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Luis Paris & Błażej Szepietowski

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Institut Henri Poincaré, 11, rue Pierre et Marie Curie
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A PRESENTATION FOR THE MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

BY LUIS PARIS & BŁAŻEJ SZEPIETOWSKI

ABSTRACT. — Let $N_{g,n}$ denote the nonorientable surface of genus g with n boundary components and $\mathscr{M}(N_{g,n})$ its mapping class group. We obtain an explicit finite presentation of $\mathscr{M}(N_{g,n})$ for $n \in \{0, 1\}$ and all g such that $g + n > 3$.

RÉSUMÉ (*Une présentation du mapping class groupe d'une surface non orientable*)
Notons $N_{g,n}$ la surface non orientable de genre g avec n composantes de bord et $\mathscr{M}(N_{g,n})$ son mapping class groupe. On obtient une présentation finie explicite de $\mathscr{M}(N_{g,n})$ pour $n \in \{0, 1\}$ et pour tout g tel que $g + n > 3$.

1. Introduction

Let F be a compact connected surface with (possibly empty) boundary and let $\mathscr{P} = \mathscr{P}_m = \{P_1, \dots, P_m\}$ be a set of m distinguished points in the interior of F, called *punctures*. We define $\mathscr{H}(F, \mathscr{P})$ to be the group of all, orientation preserving if F is orientable, homeomorphisms $h\colon F \to F$ such that $h(\mathscr{P}) = \mathscr{P}$ and h is equal to the identity on the boundary of F. The *mapping class group* $\mathscr{M}(F, \mathscr{P})$ of F relatively to $\mathscr{P}$ is the group of isotopy classes of elements of $\mathscr{H}(F, \mathscr{P})$. The *pure mapping class group* $\mathscr{PM}(F, \mathscr{P})$ is the subgroup of $\mathscr{M}(F, \mathscr{P})$

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LUIS PARIS, Université de Bourgogne, Institut de Mathématiques de Bourgogne, UMR 5584 du CNRS, B.P. 47870, 21078 Dijon cedex, France. • *E-mail :* lparis@u-bourgogne.fr
BŁAŻEJ SZEPIETOWSKI, Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland • *E-mail :* blaszep@mat.ug.edu.pl

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consisting of the isotopy classes of homeomorphisms fixing each puncture. If $\mathcal{P} = \varnothing$ then we drop it in the notation and write simply $\mathcal{M}(F)$. If $\mathcal{P} = \{P\}$ then we write $\mathcal{M}(F, P)$ instead of $\mathcal{M}(F, \{P\})$. A compact surface of genus g with n boundary components will be denoted by $S_{g,n}$ if it is orientable, or by $N_{g,n}$ if it is nonorientable.

Historically, McCool [20] gave the first algorithm for finding a finite presentation for $\mathcal{M}(S_{g,1})$ for any g. His approach is purely algebraic and no explicit presentation has been derived from this algorithm. In their ground breaking paper [13] Hatcher and Thurston gave an algorithm for computing a finite presentation for $\mathcal{M}(S_{g,1})$ from its action on a simply connected simplicial complex, the *cut system complex*. By this algorithm, Harer [10] obtained a finite, but very unwieldy, presentation for $\mathcal{M}(S_{g,1})$ for any g. This presentation was simplified by Wajnryb [29, 30], who also gave a presentation for $\mathcal{M}(S_{g,0})$. Using Wajnryb's result, Matsumoto [19] found other presentations for $\mathcal{M}(S_{g,1})$ and $\mathcal{M}(S_{g,0})$, and Gervais [9] found a presentation for $\mathcal{M}(S_{g,n})$ for arbitrary $g \geq 1$ and n. Starting from Matsumoto's presentations, Labruère and Paris [16] computed a presentation for $\mathcal{M}(S_{g,n}, \mathcal{P}_m)$ for arbitrary $g \geq 1$, n and m. Benvenuti [1] and Hirose [14] independently recovered the Gervais presentation from the action of $\mathcal{M}(S_{g,n})$ on two different variations of the Harvey's curve complex [11], instead of the cut system complex.

Until present, finite presentations of $\mathcal{M}(N_{g,n}, \mathcal{P}_m)$ were known only for a few small values of (g, n, m), with $g \leq 4$. Using results of Lickorish [17, 18], Chillingworth [7] found a finite generating set for $\mathcal{M}(N_{g,0})$ for arbitrary g. This set was extended for $m > 0$ by Korkmaz [15], and for $n + m > 0$ and $g \geq 3$ by Stukow [23]. For every nonorientable surface $N_{g,n}$ there is a covering $p\colon S_{g-1,2n} \to N_{g,n}$ of degree two. By a result of Birman and Chillingworth [4], generalized for $n > 0$ in [26], $\mathcal{M}(N_{g,n})$ is isomorphic to the subgroup of $\mathcal{M}(S_{g-1,2n})$ consisting of elements commuting with the covering involution. However, since the image of $\mathcal{M}(N_{g,n})$ has infinite index in $\mathcal{M}(S_{g-1,2n})$, it seems that it would be very hard to obtain a finite presentation for $\mathcal{M}(N_{g,n})$ from a presentation of $\mathcal{M}(S_{g-1,2n})$. In [24] an algorithm for finding a finite presentation for $\mathcal{M}(N_{g,n})$ for any g and n is given, based on a result of Brown [6] and the action of $\mathcal{M}(N_{g,n})$ on the curve complex (following the idea of [1]). By this algorithm, an explicit finite presentation for $\mathcal{M}(N_{4,0})$ was obtained in [25].

In this paper we apply the algorithm given in [24] to find an explicit finite presentation for $\mathcal{M}(N_{g,n})$ for $n \in \{0, 1\}$ and all g such that $g + n > 3$. We present $\mathcal{M}(N_{g,1})$ as a quotient of the free product $\mathcal{M}(S_{\rho,r}) * \mathcal{M}(S_{0,1}, \mathcal{P}_g)$, where $g = 2\rho + r$ and $r \in \{1, 2\}$. The factor $\mathcal{M}(S_{\rho,r})$ comes from an embedding of $S_{\rho,r}$ in $N_{g,1}$ and it is generated by Dehn twists. The factor $\mathcal{M}(S_{0,1}, \mathcal{P}_g)$, which is isomorphic to the braid group, comes from the embedding $\mathcal{M}(S_{0,1}, \mathcal{P}_g) \to \mathcal{M}(N_{g,1})$ defined in [26], and it is generated by $g - 1$ crosscap transpositions. There are three families of defining relations of $\mathcal{M}(N_{g,1})$: (A) relations from $\mathcal{M}(S_{\rho,r})$ between Dehn twists, (B) braid relations between crosscap transpositions, and (C) relations involving generators of both types. A presentation for $\mathcal{M}(N_{g,0})$ is obtained from that of $\mathcal{M}(N_{g,1})$ by adding three relations.

The presentations for $\mathcal{M}(N_{g,1})$ and $\mathcal{M}(N_{g,0})$ are given respectively in Theorems 3.5 and 3.6 in Section 3. They are proved simultaneously by induction on g. The base cases $(g,n) \in \{(3,1),(4,0)\}$ are proved in Section 4. Theorem 3.5 is proved in Section 7 under the assumption that Theorem 3.6 is true. The proof of Theorem 3.6 uses the action of $\mathcal{M}(N_{g,0})$ on the ordered complex of curves defined in Section 5, and it occupies Sections 6, 8, where presentations of stabilizers of vertices are calculated, and Sections 9, 10, where we deal with relations corresponding to simplices of dimensions 1 and 2.

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2. Preliminaries

2.1. Simple closed curves and Dehn twists. — By a *simple closed curve* in F we mean an embedding $\gamma\colon S^1 \to F\backslash\partial F$. Note that γ has an orientation; the curve with the opposite orientation but same image will be denoted by γ^{-1}. By abuse of notation, we will often identify a simple closed curve with its oriented image and also with its isotopy class. We say that γ is *generic* if it does not bound a disc nor a Möbius band and is not isotopic to a boundary component. According to whether a regular neighborhood of γ is an annulus or a Möbius strip, we call γ respectively *two-* or *one-sided.* We say that γ is *nonseparating* if $F\backslash\gamma$ is connected and *separating* otherwise.

Given a two-sided simple closed curve γ, T_γ denotes a Dehn twist about γ. On a nonorientable surface it is impossible to distinguish between right and left twists, so the direction of a twist T_γ has to be specified for each curve γ. In this paper it is usually indicated by arrows in a figure. Equivalently we may choose an orientation of a regular neighborhood of γ. Then T_γ denotes the right Dehn twist with respect to the chosen orientation. Recall that T_γ does not depend on the orientation of γ.

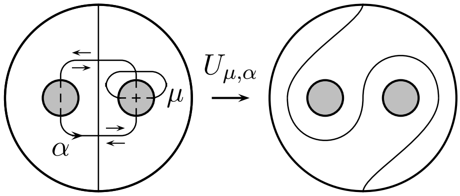

FIGURE 1. Crosscap transposition.

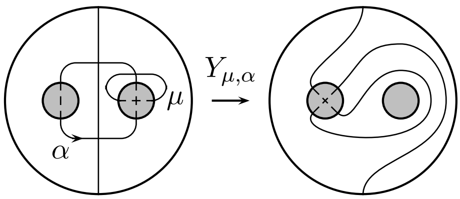

FIGURE 2. Crosscap slide.

2.2. Crosscap slides and transpositions. — We begin this subsection by describing a convention used in all figures in this paper. We explain this on the example of Figure 1. The shaded discs represent crosscaps; this means that their interiors should be removed, and then antipodal points in each resulting boundary component should be identified. The small arrows on two sides of the curve α indicate the direction of the Dehn twist T_α.

Let $N = N_{g,n}$ be a nonorientable surface of genus $g \geq 2$. Suppose that μ and α are two simple closed curves in N, such that μ is one-sided, α is two-sided and they intersect in one point. Let $K \subset N$ be a regular neighborhood of $\mu \cup \alpha$, which is homeomorphic to the Klein bottle with a hole. On Figure 1 a homeomorphism of K is shown, which interchanges the two crosscaps keeping the boundary of K fixed. It may be extended by the identity outside K to a homeomorphism of N, which we call *crosscap transposition* and denote as $U_{\mu,\alpha}$. We define *crosscap slide* $Y_{\mu,\alpha}$ to be the composition

$$Y_{\mu,\alpha} = T_\alpha U_{\mu,\alpha},$$

where T_α is the Dehn twist about α in the direction indicated by the arrows in Figure 1. If $M \subset K$ is a regular neighborhood of μ, which is a Möbius strip, then $Y_{\mu,\alpha}$ may be described as the effect of pushing M once along α (Figure 2). Observe that $Y_{\mu,\alpha}$ reverses the orientation of μ. Up to isotopy, $Y_{\mu,\alpha}$ does not depend on the choice of the regular neighborhood K. It also does not depend on the orientation of μ but does depend on the orientation of α, as $Y_{\mu,\alpha^{-1}} = Y_{\mu,\alpha}^{-1}$.

For any $h \in \mathcal{M}(N)$ we have the formula

$$hY_{\mu,\alpha}h^{-1} = Y_{h(\mu),h(\alpha)}.$$

The crosscap slide was introduced under the name Y-homeomorphism by Lickorish, who proved that $\mathcal{M}(N_{g,0})$ is generated by Dehn twists and one crosscap slide for $g \geq 2$ [17, 18].

2.3. Exact sequences. — Given an exact sequence of groups

$$1 \to K \to G \xrightarrow{\rho} H \to 1$$

and presentations $K = \langle S_K \mid R_K \rangle$ and $H = \langle S_H \mid R_H \rangle$, a presentation for G may be obtained as follows. We identify K with its image in G. For each $x \in S_H$ we choose $\widetilde{x} \in G$ such that $\rho(\widetilde{x}) = x$ and let

$$\widetilde{S_H} = \{\widetilde{x} \mid x \in S_H\}.$$

For each $r = x_1^{\epsilon_1} \cdots x_k^{\epsilon_k} \in R_H$ let $\widetilde{r} = \widetilde{x_1}^{\epsilon_1} \cdots \widetilde{x_k}^{\epsilon_k}$. Since $\rho(\widetilde{r}) = 1$, there is a word w_r over S_K representing the same element of G as $\widetilde{r}$. Let

$$R_1 = \{\widetilde{r}w_r^{-1} \mid r \in R_H\}.$$

Since K is a normal subgroup of G, for $x \in S_H$ and $y \in S_K$ we have $\widetilde{x}y\widetilde{x}^{-1} \in K$ and there is a word $w(x,y)$ over S_K representing the same element of G as $\widetilde{x}y\widetilde{x}^{-1}$. Let

$$R_2 = \{\widetilde{x}y\widetilde{x}^{-1}w(x,y)^{-1} \mid x \in S_H, y \in S_K\}.$$

Proof of the following lemma is left to the reader.

Lemma 2.1. — *The group G admits a presentation*

$$G = \left\langle S_K \cup \widetilde{S_H} \mid R_K \cup R_1 \cup R_2 \right\rangle.$$

We remark that in the above presentation we do not need the relators $\widetilde{x}^{-1}y\widetilde{x}w'(x,y)^{-1}$ for $x \in S_H$, $y \in S_K$, where $w'(x,y)$ is a word over S_K representing the same element of G as $\widetilde{x}^{-1}y\widetilde{x}$. Indeed, such a relator is conjugate to $y\widetilde{x}w'(x,y)^{-1}\widetilde{x}^{-1}$ and the last word belongs to the normal closure of $R_K \cup R_2$ in the free group on $S_K \cup \widetilde{S_H}$.

The generators S_K and $\widetilde{S_H}$ will be called *kernel* and *cokernel* generators respectively. The relators R_K, R_1 and R_2 will be called *kernel, cokernel* and *conjugation* relators respectively. In this paper we work with relations rather then relators.

The inclusion $\mathcal{P}_{m-1} \subset \mathcal{P}_m$ gives rise to a *forgetful homomorphism* $\mathfrak{f}\colon \mathcal{PM}(F,\mathcal{P}_m) \to \mathcal{PM}(F,\mathcal{P}_{m-1})$. By [2], if the Euler characteristic of $F \backslash \mathcal{P}_{m-1}$ is negative, then we have the following *Birman exact sequence*

(2.1) $$1 \to \pi_1(F\backslash\mathcal{P}_{m-1}, P_m) \xrightarrow{\mathfrak{p}} \mathcal{PM}(F,\mathcal{P}_m) \xrightarrow{\mathfrak{f}} \mathcal{PM}(F,\mathcal{P}_{m-1}) \to 1,$$

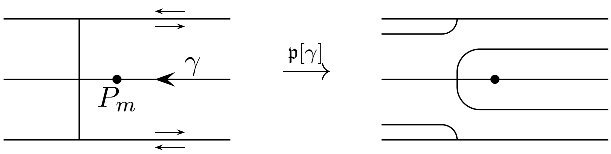

FIGURE 3. Pushing a puncture along a simple loop.

where $\mathfrak{p}$ is the *point pushing map.* Although the above result is proved in [2] for orientable F, the same proof works for nonorientable F as well. In order for $\mathfrak{p}$ to be a homomorphism, the product $\gamma\delta$ of two loops based at P_m means go along δ first and then along γ. Suppose that γ is a simple loop on $F\backslash\mathcal{P}_{m-1}$ based at P_m and let A be its regular neighborhood. If $[\gamma]$ denotes the homotopy class of γ, then $\mathfrak{p}[\gamma]$ is isotopic to a homeomorphism equal to the identity outside A, and obtained by pushing P once along γ keeping the boundary of A fixed, see Figure 3. Note that A is a Möbius band if γ is one-sided, or an annulus if γ is two-sided. In the latter case $\mathfrak{p}[\gamma]$ may be expressed in terms of Dehn twists about the boundary components of A.

LEMMA 2.2. — *Suppose that γ is a two-sided simple loop based at P_m and δ_1, δ_2 are the boundary components of a regular neighborhood of γ. Then $\mathfrak{p}[\gamma] = T_{\delta_1}T_{\delta_2}$, where the directions of the twists are determined by the orientation of γ as indicated by arrows on the left hand side of Figure 3.* □

The group $\mathcal{PM}(F,\mathcal{P}_m)$ acts on $\pi_1(F\backslash\mathcal{P}_{m-1},P_m)$ in the obvious way. The next lemma says that $\mathfrak{p}$ is $\mathcal{PM}(F,\mathcal{P}_m)$-equivariant.

LEMMA 2.3. — *For $h \in \mathcal{PM}(F,\mathcal{P}_m)$ and $[\gamma] \in \pi_1(F\backslash\mathcal{P}_{m-1},P_m)$ we have $\mathfrak{p}(h[\gamma]) = h\mathfrak{p}[\gamma]h^{-1}$.* □

Suppose that N is a nonorientable surface. We define $\mathcal{PM}^+(N,\mathcal{P}_m)$ to be the subgroup of $\mathcal{PM}(N,\mathcal{P}_m)$ consisting of the isotopy classes of homeomorphisms preserving local orientation at each puncture. Observe that it is a normal subgroup of index 2^m. For $1 \le m \le n$ choose m boundary components $\gamma_1,\dots,\gamma_m$ of $N_{g,n}$. Consider the surface $N_{g,n-m}$ as being obtained from $N_{g,n}$ by gluing a disc with a puncture P_i in its interior along γ_i for $i = 1,\dots,m$. Let $\mathcal{P}_m = \{P_1,\dots,P_m\}$. Since every homeomorphism in $\mathcal{H}(N_{g,n})$ may be extended by the identity on the discs to an element of $\mathcal{H}(N_{g,n-m},\mathcal{P}_m)$, the inclusion $\imath\colon N_{g,n} \to N_{g,n-m}$ induces a homomorphism

$$\imath_*\colon \mathcal{M}(N_{g,n}) \to \mathcal{PM}^+(N_{g,n-m},\mathcal{P}_m).$$

It is clearly surjective, and if $(g,n)\neq(1,1)$ then its kernel is the free abelian group of rank m generated by the Dehn twists T_{γ_i} for $i=1,\dots,m$ (see [22, Theorem 3.6]). Summarizing, we have the following exact sequence.

$$1\to\mathbb{Z}^m\to\mathcal{M}(N_{g,n})\xrightarrow{\imath_*}\mathcal{PM}^+(N_{g,n-m},\mathcal{P}_m)\to 1. \tag{2.2}$$

2.4. Blowup homomorphism and crosscap pushing map. — In this subsection we recall from [27] the definitions of blowup homomorphism and crosscap pushing map which will be important tools in what follows.

Let F be a surface with $m\geq 1$ punctures $\mathcal{P}_m=\{P_1,\dots,P_m\}$. Let $U=\{z\in\mathbb{C}\,|\,|z|\leq 1\}$ and for $i=1,\dots,m$ fix an embedding $e_i\colon U\to F\backslash\partial F$ such that $e_i(0)=P_i$. Let $\widetilde{F}$ be the nonorientable surface obtained by removing from F the interiors of $e_i(U)$ and then identifying $e_i(z)$ with $e_i(-z)$ for $z\in S^1=\partial U$ and $i=1,\dots,m$. Thus $\widetilde{F}=N_{g+m,n}$ if $F=N_{g,n}$ or $\widetilde{F}=N_{2g+m,n}$ if $F=S_{g,n}$.

We define a *blowup homomorphism*

$$\mathfrak{b}\colon\mathcal{M}(F,\mathcal{P}_m)\to\mathcal{M}(\widetilde{F})$$

as follows. Represent $h\in\mathcal{M}(F,\mathcal{P}_m)$ by a homeomorphism $h\colon F\to F$ such that for some permutation $\sigma\in\mathrm{Sym}_m$ we have $h(e_i(z))=e_{\sigma(i)}(z)$ or $h(e_i(z))=e_{\sigma(i)}(\overline{z})$ for $z\in U$ and $i=1,\dots,m$. Such h commutes with the identification leading to $\widetilde{F}$ and thus induces an element $\mathfrak{b}(h)\in\mathcal{M}(\widetilde{F})$. We refer the reader to [27] for a proof that $\mathfrak{b}$ is well defined (the proof in [27] is only for $m=1$ but it can be easily modified to work for $m>1$). The next proposition is proved in [26] for $F=S_{0,1}$ but the same proof works for any F.

Proposition 2.4. — *The blowup homomorphism $\mathfrak{b}\colon\mathcal{M}(F,\mathcal{P}_m)\to\mathcal{M}(\widetilde{F})$ is injective for any surface F.* □

We define the *crosscap pushing map*

$$\mathfrak{c}\colon\pi_1(F\backslash\mathcal{P}_{m-1},P_m)\to\mathcal{M}(\widetilde{F})$$

as the composition $\mathfrak{c}=\mathfrak{b}\circ\mathfrak{p}$, where $\mathfrak{p}$ is the point pushing map from the Birman exact sequence (2.1). If γ is a simple loop on $F\backslash\mathcal{P}_{m-1}$ based at P_m, then it follows immediately from the description of $\mathfrak{p}[\gamma]$, that $\mathfrak{c}[\gamma]$ is either a crosscap slide if γ is one-sided, or a product of two Dehn twists about the boundary components of a Möbius band with a hole if γ is two-sided (just replace the puncture with a crosscap on Figure 3).

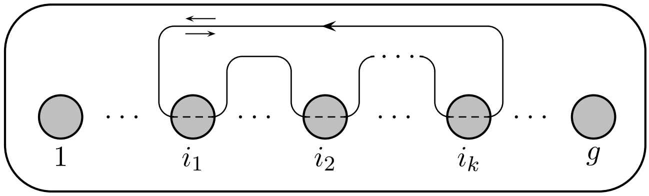

FIGURE 4. The curve γ_I for $I = \{i_1, i_2, \dots, i_k\}$.

2.5. Notation. — Let us represent $N_{g,0}$ and $N_{g,1}$ as respectively a sphere or a disc with g crosscaps. This means that interiors of g small pairwise disjoint discs should be removed from the sphere/disc, and then antipodal points in each of the resulting boundary components should be identified. Let us arrange the crosscaps as shown on Figure 4 and number them from 1 to g. For each nonempty subset $I \subseteq \{1, \dots, g\}$ let γ_I be the simple closed curve shown on Figure 4. Note that γ_I is two-sided if and only if I has even number of elements. In such case T_{γ_I} will be the Dehn twist about γ_I in the direction indicated by arrows on Figure 4. The following curves will play a special role and so we give them different names.

- $\mu_i = \gamma_{\{i\}}$ for $i = 1, \dots, g$,
- $\alpha_i = \gamma_{\{i,i+1\}}$ for $i = 1, \dots, g-1$,
- $\beta = \gamma_{\{1,2,3,4\}}$,
- $\beta_j = \gamma_{\{1,\dots,2j+2\}}$ for $2 \le 2j \le g-2$,
- $\xi = \gamma_{\{1,\dots,g\}}$.

Note that $\beta = \beta_1$ and if $g = 2\rho+2$ then $\xi = \beta_\rho$. We also give names to elements of $\mathcal{M}(N_{g,n})$ associated with these curves.

- $a_i = T_{\alpha_i}$,
- $y_i = Y_{\mu_{i+1},\alpha_i}$,
- $u_i = U_{\mu_{i+1},\alpha_i}$ for $i = 1, \dots, g-1$,
- $b = T_\beta$,
- $b_j = T_{\beta_j}$ for $2 \le 2j \le g-2$,
- $v = Y_{\mu_4,\beta}$,
- $c = T_{\gamma_{\{3,4,5,6\}}}$,
- $r_g = a_1 \cdots a_{g-1} u_{g-1} \cdots u_1$.

If the surface is closed ($n = 0$) then r_g is isotopic to the homeomorphism induced by the reflection of Figure 4 across the line containing centers of the shaded discs (see [25, Remark 2.4]).

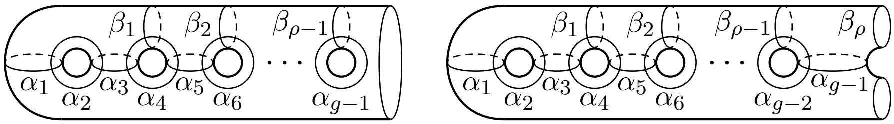

FIGURE 5. Regular neighborhood of the union of the curves α_i for $g = 2\rho + 1$ (left) and $g = 2\rho + 2$ (right).

3. Presentations

The groups $\mathcal{M}(N_{1,0})$ and $\mathcal{M}(N_{1,1})$ are trivial by [8, Theorem 3.4]. The following presentations were obtained in [17, 21, 4] respectively.

$$\mathcal{M}(N_{2,0}) = \langle a_1, y_1 \mid a_1^2 = y_1^2 = (a_1y_1)^2 = 1\rangle$$
$$\mathcal{M}(N_{2,1}) = \langle a_1, y_1 \mid a_1y_1a_1 = y_1\rangle$$
$$\mathcal{M}(N_{3,0}) = \langle a_1, a_2, y_2 \mid a_1a_2a_1 = a_2a_1a_2, y_2^2 = (a_1y_2)^2 = (a_2y_2)^2 = (a_1a_2)^6 = 1\rangle$$

In this section we describe some other known presentations of various mapping class groups and also state our main theorems which provide presentations for $\mathcal{M}(N_{g,n})$ for $n \in \{0, 1\}$ and $g + n \geq 4$.

3.1. Orientable subsurface. — Consider a regular neighborhood Σ of the union of the curves α_i for $i = 1, \ldots, g-1$. This is an orientable subsurface of $N_{g,n}$ homeomorphic to $S_{\rho,r}$, where $r \in \{1, 2\}$ and $g = 2\rho + r$ (Figure 5). The following theorem, whose proof is given in the Appendix, provides a presentation for $\mathcal{M}(S_{\rho,r})$, which will be a part of the presentation of $\mathcal{M}(N_{g,n})$.

THEOREM 3.1. — *For $r \in \{1, 2\}$, $\rho \geq 1$ and $g = 2\rho + r$, $\mathcal{M}(S_{\rho,r})$ admits a presentation with generators a_i, b_j for $1 \leq i \leq g-1$, $0 \leq 2j \leq g-2$ and relations:*

(A1) $a_ia_j = a_ja_i$ *for* $|i - j| > 1$,
(A2) $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}$ *for* $1 \leq i \leq g-2$,
(A3) $a_ib_1 = b_1a_i$ *for* $i \neq 4$ *if* $g \geq 4$,
(A4) $b_1a_4b_1 = a_4b_1a_4$ *if* $g \geq 5$,
(A5) $(a_2a_3a_4b_1)^{10} = (a_1a_2a_3a_4b_1)^6$ *if* $g \geq 5$,
(A6) $(a_2a_3a_4a_5a_6b_1)^{12} = (a_1a_2a_3a_4a_5a_6b_1)^9$ *if* $g \geq 7$,
(A7) $b_0 = a_1$,
(A8) $b_{i+1} = (b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3}b_i)^5(b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3})^{-6}$ *for* $2 \leq 2i \leq g-4$,
(A9a) $b_\rho a_{2\rho-3} = a_{2\rho-3}b_\rho$ *if* $g = 2\rho + 2 > 6$,
(A9b) $b_2b_1 = b_1b_2$ *if* $g = 6$.

It follows immediately from the above presentation that $\mathcal{M}(S_{\rho,r})$ is generated by $b = b_1$ and a_i for $i = 1, \dots, g-1$. Moreover, if g is odd, then we can drop the generators b_j for $j \neq 1$ and the relations (A7, A8). The resulting presentation is the same as the one given in [19]. However, if we wanted to do the same for even g, then in the relations (A9a, A9b) the generator b_ρ would have to be replaced by its expression in terms of b and the a_i's.

