde{U}_j(\theta)B(0) - S^*H(s_j\phi)S\| < 3\varepsilon.$$

Recall that U_j converge in $\mathcal{G}_N^\rho(\mathbb{R}, U(n))$ to U_∞ and $H(s_j\phi) \to H(\psi)$. Then for any $\varepsilon > 0$ and $d > 0$ fixed and $\theta \in [-d+1, d-1]$ we have

$$\|B(0)^*U_\infty(\theta)B(0) - S^*H(\psi)S\| < 3\varepsilon,$$

hence,

$$B(0)^*U_\infty B(0) \equiv S^*H(\psi)S.$$

In the same way we get

$$B(0)^*\widetilde{U}_\infty B(0) \equiv S^*H(\widetilde{\psi})S$$

which completes the proof. □

We have proved that $R^m(\Phi)$ has a subsequence

$$\begin{aligned}\Phi_j := R^{m_j}(\Phi) &= \{(1, A_{s_j}(\beta_{m_j-1}\cdot)), (\alpha_{m_j}, A_{\tilde{s}_j}(\beta_{m_j-1}\cdot))\} \\ &=: \{(1, U_j), (\alpha_{m_j}, \widetilde{U}_j)\}\end{aligned}$$

with $s_j = (-1)^{m_j-1}q_{m_j-1}$ and $\tilde{s}_j = (-1)^{m_j}q_{m_j}$, converging to the constant $\mathbb{Z}^2$-action

$$\Phi_\infty := \{(1, B(0)S^*H(\psi)SB(0)^*), (\alpha_\infty, B(0)S^*H(\widetilde{\psi})SB(0)^*)\}.$$

We want to show that after a conjugation this subsequence will become a sequence of normalized $\mathbb{Z}^2$-actions converging in $\mathcal{G}_{\widetilde{L}}^\rho(\mathbb{R}, U(n))$ to a constant normalized $\mathbb{Z}^2$-action. Set $Q(\theta) := S^*H(\theta\psi)S$.

Lemma 4.7. — *There is a sequence P_j in $\mathcal{G}_N^\rho(\mathbb{R}, U(n))$ converging to the identity mapping I in $\mathcal{G}_N^\rho([-2,2], U(n))$ and such that*

$$\Psi_j := \mathrm{Ad}\,(B(0)QP_j)\,.\Phi_j = \big\{(1, I), (\alpha_{m_j}, W_j)\big\} \tag{4.109}$$

for suitable $W_j \in \mathcal{G}_N^\rho(\mathbb{T}, U(n))$, where W_j converges to $S^*H(\widetilde{\psi} - \alpha_\infty\psi)S$ in $\mathcal{G}_N^\rho(\mathbb{T}, U(n))$.

Proof. — Set $\widetilde{\Phi}_j := \mathrm{Ad}\,(B(0)Q).\Phi_j = Ad(B(0)Q).R^{m_j}(\Phi)$. It follows from the definition of Φ_∞ and Q that

$$\mathrm{Ad}\,(B(0)Q).\Phi_\infty = \{(1, I), (\alpha_\infty, S^*H(\widetilde{\psi} - \alpha_\infty\psi)S)\}.$$

Thus $\widetilde{\Phi}_j(1,0)$ and $\widetilde{\Phi}_j(0,1)$ converge uniformly in $\mathcal{G}_N^\rho(\mathbb{R}, U(n))$ to $(1, I)$ and $(\alpha_\infty, S^*H(\psi - \alpha_\infty\psi)S)$ respectively. By Lemma 4.1, there exist a sequence of $P_j \in \mathcal{G}_N^\rho(\mathbb{R}, U(n))$, satisfying

$$\lim_{j\to\infty} \|P_j - I\|_{N,2}^{\mathcal{G}} = 0,$$

such that $\mathrm{Ad}\,(P_j).\widetilde{\Phi}_j$ is a sequence of normalized $\mathbb{Z}^2$-actions $\{(1,I),(\alpha_{m_j},W_j)\}$. Thus $W_j \in \mathscr{G}^\rho_N(\mathbb{T},U(n))$ (W_j is automatically $\mathbb{Z}$-periodic thanks to the commutation) and it converges to $S^*H(\widetilde{\psi}-\alpha_\infty\psi)S$, i.e.,

$$\lim_{j\to\infty}\|W_j - S^*H(\widetilde{\psi}-\alpha_\infty\psi)S\|^{\mathscr{G}}_N = 0. \qquad \square$$