PROPOSITION 3.2. — *The map $\jmath_* \colon \mathcal{M}(S_{\rho,r}) \to \mathcal{M}(N_{g,n})$ induced by the inclusion of Σ in $N_{g,n}$ is injective for $n = 1$, whereas for $n = 0$ its kernel is an infinite cyclic group generated by $(a_1 \cdots a_{g-1})^k$, where $k = g$ if g is even, or $k = 2g$ if g is odd. The composition of $\jmath_* \colon \mathcal{M}(S_{\rho,r}) \to \mathcal{M}(N_{g,1})$ with $\imath_* \colon \mathcal{M}(N_{g,1}) \to \mathcal{M}^+(N_{g,0}, P)$ is also injective.*

Proof. — Set $x = (a_1 \cdots a_{g-1})^k$. If g is odd then x is equal to a Dehn twist about the boundary of Σ, while if g is even then x is the product of twists about the two boundary components (see [16]). The complement in $N_{g,n}$ of the interior of Σ is either a Möbius band with n holes if g is odd, or an annulus with n holes if g is even. By [22, Theorem 3.6], the maps $\jmath_*$ and $\imath_* \circ \jmath_*$ are injective for $n = 1$, whereas for $n = 0$ the kernel of $\jmath_*$ is an infinite cyclic group generated by x. □

3.2. Punctured disc and sphere. — Mapping class groups of a punctured disc or sphere are very closely related to braid groups. In fact $\mathcal{M}(S_{0,1}, \mathcal{P}_g)$ is isomorphic to the Artin braid group on g strands, while $\mathcal{M}(S_{0,0}, \mathcal{P}_g)$ is isomorphic to the quotient of the group of spherical braids on g strands by its center. Both groups are generated by $g-1$ elements called elementary braids or half twists. For $n \in \{0, 1\}$ we have the blowup homomorphism

$$\mathfrak{b} \colon \mathcal{M}(S_{0,n}, \mathcal{P}_g) \to \mathcal{M}(N_{g,n})$$

defined in Subsection 2.4 which is injective and maps the elementary braids on the crosscap transpositions u_i for $i = 1, \dots, g-1$ (see [26]). From now on we will identify $\mathcal{M}(S_{0,n}, \mathcal{P}_g)$ with its image in $\mathcal{M}(N_{g,n})$. The generators u_i satisfy the well known defining relations listed in the following theorem (see [3]).

THEOREM 3.3. — *The group $\mathcal{M}(S_{0,1}, \mathcal{P}_g)$ admits a presentation with generators u_i for $i = 1, \dots, g-1$ and relations*

(B1) $u_i u_j = u_j u_i$ *for* $|i - j| > 1$,

(B2) $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ *for* $i = 1, \dots, g-2$.

The group $\mathcal{M}(S_{0,0}, \mathcal{P}_g)$ is isomorphic to the quotient of $\mathcal{M}(S_{0,1}, \mathcal{P}_g)$ by the relations

(B3) $(u_1 u_2 \cdots u_{g-1})^g = 1$,

(B4) $(u_1 u_2 \cdots u_{g-2})^{g-1} = 1$. □

For $k = 1, \ldots, g$ we define $\Delta_k \in \mathcal{M}(S_{0,1}, \mathcal{P}_g)$ as

$$\Delta_1 = 1, \qquad \Delta_k = (u_1 u_2 \cdots u_{k-1})\Delta_{k-1}.$$

The following relations hold in $\mathcal{M}(S_{0,1}, \mathcal{P}_g)$.

(B5) $\Delta_k u_i = u_{k-i}\Delta_k$ for $i = 1, \ldots, k-1$,
(B6) $\Delta_k = \Delta_{k-1}(u_{k-1} \cdots u_2 u_1)$,
(B7) $\Delta_k^2 = (u_1 u_2 \cdots u_{k-1})^k$,
(B8) $\Delta_k^2 = \Delta_{k-1}^2(u_{k-1} \cdots u_2 u_1)(u_1 u_2 \cdots u_{k-1})$.

Equalities (B6) and (B8) are straightforward consequences of (B5) and (B7), which can be found in [5]. By (B7) the relations (B3, B4) are respectively $\Delta_g^2 = 1$ and $\Delta_{g-1}^2 = 1$. It follows immediately from (B8) that one of them may be replaced in the presentation of $\mathcal{M}(S_{0,0}, \mathcal{P}_g)$ by

$$\text{(B4a)} \quad (u_{g-1} \cdots u_2 u_1)(u_1 u_2 \cdots u_{g-1}) = 1.$$

By (B5) Δ_g^2 is central in $\mathcal{M}(S_{0,1}, \mathcal{P}_g)$. Geometrically it is the right Dehn twist about the boundary of $N_{g,1}$.

Lemma 3.4. — *In $\mathcal{M}(S_{0,1}, \mathcal{P}_g)$ we have*

$$\Delta_g^2 = u_{g-1}^2(u_{g-2}u_{g-1}^2 u_{g-2}) \cdots (u_1 u_2 \cdots u_{g-1}^2 \cdots u_2 u_1).$$

Proof. — By expanding (B8) inductively we have

$$\Delta_g^2 = u_1^2(u_2 u_1^2 u_2) \cdots (u_{g-1}u_{g-2} \cdots u_1^2 \cdots u_{g-2}u_{g-1}).$$

Conjugating both sides by Δ_g we obtain the desired equality. □

3.3. Main theorems

Theorem 3.5. — *Let $g = 2\rho + r$ for $r \in \{1, 2\}$ and $\rho \geq 1$. The group $\mathcal{M}(N_{g,1})$ is isomorphic to the quotient of the free product $\mathcal{M}(S_{\rho,r}) * \mathcal{M}(S_{0,1}, \mathcal{P}_g)$ by the following relations.*

(C1) $a_1 u_i = u_i a_1$ *for* $i = 3, \ldots, g-1$,
(C2) $a_i u_{i+1} u_i = u_{i+1} u_i a_{i+1}$ *for* $i = 1, \ldots, g-2$,
(C3) $a_{i+1} u_i u_{i+1} = u_i u_{i+1} a_i$ *for* $i = 1, \ldots, g-2$,
(C4) $a_1 u_1 a_1 = u_1$,
(C5) $u_2 a_1 a_2 u_1 = a_1 a_2$,
(C6) $(u_3 b)^2 = (a_1 a_2 a_3)^2 (u_1 u_2 u_3)^2$ *if* $g \geq 4$,
(C7) $u_5 b = b u_5$ *if* $g \geq 6$,
(C8) $a_4 u_4 (a_4 a_3 a_2 a_1 u_1 u_2 u_3 u_4) b = b a_4 u_4$ *if* $g \geq 5$.

Theorem 3.6. — *Let $g = 2\rho + r$ for $r \in \{1, 2\}$. For $g \geq 4$ the group $\mathcal{M}(N_{g,0})$ is isomorphic to the quotient of the free product $\mathcal{M}(S_{\rho,r}) * \mathcal{M}(S_{0,0}, \mathcal{P}_g)$ by the relations (C1–C8) from Theorem 3.5 and*

(D) $a_1(a_2 a_3 \cdots a_{g-1} u_{g-1} \cdots u_3 u_2) a_1 = a_2 a_3 \cdots a_{g-1} u_{g-1} \cdots u_3 u_2$.

Let $g = 2\rho + r$ for $r \in \{1,2\}$. We define

$\mathscr{G}_{g,1}$ as the quotient of $\mathcal{M}(S_{\rho,r}) * \mathcal{M}(S_{0,1}, \mathscr{P}_g)$ by the relations (C1–C8),
$\mathscr{G}_{g,0}$ as the quotient of $\mathcal{M}(S_{\rho,r}) * \mathcal{M}(S_{0,0}, \mathscr{P}_g)$ by the relations (C1–C8, D).
$\mathscr{G}^1_{g,0}$ as the quotient of $\mathscr{G}_{g,1}$ by the relation (B3).

By abuse of notation, we will denote by the same symbols the generators of $\mathcal{M}(S_{\rho,r})$ and $\mathcal{M}(S_{0,n}, \mathscr{P}_g)$ and their images in $\mathscr{G}_{g,n}$ or $\mathscr{G}^1_{g,0}$. The very essential idea of the proof of above theorems is the following. In the first step we are going to show that there is a homomorphism $\varphi_{g,n}\colon \mathscr{G}_{g,n} \to \mathcal{M}(N_{g,n})$ and then the rest of the paper will be devoted to proving that it has an inverse.

PROPOSITION 3.7. — *Let $g = 2\rho + r$ for $r \in \{1,2\}$. For $n \in \{0,1\}$ the map*

$$(\jmath_* * \mathfrak{b})\colon \mathcal{M}(S_{\rho,r}) * \mathcal{M}(S_{0,n}, \mathscr{P}_g) \to \mathcal{M}(N_{g,n})$$

induces a homomorphism $\varphi_{g,n}\colon \mathscr{G}_{g,n} \to \mathcal{M}(N_{g,n})$.

Proof. — We have to show that the relations (C1–C8, D) are satisfied in $\mathcal{M}(N_{g,n})$. For $|i-j| > 1$ the crosscap transposition u_i is equal to the identity in a neighborhood of the curve α_j and thus it commutes with the twist a_j. Thus (C1) is satisfied and analogously (C7). Observe that $u_{i+1}u_i(\alpha_{i+1}) = \alpha_i$ and the local orientation used to define a_i agrees with that induced by $u_{i+1}u_i$ from the local orientation used to define a_{i+1}. Thus $(u_{i+1}u_i)a_{i+1}(u_{i+1}u_i)^{-1} = a_i$ which is equivalent to (C2) and (C3) is proved analogously. Since u_i preserves α_i but reverses orientation of its neighborhood thus

$$(*) \qquad u_i a_i u_i^{-1} = a_i^{-1} \qquad \text{for } 1 \le i \le g-1.$$

In particular (C4) is satisfied. Let $x = a_i a_{i+1}$. It can be easily checked that $x(\mu_{i+1}) = \mu_{i+2}^{-1}$ and $x(\alpha_i) = \alpha_{i+1}^{-1}$. It follows that $x y_i x^{-1} = y_{i+1}^{-1}$, hence $x a_i u_i x^{-1} = u_{i+1}^{-1} a_{i+1}^{-1} = a_{i+1} u_{i+1}^{-1}$, where the last equality follows from $(*)$. By the braid relation (A2) we have $x a_i x^{-1} = a_{i+1}$ and thus

$$(**) \qquad u_{i+1} a_i a_{i+1} u_i = a_i a_{i+1} \qquad \text{for } 1 \le i \le g-2.$$

In particular (C5) is satisfied. Let K be a regular neighborhood of $\beta \cup \alpha_3 \cup \mu_4$. It is homeomorphic to Klein bottle with two holes and one of its boundary components is isotopic to α_1 while the other one is isotopic to $y_2^{-1}u_3^{-1}y_2^{-1}(\alpha_1)$. By [24, Lemma 7.8] we have

$$(u_3 b)^2 = a_1 y_2^{-1} u_3^{-1} y_2^{-1} a_1 y_2 u_3 y_2 = a_1 \underline{u_2^{-1} a_2^{-1}} u_3^{-1} \underline{u_2^{-1} a_2^{-1}} a_1 a_2 \underline{u_2 u_3 a_2} u_2 \overset{(*,\text{C3})}{=}$$
$$a_1 a_2 \underline{u_2^{-1} u_3^{-1}} a_2 u_2^{-1} a_1 a_2 a_3 u_2 u_3 u_2 \overset{(\text{C2})}{=} a_1 a_2 a_3 \underline{u_2^{-1} u_3^{-1} u_2^{-1} a_1 a_2 a_3} u_2 u_3 u_2 \overset{(**)}{=}$$
$$(a_1 a_2 a_3)^2 u_1 u_2 u_1 u_2 u_3 u_2 \overset{(\text{B1,B2})}{=} (a_1 a_2 a_3)^2 (u_1 u_2 u_3)^2$$

which proves (C6). Let $z = a_4a_3a_2a_1u_1u_2u_3u_4$. We have

$$\begin{aligned}z &= (a_4u_4)(u_4^{-1}a_3u_3u_4)(u_4^{-1}u_3^{-1}a_2u_2u_3u_4)(u_4^{-1}u_3^{-1}u_2^{-1}a_1u_1u_2u_3u_4)\\ &= y_4(u_4^{-1}y_3u_4)(u_4^{-1}u_3^{-1}y_2u_3u_4)(u_4^{-1}u_3^{-1}u_2^{-1}y_1u_2u_3u_4)\\ &= Y_{\mu_5,\gamma_{\{4,5\}}}Y_{\mu_5,\gamma_{\{3,5\}}}Y_{\mu_5,\gamma_{\{2,5\}}}Y_{\mu_5,\gamma_{\{1,5\}}}.\end{aligned}$$

Consider the surface N' obtained by cutting $N_{g,n}$ along μ_5 and then gluing a disc with a puncture P along the resulting boundary component. Then $N_{g,n}$ may be seen as being obtained from N' by blowing up the puncture and we have the crosscap pushing map $\mathfrak{c}\colon \pi_1(N',P) \to \mathcal{M}(N_{g,n})$ whose image contains the crosscap slides $Y_{\mu_5,\gamma_{\{i,5\}}}$ for $i = 1,2,3,4$. Since this is a homomorphism, thus z is isotopic to the effect of pushing μ_5 once along $\gamma_{\{1,2,3,4,5\}}$. One of the boundary components of the regular neighborhood of $\mu_5 \cup \gamma_{\{1,2,3,4,5\}}$ is isotopic to β, while the other one is isotopic to $y_4^{-1}(\beta)$. From Lemma 2.2 we have $z = y_4^{-1}by_4b^{-1}$ which is equivalent to (C8). Finally it is easy to check that if the surface is closed (i.e., $n = 0$) then $a_2\cdots a_{g-1}u_{g-1}\cdots u_2$ preserves the curve α_1 (up to isotopy) and reverses orientation of its neighborhood, which proves the relation (D). □

Consider the exact sequence (2.2) in the case $m = 1$:

$$1 \to \mathbb{Z} \to \mathcal{M}(N_{g,1}) \xrightarrow{\imath_*} \mathcal{M}^+(N_{g,0},P) \to 1.$$

The kernel of $\imath_*$ is generated by the Dehn twist $T_{\partial N_{g,1}} = \Delta_g^2$. Recall that $\mathcal{G}_{g,0}^1$ is the quotient of $\mathcal{G}_{g,1}$ by the normal closure of Δ_g^2. Since $\imath_*(\varphi_{g,1}(\Delta_g^2)) = 1$, there is a homomorphism $\varphi_{g,0}^1\colon \mathcal{G}_{g,0}^1 \to \mathcal{M}^+(N_{g,0},P)$ such that $\varphi_{g,0}^1 \circ p = \imath_* \circ \varphi_{g,1}$, where $p\colon \mathcal{G}_{g,1} \to \mathcal{G}_{g,0}^1$ is the canonical projection. We will prove in Section 7 that $\varphi_{g,0}^1$ is an isomorphism.

3.4. Some consequences of the defining relations. — Throughout this paper we will often have to verify that some relations are satisfied in $\mathcal{G}_{g,n}$ or $\mathcal{G}_{g,0}^1$. In this subsection we prove the most useful ones.

Lemma 3.8. — *The following relations hold in $\mathcal{G}_{g,n}$ for $n = 0,1$.*

(C1a) $a_iu_j = u_ja_i$ *for* $|j-i| > 1$,
(C4a) $a_iu_ia_i = u_i$ *for* $i = 1,\dots,g-1$,
(C5a) $u_{i+1}a_ia_{i+1}u_i = a_ia_{i+1}$ *for* $i = 1,\dots,g-2$,
(C6a) $(bu_3)^2 = (u_3b)^2 = (a_1a_2a_3)^2(u_1u_2u_3)^2$,
(C7a) $u_ib = bu_i$ *for* $i = 5,\dots,g-1$.

Proof. — Fix $i > 1$ and let $x = (u_{i-1}u_i)\cdots(u_2u_3)(u_1u_2)$. By (C3) we have $xa_1x^{-1} = a_i$ and by the relations (B1,B2) we have $xu_1x^{-1} = u_i$ and

$$xu_jx^{-1} = \begin{cases} u_j \text{ for } j > i+1 \\ u_{j-2} \text{ for } 3 \leq j < i+1 \end{cases}$$

Thus (C1a) may be obtained by conjugating by x both sides of (C1). If we set $y = u_1u_2\cdots u_{g-1}$, then, for $i \in \{1,\dots,g-2\}$, we have $yu_iy^{-1} = u_{i+1}$ by (B1,B2) and $ya_iy^{-1} = a_{i+1}$ by (C1a,C3). Hence (C4a,C5a) follow from (C4,C5) by applying conjugation by y as many times as needed. We have

$$u_3(a_1a_2a_3)^2 = (a_1a_2a_3)u_2^{-1}(a_1a_2a_3) = (a_1a_2a_3)^2u_1$$

by (C1a,C5a) and $u_1(u_1u_2u_3)^2 = (u_1u_2u_3)^2u_3$ by (B1,B2). Thus u_3 commutes with $(a_1a_2a_3)^2(u_1u_2u_3)^2$, which together with (C6) proves (C6a). If $i > 5$ then for $z = (a_{i-1}a_i)\cdots(a_5a_6)$ we have $zu_5z^{-1} = u_i^{\pm1}$ by (C5a) and $zbz^{-1} = b$ by (A3). Thus (C7a) is obtained by conjugating both sides of (C7) by z. □

LEMMA 3.9. — *The following relations hold in $\mathscr{G}_{g,n}$ for $n = 0,1$.*

(E1) $\Delta_k a_i = a_{k-i}^{-1}\Delta_k$ *for $2 \leq k \leq g$ and $i = 1,\dots,k-1$,*
(E2) $r_g^2 = \Delta_g^2$,
(E3) $r_ga_i = a_ir_g$ *for $i = 2,\dots,g-1$,*
(E4) $u_ir_gu_i = r_g$ *for $i = 2,\dots,g-1$.*

In $\mathscr{G}_{g,0}^1$ and $\mathscr{G}_{g,0}$ we have

(E2a) $r_g^2 = 1$.

Proof. — We prove (E1) by induction on k. For $k = 2$ it is equivalent to (C4). Suppose (E1) is true for some $k \geq 2$. For $i \leq k-1$ we have

$$\Delta_{k+1}a_i = (u_1\cdots u_k)\Delta_ka_i = (u_1\cdots u_k)a_{k-i}^{-1}\Delta_k \overset{\text{(C1a,C3)}}{=} a_{k+1-i}^{-1}(u_1\cdots u_k)\Delta_k = a_{k+1-i}^{-1}\Delta_{k+1}$$

and for $i = k$

$$\Delta_{k+1}a_k = \Delta_k(u_k\cdots u_1)a_k \overset{\text{(C1a,C2)}}{=} \Delta_ka_{k-1}(u_k\cdots u_1) = a_1^{-1}\Delta_k(u_k\cdots u_1) = a_1^{-1}\Delta_{k+1}$$

which finishes the proof of (E1).

Let $x = a_1\cdots a_{g-1}$ and $z = u_{g-1}\cdots u_1$, so that $r_g = xz$. We are going to prove by induction on i that

$$(*) \quad xz(u_2\cdots u_{g-1})(u_2\cdots u_{g-2})\cdots(u_2\cdots u_{i+1})x = \Delta_g\Delta_i$$

for $i = 1, \dots, g-1$. If $i = 1$ then $(*)$ becomes $x\Delta_g x = \Delta_g$ and it follows from (E1). Suppose that $(*)$ holds for some $i < g-1$. By (C1a,C5a) we have $(u_2 \cdots u_{i+1})x = x(u_i \cdots u_1)^{-1}$ and

$$xz(u_2 \cdots u_{g-1}) \cdots (u_2 \cdots u_{i+2})x = \Delta_g \Delta_i (u_i \cdots u_1) = \Delta_g \Delta_{i+1}.$$

For $i = g-1$ we obtain

$$xzx = \Delta_g \Delta_{g-1} \iff (xz)^2 = \Delta_g \Delta_{g-1} z = \Delta_g^2$$

which is equivalent to (E2). For $i = 2, \dots, g-1$ we have $a_i x = x a_{i-1}$ by (A1, A2), $a_{i-1} z = z a_i$ by (C1a, C2), $u_i x = x u_{i-1}^{-1}$ by (C1a, C5a), $u_{i-1} z = z u_i$ by (B1, B2). The relations (E3, E4) follow. In $\mathscr{G}^1_{g,0}$ and $\mathscr{G}_{g,0}$ we have $\Delta_g^2 = 1$ (B3) and thus (E2a) is a consequence of (E2). □

LEMMA 3.10. — *In $\mathscr{G}_{g,0}$ we have*

(E3a) $r_g a_i = a_i r_g$ *for* $i = 1, \dots, g-1$,
(E4a) $u_i r_g u_i = r_g$ *for* $i = 1, \dots, g-1$,
(E5) $(a_1 a_2 \cdots a_{g-1})^2 = (u_{g-1} \cdots u_2 u_1)^{-2} = (u_1 u_2 \cdots u_{g-1})^2$,
(E6) $(a_1 a_2 \cdots a_{g-1})^k = 1$, *where $k = g$ if g is even or $k = 2g$ if g is odd.*

Proof. — For $i > 1$ (E3a, E4a) are the same as (E3, E4), while for $i = 1$ (E3a) follows from (D, C4), and

$$r_g u_1 r_g \overset{\text{C5}}{=} r_g a_2^{-1} a_1^{-1} u_2^{-1} a_1 a_2 r_g \overset{\text{E3a}}{=} a_2^{-1} a_1^{-1} r_g u_2^{-1} r_g a_1 a_2 \overset{\text{E4}}{=} u_1^{-1}$$

By (E2a) we have

$$1 = r_g^2 = r_g (a_1 \cdots a_{g-1} u_{g-1} \cdots u_1) \overset{\text{E3a}}{=} (a_1 \cdots a_{g-1}) r_g (u_{g-1} \cdots u_1)$$
$$= (a_1 \cdots a_{g-1})^2 (u_{g-1} \cdots u_1)^2 \overset{\text{B4a}}{=} (a_1 \cdots a_{g-1})^2 (u_1 \cdots u_{g-1})^{-2}.$$

This proves (E5), which together with (B3) implies (E6). □

LEMMA 3.11. — *In the presentation of $\mathcal{M}(N_{g,0})$ given in Theorem 3.6 the relation* (D) *may be replaced by*

$$\text{(Da)} \quad a_{g-1}(u_{g-2}u_{g-3} \cdots u_1 a_1 \cdots a_{g-3} a_{g-2}) a_{g-1} = u_{g-2} u_{g-3} \cdots u_1 a_1 \cdots a_{g-3} a_{g-2}.$$

Proof. — If we conjugate both sides of (Da) by Δ_g, then by (B5) and (E1) we obtain

$$a_1^{-1}(u_2 \cdots u_{g-1} a_{g-1}^{-1} \cdots a_2^{-1}) a_1^{-1} = u_2 \cdots u_{g-1} a_{g-1}^{-1} \cdots a_2^{-1} \overset{\text{C4}}{\iff}$$
$$a_1(u_1 u_2 \cdots u_{g-1} a_{g-1}^{-1} \cdots a_2^{-1}) a_1^{-1} = u_1 u_2 \cdots u_{g-1} a_{g-1}^{-1} \cdots a_2^{-1} \overset{\text{B4a}}{\iff}$$
$$a_1(u_1^{-1} u_2^{-1} \cdots u_{g-1}^{-1} a_{g-1}^{-1} \cdots a_2^{-1}) a_1^{-1} = u_1^{-1} u_2^{-1} \cdots u_{g-1}^{-1} a_{g-1}^{-1} \cdots a_2^{-1} \overset{\text{C4}}{\iff}$$
$$a_1^{-1}(u_2^{-1} \cdots u_{g-1}^{-1} a_{g-1}^{-1} \cdots a_2^{-1}) a_1^{-1} = u_2^{-1} \cdots u_{g-1}^{-1} a_{g-1}^{-1} \cdots a_2^{-1}$$

which is equivalent to (D). □

LEMMA 3.12. — *Suppose that w is either* (1) *a word in the generators u_i and their inverses or* (2) *a word in the generators a_i, b_j and their inverses. Then w represents the trivial element of $\mathcal{M}(N_{g,n})$ or $\mathcal{M}^+(N_{g,0},P)$ if and only if it represents the trivial element of $\mathcal{G}_{g,n}$ or $\mathcal{G}^1_{g,0}$ respectively.*

Proof. — Case (1) follows from the injectivity of $\mathfrak{b}\colon \mathcal{M}(S_{0,n},\mathcal{P}_g) \to \mathcal{M}(N_{g,n})$ (Proposition 2.4) and the fact that the kernel of $\imath_* \circ \mathfrak{b}\colon \mathcal{M}(S_{0,1},\mathcal{P}_g) \to \mathcal{M}^+(N_{g,0},P)$ is generated by Δ_g^2, which is trivial in $\mathcal{G}^1_{g,0}$. Analogously for (2), we have that $\jmath_*\colon \mathcal{M}(S_{\rho,r}) \to \mathcal{M}(N_{g,1})$ and $\imath_* \circ \jmath_*\colon \mathcal{M}(S_{\rho,r}) \to \mathcal{M}^+(N_{g,0},P)$ are injective by Proposition 3.2. For $n = 0$ we have to check that the image under the map $\mathcal{M}(S_{\rho,r}) \to \mathcal{G}_{g,0}$ of the kernel of $\jmath_*\colon \mathcal{M}(S_{\rho,r}) \to \mathcal{M}(N_{g,0})$ is trivial. This is the case by Proposition 3.2 and the relation (E6). □

LEMMA 3.13. — *In $\mathcal{G}^1_{4,0}$ we have:*

(G1) $bu_3u_2b^{-1} = (a_1a_2a_3)^3u_2u_3(a_1a_2a_3)^{-1}$,
(G2) $bu_3u_2u_1b = (a_1a_2a_3)^3(u_3u_2u_1)^{-1}(a_1a_2a_3)^3$,
(G3) $((a_1a_2a_3)^{-4}br_4)^2 = 1$.

In $\mathcal{G}_{4,0}$ we have

(G3a) $(br_4)^2 = 1$.

Proof. — Let $x = a_1a_2a_3$ and $z = u_1u_2u_3$. We have

$$\text{(C6a) } (bu_3)^2 = (u_3b)^2 = x^2z^2, \quad \text{(i) } u_2 = x^{-1}u_3^{-1}x, \quad \text{(ii) } u_1 = x^{-1}u_2^{-1}x,$$

the last two relations following from (C5a). Since b commutes with x, from (C6a, i, ii) we obtain

$$\text{(iii) } (b^{-1}u_2)^2 = x^{-1}z^{-2}x^{-1}, \qquad \text{(iv) } (bu_1)^2 = z^2x^2.$$

$$\begin{aligned}
\underline{bu_3}u_2b^{-1} &\overset{\text{C6a}}{=} x^2z^2u_3^{-1}\underline{b^{-1}u_2b^{-1}} \overset{\text{(iii)}}{=} x^2z^2u_3^{-1}x^{-1}z^{-2}\underline{x^{-1}u_2^{-1}} \\
&\overset{\text{(ii)}}{=} x^2\underline{z^2u_3^{-1}}x^{-1}\underline{z^{-2}u_1}x^{-1} \\
&= x^2u_1u_2u_3u_1u_2x^{-1}\underline{u_3^{-1}u_2^{-1}u_1^{-1}u_3^{-1}}u_2^{-1}x^{-1} \\
&\overset{\text{B1,B2}}{=} x^2u_1u_2u_3u_1u_2\underline{x^{-1}u_2^{-1}}u_3^{-1}u_2^{-1}u_1^{-1}u_2^{-1}x^{-1} \\
&\overset{\text{(ii)}}{=} x^2\underline{u_1u_2u_3u_1u_2u_1}x^{-1}u_3^{-1}u_2^{-1}u_1^{-1}u_2^{-1}x^{-1} \\
&= x^2\underline{\Delta_4x^{-1}}u_3^{-1}u_2^{-1}u_1^{-1}u_2^{-1}x^{-1} \\
&\overset{\text{E1}}{=} x^3\Delta_4u_3^{-1}u_2^{-1}u_1^{-1}u_2^{-1}x^{-1} \overset{\text{(B1,B2)}}{=} x^3u_2u_3x^{-1}.
\end{aligned}$$

$$bu_3u_2u_1b = (bu_3u_2b^{-1})(bu_1b) = (x^3\Delta_4u_3^{-1}u_2^{-1}u_1^{-1}\underline{u_2^{-1}x^{-1}})(z^2\underline{x^2u_1^{-1}})$$

$$\stackrel{\text{(i,ii)}}{=} x^3\Delta_4 u_3^{-1}u_2^{-1}u_1^{-1}x^{-1}\underline{u_3u_1u_2u_3u_1u_2}x^2 = x^3\Delta_4 u_3^{-1}u_2^{-1}u_1^{-1}x^{-1}\Delta_4 x^2$$
$$\stackrel{\text{E1,B5}}{=} x^3\Delta_4^2(u_3u_2u_1)^{-1}x^3 \stackrel{\text{B3}}{=} x^3(u_3u_2u_1)^{-1}x^3.$$
$$(x^{-4}br_4)^2 = x^{-3}\underline{bu_3u_2u_1b}x^{-3}u_3u_2u_1 \stackrel{\text{(G2)}}{=} (u_3u_2u_1)^{-1}u_3u_2u_1 = 1.$$

The relation (G3a) follows from (G3) and (E6). □

4. The base cases

In this section we deduce the main theorems for $(g,n)\in\{(3,1),(4,0)\}$ from the presentations of $\mathcal{M}(N_{g,n})$ obtained in [24, 25]. Recall from Subsection 3.3 the definitions of $\mathcal{G}_{g,n}$ and $\varphi_{g,n}$. Theorem 3.5 for $g=3$ follows from the following.