On the other hand, (α_{m_j},W_j) can be conjugated to the constant

$$(\alpha_{m_j}, S^*H((-1)^{m_j}\beta^{-1}_{m_j-1}\phi)S)$$

by a measurable conjugation. More precisely, let us consider the following sequences

$$\begin{aligned} E_j(\theta) &= S^*H((-1)^{m_j-1}q_{m_j-1}\phi\theta)S,\\ T_j(\theta) &= P_j(\theta)^{-1}Q(\theta)^{-1}B(0)^*, \end{aligned}$$

and

$$G_j(\theta) = T_j(\theta)B(\beta_{m_j-1}\theta)E_j(\theta),$$

where $Q = S^*H(\psi\theta)S$ is introduced in Lemma 4.7.

Lemma 4.8. — *The measurable functions G_j are all $\mathbb{Z}$-periodic and satisfy*

$$\liminf_{j\to+\infty}[G_j] > \frac{1}{2}n^{-3/2},$$

$$\mathrm{Ad}\,(G_j).(\alpha_{m_j},W_j) = (\alpha_{m_j}, S^*H((-1)^{m_j-1}\beta^{-1}_{m_j-1}\phi)S).$$

Proof. — One can easily check that

$$\begin{aligned} \mathrm{Ad}\,(G_j).\Psi_j &= \mathrm{Ad}\,(B(\beta_{m_j-1}\theta)E_j(\theta)).R^{m_j}(\Phi)\\ &= \{(1,I),(\alpha_{m_j}, S^*H((-1)^{m_j}\beta^{-1}_{m_j-1}\phi)S)\}. \end{aligned}$$

In particular, any G_j is $\mathbb{Z}$-periodic.

Let

$$\begin{aligned} \widetilde{G}_j(\theta) &= G_j(\theta)S^*H(l_j\theta)S\\ &= P_j(\theta)^{-1}S^*H(-\psi\theta)SB(0)^*B(\beta_{m_j-1}\theta)S^*H((s_j\phi+l_j)\theta)S \end{aligned}$$

(l_j is given in Lemma 4.6). Then for any $\theta\in\mathbb{T}$ we have the estimates

$$\begin{aligned} ||\widetilde{G}_j(\theta)-I|| \le{}& ||P_j(\theta)^{-1}-I||\\ &+||B(0)^*B(\beta_{m_j-1}\theta)-I||\\ &+||H((-\psi+s_j\phi+l_j)\theta)-I|| \end{aligned}$$

where $s_j = (-1)^{m_j} q_{m_j-1}$. By Lemma 4.6 and 4.7, we have $s_j\phi + l_j \to \psi$ and $P_j(\theta) \to I$ uniformly on $[-2,2]$. Moreover, by Lemma 4.4

$$\lim_{j\to\infty} \int_0^1 ||B(0)^* B(\beta_{m_j}\theta) - I|| d\theta = 0.$$

Thus we have

$$\begin{aligned}\lim_{j\to\infty} \int_0^1 ||G_j(\theta)^* - S^* H(-l_j\theta) S|| d\theta &= \lim_{j\to\infty} \int_0^1 ||\widetilde{G}_j(\theta)^* - I|| d\theta \\ &= \lim_{j\to\infty} \int_0^1 ||\widetilde{G}_j(\theta) - I|| d\theta = 0.\end{aligned}$$

By Lemma 2.4, $\lceil H(-l_j\theta) \rfloor = \lceil S^* H(-l_j\theta) S \rfloor \geq n^{-3/2}$. Then by Lemma 2.8, $\lceil G_j^* \rfloor > \frac{1}{2} n^{-3/2}$ as j is large enough. □

We are ready to complete the proof of Theorem 1.2. We would like to apply the local result Theorem 1.1. To this end, we need the following fact.