THEOREM 4.1. — *The map $\varphi_{3,1}\colon \mathcal{G}_{3,1}\to\mathcal{M}(N_{3,1})$ is an isomorphism.*

Proof. — By [24, Theorem 7.16] $\mathcal{M}(N_{3,1})$ admits a presentation with generators a_1, a_2, u_2, d (called respectively B, A_1, U, A_2 in [24]), where d is a Dehn twist about the curve $a_1^{-1}u_2(\alpha_1)$. The defining relations are

(i) $a_2d=da_2$ (ii) $a_2a_1a_2=a_1a_2a_1$ (iii) $da_1d=a_1da_1$

(iv) $u_2a_2u_2^{-1}=a_2^{-1}$ (v) $u_2a_1u_2^{-1}=a_1d^{-1}a_1^{-1}$ (vi) $(du_2)^2=(u_2d)^2$

(vii) $(du_2)^2=(a_2d^2a_1)^3$.

We define $\psi\colon\mathcal{M}(N_{3,1})\to\mathcal{G}_{3,1}$ on the generators as $\psi(a_1)=a_1$, $\psi(a_2)=a_2$, $\psi(u_2)=u_2$, $\psi(d)=a_1^{-1}u_2a_1^{-1}u_2^{-1}a_1$. To prove that ψ is a homomorphism we have to show that it respects the relations (i–vii). This is obvious for (v) and (ii, iv) are (A2,C4a).

$$\psi(d)=a_1^{-1}\underline{u_2a_1^{-1}u_2^{-1}}a_1 \stackrel{\text{C3}}{=} a_1^{-1}u_1^{-1}a_2^{-1}\underline{u_1a_1} \stackrel{\text{C4}}{=} a_1^{-1}\underline{u_1^{-1}a_2^{-1}a_1^{-1}}u_1$$
$$\stackrel{\text{C5}}{=} a_1^{-1}a_2^{-1}a_1^{-1}u_2u_1=(a_1a_2a_1)^{-1}u_2u_1,$$
$$\psi(a_2)\psi(d)=a_2(a_1a_2a_1)^{-1}u_2u_1 \stackrel{\text{A2}}{=} (a_1a_2a_1)^{-1}a_1u_2u_1$$
$$\stackrel{\text{C2}}{=} (a_1a_2a_1)^{-1}u_2u_1a_2=\psi(d)\psi(a_2).$$

The relation $\psi(d)\psi(a_1)\psi(d)=\psi(a_1)\psi(d)\psi(a_1)$ is equivalent to

$$(a_1a_2a_1)^{-1}u_2u_1a_1(a_1a_2a_1)^{-1}u_2u_1=a_1(a_1a_2a_1)^{-1}u_2u_1a_1 \iff$$
$$u_2\underline{u_1a_2^{-1}a_1^{-1}u_2}u_1=a_2u_2u_1a_1 \stackrel{\text{C5}}{\iff} u_2a_2^{-1}a_1^{-1}u_1=a_2u_2u_1a_1.$$

The last relation follows easily from (C4a).

$$(\psi(d)\psi(u_2))^2=(a_1a_2a_1)^{-1}u_2u_1u_2(a_1a_2a_1)^{-1}u_2u_1u_2$$

$$
\begin{aligned}
&= (a_1a_2a_1)^{-1}\Delta_3(a_1a_2a_1)^{-1}\Delta_3 \\
&\overset{\text{E1}}{=} \Delta_3^2, \\
(\psi(u_2)\psi(d))^2 &= u_2(\psi(d)\psi(u_2))^2u_2^{-1} = u_2\Delta_3^2u_2^{-1} \\
&= \Delta_3^2 = (\psi(d)\psi(u_2))^2, \\
(\psi(a_2)\psi(d)^2\psi(a_1))^3 &= (a_2(a_1a_2a_1)^{-1}u_2u_1(a_1a_2a_1)^{-1}u_2u_1a_1)^3 \\
&\overset{\text{A2}}{=} (a_2(a_2a_1a_2)^{-1}u_2u_1(a_2a_1a_2)^{-1}u_2u_1a_1)^3 \\
&= a_1^{-1}(a_2^{-1}u_2\underline{u_1a_2^{-1}a_1^{-1}}a_2^{-1}u_2u_1)^3a_1 \\
&\overset{\text{C5}}{=} a_1^{-1}(\underline{a_2^{-1}u_2a_2^{-1}}a_1^{-1}\underline{u_2^{-1}a_2^{-1}u_2}u_1)^3a_1 \\
&\overset{\text{C4a}}{=} a_1^{-1}(u_2a_1^{-1}a_2u_1)^3a_1 \\
&= a_1^{-1}(u_2a_1^{-1}\underline{a_2u_1u_2}a_1^{-1}\underline{a_2u_1u_2}a_1^{-1}a_2u_1)a_1 \\
&= a_1^{-1}(u_2a_1^{-1}u_1u_2u_1u_2a_2u_1)a_1 \\
&= a_1^{-1}(u_2a_1^{-1}\Delta_3u_2a_2u_1)a_1 \overset{\text{B5,E1}}{=} a_1^{-1}(\Delta_3u_1\underline{a_2u_2a_2}u_1)a_1 \\
&\overset{\text{C4a}}{=} a_1^{-1}(\Delta_3u_1u_2u_1)a_1 = a_1^{-1}\Delta_3^2a_1 \overset{\text{E1}}{=} \Delta_3^2 = (\psi(d)\psi(u_2))^2.
\end{aligned}
$$

Thus ψ is a homomorphism. Observe that $\mathcal{G}_{3,1}$ and $\mathcal{M}(N_{3,1})$ are generated by a_1, a_2 and u_2. Indeed, the generator u_1 of $\mathcal{G}_{3,1}$ is redundant because of (C5) and the generator d of $\mathcal{M}(N_{3,1})$ is redundant because of (v). Since for $x \in \{a_1, a_2, u_2\}$ we have $\psi(\varphi_{3,1}(x)) = x = \varphi_{3,1}(\psi(x))$ thus ψ is the inverse of $\varphi_{3,1}$. □

Corollary 4.2. — *Suppose that $g \geq 4$, $i \in \{1, \dots, g-2\}$ is fixed and $w \in \mathcal{G}_{g,n}$ is represented by a word in the generators a_i, u_i, a_{i+1}, u_{i+1} and their inverses. Then $\varphi_{g,n}(w) = 1$ if and only if $w = 1$.*

Proof. — Consider a subsurface $K \subset N_{g,n}$ which is a disc with crosscaps $i, i+1, i+2$. Thus K is a copy of $N_{3,1}$. By [22, Theorem 3.6] the map $\imath_*\colon \mathcal{M}(K) \to \mathcal{M}(N_{g,n})$ induced by the inclusion of K in $N_{g,n}$ is injective, except for the case $(g,n) = (4,0)$ where its kernel is generated by a Dehn twist about the boundary of K. As in the proof of Theorem 4.1, there is a homomorphism $\psi\colon \imath_*(\mathcal{M}(K)) \to \mathcal{G}_{g,n}$ such that $\psi(x) = x$ for $x \in \{a_i, a_{i+1}, u_i, u_{i+1}\}$. For $(g,n) = (4,0)$ we additionally have to verify that $\psi(T_{\partial K}) = 1$, which is true by (B4), because either $T_{\partial K} = \Delta_2^2$ if $i = 1$, or $T_{\partial K} = \Delta_3\Delta_2^2\Delta_3$ if $i = 2$. Since $\psi(\varphi_{g,n}(w)) = w$ the corollary is proved. □

Theorem 3.6 for $g = 4$ follows from the following.

Theorem 4.3. — *The map $\varphi_{4,0}\colon \mathcal{G}_{4,0} \to \mathcal{M}(N_{4,0})$ is an isomorphism.*

Proof. — By [25, Theorem 2.1] $\mathcal{M}(N_{4,0})$ admits the presentation with generators a_i, u_i, for $i = 1, 2, 3$, b, r_4, d (the last two generators are denoted respectively as t and a_4 in [25]), where d is a Dehn twist about the curve $a_2^{-1}u_3(\alpha_2)$. The defining relations are (A1–A3, B1, C1, C4, C5a, E2a, E3a, E4, E6, G3a) and

(i) $r_4 = a_1a_2a_3u_3u_2u_1$ (ii) $u_3a_2u_3^{-1} = a_2d^{-1}a_2^{-1}$ (iii) $u_1^2 = u_3^2$

(iv) $(u_3b)^2 = 1$ (v) $(u_3d)^2 = 1$ (vi) $da_3 = a_3d$

(vii) $da_2d = a_2da_2$ (viii) $(da_2a_3)^4 = 1$ (ix) $u_3du_3^{-1} = u_1du_1^{-1}$.

We define $\psi\colon \mathcal{M}(N_{4,0}) \to \mathcal{G}_{4,0}$ on the generators as $\psi(a_i) = a_i$, $\psi(u_i) = u_i$ for $i = 1, 2, 3$, $\psi(b) = b$, $\psi(r_4) = a_1a_2a_3u_3u_2u_1$ and $\psi(d) = a_2^{-1}u_3a_2^{-1}u_3^{-1}a_2$. To show that ψ is a homomorphism we have to show that the relations (iii–ix) are satisfied in $\mathcal{G}_{4,0}$. By Lemma 3.12 the relation (iii) is satisfied in $\mathcal{G}_{4,0}$. The relations (v,vi,vii,viii) can be rewritten using (ii) in the generators a_2, u_2, a_3, u_3 and so they hold in $\mathcal{G}_{4,0}$ by Corollary 4.2. We have

$$(u_3b)^2 \overset{(C6)}{=} (a_1a_2a_3)^2(u_1u_2u_3)^2 \overset{(E5)}{=} (a_1a_2a_3)^4 \overset{(E6)}{=} 1.$$

As in the proof of Theorem 4.1 we have $\psi(d) = (a_2a_3a_2)^{-1}u_3u_2$ and $\psi(u_3)\psi(d)\psi(u_3)^{-1} = \psi(u_1)\psi(d)\psi(u_1)^{-1}$ is equivalent to

$$\begin{aligned}
u_3(a_2a_3a_2)^{-1}u_3u_2u_1 &= u_1\underline{(a_2a_3a_2)^{-1}u_3u_2u_3} \overset{(C2,C3,C4a)}{\Longleftrightarrow}\\
u_3(a_2a_3a_2)^{-1}u_3u_2u_1 &= u_1u_3u_2u_3a_2a_3a_2 \overset{(B1)}{\Longleftrightarrow}\\
(a_2a_3a_2)^{-1}\underline{u_3u_2u_1(a_2a_3a_2)^{-1}} &= u_1u_2u_3 \overset{(C1a,C2)}{\Longleftrightarrow}\\
\underline{(a_2a_3a_2)^{-1}(a_1a_2a_1)^{-1}}u_3u_2u_1 &= u_1u_2u_3 \overset{(A1,A2)}{\Longleftrightarrow}\\
(a_1a_2a_3)^{-2} &= u_1u_2u_3(u_3u_2u_1)^{-1} \overset{(E5)}{\Longleftrightarrow}\\
(u_3u_2u_1)^2 &= u_1u_2u_3(u_3u_2u_1)^{-1}.
\end{aligned}$$

The last relation is satisfied in $\mathcal{G}_{4,0}$ by Lemma 3.12. Since $\varphi_{4,0} \circ \psi = id$, hence ψ is injective, and since $\mathcal{G}_{4,0}$ is generated by a_i, u_i and b, it is also surjective. It follows that $\varphi_{4,0}$ is an isomorphism. □

5. Curve complexes

5.1. Definitions and simple connectedness. — Let $N = N_{g,n}$. Suppose that $C = (\gamma_1, \ldots, \gamma_m)$ is an m-tuple of generic curves on N. We say that C is a *generic m-tuple of disjoint curves* if for $i \neq j$

– γ_i is disjoint from γ_j, and

– γ_i is neither isotopic to γ_j nor to γ_j^{-1}.

We denote by N_C the compact surface obtained by cutting N along C. If $C' = (\gamma'_1, \dots, \gamma'_m)$ then we say that C and C' are *equivalent* if γ_i is isotopic to ${\gamma'}_i^{\pm 1}$ for $i = 1, \dots, m$, and *equivalent up to permutation* if γ_i is isotopic to ${\gamma'}_{\tau(i)}^{\pm 1}$ for $i = 1, \dots, m$ and for some permutation $\tau \in \mathrm{Sym}_m$. We denote by $[C] = [\gamma_1, \dots, \gamma_m]$ the equivalence class of C, and by $\langle C \rangle = \langle \gamma_1, \dots, \gamma_m \rangle$ its equivalence class up to permutation.

The *complex of curves* $\mathcal{C}(N)$ is a simplicial complex whose m-simplices are the equivalence classes up to permutation of generic $(m+1)$-tuples of disjoint curves on N. We are going to use its two full subcomplexes: $\mathcal{C}_0(N)$ is the subcomplex of $\mathcal{C}(N)$ consisting of simplices $\langle C \rangle$ such that N_C is connected; $\mathcal{D}(N)$ is the subcomplex of $\mathcal{C}_0(N)$ consisting of simplices $\langle C \rangle$ such that N_C is nonorientable.

The *ordered complex of curves* $\mathcal{C}^{\mathrm{ord}}(N)$ is a Δ-complex (in the sense of [12], Chapter 2) whose m-simplices are the equivalence classes of generic $(m+1)$-tuples of disjoint curves on N. If $[\gamma_1, \dots, \gamma_{m+1}]$ is an m-simplex then its faces are the $(m-1)$-simplices $[\gamma_1, \dots, \widehat{\gamma_i}, \dots, \gamma_{m+1}]$ for $i = 1, \dots, m+1$, where $\widehat{\gamma_i}$ means that γ_i is deleted. We define the subcomplexes $\mathcal{C}_0^{\mathrm{ord}}(N)$ and $\mathcal{D}^{\mathrm{ord}}(N)$ as the ordered versions of $\mathcal{C}_0(N)$ and $\mathcal{D}(N)$.

The complex of curves was introduced by Harvey [11] and the ordered complex of curves by Benvenuti [1]. The following theorem was proved for the unordered complexes in [28, Theorems 5.4, 5.5]. To obtain the result for their ordered versions, the proof of [1, Proposition 8] can be applied.

THEOREM 5.1. — *The complexes $\mathcal{C}_0(N_{g,n})$ and $\mathcal{C}_0^{\mathrm{ord}}(N_{g,n})$ are simply connected for $g \geq 5$; $\mathcal{D}(N_{g,n})$ and $\mathcal{D}^{\mathrm{ord}}(N_{g,n})$ are simply connected for $g \geq 7$.* □

The mapping class group $\mathcal{M}(N)$ acts on the set of isotopy classes of generic curves on N, and thus it also acts on the complexes $\mathcal{C}(N)$, $\mathcal{C}_0(N)$, $\mathcal{D}(N)$ and their ordered versions by permuting their simplices. We say that two simplices σ_1, σ_2 are $\mathcal{M}(N)$-*equivalent* if $\sigma_2 = h\sigma_1$ for some $h \in \mathcal{M}(N)$. Observe that $\mathcal{C}^{\mathrm{ord}}(N)$ has a natural orientation (the vertices of every simplex are ordered) preserved by $\mathcal{M}(N)$. In particular, $\mathcal{M}(N)$ acts on the 1-simplices of $\mathcal{C}^{\mathrm{ord}}(N)$ without inversion, which simplifies the statement of Brown's theorem below. This is the only, purely technical, reason for considering $\mathcal{C}^{\mathrm{ord}}(N)$ instead of $\mathcal{C}(N)$.

5.2. The structure of a stabiliser. — Let $N = N_{g,n}$. Suppose that $C = (\gamma_1, \ldots, \gamma_m)$ is a generic m-tuple of disjoint curves on N such that N_C is connected. Suppose that γ_i are two-sided for $i \leq r$ and one-sided for $i > r$.

Let $\mathrm{Stab}[C] = \mathrm{Stab}_{\mathcal{M}(N)}[C]$ denote the stabilizer of the simplex $[C]$ with respect to the action of $\mathcal{M}(N)$ on $C_0^{\mathrm{ord}}(N)$. This is the subgroup of $\mathcal{M}(N)$ consisting of the isotopy classes of homeomorphisms fixing each curve γ_i. We define $\mathrm{Stab}^+[C] = \mathrm{Stab}^+_{\mathcal{M}(N)}[C]$ to be the subgroup of $\mathrm{Stab}[C]$ consisting of the isotopy classes of homeomorphisms fixing each curve γ_i, preserving its orientation, and preserving its sides if γ_i is two-sided. We have an exact sequence

$$1 \to \mathrm{Stab}^+[C] \to \mathrm{Stab}[C] \xrightarrow{\eta} \mathbb{Z}_2^{m+r}, \tag{5.1}$$

where $\eta(h)$ is the vector $(e_i)_{i=1}^{m+r}$ defined as follows

- for $i = 1, \ldots, m$, $e_i = 0$ if h preserves orientation of γ_i and $e_i = 1$ otherwise,
- for $j = 1, \ldots, r$, $e_{m+j} = 0$ if h preserves sides of γ_j and $e_{r+j} = 1$ otherwise.

Remark 5.2. — The map η is not surjective in general and its image depends on C. For example, if N_C is orientable and $m > 1$ then η is not onto. Indeed, suppose that $h \in \mathrm{Stab}[C]$ preserves sides of γ_j for $j = 1, \ldots, r$. Then since N_C is orientable, h either preserves orientation of each γ_i or reverses orientation of each γ_i. On the other hand, we leave it as an exercise for the reader to check that if N_C is nonorientable then η is surjective.

The gluing map $N_C \to N$ induces a surjective homomorphism

$$\rho_C \colon \mathcal{M}(N_C) \to \mathrm{Stab}^+[C].$$

For $i = 1, \ldots, r$ let δ_i, δ_i' be the boundary components of a regular neighborhood A_i of γ_i. Note that if T_{δ_i}, $T_{\delta_i'}$ are right Dehn twists with respect to some orientation of A_i, then $T_{\delta_i}^{-1} T_{\delta_i'} \in \ker \rho_C$. For $i = r+1, \ldots, m$ let ϵ_i be the boundary curve of a regular neighborhood M_i of γ_i and note that $T_{\epsilon_i} \in \ker \rho_C$. By [24, Lemma 4.1] $\ker \rho_C$ is the free abelian group of rank m generated by $T_{\delta_i}^{-1} T_{\delta_i'}$ for $i = 1, \ldots, r$ and T_{ϵ_j} for $i = r+1, \ldots, m$. Summarizing, we have the following exact sequence

$$1 \to \mathbb{Z}^m \to \mathcal{M}(N_C) \xrightarrow{\rho_C} \mathrm{Stab}^+[C] \to 1. \tag{5.2}$$

Suppose that N_C is nonorientable and C consists entirely of one-sided curves ($r = 0$). Let N' be the surface obtained by cutting N along γ_i and gluing a disc with a puncture P_i along the resulting boundary component for $i = 1, \ldots, m$. Note that N may be seen as being obtained from N' by blowing up the punctures $\mathcal{P}_m = \{P_1, \ldots, P_m\}$, and we have the blowup homomorphism $\mathfrak{b} \colon \mathcal{PM}(N', \mathcal{P}_m) \to \mathcal{M}(N)$, whose image is contained in $\mathrm{Stab}[C]$.

Lemma 5.3. — *$\mathfrak{b} \colon \mathcal{PM}(N', \mathcal{P}_m) \to \mathrm{Stab}_{\mathcal{M}(N)}[C]$ is an isomorphism.*

Proof. — Since $\mathfrak{b}$ is injective by Proposition 2.4, it suffices to show that its image is equal to $\mathrm{Stab}_{\mathcal{M}(N)}[C]$. It follows immediately from the definitions that $\rho_C = \mathfrak{b} \circ \imath_*$, where $\imath_* \colon \mathcal{M}(N_C) \to \mathcal{M}^+(N', \mathcal{P}_m)$ is the map induced by the inclusion of N_C in N'. Thus the image of $\mathfrak{b}$ contains $\mathrm{Stab}^+[C]$. As N' is nonorientable by assumption, $\pi_1(N' \backslash (\mathcal{P}_m \backslash \{P_i\}), P_i)$ contains a homotopy class of one-sided loops, whose image under the crosscap pushing map $\mathfrak{c} \colon \pi_1(N' \backslash (\mathcal{P}_m \backslash \{P_i\}), P_i) \to \mathcal{M}(N)$ is a crosscap slide reversing the orientation of γ_i and equal to the identity on γ_j for $j \neq i$. It follows that the image of $\mathfrak{b}$ is equal to $\mathrm{Stab}[C]$. □

Let us identify $N_{g-1,n}$ with the surface obtained from N' by blowing up $\mathcal{P}_{m-1}$, and by abuse of notation, treat $C' = (\gamma_1, \dots, \gamma_{m-1})$ as a generic $(m-1)$-tuple of disjoint curves on $N_{g-1,n}$. Consider the following commutative diagram

$$\begin{array}{ccccc}
\pi_1(N' \backslash \mathcal{P}_{m-1}, P_m) & \xrightarrow{\mathfrak{p}} & \mathcal{PM}(N', \mathcal{P}_m) & \xrightarrow{\mathfrak{f}} & \mathcal{PM}(N', \mathcal{P}_{m-1}) \\
\| & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} \\
\pi_1(N' \backslash \mathcal{P}_{m-1}, P_m) & \xrightarrow{\mathfrak{c}} & \mathrm{Stab}_{\mathcal{M}(N_{g,n})}[C] & \xrightarrow{\zeta} & \mathrm{Stab}_{\mathcal{M}(N_{g-1,n})}[C']
\end{array}$$

whose top row is a part of the Birman exact sequence (2.1), $\mathfrak{c}$ is the crosscap pushing map and $\zeta = \mathfrak{b} \circ \mathfrak{f} \circ \mathfrak{b}^{-1}$. As the vertical maps are isomorphisms, exactness of (2.1) implies exactness of the sequence

$$(5.3) \quad 1 \to \pi_1(N' \backslash \mathcal{P}_{m-1}, P_m) \xrightarrow{\mathfrak{c}} \mathrm{Stab}_{\mathcal{M}(N_{g,n})}[C] \xrightarrow{\zeta} \mathrm{Stab}_{\mathcal{M}(N_{g-1,n})}[C'] \to 1.$$

5.3. Orbits and a presentation of $\mathcal{M}(N)$. — For the rest of this section we fix $g \geq 5$ and $N = N_{g,0}$. Let $\widetilde{X}$ denote $\mathcal{D}^{\mathrm{ord}}(N)$ if $g \geq 7$ or $\mathcal{C}_0^{\mathrm{ord}}(N)$ if $g \in \{5, 6\}$. Let $X = \widetilde{X}/\mathcal{M}(N)$ and $p \colon \widetilde{X} \to X$ be the canonical projection. Observe that X inherits from $\widetilde{X}$ the structure of a Δ-complex. Let $\mathcal{S}_m(X)$ (resp. $\mathcal{S}_m(\widetilde{X})$) be the set of m-simplices of X (resp. $\widetilde{X}$). The simplices of dimension 0, 1 and 2 will be called *vertices*, *edges* and *triangles* respectively. Observe that the canonical projection $p \colon \widetilde{X} \to X$ induces a surjection $p \colon \mathcal{S}_m(\widetilde{X}) \to \mathcal{S}_m(X)$. In the present subsection we will determine a section to p for $m = 0, 1, 2$, that is a map $s \colon \mathcal{S}_m(X) \to \mathcal{S}_m(\widetilde{X})$ such that $p \circ s = \textit{identity}$. We also describe a presentation of $\mathcal{M}(N)$ obtained by applying Brown's theorem to the action of $\mathcal{M}(N)$ on $\widetilde{X}$.

For $C = (\gamma_1, \dots, \gamma_m)$ and $I \subseteq \{1, \dots, m\}$ let $C_I = (\gamma_i)_{i \in I}$. The following proposition is a special case of [24, Proposition 5.2].

Proposition 5.4. — *Two simplices $[C] = [\gamma_1, \dots, \gamma_m]$ and $[C'] = [\gamma'_1, \dots, \gamma'_m]$ of $\widetilde{X}$ are $\mathcal{M}(N)$-equivalent if and only if the following two conditions are satisfied.*

TABLE 1. The edges of X.

e	$s(e)$	$s(t(e))$	h_e	$N_{s(e)}$	g
e_1	$[\alpha_1, \mu_g]$	$[\mu_g]$	1	$N_{g-3,3}$	≥ 5
e_2	$[\alpha_1, \alpha_3]$	$[\alpha_1]$	$a_2a_3a_1a_2$	$N_{g-4,4}$	≥ 5
e_3	$[\mu_g, \mu_{g-1}]$	$[\mu_g]$	a_{g-1}^{-1}	$N_{g-2,2}$	≥ 5
e_4	$[\alpha_1, \xi]$	$[\xi]$	1	$S_{1,g-2}$	$5, 6$
e_5	$[\mu_5, \beta_1]$	$[\alpha_1]$	$a_4ba_3a_4a_2a_3a_1a_2$	$S_{1,3}$	5
e_6	$[\alpha_1, \gamma_{\{3,4,5,6\}}]$	$[\alpha_1]$	$a_2ca_1a_2$	$S_{1,4}$	6
e_7	$[\mu_6, \gamma_{\{1,2,3,4,5\}}]$	$[\mu_6]$	b_2^{-1}	$S_{2,2}$	6

- *For every $i \in \{1,\dots,m\}$, γ_i is one-sided if and only if γ_i' is one-sided.*
- *For every $I \subseteq \{1,\dots,m\}$, the surface N_{C_I} is orientable if and only if $N_{C_I'}$ is orientable.* □

Note that the second condition is vacuous for $\widetilde{X} = \mathscr{D}^{\mathrm{ord}}(N)$, that is if $g \geq 7$. As an immediate corollary we see that every vertex of $\widetilde{X}$ is $\mathscr{M}(N)$-equivalent to one of the following (see Subsection 2.5 for definitions).

- $[\alpha_1]$ – two-sided curve with a non-orientable complement,
- $[\mu_g]$ – one-sided curve with a non-orientable complement,
- $[\xi]$ – curve with an orientable complement, one-sided for odd g or two-sided for even g.

We define

$$v_1 = p[\alpha_1],\ v_2 = p[\mu_g],\ v_3 = p[\xi], \qquad s(v_1) = [\alpha_1],\ s(v_2) = [\mu_g],\ s(v_3) = [\xi].$$

Note that $\mathscr{S}_0(X) = \{v_1, v_2\}$ if $g \geq 7$ or $\mathscr{S}_0(X) = \{v_1, v_2, v_3\}$ if $g \in \{5, 6\}$. If e is an edge of X or $\widetilde{X}$, then we denote by $i(e)$ and $t(e)$ its initial and terminal vertices respectively, and by $\overline{e}$ the edge with the same vertices as e but with the opposite orientation. We define edges $e_i \in \mathscr{S}_1(X)$ for $i \in \{1,\dots,7\}$ as $e_i = p(s(e_i))$, where $s(e_i)$ are defined in the second column of Table 1.

PROPOSITION 5.5. — *If $g \geq 7$ then $\mathscr{S}_1(X) = \{e_1, \overline{e_1}, e_2, e_3\}$;*
If $g = 5$ then $\mathscr{S}_1(X) = \{e_1, \overline{e_1}, e_2, e_3, e_4, \overline{e_4}, e_5, \overline{e_5}\}$;
If $g = 6$ then $\mathscr{S}_1(X) = \{e_1, \overline{e_1}, e_2, e_3, e_4, \overline{e_4}, e_6, e_7\}$.

Proof. — Let $[C] = [\gamma_1, \gamma_2]$ be any edge of $\widetilde{X}$. If N_C is nonorientable, then it follows easily from Proposition 5.4 that $[C]$ is $\mathscr{M}(N)$ equivalent to one of the edges $s(e_1)$, $\overline{s(e_1)}$, $s(e_2)$, or $s(e_3)$. This finishes the proof for $g \geq 7$. Suppose that N_C is orientable. There are two cases: (1) $N_{(\gamma_1)}$ and $N_{(\gamma_2)}$ are nonorientable; (2) $N_{(\gamma_1)}$ or $N_{(\gamma_2)}$ is orientable. In the case (1) C is $\mathscr{M}(N)$-equivalent to one

TABLE 2. The triangles of X.

f	$s(f)$	ν_1,ν_2,ν_3	$\varepsilon_1,\varepsilon_2,\varepsilon_3$	g
f_1	$[\alpha_1,\alpha_3,\alpha_5]$	v_1,v_1,v_1	e_2,e_2,e_2	≥ 6
f_2	$[\alpha_1,\alpha_3,\mu_g]$	v_1,v_1,v_2	e_2,e_1,e_1	≥ 5
f_3	$[\alpha_1,\mu_g,\mu_{g-1}]$	v_1,v_2,v_2	e_1,e_3,e_1	≥ 5
f_4	$[\mu_g,\mu_{g-1},\mu_{g-2}]$	v_2,v_2,v_2	e_3,e_3,e_3	≥ 5
f_5	$[\alpha_1,\alpha_3,\xi]$	v_1,v_1,v_3	e_2,e_4,e_4	$5,6$
f_6	$[\mu_6,\mu_5,\beta]$	v_2,v_2,v_1	$e_3,\overline{e_1},\overline{e_1}$	6
f_7	$[\mu_5,\mu_4,\gamma_{\{1,2,3\}}]$	v_2,v_2,v_2	e_3,e_3,e_3	5
f_8	$[\alpha_1,\mu_6,\gamma_{\{1,\dots,5\}}]$	v_1,v_2,v_2	e_1,e_7,e_1	6
f_9	$[\alpha_1,\mu_5,\beta]$	v_1,v_2,v_1	e_1,e_5,e_2	5
f_{10}	$[\alpha_1,\alpha_3,\gamma_{\{3,4,5,6\}}]$	v_1,v_1,v_1	e_2,e_2,e_6	6

of the edges $s(e_5)$, $\overline{s(e_5)}$, $s(e_6)$, or $s(e_7)$. Suppose that we are in case (2) and $N_{(\gamma_2)}$ is orientable. Since N_C is connected, there is a curve on N disjoint from γ_1 and intersecting γ_2 in one point. As such curve must be one-sided, $N_{(\gamma_1)}$ is nonorientable and $[C]$ is $\mathcal{M}(N)$-equivalent to $s(e_4)$. □

The representatives $s(e_i)$ of the edges e_i for $i \in \{1,\dots,7\}$ have been chosen in such a way that $i(s(e_i)) = s(i(e_i))$. The elements h_{e_i} defined in the fourth column of Table 1 satisfy $h_{e_i}(s(t(e_i)) = t(s(e_i))$. For $i \in \{1,4,5\}$ we define $s(\overline{e_i}) = h_{e_i}^{-1}(\overline{s(e_i)})$ and $h_{\overline{e_i}} = h_{e_i}^{-1}$. In this way, for every $e \in \mathcal{S}_1(X)$ we have $i(s(e)) = s(i(e))$ and $h_e(s(t(e)) = t(s(e))$. The conjugation map c_e defined as $c_e(x) = h_e^{-1}xh_e$ maps $\mathrm{Stab}\,t(s(e))$ onto $\mathrm{Stab}\,s(t(e))$; in particular $c_e(\mathrm{Stab}\,s(e)) \subset \mathrm{Stab}\,s(t(e))$.

Suppose that $\widetilde{f} = [\gamma_1,\gamma_2,\gamma_3] \in \mathcal{S}_2(\widetilde{X})$ and $f = p(\widetilde{f}) \in \mathcal{S}_2(X)$. For a permutation $\sigma \in \mathrm{Sym}_3$ we define $\widetilde{f}^\sigma = [\gamma_{\sigma(1)},\gamma_{\sigma(2)},\gamma_{\sigma(3)}]$ and $f^\sigma = p(\widetilde{f}^\sigma)$. We say that $\widetilde{f}^\sigma$ (resp. f^σ) is a permutation of $\widetilde{f}$ (resp. f). We also define $\varepsilon_1(f) = p[\gamma_1,\gamma_2]$, $\varepsilon_2(f) = p[\gamma_2,\gamma_3]$, $\varepsilon_3(f) = p[\gamma_1,\gamma_3]$, and $\nu_i(f) = p[\gamma_i]$ for $i = 1,2,3$.

We define triangles $f_i \in \mathcal{S}_2(X)$ for $i \in \{1,\dots,10\}$ as $f_i = p(s(f_i))$, where $s(f_i)$ are defined in the second column of Table 2.

PROPOSITION 5.6. — *Every triangle of X is a permutation of f_i for some $i \in \{1,\dots,10\}$.*

Proof. — Suppose that $f = p[C]$ for $C = (\gamma_1,\gamma_2,\gamma_3)$. If N_C is nonorientable, then by Proposition 5.4, f is determined up to permutation by the number of one-sided vertices. It follows that f is a permutation of one of the triangles: f_1

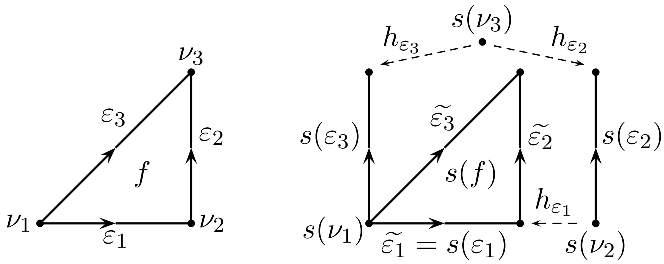

FIGURE 6. A triangle in X and its representative in $\widetilde{X}$.

(if $g \geq 7$), f_2 (if $g \geq 6$), f_3 or f_4. We assume that N_C is orientable, $g \in \{5, 6\}$. There are three cases.

Case 1: $N_{(\gamma_i,\gamma_j)}$ are nonorientable for all $1 \leq i < j \leq 3$. Then again f is determined up to permutation by the number of one-sided vertices, and it is a permutation of one of the triangles: f_1, f_2, f_6 or f_7.

Case 2: $N_{(\gamma_i)}$ is orientable for some $i \in \{1, 2, 3\}$. Assume $i = 3$. By the same argument as in the proof of Proposition 5.5 (case (2)), $N_{(\gamma_1,\gamma_2)}$ is nonorientable and $f = f_5$ by Proposition 5.4.

Case 3: f contains one of the edges: e_5, e_6 or e_7. Assume $p[\gamma_1, \gamma_2] = e_i$ for $i \in \{5, 6, 7\}$. Since $N_{(\gamma_1,\gamma_2)}$ is orientable, thus γ_3 is two-sided and it is easy to see, by similar argument as in the proof of Proposition 5.5 (case (2)), that $N_{(\gamma_i,\gamma_3)}$ are nonorientable for $i = 1, 2$. It follows that f is a permutation of one of the triangles f_8, f_9 or f_{10}. □

Let $f = f_i$ for some $i \in \{1, \dots, 10\}$. For $j = 1, 2, 3$ we let $\nu_j = \nu_j(f)$, $\varepsilon_j = \varepsilon_j(f)$ and define $\widetilde{\varepsilon_j}$ to be the edge of $s(f)$ such that $p(\widetilde{\varepsilon_j}) = \varepsilon_j$ (Figure 6). The representatives $s(f)$ have been chosen in such a way that $\widetilde{\varepsilon_1} = s(\varepsilon_1)$. For $j = 1, 2, 3$ we choose $x_j = x_j(f) \in \operatorname{Stab} s(\nu_j)$ such that

$$x_1(s(\varepsilon_3)) = \widetilde{\varepsilon_3}, \quad h_{\varepsilon_1} x_2(s(\varepsilon_2)) = \widetilde{\varepsilon_2}, \quad h_{\varepsilon_1} x_2 h_{\varepsilon_2} x_3 h_{\varepsilon_3}^{-1} = x_1. \tag{5.4}$$

For $\sigma \in \mathrm{Sym}_3$ we define $s(f^\sigma) = z_\sigma(s(f))^\sigma$ and $x_j(f^\sigma)$ according to the following table:

σ	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$	$(1,3,2)$
z_σ	$h_{\varepsilon_1}^{-1}$	$h_{\varepsilon_2}^{-1} x_2^{-1} h_{\varepsilon_1}^{-1}$	x_1^{-1}	$x_2^{-1} h_{\varepsilon_1}^{-1}$	$h_{\varepsilon_3}^{-1} x_1^{-1}$
$x_1(f^\sigma)$	x_2	x_3	x_1^{-1}	x_2^{-1}	x_3^{-1}
$x_2(f^\sigma)$	x_1	x_2^{-1}	x_3^{-1}	x_3	x_1^{-1}
$x_3(f^\sigma)$	x_3^{-1}	x_1	x_2^{-1}	x_1^{-1}	x_2

In this way the equations $\widetilde{\varepsilon_1} = s(\varepsilon_1)$ and (5.4) are satisfied for every $f \in \mathcal{S}_2(X)$. We check this for $\sigma = (1,2)$, the other cases can be checked similarly. Let $f' = f^{(1,2)}$, and for $j = 1,2,3$, $x'_j = x_j(f')$, $\varepsilon'_j = \varepsilon_j(f')$. We have $\varepsilon'_1 = \overline{\varepsilon_1}$, $\varepsilon'_2 = \varepsilon_3$, $\varepsilon'_3 = \varepsilon_2$, the edges of $s(f')$ are $\widetilde{\varepsilon'_1} = h_{\varepsilon_1}^{-1}(\widetilde{\overline{\varepsilon_1}})$, $\widetilde{\varepsilon'_2} = h_{\varepsilon_1}^{-1}(\widetilde{\varepsilon_3})$, $\widetilde{\varepsilon'_3} = h_{\varepsilon_1}^{-1}(\widetilde{\varepsilon_2})$, and $s(\varepsilon'_1) = s(\overline{\varepsilon_1}) = h_{\varepsilon_1}^{-1}(\overline{s(\varepsilon_1)}) = \widetilde{\varepsilon'_1}$,

$$\begin{aligned} x'_1(s(\varepsilon'_3)) = x_2(s(\varepsilon_2)) = h_{\varepsilon_1}^{-1}(\widetilde{\varepsilon_2}) = \widetilde{\varepsilon'_3}, \\ h_{\varepsilon'_1} x'_2(s(\varepsilon'_2)) = h_{\varepsilon_1}^{-1} x_1(s(\varepsilon_3)) = h_{\varepsilon_1}^{-1}(\widetilde{\varepsilon_3}) = \widetilde{\varepsilon'_2}, \\ h_{\varepsilon'_1} x'_2 h_{\varepsilon'_2} x'_3 h_{\varepsilon'_3}^{-1} = h_{\varepsilon_1}^{-1} x_1 h_{\varepsilon_3} x_3^{-1} h_{\varepsilon_2}^{-1} = x_2 = x'_1. \end{aligned}$$

The following theorem is a special case of a general result of Brown [6] (cf. [24, Theorem 6.3]).

THEOREM 5.7. — *Suppose that:*

(1) *for each* $v \in \mathcal{S}_0(X)$ *the stabilizer* $\operatorname{Stab} s(v)$ *admits a presentation* $\langle S_v \mid R_v \rangle$,
(2) *for each* $e \in \mathcal{S}_1(X)$ *the stabilizer* $\operatorname{Stab} s(e)$ *is generated by* G_e.

Then $\mathcal{M}(N)$ *admits a presentation with generators*

$$\bigcup_{v \in \mathcal{S}_0(X)} S_v \cup \{h_e \mid e \in \mathcal{S}_1(X)\}$$

and relations:

- $\bigcup_{v \in \mathcal{S}_0(X)} R_v$,
- $h_{e_1} = 1$ *and (if* $g \in \{5,6\}$*)* $h_{e_4} = 1$,
- $h_e^{-1} \imath_e(x) h_e = c_e(x)$ *for* $e \in \mathcal{S}_1(X)$ *and* $x \in G_e$, *where* $\imath_e \colon \operatorname{Stab} s(e) \to \operatorname{Stab} s(i(e))$ *is the inclusion and* $c_e \colon \operatorname{Stab} s(e) \to \operatorname{Stab} s(t(e))$ *is the conjugation map defined above,*
- $h_{\varepsilon_1(f)} x_2(f) h_{\varepsilon_2(f)} x_3(f) h_{\varepsilon_3(f)}^{-1} = x_1(f)$ *for* $f \in \mathcal{S}_2(X)$. □

Recall that in order to prove Theorem 3.6 we want to define a homomorphism $\psi \colon \mathcal{M}(N_{g,0}) \to \mathcal{G}_{g,0}$, which will be the inverse of $\varphi_{g,0}$ from Proposition 3.7. We will use the presentation from Theorem 5.7 and define ψ on the generators and prove that it respects the defining relations. To this end we need presentations of $\operatorname{Stab} s(v)$ and generators of $\operatorname{Stab} s(e)$ which will be obtained by using the exact sequences defined in the previous subsection and the induction hypothesis.

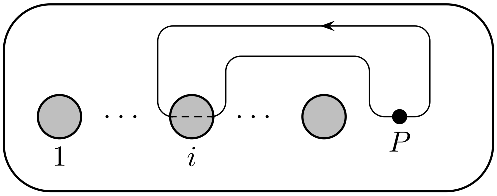

FIGURE 7. The loop η_i.

6. The stabilizer of $[\mu_g]$

In this section we are assuming $n \in \{0,1\}$, $g+n \geq 4$ and that Theorems 3.5 and 3.6 are true for $g-1$. Recall from Subsection 3.3 the definitions of $\mathscr{G}_{g,n}$ and $\varphi_{g,n}$.

THEOREM 6.1. — *The stabilizer* $\mathrm{Stab}[\mu_g] = \mathrm{Stab}_{\mathcal{M}(N_{g,n})}[\mu_g]$ *is generated by* u_i*,* a_i*,* b_j *for* $1 \leq i \leq g-2$*,* $2 \leq 2j \leq g-3$ *and* $a_{g-1}u_{g-1}$*. There is a homomorphism* $\psi_{v_2}\colon \mathrm{Stab}[\mu_g] \to \mathscr{G}_{g,n}$ *such that* $\varphi_{g,n} \circ \psi_{v_2} = \mathrm{id}_{\mathrm{Stab}[\mu_g]}$ *and* $\psi_{v_2}(x) = x$ *for each generator* x *of* $\mathrm{Stab}[\mu_g]$.

Proof. — We will obtain a presentation of $\mathrm{Stab}[\mu_g]$ by applying Lemma 2.1 to the exact sequence (5.3), which in this case is

$$1 \to \pi_1(N_{g-1,n}, P) \xrightarrow{\mathfrak{c}} \mathrm{Stab}[\mu_g] \xrightarrow{\zeta} \mathcal{M}(N_{g-1,n}) \to 1,$$

where we assume that $N_{g-1,n}$ was obtained by cutting $N_{g,n}$ along μ_g and then gluing a disc with puncture P along the resulting boundary component.

The kernel $\pi_1(N_{g-1,n}, P)$ is generated by the homotopy classes of the loops η_i in Figure 7 for $i = 1, \ldots, g-1$. Let $\sigma_i = \mathfrak{c}[\eta_i] = Y_{\mu_g, \gamma_{\{i,g\}}}$. If $n = 1$ then $\pi_1(N_{g-1,n}, P)$ is free, while if $n = 0$, then there is a single kernel relation:

$$\text{(K)} \qquad \sigma_{g-1}^2 \cdots \sigma_1^2 = 1.$$

By the induction hypothesis $\mathcal{M}(N_{g-1,n})$ admits the presentation given in Theorem 3.5 if $n = 1$ or 3.6 if $n = 0$. In the latter case we replace the relation (D) by (Da) (see Lemma 3.11). For the cokernel generators we take u_i, a_i and b_j for $1 \leq i \leq g-2$, $0 \leq 2j \leq g-3$. Observe that all defining relations of $\mathcal{M}(N_{g-1,n})$ are satisfied also in $\mathcal{M}(N_{g,n})$, except (B4) and (Da) if $n = 0$, in which case we have instead

($\widetilde{\text{B4}}$) $(u_1 \cdots u_{g-3})^{g-2} = \sigma_{g-1}^2$

($\widetilde{\text{Da}}$) $a_{g-2}(u_{g-3} \cdots u_1 a_1 \cdots a_{g-3}) a_{g-2} (u_{g-3} \cdots u_1 a_1 \cdots a_{g-3})^{-1} = \sigma_{g-1}^2 \sigma_{g-2} \sigma_{g-1}^{-1}$.

These relations hold in $\mathcal{M}(N_{g,0})$ because the corresponding relations, with each σ_i replaced by $\mathfrak{p}[\eta_i]$, hold in $\mathcal{M}(N_{g-1,0}, P)$. Indeed, by Lemma 2.2, $\mathfrak{p}[\eta_{g-1}^2]$ is equal to a product of two Dehn twists about the boundary curves of a regular neighborhood of a simple loop homotopic to η_{g-1}^2. One of these curves bounds a Möbius strip, and hence the twist is trivial, while the other curve bounds $N_{g-2,1}$ and the twist is equal to $\Delta_{g-2}^2 = (u_1 \cdots u_{g-3})^{g-2}$. Analogously, $\mathfrak{p}[\eta_{g-1}^2 \eta_{g-2} \eta_{g-1}^{-1}]$ is equal to the product of Dehn twists about the curves α_{g-2} and $u_{g-3} \cdots u_1 a_1 \cdots a_{g-3}(\alpha_{g-2})$.

To determine the conjugation relations we have to express $x\sigma_i x^{-1}$ in terms of the kernel generators for $i = 1, \dots, g-1$ and each cokernel generator x (it is not necessary to consider $x^{-1}\sigma_i x$ by the remark after Lemma 2.1). This can be done by first expressing $x[\eta_i]$ in the generators of $\pi_1(N_{g-1,n}, P)$, and then applying $\mathfrak{c}$ together with Lemma 2.3. Since every cokernel generator can be expressed in terms of u_i for $i = 1, \dots, g-2$, a_{g-2} and b, we only have to use these cokernel generators to produce the conjugation relations. As a result we obtain:

(1) $u_i \sigma_{i+1} u_i^{-1} = \sigma_i$,

(2) $u_i \sigma_i u_i^{-1} = \sigma_i^{-2} \sigma_{i+1} \sigma_i^2$,

(3) $u_i \sigma_j u_i^{-1} = \sigma_j$ for $j \neq i, i+1$,

(4) $a_{g-2} \sigma_{g-1} a_{g-2}^{-1} = \sigma_{g-1}^2 \sigma_{g-2}$,

(5) $a_{g-2} \sigma_{g-2} a_{g-2}^{-1} = \sigma_{g-2}^{-1} \sigma_{g-1}^{-1} \sigma_{g-2}$,

(6) $a_{g-2} \sigma_j a_{g-2}^{-1} = \sigma_j$ for $j < g-2$,

(7) $b \sigma_j b^{-1} = \sigma_j$ for $j > 4$,

(8) $b \sigma_4 b^{-1} = \sigma_4 \delta$,

(9) $b \sigma_4^2 \sigma_3 b^{-1} = \sigma_4^2 \sigma_3 \delta$,

(10) $b \sigma_4^2 \sigma_3^2 \sigma_2 b^{-1} = \sigma_4^2 \sigma_3^2 \sigma_2 \delta$,

(11) $b \sigma_4^2 \sigma_3^2 \sigma_2^2 \sigma_1 b^{-1} = \sigma_4^2 \sigma_3^2 \sigma_2^2 \sigma_1 \delta$,

where $\delta = \sigma_4 \sigma_3 \sigma_2 \sigma_1$.

We also have the following relations:

$$a_3 \sigma_4 a_3^{-1} = \sigma_4^2 \sigma_3, \quad a_2 a_3 \sigma_4 a_3^{-1} a_2^{-1} = \sigma_4^2 \sigma_3^2 \sigma_2$$
$$a_1 a_2 a_3 \sigma_4 a_3^{-1} a_2^{-1} a_1^{-1} = \sigma_4^2 \sigma_3^2 \sigma_2^2 \sigma_1, \quad a_j \delta a_j^{-1} = \delta, \text{ for } j = 1, 2, 3,$$

but since every a_i can be expressed in terms of a_{g-2} and u_j's by (C2), these relations are consequences of (C2), (1–6) and the kernel relation (K). Since b commutes with a_j for $j = 1, 2, 3$ (A3), the relations above together with (8) imply (9, 10, 11). This shows that (9, 10, 11) are redundant, they follow from other relations.

For $i > 1$, if we conjugate both sides of (2) by $u_i u_{i-1}$, then by using (B2), (1), (3) we obtain the relation $u_{i-1} \sigma_{i-1} u_{i-1}^{-1} = \sigma_{i-1}^{-2} \sigma_i \sigma_{i-1}^2$. If follows that we only need (2) with $i = g-2$. Therefore we replace (2) by

$$(2') \ u_{g-2} \sigma_{g-2} u_{g-2}^{-1} = \sigma_{g-2}^{-2} \sigma_{g-1} \sigma_{g-2}^2.$$

We claim that we can also replace (3) by

$$(3') \ u_i \sigma_{g-1} u_i^{-1} = \sigma_{g-1} \quad \text{for } i \leq g-3.$$

Indeed, if we set $x = u_j u_{j+1} \cdots u_{g-2}$, then $x \sigma_{g-1} x^{-1} = \sigma_j$ by (1), for $i < j-1$ we have $x u_i x^{-1} = u_i$ by (B1), and for $i > j$ we have $x u_{i-1} x^{-1} = u_i$ by (B1,B2).

Thus (3) follows from (3′) by applying conjugation by x. Similarly, it can be easily proved, using (1) and (C1a, C7a), that (6, 7) can be replaced by

$$(6')\ a_{g-2}\sigma_{g-3}a_{g-2}^{-1}=\sigma_{g-3}\qquad (7')\ b\sigma_{g-1}b^{-1}=\sigma_{g-1}\text{ if } g>5.$$

We have $\sigma_{g-1}=y_{g-1}=a_{g-1}u_{g-1}$, and $\sigma_i=(u_i\cdots u_{g-2})a_{g-1}u_{g-1}(u_i\cdots u_{g-2})^{-1}$ for $i<g-1$ by (1). It follows that Stab$[\mu_g]$ is generated by the elements listed in the theorem. To prove that the mapping $\psi_{v_2}(x)=x$ for x a generator of Stab$[\mu_g]$ extends to a homomorphism, we have to check that (K, $\widetilde{\text{B4}}$, $\widetilde{\text{Da}}$, 2′, 3′, 4, 5, 6′, 7′, 8) are satisfied in $\mathscr{G}_{g,n}$.

By (1), the kernel relation (K) can be rewritten using only the generators u_i and σ_{g-1}^2. Since $\sigma_{g-1}^2=(a_{g-1}u_{g-1})^2=u_{g-1}^2$ by (C4a), (K) is satisfied in $\mathscr{G}_{g,n}$ by Lemma 3.12, and so is ($\widetilde{\text{B4}}$). By (Da) in $\mathscr{G}_{g,n}$ we have

$$a_{g-1}^{-1}=(u_{g-2}\cdots u_1\underline{a_1\cdots a_{g-2})a_{g-1}(a_{g-2}^{-1}}\cdots a_1^{-1}u_1^{-1}\cdots u_{g-2}^{-1})\overset{(\text{B1,B2,C1})}{=}$$
$$u_{g-2}a_{g-1}^{-1}(u_{g-3}\cdots u_1a_1\cdots a_{g-3})a_{g-2}(a_{g-3}^{-1}\cdots a_1^{-1}u_1^{-1}\cdots u_{g-3}^{-1})a_{g-1}u_{g-2}^{-1}$$

and after substitution ($\widetilde{\text{Da}}$) becomes

$$a_{g-2}(u_{g-2}a_{g-1}^{-1})^{-1}a_{g-1}^{-1}(u_{g-2}a_{g-1}^{-1})=\sigma_{g-1}^2\sigma_{g-2}\sigma_{g-1}^{-1}.$$

Since σ_{g-1} and σ_{g-2} can be expressed in the generators a_{g-1}, a_{g-2}, u_{g-1}, u_{g-2}, the last relation holds in $\mathscr{G}_{g,n}$ by Corollary 4.2 and so do (2′, 4, 5). (3′) follows from (B1, C1a), (6′) follows from (C3, A1, C1a), (7′) follows from (A3, C7a). By (B1, B2, C1a, C2) we have

$$\sigma_4=(u_4\cdots u_{g-2})a_{g-1}u_{g-1}(u_4\cdots u_{g-2})^{-1}=(u_5\cdots u_{g-1})^{-1}a_4u_4(u_5\cdots u_{g-1})$$
$$\delta=(u_5\cdots u_{g-1})^{-1}a_4a_3a_2a_1u_1u_2u_3u_4(u_5\cdots u_{g-1})$$

and we see that (8) follows from (C8) and the fact that $(u_5\cdots u_{g-1})$ commutes with b (C7a). Thus ψ_{v_2} is a homomorphism and obviously $\varphi_{g,n}\circ\psi_{v_2}=\mathrm{id}_{\mathrm{Stab}[\mu_g]}$. □

LEMMA 6.2. — *If $g\geq 5$ then the following relation holds in $\mathscr{G}_{g,1}$.*

$$(\text{C9})\quad b(a_4a_3a_2a_1u_1u_2u_3u_4)=(a_4a_3a_2a_1u_1u_2u_3u_4)b.$$

Proof. — By (C8) we have $a_4a_3a_2a_1u_1u_2u_3u_4=(a_4u_4)^{-1}b(a_4u_4)b^{-1}$. (C9) is satisfied in $\mathscr{M}(N_{g,1})$ because $(a_4u_4)^{-1}b(a_4u_4)$ is a Dehn twist about the curve $(a_4u_4)^{-1}(\beta)$, which is disjoint from β up to isotopy. Since b, $a_4u_4\in$ Stab$[\mu_g]$, (C9) also holds in $\mathscr{G}_{g,1}$ by Theorem 6.1. □

We define $\mathcal{S}_{g,n}(v_2)$ to be the image of ψ_{v_2} in $\mathcal{G}_{g,n}$. By Theorem 6.1 ψ_{v_2} is an isomorphism onto $\mathcal{S}_{g,n}(v_2)$, whose inverse is the restriction of $\varphi_{g,n}$.

Recall from Subsection 3.3 that $\mathcal{G}^1_{g,0}$ is the quotient of $\mathcal{G}_{g,1}$ by the normal closure of Δ_g^2 and $\varphi^1_{g,0}\colon \mathcal{G}^1_{g,0} \to \mathcal{M}^+(N_{g,0}, P)$ is a homomorphism induced by $\varphi_{g,1}$.

Lemma 6.3. — *Δ_g^2 is central in $\mathcal{G}_{g,1}$.*

Proof. — Since $u_{g-1}^2 = (a_{g-1}u_{g-1})^2$ by (C4a), $u_{g-1}^2 \in \mathcal{S}_{g,n}(v_2)$. By Lemma 3.4, $\Delta_g^2 \in \mathcal{S}_{g,1}(v_2)$. Since $\varphi_{g,1}(\Delta_g^2)$ is a Dehn twist about the boundary of $N_{g,1}$, it is central in $\mathcal{M}(N_{g,1})$, and since the restriction of $\varphi_{g,1}$ to $\mathcal{S}_{g,1}(v_2)$ is an isomorphism, Δ_g^2 is central in $\mathcal{S}_{g,1}(v_2)$. Note that $\mathcal{G}_{g,1}$ is generated by $\mathcal{S}_{g,1}(v_2)$, u_{g-1}, and if $g = 4$ then also b. By (B5) Δ_g^2 commutes with u_{g-1} and it remains to prove that it commutes with b if $g = 4$. By (C6a) b commutes with $(a_1a_2a_3)^2(u_1u_2u_3)^2$, and by (A3) it commutes with $(a_1a_2a_3)^2$, hence it commutes with $(u_1u_2u_3)^2$ and with $\Delta_4^2 = (u_1u_2u_3)^4$. ☐

Corollary 6.4. — *$\varphi_{g,1}$ is an isomorphism if and only if $\varphi^1_{g,0}$ is an isomorphism.* ☐

Proof. — By Lemma 6.3 the normal closure of Δ_g^2 is a cyclic subgroup of $\mathcal{G}_{g,1}$. Moreover, as $\ker \imath_*$ is infinite cyclic, the restriction of $\varphi_{g,1}$ to the subgroup of $\mathcal{G}_{g,1}$ generated by Δ_g^2 is an isomorphism onto $\ker \imath_*$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{G}_{g,1} & \longrightarrow & \mathcal{G}^1_{g,0} & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \varphi_{g,1}} & & \downarrow{\scriptstyle \varphi^1_{g,0}} & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{M}(N_{g,1}) & \xrightarrow{\imath_*} & \mathcal{M}^+(N_{g,0}, P) & \longrightarrow & 1. \end{array}$$

Since the rows are exact, the corollary follows from the five lemma. ☐

Let $\mathcal{S}^1_{g,0}(v_2)$ be the image of $\mathcal{S}_{g,1}(v_2)$ under the canonical projection $p\colon \mathcal{G}_{g,1} \to \mathcal{G}^1_{g,0}$. Since $\mathrm{Stab}_{\mathcal{M}^+(N_{g,0},P)}[\mu_g] = \imath_*(\mathrm{Stab}_{\mathcal{M}(N_{g,1})}[\mu_g])$, there is an isomorphism

$$\psi^1_{v_2}\colon \mathrm{Stab}_{\mathcal{M}^+(N_{g,0},P)}[\mu_g] \to \mathcal{S}^1_{g,0},$$

such that $\imath_* \circ \psi^1_{v_2} = \psi_{v_2} \circ p$, and whose inverse is the restriction of $\varphi^1_{g,0}$.