Lemma 4.9. — *Let $\phi = (\phi_1, \ldots, \phi_n) \in \Upsilon(\alpha)$, where $\Upsilon(\alpha)$ is defined in* (1.2). *Then for all $m \in \mathbb{N}$ one has $(-1)^m \beta_{m-1}^{-1} \phi \in \Upsilon(\alpha_m)$.*

Proof. — Without loss of generality, we assume

$$|\phi_t - \phi_{\widetilde{t}}| \leq 2, \quad t \neq \widetilde{t} \in \{1, \ldots, n\}.$$

We consider two cases. If $|l| > 2|k| + 3$ and $t \neq \widetilde{t} \in \{1, \ldots, n\}$ we have

$$|k\alpha - (-1)^m \beta_{m-1}^{-1} (\phi_t - \phi_{\widetilde{t}}) - l| \geq |l| - |k| - 2 \geq 2|k| + 3 - |k| - 2 = |k| + 1.$$

Let $|l| \leq 2|k| + 3$ and $t \neq \widetilde{t} \in \{1, \ldots, n\}$. There exist $\sigma, \nu > 0$, such that for any $k, l \in \mathbb{Z}$ and $t \neq \widetilde{t} \in \{1, \ldots, n\}$

$$|k\alpha - (\phi_t - \phi_{\widetilde{t}}) - l| \geq \frac{\sigma}{(1 + |k|)^\nu}$$

and then

$$\begin{aligned}&|k\alpha_m - (-1)^m \beta_{m-1}^{-1} (\phi_t - \phi_{\widetilde{t}}) - l| \\ &\quad = \beta_{m-1}^{-1} |(-1)^m k\beta_m - (\phi_t - \phi_{\widetilde{t}}) - (-1)^m l\beta_{m-1}| \\ &\quad = \beta_{m-1}^{-1} |(q_m\alpha - p_m)k - (\phi_t - \phi_{\widetilde{t}}) + (q_{m-1}\alpha - p_{m-1})l| \\ &\quad = \beta_{m-1}^{-1} |(q_m k + q_{m-1} l)\alpha - (\phi_t - \phi_{\widetilde{t}}) - (p_m + p_{m-1})| \\ &\quad \geq \frac{\beta_{m-1}^{-1}\sigma}{(1 + q_m|k| + q_{m-1}|l|)^\nu}.\end{aligned}$$

Since $|l| \leq 2|k|+3$ and $t \neq \widetilde{t} \in \{1,\dots,n\}$ we have

$$\begin{aligned}|k\alpha_m - (-1)^m \beta_{m-1}^{-1}(\phi_t - \phi_{\widetilde{t}}) - l| \\ &\geq \frac{\beta_{m-1}^{-1}\sigma}{(1+q_m|k|+q_{m-1}|l|)^\nu} \\ &\geq \frac{\beta_{m-1}^{-1}\sigma}{(1+q_m|k|+2q_m|k|+3q_m)^\nu} \\ &\geq \frac{\beta_{m-1}^{-1}\sigma}{(4q_m)^\nu(|k|+1)^\nu}.\end{aligned}$$

So there exists $\sigma_m > 0$, such that for any $k, l \in \mathbb{Z}$ and $t \neq \widetilde{t} \in \{1,\dots,n\}$

$$|k\alpha - (-1)^m \beta_{m-1}^{-1}(\phi_t - \phi_{\widetilde{t}}) - l| \geq \frac{\sigma_m}{(1+|k|)^\nu}. \qquad \square$$

Now we go back to the proof of Theorem 1.2. Lemma 4.8 , 4.9 allows us to apply Theorem 1.1. For j is sufficiently large, there exists B_j in $\mathcal{G}^\rho(\mathbb{T}, U(n))$ satisfying

$$B_j(\theta) = G_j(\theta)$$

for a.e. $\theta \in \mathbb{T}$. Since E_j and T_j are analytic,

$$\widetilde{B}_j(\theta) := T_j(\beta_{m_j-1}^{-1}\theta)^{-1} B_j(\beta_{m_j-1}^{-1}\theta) E_j(\beta_{m_j-1}^{-1}\theta)^{-1}$$

is in $\mathcal{G}^\rho(\mathbb{R}, U(n))$. Moreover,

$$\widetilde{B}_j(\theta) = T_j(\beta_{m_j-1}^{-1}\theta)^{-1} G_j(\beta_{m_j-1}^{-1}\theta) E_j(\beta_{m_j-1}^{-1}\theta)^{-1} = B(\theta)$$

for a.e. $\theta \in \mathbb{R}$. Recall that B is $\mathbb{Z}$-periodic for a.e. $\theta \in \mathbb{R}$, so $\widetilde{B}_j$ is also $\mathbb{Z}$-periodic for a.e. $\theta \in \mathbb{R}$ and it is then $\mathbb{Z}$-periodic for all $\theta \in \mathbb{R}$ (thanks to the continuity of $\widetilde{B}_j$).