7. Proof of Theorem 3.5

In this section we assume that $g \geq 4$ is fixed, Theorem 3.6 is true for g and Theorem 3.5 is true for $g-1$. The last assumption implies that Theorem 6.1 is true for g, as well as Lemma 6.2 and Corollary 6.4. Theorem 3.5 for g will follow from Corollary 6.4 and the following theorem.

THEOREM 7.1. — *$\varphi^1_{g,0}\colon \mathscr{G}^1_{g,0} \to \mathscr{M}^+(N_{g,0}, P)$ is an isomorphism.*

Proof. — First we will obtain a presentation for $\mathscr{M}(N_{g,0}, P)$ by applying Lemma 2.1 to the Birman exact sequence (2.1)

$$1 \to \pi_1(N_{g,0}, P) \xrightarrow{\mathfrak{p}} \mathscr{M}(N_{g,0}, P) \xrightarrow{\mathfrak{f}} \mathscr{M}(N_{g,0}) \to 1$$

and the presentation of $\mathscr{M}(N_{g,0})$ given in Theorem 3.6. Then we will apply the Reidemeister-Schreier method to find a presentation of $\mathscr{M}^+(N_{g,0}, P)$, which is an index 2 subgroup of $\mathscr{M}(N_{g,0}, P)$.

To obtain a presentation for $\mathscr{M}(N_{g,0}, P)$ we proceed in the same way as we did in the proof of Theorem 6.1, with the following differences: (1) we use the sequence (2.1) instead of (5.3); (2) in the presentation of $\mathscr{M}(N_{g,0})$ we use (D) instead of (Da), and we replace the relation (B4) by the equivalent relation

$$\text{(B4b)}\ (u_2 \cdots u_{g-1})^{g-1} = 1,$$

obtained by conjugating (B4) by Δ_g; (3) to produce the conjugation relations we use the cokernel generators u_i for $i = 1, \dots, g-1$, b and a_1^{-1} (instead of a_{g-1}). As a result we obtain a presentation with kernel generators $\sigma_i = \mathfrak{p}[\eta_i]$ for $i \in \{1, \dots, g\}$ and cokernel generators u_i, a_i and b_j for $1 \leq i \leq g-1$, $0 \leq 2j \leq g-2$. There is the single kernel relation

$$\text{(K)} \qquad \sigma_g^2 \cdots \sigma_1^2 = 1,$$

the cokernel relations are the defining relations of $\mathscr{M}(N_{g,0})$ except for (B4b) and (D), instead of which we have

$(\widetilde{\text{B4b}})$ $(u_2 \cdots u_{g-1})^{g-1} = \sigma_1^2$

$(\widetilde{\text{D}})$ $a_1(a_2 \cdots a_{g-1}u_{g-1} \cdots u_2)a_1(a_2 \cdots a_{g-1}u_{g-1} \cdots u_2)^{-1} = (\sigma_2\sigma_1)^{-1}$.

By Lemma 6.2, if $g \geq 5$ then (C9) is a consequence of the cokernel relations.

The conjugation relations are

(1) $u_i\sigma_{i+1}u_i^{-1} = \sigma_i$, (2) $u_i\sigma_iu_i^{-1} = \sigma_i^{-2}\sigma_{i+1}\sigma_i^2$,

(3) $u_i\sigma_ju_i^{-1} = \sigma_j$ for $j \neq i, i+1$, (4) $a_1^{-1}\sigma_1a_1 = \sigma_2\sigma_1^2$,

(5) $a_1^{-1}\sigma_2a_1 = \sigma_2\sigma_1^{-1}\sigma_2^{-1}$, (6) $a_1^{-1}\sigma_ia_1 = \sigma_i$ for $i > 2$,

(7) $b^{-1}\sigma_jb = \sigma_j$ for $j > 4$, (8) $b^{-1}\sigma_1b = \delta\sigma_1$,

(9) $b^{-1}\sigma_2\sigma_1^2b = \delta\sigma_2\sigma_1^2$, (10) $b^{-1}\sigma_3\sigma_2^2\sigma_1^2b = \delta\sigma_3\sigma_2^2\sigma_1^2$,

(11) $b^{-1}\sigma_4\sigma_3^2\sigma_2^2\sigma_1^2 b = \delta\sigma_4\sigma_3^2\sigma_2^2\sigma_1^2$, where $\delta = \sigma_4\sigma_3\sigma_2\sigma_1$.

For $i > 1$ and $j \notin \{i, i+1\}$ we have

$$a_i^{-1}\sigma_i a_i = \sigma_{i+1}\sigma_i^2, \quad a_i^{-1}\sigma_{i+1}a_i = \sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}, \quad a_i^{-1}\sigma_j a_i = \sigma_j.$$

Since a_i can be expressed in terms of a_1 and the u_k's by (C2) or (C3), these relations follow from (C2,C3), (1–6) and (K). As a consequence of the above relations we have

$$a_1^{-1}\sigma_1 a_1 = \sigma_2\sigma_1^2, \quad a_2^{-1}a_1^{-1}\sigma_1 a_1 a_2 = \sigma_3\sigma_2^2\sigma_1^2$$
$$a_3^{-1}a_2^{-1}a_1^{-1}\sigma_1 a_1 a_2 a_3 = \sigma_4\sigma_3^2\sigma_2^2\sigma_1^2, \quad a_j^{-1}\delta a_j = \delta, \text{ for } j = 1, 2, 3.$$

These relations together with (8) and (A3) imply (9,10,11). Hence the last relations are redundant. We also have

$$(a_2a_1a_3a_2)^{-1}\sigma_2\sigma_1(a_2a_1a_3a_2) = (a_1a_3a_2)^{-1}(\sigma_3\sigma_2^2\sigma_1)(a_1a_3a_2) =$$
$$(a_3a_2)^{-1}(\sigma_3\sigma_2)(a_3a_2) = a_2^{-1}(\sigma_4\sigma_3^2\sigma_2)a_2 = \sigma_4\sigma_3.$$

It follows that (8) can be replaced by

$$(8')\ b^{-1}\sigma_1 b = (a_2a_1a_3a_2)^{-1}\sigma_2\sigma_1(a_2a_1a_3a_2)\sigma_2\sigma_1^2.$$

Similarly as in the proof of Theorem 6.1 it can be proved that (2,3,6,7) can be replaced by

$$(2')\ u_1\sigma_1u_1^{-1} = \sigma_1^{-2}\sigma_2\sigma_1^2, \quad (3')\ u_i\sigma_1u_i^{-1} = \sigma_1 \text{ for } i \geq 2,$$
$$(6')\ a_1^{-1}\sigma_3 a_1 = \sigma_3, \quad (7')\ b\sigma_5 b^{-1} = \sigma_5.$$

We have

$$a_1^{-1}\sigma_3 a_1 \overset{(1)}{=} a_1^{-1}u_2^{-1}u_1^{-1}\sigma_1 u_1 u_2 a_1 \overset{(C3)}{=} u_2^{-1}u_1^{-1}a_2^{-1}\sigma_1 a_2 u_1 u_2.$$

Therefore we can replace $(6')$ by $(6'')$ $a_2^{-1}\sigma_1 a_2 = \sigma_1$. By (1), the relation (5) is equivalent to

$$a_1^{-1}u_1^{-1}\sigma_1 u_1 a_1 = u_1^{-1}\sigma_1 u_1\sigma_1^{-1}u_1^{-1}\sigma_1^{-1}u_1 \overset{(C4)}{\Longleftrightarrow}$$
$$a_1\sigma_1 a_1^{-1} = \sigma_1 u_1\sigma_1^{-1}u_1^{-1}\sigma_1^{-1} \overset{(1)}{\Longleftrightarrow} a_1\sigma_1 a_1^{-1} = \sigma_1 u_1\sigma_1^{-1}\sigma_2^{-1}u_1^{-1}\ (5').$$

Let $z = a_4a_3a_2a_1u_1u_2u_3u_4$. We have

$$\sigma_5 \overset{(1)}{=} (u_1u_2u_3u_4)^{-1}\sigma_1(u_1u_2u_3u_4) = z^{-1}(a_4a_3a_2a_1)\sigma_1(a_4a_3a_2a_1)^{-1}z.$$

Since z commutes with b by (C9), we can replace $(7')$ by

$$(7'')\ b^{-1}(a_4a_3a_2a_1)\sigma_1(a_4a_3a_2a_1)^{-1}b = (a_4a_3a_2a_1)\sigma_1(a_4a_3a_2a_1)^{-1}.$$

Summarising, we have reduced the conjugation relations to the following ones, which are rewritten in a convenient way.

(R1) $u_i\sigma_{i+1}u_i^{-1} = \sigma_i$ for $i = 1, \ldots, g-1$,

(R2) $\sigma_1 u_1 \sigma_1^{-1} = (\sigma_2\sigma_1)^{-1}\sigma_1^2 u_1$,
(R3) $\sigma_1 u_i \sigma_1^{-1} = u_i$ for $i \geq 2$,
(R4) $\sigma_1 a_1 \sigma_1^{-1} = a_1(\sigma_2\sigma_1)$,
(R5) $\sigma_1^{-1} a_1 \sigma_1 = u_1(\sigma_2\sigma_1)^{-1}u_1^{-1}a_1$,
(R6) $\sigma_1 a_2 \sigma_1^{-1} = a_2$,
(R7) $\sigma_1(a_4a_3a_2a_1)^{-1}b(a_4a_3a_2a_1)\sigma_1^{-1} = (a_4a_3a_2a_1)^{-1}b(a_4a_3a_2a_1)$,
(R8) $\sigma_1 b \sigma_1^{-1} = b(a_2a_1a_3a_2)^{-1}\sigma_2\sigma_1(a_2a_1a_3a_2)\sigma_2\sigma_1$.

Since all defining relations of $\mathscr{G}^1_{g,0}$ appear as cokernel relations in our presentation, the relations (E2a, E3) from Lemma 3.9 are consequences of the cokernel relations. Let $r = r_g$ and note that $(\widetilde{\mathrm{D}})$ can be rewritten as

$$\sigma_2\sigma_1 = a_1^{-1}ru_1^{-1}a_1^{-1}u_1r^{-1} \overset{\text{C4,E2a}}{=} a_1^{-1}ra_1r.$$

Claim 1: (R7) *is redundant*

Proof of Claim 1. — By (A1–A4) we have

$$(a_4a_3a_2a_1)^{-1}b(a_4a_3a_2a_1) = (ba_4a_3a_2)a_1(ba_4a_3a_2)^{-1},$$

thus (R7) is equivalent to

$$\sigma_1(ba_4a_3a_2)a_1(ba_4a_3a_2)^{-1}\sigma_1^{-1} = (ba_4a_3a_2)a_1(ba_4a_3a_2)^{-1}.$$

It is a consequence of (R3), (R6) and (C3) that σ_1 commutes with a_2, a_3 and a_4. It follows that the relation above is equivalent to

$$(b^{-1}\sigma_1 b\sigma_1^{-1})a_4a_3a_2(\sigma_1a_1\sigma_1^{-1})(a_4a_3a_2)^{-1}(\sigma_1b^{-1}\sigma_1^{-1}b) = (a_4a_3a_2)a_1(a_4a_3a_2)^{-1}.$$

Let L denote the left hand side of the last relation. By $(\widetilde{D}$, R4, R8) we have

$$\begin{aligned}L = ((a_2a_1a_3a_2)^{-1}a_1^{-1}ra_1r(a_2a_1a_3a_2)a_1^{-1}ra_1r)a_4a_3a_2(ra_1r)\cdot\\ a_2^{-1}a_3^{-1}a_4^{-1}(ra_1^{-1}ra_1(a_2a_1a_3a_2)^{-1}ra_1^{-1}ra_1(a_2a_1a_3a_2)),\end{aligned}$$

and since

$$\begin{aligned}&(a_2a_1a_3a_2)a_1^{-1}ra_1\underline{ra_4a_3a_2r}a_1\underline{ra_2^{-1}a_3^{-1}a_4^{-1}r}a_1^{-1}ra_1(a_2a_1a_3a_2)^{-1}\\ &\overset{\text{(E2a,E3)}}{=} (a_2a_1a_3a_2)a_1^{-1}\underline{ra_1a_4a_3a_2a_1a_2^{-1}a_3^{-1}a_4^{-1}a_1^{-1}}ra_1(a_2a_1a_3a_2)^{-1}\\ &\overset{\text{(A1,A2)}}{=} (a_2a_1a_3a_2)a_1^{-1}\underline{ra_2^{-1}a_3^{-1}a_4a_3a_2r}a_1(a_2a_1a_3a_2)^{-1}\\ &\overset{\text{(E2a,E3)}}{=} (a_2a_1a_3a_2)a_1^{-1}a_2^{-1}a_3^{-1}a_4a_3a_2a_1(a_2a_1a_3a_2)^{-1}\\ &\overset{\text{(A1,A2)}}{=} a_3^{-1}a_4a_3,\end{aligned}$$

thus

$$\begin{aligned}L &= a_2^{-1}a_3^{-1}a_1^{-1}a_2^{-1}\underline{a_1^{-1}ra_1ra_3^{-1}a_4a_3ra_1^{-1}ra_1}a_2a_1a_3a_2\\ &\overset{\text{(E2a,E3,A1)}}{=} a_2^{-1}a_3^{-1}a_1^{-1}a_2^{-1}a_3^{-1}a_4a_3a_2a_1a_3a_2\end{aligned}$$

$$\overset{(A1,A2)}{=} (a_4a_3a_2)a_1(a_4a_3a_2)^{-1}.$$

This ends the proof of Claim 1, which allows us to rule out (R7) from the presentation. Then we make the following transformations.

(1) By using the relations (R1) and ($\widetilde{\text{B4b}}$) we rewrite (K) in the generators u_i. By Lemma 3.12, (K) can be removed from the presentation.

(2) In (C8) we replace $a_4a_3a_2a_1u_1u_2u_3u_4$ by the right hand side of the following equation, which is a consequence of (B1, B2, C2).

$$a_4u_4(u_4^{-1}a_3u_3u_4)(u_4^{-1}u_3^{-1}a_2u_2u_3u_4)(u_4^{-1}u_3^{-1}u_2^{-1}a_1u_1u_2u_3u_4) =$$
$$a_4u_4(u_3a_4u_4u_3^{-1})(u_2u_3a_4u_4u_3^{-1}u_2^{-1})(u_1u_2u_3a_4u_4u_3^{-1}u_2^{-1}u_1^{-1})$$

In this way (C8) is expressed in the generators of $\text{Stab}[\mu_g]$.

(3) In (C6) we replace $(a_1a_2a_3)^2(u_1u_2u_3)^2$ by its expression in the generators of $\text{Stab}[\mu_g]$ resulting from the following equations.

$$(a_1a_2a_3)^2(u_1u_2u_3)^2 \overset{(C1a,C3)}{=} a_1a_2a_3a_1a_2(u_1u_2u_3)^2a_1 \overset{(A1,B1,B2)}{=}$$
$$a_1a_2a_1a_3a_2(u_2u_3u_1u_2u_3^2)a_1 = a_1a_2a_1a_3u_3(u_3^{-1}a_2u_2u_3)u_1u_2u_3^2a_1 \overset{(B2,C2)}{=}$$
$$a_1a_2a_1a_3u_3(u_2a_3u_3u_2^{-1})u_1u_2u_3^2a_1 \overset{(C4a)}{=} a_1a_2a_1(a_3u_3)u_2(a_3u_3)u_2^{-1}u_1u_2(a_3u_3)^2a_1.$$

(4) Using Lemma 3.4, we replace $(u_1\cdots u_{g-1})^g$ by

$$u_{g-1}^2(u_{g-2}u_{g-1}^2u_{g-2})\cdots(u_1\cdots u_{g-1}^2\cdots u_1)$$

in (B3), and $(u_2\cdots u_{g-1})^{g-1}$ by

$$u_{g-1}^2(u_{g-2}u_{g-1}^2u_{g-2})\cdots(u_2\cdots u_{g-1}^2\cdots u_2)$$

in ($\widetilde{\text{B4b}}$). Note that these are expressions in the generators of $\text{Stab}[\mu_g]$.

(5) We replace $(\sigma_2\sigma_1)$ by $a_1^{-1}ra_1r$ in (R2, R4, R5, R8) and by $u_1^{-1}\sigma_1u_1\sigma_1^{-1}\sigma_1^2$ in ($\widetilde{\text{D}}$), then we rule out the generators σ_i for $i > 1$ together with (R1).

(6) If $g = 4$, then we replace the right hand side of (R8) by $b(a_1a_2a_3)^{-4}$. We have to prove that this yields an equivalent relation. Let $w = a_1a_2a_3$, $z = u_3u_2u_1$. We have $r = wz$ and

$$ra_1^{-1}ra_1 = wza_2a_3za_1 \overset{(C1a,C2)}{=} wa_1a_2z^2a_1 \overset{(B1,B2)}{=} wa_1a_2(u_3^2u_2u_3u_1u_2)a_1$$
$$\overset{(C1a,C3)}{=} wa_1a_2a_3(u_3^2u_2u_3u_1u_2) = w^2z^2.$$

Hence, the inverse of the right hand side of (R8) times b equals

$$w^2z^2\underline{(a_2a_1a_3a_2)^{-1}w^2}z^2(a_2a_1a_3a_2) \overset{(A1,A2)}{=} w^2z^2a_3^2z^2(a_2a_1a_3a_2)$$
$$\overset{(B1,B2)}{=} w^2\underline{(u_3^2u_2u_1u_3u_2)a_3^2}z^2(a_2a_1a_3a_2)$$
$$\overset{(C1,C2)}{=} w^2a_1^2z^4a_2a_1a_3a_2$$

$$\overset{(B3)}{=} w^2a_1^2a_2a_1a_3a_2 \overset{(A1,A2)}{=} (a_1a_2a_3)^4.$$

(7) In (A9) we replace b_ρ by the expression in the generators b and a_i resulting from (A7,A8). Then we rule out the generators b_j for $j \neq 1$ together with (A7, A8).

As a result of these transformations we obtain a presentation with generators a_i, u_i for $i = 1, \ldots, g-1$, b and σ_1, which we will denote simply as σ. Until the end of this proof, we will denote $\mathscr{M}(N_{g,0}, P)$ as $\mathscr{M}$, $\mathscr{M}^+(N_{g,0}, P)$ as $\mathscr{M}^+$, $\mathscr{G}^1_{g,0}$ as $\mathscr{G}$, and $\varphi^1_{g,0}$ as φ.

In the next step we obtain a presentation for $\mathscr{M}^+$ by the Reidemeister-Shreier method, for which we take $\{1, \sigma\}$ as a transversal of $\mathscr{M}/\mathscr{M}^+$. The generators of $\mathscr{M}^+$ are:

$$b,\ a_i,\ u_i,\ b' = \sigma b \sigma^{-1},\ a_i' = \sigma a_i \sigma^{-1},\ u_i' = \sigma u_i \sigma^{-1},\ \sigma^2,$$

for $i = 1, \ldots, g-1$. Note that every defining relation (Rel) of $\mathscr{M}$ can be regarded as a relation in the generators of $\mathscr{M}^+$ after replacing every sub-word $\sigma x \sigma^{-1}$ by the generator x' for $x \in \{a_i, u_i, b\}$, and every $\sigma^{-1} x \sigma$ by $\sigma^{-2} x' \sigma^2$. Conjugating (Rel) by σ yields another relation (Rel′), obtained by replacing in (Rel) every $x \in \{a_i, u_i, b\}$ by x', and every x' by $\sigma^2 x \sigma^{-2}$. The relations (Rel) and (Rel′), for all defining relations (Rel) of $\mathscr{M}$, are the defining relations of $\mathscr{M}^+$.

Let $G_{\mathscr{M}^+} = \{a_i, u_i, a_i', u_i' \,|\, 1 \leq i \leq g-1\} \cup \{b, b', \sigma^2\}$ denote the set of generators of $\mathscr{M}^+$. We define $\psi\colon G_{\mathscr{M}^+} \to \mathscr{G}$ as

$$\begin{aligned}
&\psi(x) = x \quad \text{for } x \in \{a_i, u_i \,|\, i = 1, \ldots, g-1\} \cup \{b\},\\
&\psi(a_i') = a_i, \quad \psi(u_i') = u_i \quad \text{for } i = 2, \ldots, g-1,\\
&\psi(\sigma^2) = u_{g-1}^2(u_{g-2}u_{g-1}^2u_{g-2}) \cdots (u_2 \cdots u_{g-1}^2 \cdots u_2),\\
&\psi(a_1') = ra_1r,\\
&\psi(u_1') = ra_1^{-1}ra_1\psi(\sigma^2)u_1,\\
&\psi(b') = b(a_2a_1a_3a_2)^{-1}a_1^{-1}ra_1r(a_2a_1a_3a_2)a_1^{-1}ra_1r.
\end{aligned}$$

Observe that for $x \in G_{\mathscr{M}^+}$ we have $\varphi(\psi(x)) = x$. In order to show that ψ can be extended to a homomorphism $\psi\colon \mathscr{M}^+ \to \mathscr{G}$, we have to show that it respects the defining relations of $\mathscr{M}^+$.

It is obvious that ψ respects the relations (A1–A6, A9, B1–B3, C1–C8, $\widetilde{\text{B4}}$b, R2, R3, R4, R6, R8).

Let $\mathscr{S} = \mathscr{S}^1_{g,0}(v_2)$ and recall that this is the subgroup of $\mathscr{G}$ generated by a_i, u_i for $i \in \{1, \ldots, g-2\}$, $a_{g-1}u_{g-1}$ and if $g > 4$ then also b. Suppose that $w = x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k}$, where $x_i \in G_{\mathscr{M}^+}$ and $\varepsilon_i \in \{1, -1\}$. We say that w *is expressible in* $\mathscr{S}$ if $\psi(x_1)^{\varepsilon_1} \cdots \psi(x_k)^{\varepsilon_k} \in \mathscr{S}$, and a relation in $\mathscr{M}^+$ is expressible in $\mathscr{S}$ if its both sides are expressible in $\mathscr{S}$. If $x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k} = 1$ is a relation expressible in $\mathcal{S}$, then since $\varphi(\psi(x_i)) = x_i$ and the restriction of φ to $\mathcal{S}$ is an isomorphism, $\psi(x_1)^{\varepsilon_1} \cdots \psi(x_k)^{\varepsilon_k} = 1$. Thus ψ respects the relations expressible in $\mathcal{S}$.

Set $G_{\mathcal{S}+} = \{a_i, u_i, a_i', u_i' \,|\, 1 \le i \le g-2\} \cup \{\sigma^2, a_{g-1}u_{g-1}, a'_{g-1}u'_{g-1}, u_{g-1}^2, {u'}_{g-1}^2\}$ plus $\{b, b'\}$ if $g > 4$. Observe that every element of $G_{\mathcal{S}+}$ is expressible in $\mathcal{S}$, hence every word on $G_{\mathcal{M}+} \cup G_{\mathcal{M}+}^{-1}$ obtained as a concatenation of elements of $G_{\mathcal{S}+} \cup G_{\mathcal{S}+}^{-1}$ is expressible in $\mathcal{S}$. It follows that the following relations are expressible in $\mathcal{S}$, hence are respected by ψ: (R2′, R4′, R5, R5′, R6′, $\widetilde{\mathrm{D}}$, $\widetilde{\mathrm{D}}'$, B3′, $\widetilde{\mathrm{B4b}}$', C4′, C5′, C8′) and (R8′) if $g > 4$. (R3′) is expressible in $\mathcal{S}$ for $i < g-1$, and for $i = g-1$ it is $\sigma^2 u_{g-1} \sigma^{-2} = u'_{g-1}$. Since $\psi(u'_{g-1}) = \psi(u_{g-1}) = u_{g-1}$ and $\psi(\sigma^2)$ is a word in the u_i's, thus ψ respects (R3′) for $i = g-1$ by Lemma 3.12. □

Claim 2. (C6′) *is expressible in* $\mathcal{S}$

Proof of Claim 2. — The right hand side of (C6′) (after transformation (3)) is a concatenation of elements of $G_{\mathcal{S}+} \cup G_{\mathcal{S}+}^{-1}$, thus it is expressible in $\mathcal{S}$. It suffices to show that the same is true for the left hand side. There is nothing to do if $g > 4$, so we assume $g = 4$. Let $w = (a_1a_2a_3)$.

$$\begin{aligned}
(\psi(u_3')\psi(b'))^2 &= (u_3bw^{-4})^2 \overset{\text{(A3)}}{=} u_3w^{-4}(bu_3)^2u_3^{-1}w^{-4} \\
&\overset{\text{(C6a)}}{=} u_3w^{-2}(u_1u_2u_3u_1u_2)w^{-4} = u_3\underline{w^{-2}\Delta_4}u_1^{-1}w^{-4} \\
&\overset{\text{(E1)}}{=} u_3\Delta_4\underline{w^2u_1^{-1}w^{-4}} \\
&\overset{\text{(C1a,C5a)}}{=} u_3\Delta_4u_3^{-1}w^{-2} = u_3^2u_2u_1u_3u_2w^{-2} \\
&\overset{\text{(A1,A2)}}{=} u_3^2u_2u_1(u_3u_2a_2^{-1}a_3^{-1})(a_1a_2)^{-2} \\
&= u_3^2u_2u_1(u_3u_2a_2^{-1}u_3^{-1})u_3a_3^{-1}(a_1a_2)^{-2} \\
&\overset{\text{(B2,C3,C4a)}}{=} u_3^2u_2u_1(u_2^{-1}a_3u_3u_2)a_3u_3(a_1a_2)^{-2} \in \mathcal{S}.
\end{aligned}$$
□

Claim 3. We have $\psi(b') = rb^{-1}(a_1a_2a_3)^4r$

Proof of Claim 3. — If $g = 4$ then we have $\psi(b') = b(a_1a_2a_3)^{-4}$ and the claim follows from (G3) in Lemma 3.13. By looking at the effect of $r\sigma$ on the curve β it can be checked that $r\sigma(\beta)$ and β are isotopic to the boundary curves of a regular neighborhood of the union of α_i for $i \in \{1,2,3\}$. Hence the chain relation $brb'r = (a_1a_2a_3)^4$ holds in $\mathcal{M}^+(N_{g,0}, P)$. If $g > 4$ then the last relation is expressible in $\mathcal{S}$ and the claim follows.

It follows from Claim 3 and $\psi(a_i') = ra_ir$ (E3), that if w' is a word in a_i', b' and their inverses, then $\psi(w') = rwr$, where w is a word in a_i, b and their inverses. If w' represents the trivial element of $\mathcal{M}^+$, then so does w, and by Lemma 3.12 $\psi(w') = 1$ in $\mathcal{G}$. Thus ψ respects the relations (A1′–A6′,A9′).

We check that ψ respects the remaining defining relations of $\mathscr{M}^+$.

(B1′): $u'_i u'_j = u'_j u'_i$. If $i, j > 1$ then (B1′) is the same as (B1), and if $i, j < g-1$ then it is expressible in $\mathscr{S}$. It remains to show that $\psi(u'_{g-1}) = u_{g-1}$ commutes in $\mathscr{G}$ with $\psi(u'_1) = ra_1^{-1}ra_1\psi(\sigma^2)u_1$, which is true, because $ru_{g-1} = u_{g-1}^{-1}r$ (E4), u_{g-1} commutes with a_1, u_1 (B1, C1) and $\psi(\sigma^2) = (u_2 \cdots u_{g-1})^{g-1}$ is central in the subgroup of $\mathscr{G}$ generated by u_i for $i > 1$ (B5).

(C1′): $a'_1 u'_i = u'_i a'_1$ for $i > 1$. This relation is respected, because $\psi(u'_i) = u_i$ commutes with $\psi(a'_1) = ra_1 r$ by (C1, E4)

(B2′, C2′, C3′) are either the same as (B2, C2, C3) if $i > 1$, or they are expressible in $\mathscr{S}$ if $i = 1$.

(C7′): $u'_5 b' = b' u'_5$. This relation is respected, because $\psi(u'_5) = u_5$ commutes with $\psi(b') = rb^{-1}(a_1a_2a_3)^4 r$ by (C1a, C7, E4).

(R8′) for $g = 4$ is $\sigma^2 b \sigma^{-2} = b'(a'_1a'_2a'_3)^{-4}$. Let $w = (a_1a_2a_3)$. Because ψ respects (A3′), we have

$$\psi(b')(\psi(a'_1)\psi(a'_2)\psi(a'_3))^{-4} = (\psi(a'_1)\psi(a'_2)\psi(a'_3))^{-4}\psi(b') = rw^{-4}rw^{-4}b,$$

$\psi(\sigma^2) = (u_3u_2)^3$, and

$$\begin{aligned}
&(u_3u_2)^3 b (u_3u_2)^{-3} = rw^{-4}rw^{-4}b \\
&\iff w^4rw^4r(u_3u_2)^3 = b(u_3u_2)^3b^{-1} \\
&\overset{(G1)}{\iff} w^4rw^4r(u_3u_2)^3 = (w^3u_2u_3w^{-1})^3 \\
&\iff w^4rw^4r(u_3u_2)^3 = w^3u_2u_3w^2u_2u_3w^2u_2u_3w^{-1} \\
&\iff wrw^5(u_3u_2u_1u_3u_2u_3)\underline{u_2u_3u_2w} = u_2u_3w^2u_2\underline{u_3w^2}u_2u_3 \\
&\overset{(C1a,C5a)}{\iff} wrw^5\underline{\Delta_4 w}u_1^{-1}u_2^{-1}u_1^{-1} = u_2u_3w^2u_2w^2u_1u_2u_3 \\
&\overset{(E1)}{\iff} wrw^4\Delta_4 = u_2u_3w^2u_2w^2\Delta_4 \iff \\
&wrw^2 = u_2u_3w^2u_2 \overset{(C1a,C5a)}{\iff} wrw^2 = wu_1^{-1}u_2^{-1}u_3^{-1}w \iff r^2 = 1.
\end{aligned}$$

Since ψ respects the defining relations of $\mathscr{M}^+$, it extends to a homomorphism, which is the inverse of φ. □

8. The stabilizers of $[\alpha_1]$ and $[\xi]$

In this section we assume that $g \geq 5$ is fixed, Theorems 3.5 and 3.6 are true for genera less then g and consequently Theorem 6.1 and Lemma 6.2 are true for g. Recall from Subsection 3.3 the definitions of $\mathscr{G}_{g,n}$ and $\varphi_{g,n}$.

8.1. The stabilizer of $[\xi]$

LEMMA 8.1. — *Let $g > 4$. In $\mathscr{G}_{g,0}$ we have $b^{-1}(a_1a_2a_3)^4 = \Delta_4 b^{-1}\Delta_4^{-1} = r_g b r_g$.*

Proof. — First we show that these relations hold in $\mathscr{M}(N_{g,0})$. Let Σ be a regular neighborhood of the union of the curves α_i for $i \in \{1,2,3\}$ oriented so that a_i are right Dehn twists. Σ is a two holed torus, whose one boundary component is isotopic to β. Let β' be the other boundary component and denote as b' the right Dehn twist about β'. We have the well known relation $(a_1a_2a_3)^4 = bb'$ (called two holed torus or 3-chain relation). It can be checked that Δ_4 and r_g preserve Σ up to isotopy and map β on β'. Moreover, Δ_4 reverses and r_g preserves the orientation of Σ. Thus $b' = \Delta_4 b^{-1}\Delta_4^{-1} = r_g b r_g$ (recall that r_g has order two in $\mathscr{M}(N_{g,0})$ by (E2a)) and the relations from the lemma are satisfied in $\mathscr{M}(N_{g,0})$. To see that they are also satisfied in $\mathscr{G}_{g,0}$ note that they are composed of elements of $\mathrm{Stab}_{\mathscr{M}(N_{g,0})}[\mu_g]$ (because $g > 4$) and hence it suffices to apply the homomorphism ψ_{v_2} from Theorem 6.1. □

THEOREM 8.2. — *Let $g \in \{5,6\}$. The stabilizer $\mathrm{Stab}[\xi] = \mathrm{Stab}_{\mathscr{M}(N_{g,0})}[\xi]$ is generated by a_i, b_j for $1 \le i \le g-1$, $2 \le 2j \le g-2$ and Δ_4 if $g = 5$, or $u_5^{-1}\Delta_4$ and r_6 if $g = 6$. There is a homomorphism $\psi_{v_3}\colon \mathrm{Stab}[\xi] \to \mathscr{G}_{g,0}$ such that $\varphi_{g,0} \circ \psi_{v_3} = \mathrm{id}_{\mathrm{Stab}[\xi]}$ and $\psi_{v_3}(x) = x$ for each generator x of $\mathrm{Stab}[\xi]$.*

Proof. — Let $\mathrm{Stab} = \mathrm{Stab}[\xi]$, $\mathrm{Stab}^+ = \mathrm{Stab}^+[\xi]$, $\mathscr{G} = \mathscr{G}_{g,0}$, $\varphi = \varphi_{g,0}$. Recall from Subsection 5.2 that $\mathrm{Stab}^+ = \rho_\xi(\mathscr{M}(N_\xi))$, where N_ξ is obtained by cutting $N_{g,0}$ along ξ and ρ_ξ is the homomorphism induced by the gluing map. N_ξ is homeomorphic to $S_{2,g-4}$ and it is easy to see that $\mathrm{Stab}^+ = \rho_\xi(\mathscr{M}(N_\xi)) = \jmath_*(\mathscr{M}(S_{2,g-4}))$, where $\jmath_*$ is the map induced by the inclusion in $N_{g,0}$ of a regular neighborhood of the union of the curves α_i for $i \in \{1,\dots,g-1\}$. It follows that Stab^+ is generated by a_i, b_j for $1 \le i \le g-1$, $2 \le 2j \le g-2$. Moreover, by Lemma 3.12 every relation in Stab^+ between these generators is also a relation in $\mathscr{G}$. By applying Lemma 2.1 to the sequence (5.1)

$$1 \to \mathrm{Stab}^+ \to \mathrm{Stab} \to \mathbb{Z}_2^{g-4} \to 1,$$

we see that Stab is generated by Stab^+ and $g-4$ cokernel generators. If $g = 5$, then for the cokernel generator we take Δ_4, which preserves the curve ξ and reverses its orientation. If $g = 6$, then for the cokernel generators we take $u_5^{-1}\Delta_4$, which reverses orientation of ξ and preserves its sides, and $r = r_6$ which preserves orientation of ξ and swaps its sides. It remains to check that the mapping $\psi_{v_3}(x) = x$ for x a generator of $\mathrm{Stab}[\xi]$ respects the cokernel and conjugation relations.

Case $g=5$. — The cokernel relation is (B4) $\Delta_4^2=1$. The conjugation relations are (E1) $\Delta_4 a_i \Delta_4 = a_{4-i}^{-1}$ for $i\in\{1,2,3\}$ and

$$(1)\ \Delta_4 b\Delta_4 = b(a_1a_2a_3)^{-4},\quad (2)\ \Delta_4 a_4\Delta_4 = (a_1a_2a_3)a_4^{-1}(a_1a_2a_3)^{-1}.$$

These relations are satisfied in $\mathscr{G}$, (1) by Lemma 8.1 and (2) because

$$(a_1a_2a_3)a_4^{-1}(a_1a_2a_3)^{-1} \overset{\text{(Da)}}{=} (u_3u_2u_1)^{-1}a_4(u_3u_2u_1)$$
$$\overset{\text{(C1a)}}{=} (u_3u_2u_1)^{-1}\Delta_3^{-1}a_4\Delta_3(u_3u_2u_1) = \Delta_4a_4\Delta_4.$$

Case $g=6$. — The cokernel relations are (E2a) $r^2=1$ and

$$(3)\quad (u_5^{-1}\Delta_4)^2=1,\qquad (4)\quad (ru_5^{-1}\Delta_4)^2=1.$$

They are satisfied in $\mathscr{G}$ because u_5 commutes with Δ_4 by (B1) and

$$u_5^{-2} \overset{\text{(B4a)}}{=} (u_4\cdots u_1)(u_1\cdots u_4) \overset{\text{(B8)}}{=} \Delta_5^2\Delta_4^{-2} \overset{\text{(B3)}}{=} \Delta_4^{-2}$$
$$(ru_5^{-1}\Delta_4)^2 \overset{\text{(E4a)}}{=} u_5\Delta_4^{-1}u_5^{-1}\Delta_4 = 1.$$

The conjugation relations are (E3a) $ra_ir=a_i$ for $i\in\{1,\dots,5\}$ and

(5) $rbr = b^{-1}(a_1a_2a_3)^4$, (6) $u_5^{-1}\Delta_4b\Delta_4^{-1}u_5 = b(a_1a_2a_3)^{-4}$,
(7) $u_5^{-1}\Delta_4a_i\Delta_4^{-1}u_5 = a_{4-i}^{-1}$ for $i\in\{1,2,3\}$, (8) $u_5^{-1}\Delta_4a_5\Delta_4^{-1}u_5=a_5^{-1}$,
(9) $u_5^{-1}\Delta_4a_4\Delta_4^{-1}u_5 = (a_1a_2a_3a_4)a_5^{-1}(a_1a_2a_3a_4)^{-1}$.

(5, 6) follow from Lemma 8.1 and (C1a, C7), (7) from (C1a, E1) and (8) from (C1a, C4a). We have

$$u_5^{-1}\Delta_4a_4\Delta_4^{-1}u_5 \overset{(3)}{=} \Delta_4^{-1}u_5a_4u_5^{-1}\Delta_4 \overset{\text{(C3)}}{=} \Delta_4^{-1}u_4^{-1}a_5u_4\Delta_4 \overset{\text{(B1,C1a)}}{=}$$
$$(u_4u_3u_2u_1)^{-1}a_5(u_4u_3u_2u_1) \overset{\text{(Da)}}{=} (a_1a_2a_3a_4)a_5^{-1}(a_1a_2a_3a_4)^{-1}$$

which proves that (9) holds in $\mathscr{G}$. We do not need to check that ψ_{v_3} respects the conjugation relations expressing rb_2r and $u_5^{-1}\Delta_4b_2\Delta_4^{-1}u_5$ in the generators of Stab^+, because they follow from (A8) and the remaining conjugation relations. □

8.2. The stabilizer of $[\alpha_1]$

THEOREM 8.3. — *Let $g\geq 5$. The stabilizer* $\mathrm{Stab}[\alpha_1]=\mathrm{Stab}_{\mathscr{M}(N_{g,0})}[\alpha_1]$ *is generated by u_i, a_i for $i\in\{1,3,\dots,g-1\}$, b_j for $2\leq 2j\leq g$, $c=T_{\gamma_{\{3,4,5,6\}}}$ (if $g\geq 6$), $v=Y_{\mu_4,\beta}$ and r_g. There is a homomorphism $\psi_{v_1}:\mathrm{Stab}[\alpha_1]\to\mathscr{G}_{g,0}$ such that $\varphi_{g,0}\circ\psi_{v_1}=\mathrm{id}_{\mathrm{Stab}[\alpha_1]}$ and $\psi_{v_1}(c)=(a_1\cdots a_5)^2b(a_1\cdots a_5)^{-2}$, $\psi_{v_1}(v)=a_3a_2a_1u_1u_2u_3$ and $\psi_{v_1}(x)=x$ for the remaining generators of* $\mathrm{Stab}[\alpha_1]$.

Proof. — Let Stab $=$ Stab$[\alpha_1]$, Stab$^+ =$ Stab$^+[\alpha_1]$, $\mathscr{G} = \mathscr{G}_{g,0}$ and $\varphi = \varphi_{g,0}$. First we are going to define a homomorphism $\psi^+\colon \mathrm{Stab}^+ \to \mathscr{G}$ such that $\varphi \circ \psi^+ = \mathrm{id}_{\mathrm{Stab}^+}$. Recall the exact sequence (5.2)

$$1 \to \mathbb{Z} \to \mathscr{M}(N_{\alpha_1}) \xrightarrow{\rho} \mathrm{Stab}^+ \to 1,$$

where N_{α_1} is obtained by cutting $N_{g,0}$ along α_1, and $\rho = \rho_{\alpha_1}$ is induced by the gluing. In order to define ψ^+ it suffices to construct a homomorphism $\psi'\colon \mathscr{M}(N_{\alpha_1}) \to \mathscr{G}$ satisfying $\psi'(\ker\rho) = 1$ and $\varphi \circ \psi' = \rho$.

Since N_{α_1} is homeomorphic to $N_{g-2,2}$, we need a presentation for $\mathscr{M}(N_{g-2,2})$, which can be obtained from the sequence (2.2)

$$1 \to \mathbb{Z} \to \mathscr{M}(N_{g-2,2}) \to \mathscr{M}^+(N_{g-2,1}, P) \to 1,$$

provided that we have a presentation for $\mathscr{M}^+(N_{g-2,1}, P)$.

Step 1: A presentation of $\mathscr{M}^+(N_{g-2,1}, P)$. — We proceed in the same way as in the proof of Theorem 7.1. First we apply Lemma 2.1 to the Birman exact sequence

$$1 \to \pi_1(N_{g-2,1}, P) \to \mathscr{M}(N_{g-2,1}, P) \to \mathscr{M}(N_{g-2,1}) \to 1$$

and the presentation of $\mathscr{M}(N_{g-2,1})$ given in Theorem 3.5. As a result we obtain a presentation of $\mathscr{M}(N_{g-2,1}, P)$ on generators a_i, u_i for $1 \le i \le g-3$, b_j for $2 \le 2j \le g-4$ and σ_l for $1 \le l \le g-2$. Since $\pi_1(N_{g-2,1}, P)$ is free, there are no kernel relations. The cokernel relations are (A1–A9, B1, B2, C1–C8) and the conjugation relations are (1–11), the same as in the proof of Theorem 7.1, and by repeating the arguments from that proof we can reduce the conjugation relations to (R1–R8). Then we make the transformations (2, 3, 7) from the proof of Theorem 7.1 and instead of (5) we replace $(\sigma_2\sigma_1)$ by $a_1^{-1}\sigma_1 a_1 \sigma_1^{-1}$ in (R2, R5, R8) and by $u_1^{-1}\sigma_1 u_1 \sigma_1^{-1}\sigma_1^2$ in (R4). Then we rule out the generators σ_i for $i > 1$ together with (R1). By the Reidemeister-Schreier method we obtain a presentation for $\mathscr{M}^+(N_{g-2,1}, P)$ on the generators b, a_i, u_i, $b' = \sigma b \sigma^{-1}$, $a'_i = \sigma a_i \sigma^{-1}$, $u'_i = \sigma u_i \sigma^{-1}$ and σ^2, where $\sigma = \sigma_1$. The relations are the defining relations of $\mathscr{M}(N_{g-2,1}, P)$ (Rel) and their conjugates by σ (Rel$'$). Note that $a'_i = a_i$ and $u'_i = u_i$ for $i > 1$ while b' and σ^2 may be expressed in the remaining generators by (R8) and (R2) respectively. We record for future reference the following remark.

Remark 8.4. — $\mathscr{M}^+(N_{g-2,1}, P)$ is generated by a_i for $i = 1, \dots, g-3$, u_j for $j = 1, \dots, g-4$, $a_{g-3}u_{g-3}$, a'_1, $a'_1 u'_1$, and b if $g-2 \ge 4$.

Step 2: A presentation for $\mathcal{M}(N_{g-2,2})$. — We apply Lemma 2.1 to the sequence (2.2). Let d_1 and d_2 be Dehn twists about the boundary components of $\mathcal{M}(N_{g-2,2})$, such that d_1 generates the kernel of the map $\mathcal{M}(N_{g-2,2}) \to \mathcal{M}^+(N_{g-2,1},P)$. We are assuming that $N_{g-2,2} = N_{g-2,1}\backslash U$, where U is a small open neighborhood of P, and we treat b, b', a_i, a_i' (resp. u_i, u_i') for $i \in \{1,\dots,g-3\}$ as Dehn twists (resp. crosscap transpositions) on $N_{g-2,2}$. These will be our cokernel generators of $\mathcal{M}(N_{g-2,2})$, together with σ^2 defined according to (R2) as $\sigma^2 = a_1^{-1}a_1'u_1'u_1^{-1}$. There are two types of defining relations of $\mathcal{M}(N_{g-2,2})$: (I) The conjugation relations: $d_1x = xd_1$, for which it suffices to take the cokernel generators x from Remark 8.4. (II) The cokernel relations: for every defining relation $w = 1$ of $\mathcal{M}^+(N_{g-2,1},P)$ we have a relation $\widetilde{w} = d_1^k$ in $\mathcal{M}(N_{g-2,2})$, where k is some integer depending on w. We denote as $(\widetilde{\mathrm{Rel}})$ and $(\widetilde{\mathrm{Rel}'})$ the cokernel relations corresponding to (Rel) and (Rel$'$) respectively.

Step 3: Definition of ψ'. — There is a homeomorphism $f\colon N_{g-2,2} \to N_{\alpha_1}$ inducing an isomorphism $f_*\colon \mathcal{M}(N_{g-2,2}) \to \mathcal{M}(N_{\alpha_1})$ such that $f_*(a_i) = a_{i+2}$, $f_*(u_i) = u_{i+2}$, for $i \in \{1,\dots,g-3\}$, $f_*(b) = T_{\gamma_{\{3,4,5,6\}}} = c$, $f_*(a_1') = b$, $f_*(b') = b_2$ and $f_*(u_1') = U_{\mu_4,\beta} = T_\beta^{-1}Y_{\mu_4,\beta} = b^{-1}v$. We assume that the Dehn twists d_1, d_2 are such that $\rho(f_*(d_1)) = a_1 = \rho(f_*(d_2^{-1}))$.

We define $\psi'\colon \mathcal{M}(N_{\alpha_1}) \to \mathcal{G}$ as $\psi' = \theta \circ f_*^{-1}$, where $\theta\colon \mathcal{M}(N_{g-2,2}) \to \mathcal{G}$ is defined on the generators as:

$$\begin{aligned}
&\theta(d_1) = a_1, \quad \theta(a_i) = a_{i+2}, \quad \theta(u_i) = u_{i+2} \quad \text{for } i \in \{1,\dots,g-3\},\\
&\theta(a_1') = b, \quad \theta(u_1') = b^{-1}a_3a_2a_1u_1u_2u_1,\\
&\theta(b) = (a_1\cdots a_5)^2 b (a_1\cdots a_5)^{-2}, \quad \theta(b') = b_2,\\
&\theta(\sigma^2) = \theta(a_1)^{-1}\theta(a_1')\theta(u_1')\theta(u_1)^{-1} = a_2a_1u_1u_2.
\end{aligned}$$

For every generator x of $\mathcal{M}(N_{g-2,2})$ we have $\varphi(\theta(x)) = \rho(f_*(x))$. This is obvious for all generators except b and u_1'. We have $\varphi(\theta(b)) = (a_1\cdots a_5)^2 b (a_1\cdots a_5)^{-2}$, $\rho(f_*(b)) = c$, and since $(a_1\cdots a_5)^2$ maps β on $\gamma_{\{3,4,5,6\}}$, the equality holds. We have $\rho(f_*(u_1')) = b^{-1}v$, $\varphi(\theta(u_1')) = b^{-1}a_3a_2a_1u_1u_2u_1$ and since the crosscap pushing map is a homomorphism, thus

$$\begin{aligned}
v = Y_{\mu_4,\beta} &= Y_{\mu_4,\gamma_{\{3,4\}}}Y_{\mu_4,\gamma_{\{2,4\}}}Y_{\mu_4,\gamma_{\{1,4\}}} = y_3(u_3^{-1}y_2u_3)(u_3^{-1}u_2^{-1}y_1u_2u_3)\\
&= a_3u_3(u_3^{-1}a_2u_2u_3)(u_3^{-1}u_2^{-1}a_1u_1u_2u_3) = a_3a_2a_1u_1u_2u_3.
\end{aligned}$$

By abuse of notation we are going to denote $\theta(b)$ by c and $b\theta(u_1')$ by v. We also set $e = \theta(\sigma^2) = a_2a_1u_1u_2$. In order to prove that θ is a homomorphism, we have to check that it respects the defining relations of $\mathcal{M}(N_{g-2,2})$.

Step 4: Proof that θ is a homomorphism. — Let $\mathscr{S} = \mathscr{S}_{g,0}(v_2)$ and let $\mathscr{A}$ be the subgroup of $\mathscr{G}$ generated by b and the a_i's. Suppose that w is a word in the generators of $\mathscr{M}(N_{g-2,2})$ and their inverses. We say that w is expressible in $\mathscr{S}$ or in $\mathscr{A}$ if it is mapped by θ on an element of $\mathscr{S}$ or $\mathscr{A}$ respectively. By similar argument as in the proof of Theorem 7.1, θ respects the relations expressible in $\mathscr{S}$, and by Lemma 3.12 it also respects the relations expressible in $\mathscr{A}$. Since $\theta(d_1) = a_1 \in \mathscr{S} \cap \mathscr{A}$, a cokernel relation $w = d_1^k$ is expressible in $\mathscr{S}$ or $\mathscr{A}$ if and only if w is. By this observation we will be able to deduce that θ respects some cokernel relations without having to determine the exponent k.

The conjugation relations are mapped by θ on $a_1\theta(x) = \theta(x)a_1$ for each cokernel generator x from Remark 8.4. Note that $\theta(x) \in \mathscr{S} \cup \mathscr{A}$ and since $a_1 \in \mathscr{S} \cap \mathscr{A}$, thus θ respects the conjugation relations.

Let $s = (u_1 \cdots u_{g-1})$. By (C1a, C3) we have $s^2 a_i s^{-2} = a_{i+2} = \theta(a_i)$, and by (B1, B2) $s^2 u_i s^{-2} = u_{i+2} = \theta(u_i)$ for $i \in \{1, \dots, g-3\}$. Also

$$s^2 b s^{-2} \overset{(E5)}{=} (a_1 \cdots a_{g-1})^2 b (a_1 \cdots a_{g-1})^{-2} \overset{(A1,A3)}{=} c = \theta(b).$$

If (Rel) is one of the relations (A1–A6, A9, B1, B2, C1–C8) then $(\widetilde{\mathrm{Rel}})$ is the same as (Rel) and it is mapped by θ on its conjugate by s^2.

If (Rel) is one of the relations (R2, R3, R6) then $(\widetilde{\mathrm{Rel}})$ is the same as (Rel) and it is trivially preserved by θ. $(\widetilde{\mathrm{R2}'}, \widetilde{\mathrm{R4}}, \widetilde{\mathrm{R4}'}, \widetilde{\mathrm{R5}}, \widetilde{\mathrm{R5}'}, \widetilde{\mathrm{C4}'}, \widetilde{\mathrm{C8}'})$ are expressible in $\mathscr{S}$. $(\widetilde{\mathrm{A1}'} - \widetilde{\mathrm{A6}'}, \widetilde{\mathrm{A9}'}, \widetilde{\mathrm{R7}}, \widetilde{\mathrm{R8}})$ are expressible in $\mathscr{A}$.

Note that $(\widetilde{\mathrm{R7}})$ is

$$(a_4 a_3 a_2 a_1')^{-1} b' (a_4 a_3 a_2 a_1') = (a_4 a_3 a_2 a_1)^{-1} b (a_4 a_3 a_2 a_1),$$

and $(\widetilde{\mathrm{R7'}})$ is

$$\sigma^2 (a_4 a_3 a_2 a_1)^{-1} b (a_4 a_3 a_2 a_1) \sigma^{-2} = (a_4 a_3 a_2 a_1')^{-1} b' (a_4 a_3 a_2 a_1').$$

We already know that θ respects $(\widetilde{\mathrm{R7}})$ and to prove the same for $(\widetilde{\mathrm{R7}'})$ it suffices to show that $e = \theta(\sigma^2)$ commutes in $\mathscr{G}$ with $(a_6 a_5 a_4 a_3)^{-1} c (a_6 a_5 a_4 a_3)$. It follows from earlier part of the proof that in $\mathscr{G}$ we have

$$(a_6 a_5 a_4 a_3)^{-1} c (a_6 a_5 a_4 a_3) = s^2 (a_4 a_3 a_2 a_1)^{-1} b (a_4 a_3 a_2 a_1) s^{-2},$$

where $s = (u_1 \dots u_{g-1})$. Setting $w = (a_4 a_3 a_2 a_1)^{-1} b (a_4 a_3 a_2 a_1)$, it suffices to show that it commutes with $s^{-2} e s^2$. We have

$$e = a_2 a_1 u_1 u_2 = a_2 u_2 (u_2^{-1} a_1 u_1 u_2) \overset{(B1,C2)}{=} (a_2 u_2) u_1 (a_2 u_2) u_1^{-1}.$$

By (B1, B2, B3) $s^{-2} u_1 s^2 = s^{g-2} u_1 s^{2-g} = u_{g-1}$, and as (R7) is valid only for $g - 2 \geq 5$, u_{g-1} commutes with w by (C1a, C7a). By (B1, B2, C1a, C3) we have

$$s^{-2} a_2 u_2 s^2 = s^{-1} a_1 u_1 s = u_{g-1}^{-1} (u_1 \cdots u_{g-2})^{-1} a_1 u_1 (u_1 \cdots u_{g-2}) u_{g-1}.$$

Since w and $(u_1\cdots u_{g-2})^{-1}a_1u_1(u_1\cdots u_{g-2})$ are in $\mathcal{S}$ and they commute in $\mathcal{M}(N_{g,0})$, they also commute in $\mathcal{G}$. We already proved that w commutes with u_{g-1}, hence it commutes with $s^{-2}es^2$ and θ respects $(\widetilde{\mathrm{R7}'})$.

$(\widetilde{\mathrm{R3}'})$ and $(\widetilde{\mathrm{R6}'})$ are $\sigma^2u_i=u_i\sigma^2$ for $i\geq 2$ and $\sigma^2a_2=a_2\sigma^2$ respectively, and they are mapped on $eu_{i+2}=u_{i+2}e$, $ea_4=a_4e$, which follow from (A1, B1, C1a).