By assumption, for a.e. $\theta \in \mathbb{T}$,

$$B(\theta+\alpha)^{-1} A(\theta) B(\theta) = C,$$

hence, for a.e. $\theta \in \mathbb{T}$,

$$\widetilde{B}_j(\theta+\alpha)^{-1} A(\theta) \widetilde{B}_j(\theta) = C,$$

which implies that for all $\theta \in \mathbb{T}$

$$\widetilde{B}_j(\theta+\alpha)^{-1} A(\theta) \widetilde{B}_j(\theta) = C$$

(thanks to the continuity of $\widetilde{B}_j$). This completes the proof of Theorem 1.2. $\square$

BIBLIOGRAPHY

[1] A. Avila & R. Krikorian – "Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles", *Ann. of Math.* **164** (2006), p. 911–940.

[2] C. Chavaudret – "Strong almost reducibility for analytic and Gevrey quasi-periodic cocycles", *Bull. Soc. Math. France* **141** (2013), p. 47–106.

[3] L. H. Eliasson – "Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation", *Comm. Math. Phys.* **146** (1992), p. 447–482.

[4] ———, "Almost reducibility of linear quasi-periodic systems", in *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, Proc. Sympos. Pure Math., vol. 69, Amer. Math. Soc., Providence, RI, 2001, p. 679–705.

[5] H.-L. Her & J. You – "Full measure reducibility for generic one-parameter family of quasi-periodic linear systems", *J. Dynam. Differential Equations* **20** (2008), p. 831–866.

[6] X. Hou & J. You – "The rigidity of reducibility of cocycles on $\mathrm{SO}(N,\mathbb{R})$", *Nonlinearity* **21** (2008), p. 2317–2330.

[7] ———, "Local rigidity of reducibility of analytic quasi-periodic cocycles on $\mathrm{U}(n)$", *Discrete Contin. Dyn. Syst.* **24** (2009), p. 441–454.

[8] ———, "Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems", *Invent. Math.* **190** (2012), p. 209–260.

[9] R. Krikorian – "Réductibilité des systèmes produits-croisés à valeurs das des groupes compacts", *Astérisque* **259** (1999).

[10] ———, "Réductibilité presque partout des flots fibrés quasi-périodiques à valeurs dans des groupes compacts", *Ann. Sci. École Norm. Sup.* **32** (1999), p. 187–240.

[11] ———, "Global density of reducible quasi-periodic cocycles on $\mathbf{T}^1 \times \mathrm{SU}(2)$", *Ann. of Math.* **154** (2001), p. 269–326.

[12] ———, "Reducibility, differentiable rigidity and Lyapunov exponents for quasi-periodic cocycles on $\mathbb{T} \times SL(2,\mathbb{R})$", preprint arXiv:math/0402333.

[13] J.-L. Lions & E. Magenes – *Problèmes aux limites non homogènes et applications. Vol. 3*, Travaux et Recherches Mathématiques, vol. 20, Dunod, 1970.

[14] J.-P. Marco & D. Sauzin – "Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems", *Publ. Math. IHÉS* **96** (2002), p. 199–275 (2003).

[15] T. Mitev & G. Popov – "Gevrey normal form and effective stability of Lagrangian tori", *Discrete Contin. Dyn. Syst. Ser. S* **3** (2010), p. 643–666.

[16] G. Popov – "Invariant tori, effective stability, and quasimodes with exponentially small error terms. I. Birkhoff normal forms", *Ann. Henri Poincaré* **1** (2000), p. 223–248.

[17] ———, "KAM theorem for Gevrey Hamiltonians", *Ergodic Theory Dynam. Systems* **24** (2004), p. 1753–1786.