$(\widetilde{\mathrm{B1}'})$ is either the same as (B1) or $u_1'u_j=u_ju_1'$ for $j>2$. The last relation is mapped by θ on $b^{-1}vu_i=u_ib^{-1}v$ for $i>4$, which follows from (B1, C1a, C7a). $(\widetilde{\mathrm{B2}'})$ is either the same as (B2) or $u_2u_1'u_2=u_1'u_2u_1'$, which is mapped on

$$\begin{aligned}&u_4b^{-1}vu_4=b^{-1}vu_4b^{-1}v\iff u_4b^{-1}a_4^{-1}(a_4vu_4)=b^{-1}a_4^{-1}(a_4vu_4)b^{-1}v\\&\iff ba_4bu_4b^{-1}a_4^{-1}=b(a_4vu_4)b^{-1}v(a_4vu_4)^{-1}.\end{aligned}$$

Since $b,v\in\mathcal{S}$ and $a_4vu_4\in\mathcal{S}$ (see transformation (5) in the proof of Theorem 7.1), thus it suffices to show that the left hand side of the last relation is also in $\mathcal{S}$. This is true because

$$\underline{ba_4b}u_4b^{-1}a_4^{-1}\overset{(\mathrm{A4})}{=}a_4\underline{ba_4u_4b^{-1}}a_4^{-1}\overset{(\mathrm{C8})}{=}a_4a_4u_4a_4vu_4a_4^{-1}\overset{(\mathrm{C4a})}{=}a_4u_4va_4u_4.$$

$(\widetilde{\mathrm{C1}'})$ is $a_1'u_i=u_ia_1'$ for $i>2$ and it is mapped on (C7a) $bu_j=u_jb$ for $j>4$. $(\widetilde{\mathrm{C2}'})$ is either the same as (C2) or $a_1'u_2u_1'=u_2u_1'a_2$, which is mapped on

$$bu_4b^{-1}v=u_4b^{-1}va_4\iff ba_4bu_4b^{-1}a_4^{-1}(a_4vu_4)=ba_4u_4b^{-1}va_4u_4.$$

Both sides of the last relation are in $\mathcal{S}$ because we showed above that $ba_4bu_4b^{-1}a_4^{-1}\in\mathcal{S}$. $(\widetilde{\mathrm{C3}'})$ is either the same as (C3) or $a_2u_1'u_2=u_1'u_2a_1'$, which is mapped on

$$a_4b^{-1}vu_4=b^{-1}vu_4b\overset{(\mathrm{A4})}{\iff}ba_4vu_4=a_4vu_4b.$$

The last relation is equivalent to (C9) from Lemma 6.2. $(\widetilde{\mathrm{C5}'})$ is $u_2a_1'a_2u_1'=a_1'a_2$ and it is mapped on

$$u_4ba_4b^{-1}v=ba_4\overset{(\mathrm{A4})}{\iff}u_4a_4^{-1}ba_4v=ba_4\overset{(\mathrm{C4a})}{\iff}a_4u_4ba_4vu_4=ba_4u_4$$

The last relation follows from (C8, C9). $(\widetilde{\mathrm{C7}'})$ is $b'u_5=u_5b'$ and it is mapped on $b_2u_7=u_7b_2$, which follows from (A8, C1a, C7a).

It remains to show that θ respects the relations $(\widetilde{\mathrm{C6}'})$ and $(\widetilde{\mathrm{R8}'})$. This follows from the next lemma, whose proof will be given in the next subsection. Recall that a relation in $\mathcal{M}(N_{g-2,2})$ whose both sides are mapped by θ on elements of $\mathcal{S}$ is called expressible in $\mathcal{S}$.

LEMMA 8.5. — *$(\widetilde{\mathrm{C6}'})$ and $(\widetilde{\mathrm{R8}'})$ are expressible in $\mathcal{S}$.*

Step 5: Checking that $\psi'(\ker\rho)=1$. — Recall that d_1, d_2 are Dehn twists about the boundary components of $N_{g-2,2}$ such that $\rho(f_*(d_1d_2))=1$, and so $f_*(d_1d_2)$ is a generator of $\ker\rho$. We have $\psi'(f_*(d_1d_2))=\theta(d_1d_2)$ and to prove $\theta(d_1d_2)=1$ it suffices to show $\theta(d_1d_2)\in\mathcal{J}$. Let $z=\sigma_{g-2}^2\cdots\sigma_1^2\in\mathfrak{p}(\pi_1(N_{g-2,1},P))$. By Lemma 2.2, in $\mathcal{M}^+(N_{g-2,1},P)$ we have the relation

$$z=d_2^{\varepsilon}(u_1\cdots u_{g-3})^{g-2},$$

where $\varepsilon\in\{-1,1\}$, which gives in $\mathcal{M}(N_{g-2,2})$

$$z(u_1\cdots u_{g-3})^{2-g}=d_2^{\varepsilon}d_1^k$$

for some $k\in\mathbb{Z}$. We have

$$\theta(d_1d_2)^{\varepsilon}=\theta(z)(u_3\cdots u_{g-1})^{2-g}a_1^{\varepsilon-k}.$$

Since $(u_3\cdots u_{g-1})^{2-g}\in\mathcal{J}$ by Lemma 3.4, it suffices to prove $\theta(z)\in\mathcal{J}$ and clearly it is enough to show that $\theta(\sigma_{g-2}^2)\in\mathcal{J}$. We have

$$\begin{aligned}&\theta(\sigma_{g-2}^2)=\theta((u_1\cdots u_{g-3})^{-1}\sigma^2(u_1\cdots u_{g-3}))=\\&(u_3\cdots u_{g-1})^{-1}a_2a_1u_1u_2(u_3\cdots u_{g-1})=\\&(u_3\cdots u_{g-1})^{-1}a_2u_2(u_1a_2u_2u_1^{-1})(u_3\cdots u_{g-1})=\\&(u_2\cdots u_{g-2})a_{g-1}u_{g-1}(u_2\cdots u_{g-2})^{-1}(u_1\cdots u_{g-2})a_{g-1}u_{g-1}(u_1\cdots u_{g-2})^{-1}\end{aligned}$$

where the last equality follows from (B1, B2, C1a, C3).

Step 6: Extending ψ^+. — We have a homomorphism $\psi^+\colon\mathrm{Stab}^+\to\mathcal{G}$ defined as $\psi^+(\rho(x))=\psi'(x)$ for $x\in\mathcal{M}(N_{\alpha_1})$. Since $\rho\circ f_*\colon\mathcal{M}(N_{g-2,2})\to\mathrm{Stab}^+$ is an epimorphism, Stab^+ is generated by u_i, a_i for $i\in\{3,\dots,g-1\}$, b, c, v and a_1. By the definition of ψ' (Step 3) we have $\psi^+(c)=(a_1\cdots a_5)^2b(a_1\cdots a_5)^{-2}$, $\psi^+(v)=a_3a_2a_1u_1u_2u_3$ and $\psi^+(x)=x$ for the remaining generators of Stab^+. Also $\psi^+(b_2)=b_2$, and by (A8) $\psi^+(b_j)=b_j$ for $j\geq 3$.

By applying Lemma 2.1 to the sequence (5.1) we see that Stab is generated by Stab^+ and two cokernel generators, for which we take u_1 (preserves orientation of α_1 and swaps its sides) and $r=r_g$ (reverses orientation of α_1 and preserves its sides). We let ψ_{v_1} be equal to ψ^+ on Stab^+ and $\psi_{v_1}(u_1)=u_1$, $\psi_{v_1}(r)=r=a_1\cdots a_{g-1}u_{g-1}\cdots u_1$. Note that $\varphi(\psi_{v_1}(x))=x$ for every generator x of Stab. It remains to check that ψ_{v_1} respects the cokernel and conjugation relations. The cokernel relations are (E2a) $r^2=1$, (E4a) $(ru_1)^2=1$ and $u_1^2=(u_3\cdots u_{g-1})^{g-2}$ which holds in $\mathcal{G}$ by Lemma 3.12.

By Remark 8.4, Stab^+ is generated by u_{g-1}, b, c, v and a_i for $i=1,3,\dots,g-1$. Set

$$w=(u_4\cdots u_{g-1})(a_4\cdots a_{g-1})^{-1}=(u_4\cdots u_{g-2})a_{g-1}u_{g-1}(a_4\cdots a_{g-2})^{-1}.$$

The conjugation relations are (E3a) $ra_ir = a_i$, (E4) $ru_{g-1}r = u_{g-1}^{-1}$, (C1a) $u_1a_iu_1^{-1} = a_i$ for $i = 3, \ldots, g-1$, (C4) $u_1a_1u_1^{-1} = a_1^{-1}$, (B1) $u_1u_{g-1}u_1^{-1} = u_{g-1}$ and

(1) $rbr = w^{-1}b^{-1}w$,
(2) $rvr = a_3w^{-1}a_3^{-1}va_3wa_3^{-1}$,
(3) $rcr = c^{-1}(a_3a_4a_5)^4$,
(4) $u_1^{-1}bu_1 = u_3^{-1}rb^{-1}ru_3$,
(5) $u_1^{-1}vu_1 = u_3^{-1}a_3^{-1}rv^{-1}ra_3u_3$,
(6) $u_1^{-1}cu_1 = c$.

It can be checked that wr and $a_3wa_3^{-1}r$ preserve the curve β, preserve its orientation and reverse local orientation of its neighborhood. Additionally $a_3wa_3^{-1}r$ preserves μ_4. Since $b = T_\beta$ and $v = Y_{\mu_4,\beta}$, thus (1, 2) are satisfied in $\mathscr{M}(N_{g,0})$. Since $w \in \mathscr{S}$, they are also satisfied in $\mathscr{G}$. Similarly, $u_1u_3^{-1}r$ and $u_1u_3^{-1}a_3^{-1}r$ preserve β, reverse its orientation and local orientation of its neighborhood. Additionally $u_1u_3^{-1}a_3^{-1}r$ preserves μ_4. It follows that (4, 5) are satisfied in $\mathscr{M}(N_{g,0})$ and also in $\mathscr{G}$. By Lemma 8.1, $rbr = b^{-1}(a_1a_2a_3)^4$. Conjugating the last relation by $(a_1 \cdots a_6)^2$ we obtain (3). Recall from Step 4 that in $\mathscr{G}$ we have $c = s^2bs^{-2}$ and $s^2u_{g-1}s^{-2} = u_1$ where $s = (u_1 \cdots u_{g-1})$. Conjugating the relation $bu_{g-1} = u_{g-1}b$ (C7a) by s^2 we obtain (6). □

8.3. Proof of Lemma 8.5. — To finish the proof of Theorem 8.3, we yet have to prove that the relations

$$(\widetilde{\mathrm{C6}'}) \quad (u_3b')^2 = a_1'a_2a_1'(a_3u_3)u_2(a_3u_3)u_2^{-1}u_1'u_2(a_3u_3)^2a_1'd_1^{k_1},$$

$$(\widetilde{\mathrm{R8}'}) \quad \sigma^2b\sigma^{-2} = b'(a_2a_1'a_3a_2)^{-1}a_1'^{-1}\sigma^2a_1\sigma^{-2}(a_2a_1'a_3a_2)a_1'^{-1}\sigma^2a_1\sigma^{-2}d_1^{k_2},$$

where k_1, k_2 are some integers, are expressible in $\mathscr{S} = \mathscr{S}_{g,0}(v_2)$. This is obvious if $g > 6$, therefore in this subsection we assume $g = 6$. We denote $\mathscr{M}(N_{6,0})$ by $\mathscr{M}$ and $\mathscr{G}_{6,0}$ by $\mathscr{G}$.

LEMMA 8.6. — *The following relations hold in* $\mathscr{G}$.

$$(1) \quad b_2a_1a_3a_5 = cdb \qquad (2) \quad c^{-1}u_5^{-1}du_5c = u_5^{-1}du_5,$$

where $d = (a_4a_3a_5a_4)^{-1}b(a_4a_3a_5a_4)$.

Proof. — It is easy to check that $(a_4a_3a_5a_4)$ maps the curve $\gamma_{\{1,2,5,6\}}$ on β and so $d = T_{\gamma_{\{1,2,5,6\}}}$. Observe that β_2, α_1, α_3 and α_5 bound a 4-holed sphere in $N_{6,0}$, and so in $\mathscr{M}$ we have the well known lantern relation, which is (1). The same relation holds also in $\mathscr{G}$ by Lemma 3.12.

Since the curves $u_5(\gamma_{\{3,4,5,6\}})$ and $\gamma_{\{1,2,5,6\}}$ are disjoint up to isotopy, (2) holds in $\mathscr{M}$. It can be checked that $w = (u_4u_3u_5u_4)(a_4a_3a_5a_4)^{-1}$ preserves the curve β and preserves local orientation of its neighborhood. It follows that b commutes with w in $\mathcal{M}$ and also in $\mathcal{G}$, because by (B1, C3,C4a)

$$w = u_4u_3(u_5a_4u_4u_5^{-1})a_5u_5a_3^{-1}a_4^{-1} = u_4u_3(u_4^{-1}a_5u_5u_4)a_5u_5a_3^{-1}a_4^{-1} \in \mathcal{S}.$$

It follows that in $\mathcal{G}$ we have $d = (u_4u_3u_5u_4)^{-1}b(u_4u_3u_5u_4)$ and

$$\begin{aligned} c &= (a_1\cdots a_5)^2 b(a_1\cdots a_5)^{-2} \stackrel{(E5)}{=} (u_5\cdots u_1)^{-2}b(u_5\cdots u_1)^2 \\ &\stackrel{(C7,B1)}{=} (u_4u_3u_5u_4u_2u_3u_1u_2u_1)^{-1}b(u_4u_3u_5u_4u_2u_3u_1u_2u_1) \\ &= (u_2u_3u_1u_2u_1)^{-1}d(u_2u_3u_1u_2u_1) = \Delta_4^{-1}u_1du_1^{-1}\Delta_4. \end{aligned}$$

Since u_1 commutes in $\mathcal{G}$ with u_5 and c (see relation (6) in Step 6 above), (2) is equivalent in $\mathcal{G}$ to

$$c^{-1}u_5^{-1}\Delta_4c\Delta_4^{-1}u_5c = u_5^{-1}\Delta_4c\Delta_4^{-1}u_5.$$

The last relation holds in $\mathcal{G}$ by Theorem 8.2, because c and $u_5^{-1}\Delta_4$ are in the image of $\psi_{v_3}\colon \mathrm{Stab}[\xi] \to \mathcal{G}$. □

Clearly the right hand side of $(\widetilde{\mathrm{C6}'})$ is expressible in $\mathcal{S}$ and so it suffices to show $(\theta(u_3)\theta(b'))^2 = (u_5b_2)^2 \in \mathcal{S}$. We have

$$\begin{aligned} &(u_5b_2)^2 \stackrel{(1)}{=} (u_5cdba_1^{-1}a_3^{-1}a_5^{-1})^2 \stackrel{(A1,C1a)}{=} \\ &u_5cdb\underline{a_5^{-1}u_5a_5^{-1}}cdba_1^{-2}a_3^{-2} \stackrel{(C4a)}{=} (u_5cdb)^2a_1^{-2}a_3^{-2}. \end{aligned}$$

Since b commutes with b_2, a_1, a_2, a_3 (A3, A9b), it also commutes with cd by (1). By (C7) b commutes with u_5 and we have

$$\begin{aligned} (u_5b_2)^2 &= (u_5cd)^2b^2a_1^{-2}a_3^{-2} = (u_5c)^2(c^{-1}u_5^{-1}du_5cd)b^2a_1^{-2}a_3^{-2} \\ &\stackrel{(2)}{=} (u_5c)^2u_5^{-2}(u_5d)^2b^2a_1^{-2}a_3^{-2}. \end{aligned}$$

By (C6) and the transformation (3) from the proof of Theorem 7.1, $(u_3b)^2$ can be expressed in $\mathcal{G}$ in terms of a_1, u_1, a_2, u_2, a_3u_3 as

$$(u_3b)^2 = a_1a_2a_1(a_3u_3)u_2(a_3u_3)u_2^{-1}u_1u_2(a_3u_3)^2a_1.$$

Conjugating this relation by $(a_1\cdots a_5)^2$ we obtain an expression of $(u_5c)^2$ in terms of a_3, u_3, a_4, u_4, a_5u_5, and conjugating by $(u_4u_3u_5u_4)^{-1}$ we obtain an expression of $(u_5d)^2$ in terms of a_1, u_1, $(u_3u_4)^{-1}a_2(u_3u_4)$, $(u_3u_4)^{-1}u_2(u_3u_4)$, a_5u_5. Hence $(u_5c)^2$ and $(u_5d)^2$ are in $\mathcal{S}$ and so is $(u_5b_2)^2$.

The relation $(\widetilde{\mathrm{R8}'})$ is mapped by θ on

$$ece^{-1} = b_2(a_4ba_5a_4)^{-1}b^{-1}ea_3e^{-1}(a_4ba_5a_4)b^{-1}ea_3e^{-1}a_1^k,$$

where $e = \theta(\sigma^2) = a_2a_1u_1u_2$. Since e commutes with a_1 by (B1,C3), a_2a_1 commutes with b_2, b, a_4, a_5 by Lemma 3.12 and $b(a_4ba_5a_4) = (a_4ba_5a_4)a_5$ by (B2, B4), the relation is equivalent to

$$u_1u_2c = b_2a_5^{-1}(a_4ba_5a_4)^{-1}u_1u_2a_3u_2^{-1}u_1^{-1}(a_4ba_5a_4)b^{-1}u_1u_2a_3a_1^k.$$

We have to show that $w \in \mathcal{S}$ for w defined as

$$w = c^{-1}u_2^{-1}u_1^{-1}b_2a_5^{-1}(a_4ba_5a_4)^{-1}u_1u_2a_3u_2^{-1}u_1^{-1}(a_4ba_5a_4)b^{-1}u_1u_2a_3a_1^k.$$

We define in $\mathcal{G}$ three equivalence relations $\sim_L$, $\sim_R$ and $\approx$ as follows. We set $w_1 \sim_L w_2$ if there exists $u \in \mathcal{S}$ such that $w_2 = uw_1$. Similarly, we set $w_1 \sim_R w_2$ if there exists $u \in \mathcal{S}$ such that $w_2 = w_1u$. Finally, we set $w_1 \approx w_2$ if there exist $u, u' \in \mathcal{S}$ such that $w_2 = uw_1u'$. Observe that the equivalence class of w for the relation $\sim_L$ is the coset $\mathcal{S}w$, its equivalence class for the relation $\sim_R$ is the coset $w\mathcal{S}$, and its equivalence class for the relation $\approx$ is the double-coset $\mathcal{S}w\mathcal{S}$. Observe also that $\approx$ is the equivalence relation generated by the union of $\sim_L$ and $\sim_R$. Moreover, we have $w \in \mathcal{S}$ if and only if the equivalence class of w for the relation $\approx$ is $\mathcal{S}$.

By (A1–A4) and (C1a) we have

$$\begin{aligned}
&(a_4ba_5a_4)^{-1}u_1u_2a_3u_2^{-1}u_1^{-1}(a_4ba_5a_4) = \\
&a_4^{-1}b^{-1}\underline{a_5^{-1}a_4^{-1}u_1u_2}a_3\underline{u_2^{-1}u_1^{-1}a_4a_5}ba_4 = \\
&a_4^{-1}b^{-1}u_1u_2\underline{a_5^{-1}a_4^{-1}a_3a_4a_5}u_2^{-1}u_1^{-1}ba_4 = \\
&a_4^{-1}b^{-1}u_1u_2a_3a_4a_5a_4^{-1}a_3^{-1}u_2^{-1}u_1^{-1}ba_4 \sim_R a_4^{-1}b^{-1}u_1u_2a_3a_4a_5.
\end{aligned}$$

Thus

$$\begin{aligned}
&w \sim_R c^{-1}u_2^{-1}u_1^{-1}b_2a_5^{-1}a_4^{-1}b^{-1}u_1u_2a_3a_4a_5 \overset{(1)}{=} \\
&c^{-1}u_2^{-1}u_1^{-1}\underline{(cdba_5^{-1}a_3^{-1}a_1^{-1})a_5^{-1}a_4^{-1}b^{-1}}u_1u_2a_3a_4a_5 = \\
&c^{-1}u_2^{-1}u_1^{-1}ca_5^{-2}a_3^{-1}(a_4a_5a_3a_4)^{-1}b(a_4a_5a_3a_4)ba_4^{-1}b^{-1}a_1^{-1}u_1u_2a_3a_4a_5.
\end{aligned}$$

We have

$$\begin{aligned}
&(a_4a_5a_3\underline{a_4)ba_4^{-1}b^{-1}}a_1^{-1}u_1u_2a_3a_4a_5 = a_4a_5a_3b^{-1}\underline{a_4a_1^{-1}u_1u_2a_3a_4a_5} = \\
&a_4a_5a_3b^{-1}a_1^{-1}u_1u_2a_3a_4a_5a_3 \sim_R a_4\underline{a_5a_3b^{-1}}a_1^{-1}u_1u_2a_3a_4a_5 = \\
&a_4b^{-1}\underline{a_5a_3a_1^{-1}u_1u_2a_3a_4a_5} = a_4b^{-1}a_3a_1^{-1}u_1u_2a_3a_4a_5a_4.
\end{aligned}$$

LEMMA 8.7. — $a_5a_4a_3a_2^{-1}a_1^{-1}u_1u_2a_3a_4a_5 \in \mathcal{S}$.

Proof. — By (Da) and (C4a) we have

$$u_5u_4u_3u_2u_1a_1a_2a_3a_4a_5 = a_5u_5u_4u_3u_2u_1a_1a_2a_3a_4 \in \mathcal{S}$$

and by (B1, B2, C3, C4a)

$$u_5u_4u_3(a_5a_4a_3)^{-1} = u_3^{-1}u_4^{-1}(a_5u_5)u_4u_3u_4^{-1}(a_5u_5)u_4(a_5u_5) \in \mathcal{S}.$$

It follows that

$$\begin{aligned} a_5a_4a_3a_2^{-1}a_1^{-1}u_1u_2a_3a_4a_5 &\approx (u_5u_4u_3)a_2^{-1}a_1^{-1}u_1u_2(u_5u_4u_3u_2u_1a_1a_2)^{-1} \\ &= (u_5u_4u_3)(u_2u_1a_1\underline{a_2u_2^{-1}}u_1^{-1}a_1a_2)^{-1}(u_5u_4u_3)^{-1} \\ &= (u_5u_4u_3)(\underline{u_2u_1a_1u_2^{-1}}a_2^{-1}\underline{u_1^{-1}a_1}a_2)^{-1}(u_5u_4u_3)^{-1} \\ &= (u_5u_4u_3)(u_1^{-1}u_2a_2\underline{u_1a_2^{-1}a_1^{-1}}u_1^{-1}a_2)^{-1}(u_5u_4u_3)^{-1} \\ &= (u_5u_4u_3)(u_1^{-1}u_2a_1^{-1}u_2^{-1}u_1^{-1}a_2)^{-1}(u_5u_4u_3)^{-1} \\ &= (u_5u_4u_3)u_1^2(u_5u_4u_3)^{-1} = u_1^2 \approx 1. \end{aligned}$$

□

By Lemma 8.7 $a_1^{-1}u_1u_2a_3a_4a_5 \sim_R a_2a_3^{-1}a_4^{-1}a_5^{-1}$ and

$$w \sim_R c^{-1}u_2^{-1}u_1^{-1}ca_5^{-2}a_3^{-1}(a_4a_5a_3a_4)^{-1}ba_4b^{-1}a_3a_2a_3^{-1}a_4^{-1}a_5^{-1}.$$

We have

$$\underline{ba_4b^{-1}a_3a_2a_3^{-1}}a_4^{-1}a_5^{-1} = a_4^{-1}ba_4a_2^{-1}a_3a_2a_4^{-1}a_5^{-1} = a_4^{-1}a_2^{-1}b\underline{a_4a_3a_4^{-1}}a_5^{-1}a_2 = a_4^{-1}a_2^{-1}ba_3^{-1}a_4a_3a_5^{-1}a_2 = a_4^{-1}a_2^{-1}a_3^{-1}ba_4a_5^{-1}a_3a_2 \sim_R a_4^{-1}a_2^{-1}a_3^{-1}ba_4a_5^{-1}$$

and thus

$$w \sim_R c^{-1}u_2^{-1}u_1^{-1}ca_5^{-2}a_3^{-1}(a_4a_5a_3a_4)^{-1}a_4^{-1}a_2^{-1}a_3^{-1}ba_4a_5^{-1}.$$

Let $s = a_1 \cdots a_5$. By (A1, A2, C1a, C5a) for $i > 1$ we have $a_is = sa_{i-1}$ and $u_is = su_{i-1}^{-1}$. We also have $c = s^2bs^{-2} \overset{(E6)}{=} s^{-4}bs^4$ and

$$c^{-1}u_2^{-1}u_1^{-1}c = s^{-4}b^{-1}\underline{s^4u_2^{-1}u_1^{-1}s^{-4}}bs^4 = s^{-4}b^{-1}su_5u_4s^{-1}bs^4 = s^{-3}a_5^{-1}a_4^{-1}b^{-1}a_4a_5u_5u_4a_5^{-1}a_4^{-1}ba_4a_5s^3 = s^{-3}a_5^{-1}a_4^{-1}a_1a_2a_1b^{-1}a_4a_5u_5u_4a_5^{-1}a_4^{-1}ba_4a_5a_3a_4a_5a_2a_3a_4a_5a_1a_2a_3a_4a_5.$$

Write $w \approx ABC$ for $A = s^{-3}a_5^{-1}a_4^{-1}a_1a_2a_1$, $B = b^{-1}a_4a_5u_5u_4a_5^{-1}a_4^{-1}b$ and $C = (a_4a_5a_3a_4a_5a_2a_3a_4a_5a_1a_2a_3a_4a_5)(a_5^{-2}a_3^{-1}(a_4a_5a_3a_4)^{-1}a_4^{-1}a_2^{-1}a_3^{-1}ba_4a_5^{-1})$.

We have

$$\begin{aligned} A &= a_2^{-1}a_1^{-1}s^{-3}a_1a_2a_1 \sim_L s^{-3}a_1a_2a_1 = s^{-1}(a_3a_4a_5a_2a_3a_4a_5)^{-1} \\ &= (a_2a_3a_4a_1a_2a_3a_4)^{-1}s^{-1} \sim_L s^{-1}, \\ C &= a_4a_5a_3a_4\underline{a_5a_2a_3a_4a_5a_1a_2a_3a_4a_5^{-1}a_3^{-1}}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-2}a_2^{-1}a_3^{-1}ba_4a_5^{-1} \\ &= a_4a_5a_3a_4a_2a_3a_4\underline{a_5a_1a_2a_3a_4a_5^{-1}a_4^{-1}a_3^{-1}a_5^{-1}}a_4^{-2}a_2^{-1}a_3^{-1}ba_4a_5^{-1} \\ &= a_4a_5a_3a_4a_2a_3a_4a_1a_2a_3\underline{a_5a_4a_5^{-1}a_4^{-1}a_5^{-1}}a_3^{-1}a_4^{-2}a_2^{-1}a_3^{-1}ba_4a_5^{-1} \end{aligned}$$

$$= a_4a_5a_3a_4a_2a_3a_1a_2\underline{a_4a_3a_4^{-1}a_3^{-1}a_4^{-2}}a_2^{-1}a_3^{-1}ba_4a_5^{-1}$$
$$= a_4a_5a_3\underline{a_4a_2a_3a_1}a_2a_3^{-1}a_4^{-1}a_2^{-1}a_3^{-1}ba_4a_5^{-1}$$
$$= a_4a_5a_3a_2a_1\underline{a_4a_3a_2a_3^{-1}a_4^{-1}a_2^{-1}a_3^{-1}}ba_4a_5^{-1} = a_4a_5a_3\underline{a_2a_1a_2^{-1}}a_3^{-1}\underline{a_4ba_4}a_5^{-1}$$
$$= a_4a_5a_3a_1^{-1}a_2a_1a_3^{-1}ba_4a_5^{-1}b \sim_R a_1^{-1}a_4a_5\underline{a_3a_2a_3^{-1}}ba_4a_5^{-1}$$
$$= a_1^{-1}a_4a_5a_2^{-1}a_3a_2ba_4a_5^{-1} \sim_R a_1^{-1}a_2^{-1}a_4a_5a_3ba_4a_5^{-1}$$
$$= a_1^{-1}a_2^{-1}a_4a_3ba_5a_4a_5^{-1} \sim_R a_1^{-1}a_2^{-1}a_4a_3ba_4^{-1}a_5,$$
$$w \approx s^{-1}(b^{-1}a_4a_5u_5u_4a_5^{-1}a_4^{-1}b)(a_1^{-1}a_2^{-1}a_4a_3ba_4^{-1}a_5)$$
$$= s^{-1}a_1^{-1}a_2^{-1}b^{-1}a_4a_5u_5u_4a_5^{-1}a_4^{-1}\underline{ba_4b}a_3a_4^{-1}a_5$$
$$= s^{-1}a_1^{-1}a_2^{-1}b^{-1}a_4a_5u_5u_4ba_5^{-1}\underline{a_4a_3a_4^{-1}}a_5$$
$$= s^{-1}a_1^{-1}a_2^{-1}b^{-1}a_4a_5u_5u_4ba_3^{-1}\underline{a_5^{-1}a_4a_5}a_3$$
$$= s^{-1}a_1^{-1}a_2^{-1}b^{-1}a_4a_5u_5u_4ba_3^{-1}a_4a_5a_4^{-1}a_3$$
$$\sim_R s^{-1}a_1^{-1}a_2^{-1}(b^{-1}a_4a_5u_5u_4b)a_3^{-1}a_4a_5.$$

LEMMA 8.8. — *In $\mathcal{G}$ we have*
$$b^{-1}(a_4a_5u_5u_4)^{-1}b = a_4a_5u_4^{-1}vu_4va_5^{-1}a_4^{-1},$$
where $v = a_3a_2a_1u_1u_2u_3$.

Proof. — Let $y_4 = a_4u_4$, $x = u_5y_4u_5^{-1}$, $z = a_4vu_4$. By (B1,C3) we have
$$a_4a_5u_5u_4 = a_4u_4(u_4^{-1}a_5u_5u_4) = a_4u_4(u_5a_4u_4u_5^{-1}) = y_4x$$
and by (C8, C9) $y_4^{-1}by_4 = bz$. Conjugating the last relation by u_5 and by x^{-1} we obtain, using (C7)
$$u_5y_4^{-1}by_4u_5^{-1} = bu_5zu_5^{-1} \iff x^{-1}bx = bu_5zu_5^{-1}$$
$$x^{-1}y_4^{-1}by_4x = x^{-1}bzx = bu_5zu_5^{-1}x^{-1}zx.$$
The last relation is equivalent to
$$b^{-1}(y_4x)^{-1}b = u_5zu_5^{-1}x^{-1}zy_4^{-1} = u_5zy_4^{-1}u_5^{-1}zy_4^{-1} = u_5a_4v\underline{a_4^{-1}u_5^{-1}a_4}va_4^{-1}$$
$$\overset{(C5a)}{=} u_5a_4\underline{va_5}u_4\underline{a_5^{-1}va_4^{-1}} \overset{(A1,C1a)}{=} \underline{u_5a_4a_5}vu_4va_5^{-1}a_4^{-1}$$
$$\overset{(C5a)}{=} a_4a_5u_4^{-1}vu_4va_5^{-1}a_4^{-1}.$$
□

By Lemma 8.8
$$w^{-1} \approx \underline{a_5^{-1}a_4^{-1}a_3a_4a_5u_4^{-1}}vu_4v\underline{a_5^{-1}a_4^{-1}a_2a_1s}$$
$$= a_3a_4a_5\underline{a_4^{-1}a_3^{-1}u_4^{-1}}vu_4va_2a_1sa_4^{-1}a_3^{-1}$$
$$\overset{(C5a)}{=} a_3a_4a_5u_3a_4^{-1}a_3^{-1}vu_4va_2a_1sa_4^{-1}a_3^{-1}$$

$$\approx a_5a_4^{-1}a_2a_1u_1u_2u_3u_4a_3a_2a_1u_1u_2u_3a_2a_1s$$
$$\sim_L a_5\underline{a_4^{-1}u_3u_4a_3}a_2a_1\underline{u_1u_2u_3a_2a_1}s$$
$$= \underline{a_5u_3u_4a_2a_1}a_3a_2u_1\underline{u_2u_3s} = u_3a_2a_1a_5u_4a_3a_2u_1su_1^{-1}u_2^{-1} \approx a_5u_4a_3a_2u_1s.$$

Since $s \sim_R (u_5u_4u_3u_2u_1)^{-1}$ (see the proof of Lemma 8.7),

$$\begin{aligned} w &\approx a_5u_4a_3a_2u_2^{-1}u_3^{-1}u_4^{-1}u_5^{-1} \\ &= a_5u_5^{-1}(u_5u_4a_3u_3^{-1}u_4^{-1}u_5^{-1})(u_5u_4u_3a_2u_2^{-1}u_3^{-1}u_4^{-1}u_5^{-1}) \\ &= (a_5u_5)^{-1}u_3^{-1}u_4^{-1}(a_5u_5)^{-1}u_4u_3u_2^{-1}u_3^{-1}u_4^{-1}(a_5u_5)^{-1}u_4u_3u_2 \approx 1. \end{aligned}$$

Thus $(\widetilde{\mathrm{R8'}})$ is expressible in $\mathcal{S}$, which completes the proof of Lemma 8.5 and the proof of Theorem 8.3.

9. Edges

In this section we assume that $g \geq 5$ is fixed and denote $\mathcal{M}(N_{g,0})$ as $\mathcal{M}$, $\mathcal{G}_{g,0}$ as $\mathcal{G}$, $\varphi_{g,0}$ as φ (see Subsection 3.3 for the definitions), and $\mathrm{Stab}_{\mathcal{M}}\sigma$ as $\mathrm{Stab}\,\sigma$ for each simplex σ of $\widetilde{X}$. We are ready to define $\psi\colon \mathcal{M} \to \mathcal{G}$ on the generators of $\mathcal{M}$ given in Theorem 5.7. In previous sections we defined homomorphisms $\psi_{v_i}\colon \mathrm{Stab}\,s(v_i) \to \mathcal{G}_{g,0}$ and we let ψ be equal to ψ_{v_i} on $\mathrm{Stab}\,s(v_i)$ for $i \in \{1,2,3\}$. For $j \in \{1,\dots,7\}$ we define $\psi(h_{e_i})$ to be the element of $\mathcal{G}$ represented by the word in the generators of $\mathcal{G}$ given in the fourth column of Table 1, and $\psi(h_{\overline{e_i}}) = \psi(h_{e_i})^{-1}$. Observe that $\varphi \circ \psi$ is the identity on the generators of $\mathcal{M}$. In this section we show that ψ respects the relations associated to the edges of X. Namely, we show that for $e \in \mathcal{S}_1(X)$ we have

$$(*) \quad \psi(h_e)^{-1}\psi_{i(e)}(x)\psi(h_e) = \psi_{t(e)}(h_e^{-1}xh_e)$$

for $x \in \mathrm{Stab}\,s(e)$. Since $\mathrm{Stab}\,s(\overline{e}) = h_e^{-1}\mathrm{Stab}\,s(e)h_e$ and $h_{\overline{e}} = h_e^{-1}$, thus it suffices to check $(*)$ for $e = e_i$, $i \in \{1,\dots,7\}$.

To prove $(*)$ it suffices to show that its left hand side is equal in $\mathcal{G}$ to $\psi_{t(e)}(z)$ for some $z \in \mathrm{Stab}\,s(t(e))$, because then by applying φ to both sides we get $z = h_e^{-1}xh_e$.

Lemma 9.1. — *For $x \in \mathrm{Stab}[\alpha_1,\mu_g]$ we have $(*)$ $\psi_{v_1}(x) = \psi_{v_2}(x)$.*

Proof. — By the proof of Theorem 8.3, $\mathrm{Stab}[\alpha_1]$ is generated by $\mathrm{Stab}^+[\alpha_1]$ and $\{u_1, r_g\}$. Note that $\{u_1,r_g\} \subset \mathrm{Stab}[\mu_g]$ and $\psi_{v_1}(x) = \psi_{v_2}(x)$ for $x \in \{u_1,r_g\}$. It remains to show that the same is true for $x \in H = \mathrm{Stab}^+[\alpha_1] \cap \mathrm{Stab}[\mu_g]$. Let N' be the surface obtained from $N = N_{g,0}$ by cutting along μ_g and gluing a disc with puncture P along the resulting boundary component. We have the exact sequence (5.3):

$$1 \to \pi_1(N'_{\alpha_1}, P) \xrightarrow{\mathfrak{c}} \mathrm{Stab}_{\mathcal{M}(N_{\alpha_1})}[\mu_g] \xrightarrow{\zeta} \mathcal{M}(N'_{\alpha_1}) \to 1,$$

where N'_{α_1} and N_{α_1} are the surfaces obtained respectively from N' and N by cutting along α_1. Set $G = \mathrm{Stab}_{\mathcal{M}(N_{\alpha_1})}[\mu_g]$ and note that $\rho_{\alpha_1}(G) = H$ and $\rho_{\alpha_1}(\mathcal{M}(N'_{\alpha_1})) = \mathrm{Stab}^+_{\mathcal{M}(N')}[\alpha_1]$. Observe that ζ maps $\ker \rho_{\alpha_1} \subset G$ isomorphically onto $\ker \rho_{\alpha_1} \subset \mathcal{M}(N'_{\alpha_1})$. It follows that ζ induces a map $\zeta' \colon H \to \mathrm{Stab}^+_{\mathcal{M}(N')}[\alpha_1]$, which fits in the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(N'_{\alpha_1}, P) & \xrightarrow{\mathfrak{c}} & G & \xrightarrow{\zeta} & \mathcal{M}(N'_{\alpha_1}) & \longrightarrow & 1 \\
 & & \Vert & & \downarrow{\scriptstyle \rho_{\alpha_1}} & & \downarrow{\scriptstyle \rho_{\alpha_1}} & & \\
1 & \longrightarrow & \pi_1(N'_{\alpha_1}, P) & \xrightarrow{\rho_{\alpha_1}\circ\mathfrak{c}} & H & \xrightarrow{\zeta'} & \mathrm{Stab}^+_{\mathcal{M}(N')}[\alpha_1] & \longrightarrow & 1,
\end{array}$$

whose both rows are exact. We can obtain generators of H from the bottom sequence. Note that N' is homeomorphic to $N_{g-1,0}$ and by the proof of Theorem 8.3 (see Step 6), $\mathrm{Stab}^+_{\mathcal{M}(N')}[\alpha_1]$ is generated by u_i, a_i for $i = 3, \ldots, g-2$, a_1, b, v and c (if $g \geq 7$). The standard generators of $\pi_1(N'_{\alpha_1}, P)$ are mapped by $\rho_{\alpha_1} \circ \mathfrak{c}$ on the crosscap slides $Y_{\mu_g,\alpha_{g-1}} = a_{g-1}u_{g-1}$, $Y_{\mu_g,\gamma_{\{i,g\}}} = u_i Y_{\mu_g,\gamma_{\{i+1,g\}}} u_i^{-1}$ for $i = 3, \ldots, g-2$ and $Y_{\mu_g,\gamma_{\{1,2,g\}}}$. It follows that H is generated by $Y_{\mu_g,\gamma_{\{1,2,g\}}}$, $a_{g-1}u_{g-1}$, u_i, a_i for $i = 3, \ldots, g-2$, a_1, b, v and c (if $g \geq 7$). By the definitions of ψ_{v_1} and ψ_{v_2} given in Theorems 8.3 and 6.1, it is easy to check that $\psi_{v_1}(x) = \psi_{v_2}(x)$ for every generator x of H, except for $x = Y_{\mu_g,\gamma_{\{1,2,g\}}}$. We have $Y_{\mu_g,\gamma_{\{1,2,g\}}} = (u_4 \cdots u_{g-1})^{-1} Y_{\mu_4,\gamma_{\{1,2,4\}}} (u_4 \cdots u_{g-1})$ and $Y_{\mu_4,\gamma_{\{1,2,4\}}} = Y^{-1}_{\mu_4,\alpha_3} Y_{\mu_4,\beta} = (a_3u_3)^{-1}v$. Thus, by Theorem 8.3 we have

$$\begin{aligned}
\psi_{v_1}(Y_{\mu_g,\gamma_{\{1,2,g\}}}) &= (u_4 \cdots u_{g-1})^{-1}(a_3u_3)^{-1} a_3a_2a_1u_1u_2u_3(u_4 \cdots u_{g-1}) \\
&= (u_3 \cdots u_{g-1})^{-1} a_2a_1u_1u_2(u_3 \cdots u_{g-1}) \\
&= (u_3 \cdots u_{g-1})^{-1}(a_2u_2)u_2^{-1}(a_1u_1)u_2(u_3 \cdots u_{g-1})
\end{aligned}$$

and it follows from (B1, B2, C3) that

$$\begin{aligned}
&\psi_{v_1}(Y_{\mu_g,\gamma_{\{1,2,g\}}}) = (u_2 \cdots u_{g-2})a_{g-1}u_{g-1}(u_2 \cdots u_{g-2})^{-1} \cdot \\
&(u_1 \cdots u_{g-2})a_{g-1}u_{g-1}(u_1 \cdots u_{g-2})^{-1}.
\end{aligned}$$

It follows that $\psi_{v_1}(Y_{\mu_g,\gamma_{\{1,2,g\}}})$ is in the image of ψ_{v_2} and thus it is equal to $\psi_{v_2}(Y_{\mu_g,\gamma_{\{1,2,g\}}})$ by the remark before Lemma 9.1. □

LEMMA 9.2. — *For* $x \in \mathrm{Stab}[\mu_{g-1}, \mu_g]$ *we have*

$$(*) \quad \psi(h_{e_3})^{-1}\psi_{v_2}(x)\psi(h_{e_3}) = \psi_{v_2}(h_{e_3}^{-1} x h_{e_3}).$$

Proof. — To obtain generators of $\mathrm{Stab}[\mu_{g-1},\mu_g]$ we use the exact sequence (5.3)

$$1 \to \pi_1(N'\backslash\{P_1\},P_2) \to \mathrm{Stab}[\mu_{g-1},\mu_g] \to \mathrm{Stab}_{\mathcal{M}(N_{g-1,0})}[\mu_{g-1}] \to 1$$

where N' is obtained from $N_{g,0}$ by cutting along μ_{g-2+i} and gluing a disc with puncture P_i along the resulting boundary component for $i \in \{1,2\}$. By Lemma 2.1 and the proof of Theorem 6.1, $\mathrm{Stab}[\mu_{g-1},\mu_g]$ is generated by

$$Z = \{u_i, a_i \mid i = 1,\dots,g-3\} \cup \{b, a_{g-2}u_{g-2}, u_{g-2}a_{g-1}u_{g-1}u_{g-2}^{-1}\}.$$

We have $h_{e_3} = a_{g-1}^{-1}$ and

$$\psi(h_{e_3})^{-1}\psi_{v_2}(u_2)\psi(h_{e_3}) = a_{g-1}u_2a_{g-1}^{-1} = u_2 = \psi_{v_2}(u_2) = \psi_{v_2}(h_{e_3}^{-1}u_2h_{e_3}).$$

Analogously $\psi(h_{e_3})^{-1}\psi_{v_2}(a_2)\psi(h_{e_3}) = a_2 = \psi_{v_2}(h_{e_3}^{-1}u_2h_{e_3})$. For $x \in Z\backslash\{a_2,u_2\}$ we have $x \in \mathrm{Stab}[\alpha_1]$ and $\psi_{v_2}(x) = \psi_{v_1}(x)$ by Lemma 9.1. Since also $h_{e_3} \in \mathrm{Stab}[\alpha_1]$ and $\psi(h_{e_3}) = \psi_{v_1}(h_{e_3})$, thus

$$\psi(h_{e_3})^{-1}\psi_{v_2}(x)\psi(h_{e_3}) = \psi_{v_1}(h_{e_3}^{-1}xh_{e_3}) = \psi_{v_2}(h_{e_3}^{-1}xh_{e_3}). \qquad \square$$

The next lemma follows from [16, Proposition 2.10].

LEMMA 9.3. — *Let $S = S_{1,r}$ be a torus with $r > 1$ boundary components $\delta_1,\dots,\delta_r$. Suppose that $\alpha_1,\dots,\alpha_r$ and β are simple closed curves on S such that (1) α_i, α_{i+1}, δ_i bound a pair of pants for $i = 1,\dots,r$ and $\alpha_{r+1} = \alpha_1$; (2) β intersects each of the curves α_i in one point. Then $\mathcal{M}(S)$ is generated by Dehn twists about β, α_i, δ_i for $i = 1,\dots,r$.* $\square$

LEMMA 9.4. — *If $g \in \{5,6\}$ then $(*)$ $\psi_{v_1}(x) = \psi_{v_3}(x)$ for $x \in \mathrm{Stab}[\alpha_1,\xi]$.*

Proof. — Denote by S the surface obtained by cutting $N_{g,0}$ along $\alpha_1 \cup \xi$. Note that S is homeomorphic to $S_{1,g-2}$. Recall the exact sequence (5.1)

$$1 \to \mathrm{Stab}^+[\alpha_1,\xi] \to \mathrm{Stab}[\alpha_1,\xi] \xrightarrow{\eta} \mathbb{Z}_2^{g-2}.$$

By Remark 5.2 η is not onto and hence its image has rank at most $g-3$. It follows that this image is spanned by the images of the following elements of $\mathrm{Stab}[\alpha_1,\xi]$: $x_1 = a_2a_1^2a_2$ (swaps sides and reverses orientation of α_1, preserves sides and orientation of ξ), $x_2 = a_2a_1a_3a_2\Delta_4$ for $g = 5$ or $x_2 = a_2a_1a_3a_2(a_5u_5)^{-1}\Delta_4$ for $g = 6$ (swaps sides and preserves orientation of α_1, reverses orientation of ξ and swaps its sides if $g = 6$), $x_3 = r_6$ for $g = 6$ (swaps sides and reverses orientation of α_1, swaps sides and preserves orientation of ξ). By Lemma 2.1 $\mathrm{Stab}[\alpha_1,\xi]$ is generated by x_i for $i = 1,2,3$ and $\mathrm{Stab}^+[\alpha_1,\xi]$. It follows from Lemma 9.3 that $\mathrm{Stab}^+[\alpha_1,\xi]$ is generated by a_i for $i = 1,3,\dots,g-1$, b, $x_2bx_2^{-1}$ and if $g = 6$ then also b_2. The generator $x_2bx_2^{-1}$

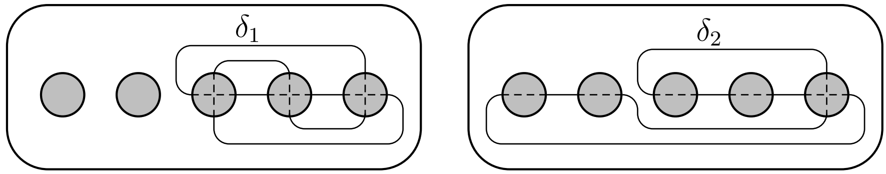

FIGURE 8. Curves from the proof of Lemma 9.5.

is redundant and it is trivial to check that ψ_{v_1} and ψ_{v_3} are equal on the remaining generators of $\mathrm{Stab}^+[\alpha_1,\xi]$. For $i=1,2,3$ we have $x_i\in\mathrm{Stab}[\mu_g]$, it is easy to check that $\psi_{v_3}(x_i)=\psi_{v_2}(x_i)$ and by Lemma 9.1 we have $\psi_{v_2}(x_i)=\psi_{v_1}(x_i)$. □

LEMMA 9.5. — $\mathrm{Stab}[\alpha_1,\alpha_3]$ *is generated by*

- $\mathrm{Stab}[\alpha_1,\alpha_3,\mu_g]\cup\mathrm{Stab}[\alpha_1,\alpha_3,\xi]$ *if* $g=5$ *or* $g=6$,
- $\mathrm{Stab}[\alpha_1,\alpha_3,\mu_g]\cup\{a_{g-1}\}$ *if* $g=7$ *or* $g\geq 9$.
- $\mathrm{Stab}[\alpha_1,\alpha_3,\mu_g]\cup\{a_7,b_3,T_{\gamma_{\{5,6,7,8\}}},T_{\gamma_{\{1,2,5,6,7,8\}}}\}$ *if* $g=8$.

Proof. — Let $\mathrm{Stab}=\mathrm{Stab}[\alpha_1,\alpha_3]$. By Lemma 2.1 applied to the sequence (5.1) Stab is generated by Stab^+ and 4 cokernel generators, for which we take u_1, u_3, r_g and $a_4a_3^2a_4$. Note that all of them are in $\mathrm{Stab}[\mu_g]$ if $g>5$ and if $g=5$ then the last one is in $\mathrm{Stab}[\xi]$. Let N' be the surface obtained by cutting $N_{g,0}$ along $\alpha_1\cup\alpha_3$. Note that N' is homeomorphic to $N_{g-4,4}$. Denote by α_1' and α_1'' (resp. α_3' and α_3'') the boundary components of N' resulting from cutting along α_1 (resp. α_3), where α_1', α_3' and β bound a pair of pants in N'. To obtain generators of Stab^+ we use the epimorphism $\rho\colon\mathcal{M}(N')\to\mathrm{Stab}^+$ induced by the gluing map. We consider cases according to the genus.

Case $g=5$. — In [25, Theorem 7.6] a presentation of $\mathcal{M}(N_{1,4})$ is given, from which we deduce that Stab^+ is generated by Dehn twists about curves disjoint from μ_g and T_{δ_1}, T_{δ_2}, $u_3^{-1}T_{\delta_2}u_3$, where δ_1, δ_2 are shown on Figure 8. Since δ_1 and δ_2 are disjoint from ξ, the lemma is proved in this case.

Case $g>5$. — Let N'' be the surface obtained from N' by gluing a disc with puncture P_1 along α_1', a disc with puncture P_2 along α_3', and a disc with puncture P_3 along α_3''. For $i=1,2,3$ we set $\mathcal{P}_i=\{P_1,\ldots,P_i\}$ and $H_i=\mathcal{PM}^+(N'',\mathcal{P}_i)$. For $i=2,3$ we set $K_i=\pi_1(N''\backslash\mathcal{P}_{i-1},P_i)$ and define G_i to be the subgroup of $\mathcal{PM}(N'',\mathcal{P}_i)$ consisting of the isotopy classes of homeomorphisms preserving local orientation at each puncture in $\mathcal{P}_{i-1}$. Note that H_i is an index-two subgroup of G_i and we have the following short exact sequence

$$1\to K_i\xrightarrow{\mathfrak{p}}G_i\xrightarrow{\mathfrak{f}}H_{i-1}\to 1 \tag{9.1}$$

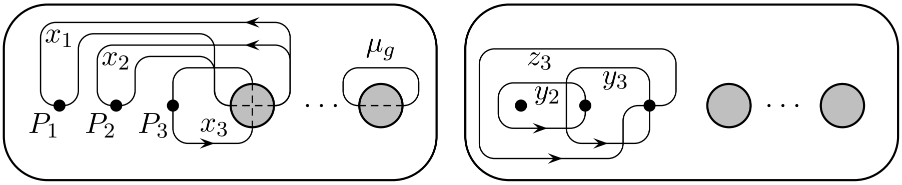

FIGURE 9. Loops from the proof of Lemma 9.5.

which is a restriction of the Birman sequence (2.1). We also have the exact sequence (2.2)

$$1 \to \mathbb{Z}^3 \to \mathcal{M}(N') \xrightarrow{\imath_*} H_3 \to 1.$$

The kernel generators of $\mathcal{M}(N')$ are Dehn twists about the boundary components and they are mapped by ρ on a_1 and a_3, which are in $\mathrm{Stab}[\mu_g]$. Also observe that if $\imath_*(x) \in \mathrm{Stab}_{H_3}[\mu_g]$ then $\rho(x) \in \mathrm{Stab}[\mu_g]$. Our next goal is to find generators for H_3. In fact we will obtain generators for the groups H_i, G_i for $i = 2, 3$ by using the sequence (9.1) and the method from Step 1 of the proof of Theorem 8.3. We set

$$w = Y_{\mu_4,\alpha_4} Y_{\mu_3,\gamma_{\{3,5\}}} = Y_{\mu_4,\alpha_4} u_3 Y_{\mu_4,\alpha_4} u_3^{-1} = u_4 a_4 u_3 u_4 a_4 u_3^{-1} = a_4^{-1} a_3^{-1} u_3 u_4.$$

Let x_1, x_2, x_3 and y_2, y_3, z_3 be the elements of $\mathcal{PM}(N'', \mathcal{P}_i)$ obtained by pushing the punctures once along the loops represented in Figure 9, which we take so that for each $I \subseteq \{5, \ldots, g\}$ such that $5 \in I$, the following equalities are satisfied, up to isotopy on $N'' \backslash \mathcal{P}_3$.

$$x_1(\gamma_I) = \gamma_{\{1,2\}\cup I}, \qquad x_2(\gamma_I) = \gamma_{\{3,4\}\cup I}, \qquad x_3(\gamma_I) = u_3^{-1} x_2(\gamma_I),$$
$$x_2 x_1(\gamma_I) = \gamma_{\{1,2,3,4\}\cup I}, \quad x_3 x_1(\gamma_I) = u_3^{-1} x_2 x_1(\gamma_I), \quad x_3 x_2(\gamma_I) = w(\gamma_I)$$
$$x_3 x_2 x_1(\gamma_I) = w x_1(\gamma_I).$$

By Remark 8.4 $H_1 = \mathcal{M}^+(N'', P_1)$ is generated by a_i for $i = 5, \ldots, g-1$, u_j for $j = 5, \ldots, g-2$, $a_{g-1}u_{g-1}$, $x_1 a_5 x_1^{-1}$, $x_1 a_5 u_5 x_1^{-1}$, and $T_{\gamma_{\{5,6,7,8\}}}$ if $g \geq 8$. We denote this set of generators by Z_1 and, by abuse of notation, we will treat it as a subset of G_i for $i = 1, 2, 3$. The image of K_2 in G_2 is generated by y_2 and t_j for $j = 5, \ldots, g$ defined as $t_5 = x_2$ and $t_{j+1} = u_j^{-1} t_j u_j$. From the sequence (9.1) we obtain that G_2 is generated by y_2, x_2 and Z_1, and its index-two subgroup H_2 is generated by

$$Z_2 = \{y_2, x_2 y_2 x_2^{-1}, x_2^2\} \cup Z_1 \cup x_2 Z_1 x_2^{-1}.$$

Similarly, G_3 is generated by y_3, z_3, x_3 and Z_2, and H_3 is generated by

$$Z_3 = \{y_3, z_3, x_3 y_3 x_3^{-1}, x_3 z_3 x_3^{-1}, x_3^2\} \cup Z_2 \cup x_3 Z_2 x_3^{-1}.$$

Set $Z_1' = Z_1 \backslash \mathrm{Stab}_{H_3}[\mu_g]$ and observe that H_3 is generated by

$$\mathrm{Stab}_{H_3}[\mu_g] \cup Z_1' \cup x_2 Z_1' x_2^{-1} \cup x_3 Z_1' x_3^{-1} \cup x_3 x_2 Z_1' x_2^{-1} x_3^{-1}.$$

Subcase $g = 7$ or $g \geq 9$. — We have $Z_1' = \{a_{g-1}\}$ and a_{g-1} commutes with x_2, x_3. The lemma follows.

Subcase $g = 6$. — We have $Z_1' = \{a_5, x_1 a_5 x_1^{-1}\}$ and

$$\begin{aligned}
&x_1 a_5 x_1^{-1} = T_{\gamma_{\{1,2,5,6\}}}, \quad x_2 a_5 x_2^{-1} = T_{\gamma_{\{3,4,5,6\}}} = c, \quad x_3 a_5 x_3^{-1} = u_3^{-1} c u_3,\\
&(x_2 x_1) a_5 (x_2 x_1)^{-1} = b_2, \quad (x_3 x_1) a_5 (x_3 x_1)^{-1} = u_3^{-1} b_2 u_3,\\
&(x_3 x_2) a_5 (x_3 x_2)^{-1} = w a_5 w^{-1} \quad (x_3 x_2 x_1) a_5 (x_3 x_2 x_1)^{-1} = w T_{\gamma_{\{1,2,5,6\}}} w^{-1}.
\end{aligned}$$

Since a_5, c, b_2 and $T_{\gamma_{\{1,2,5,6\}}}$ are in $\mathrm{Stab}[\alpha_1, \alpha_3, \xi]$ and u_3, w are in $\mathrm{Stab}[\alpha_1, \alpha_3, \mu_g]$, the lemma is proved for $g = 6$.

Subcase $g = 8$. — We have $Z_1' = \{a_7, T_{\gamma_{\{5,6,7,8\}}}\}$, a_7 commutes with x_2, x_3 and

$$\begin{aligned}
&x_2 T_{\gamma_{\{5,6,7,8\}}} x_2^{-1} = T_{\gamma_{\{3,4,5,6,7,8\}}}, \quad x_3 T_{\gamma_{\{5,6,7,8\}}} x_3^{-1} = u_3^{-1} T_{\gamma_{\{3,4,5,6,7,8\}}} u_3,\\
&(x_3 x_2) T_{\gamma_{\{5,6,7,8\}}} (x_3 x_2)^{-1} = w T_{\gamma_{\{5,6,7,8\}}} w^{-1}.
\end{aligned}$$

We have $u_3, w \in \mathrm{Stab}[\alpha_1, \alpha_3, \mu_g]$, and to finish the proof it suffices to express $T_{\gamma_{\{3,\ldots,8\}}}$ in terms of b_3 and the remaining generators. Observe that β_3, α_1, α_3 and $\gamma_{\{5,6,7,8\}}$ bound a four holed sphere and we have the lantern relation:

$$b_3 a_1 a_3 T_{\gamma_{\{5,6,7,8\}}} = b T_{\gamma_{\{3,4,5,6,7,8\}}} T_{\gamma_{\{1,2,5,6,7,8\}}}$$

which does the job, because a_1, a_3 and b are in $\mathrm{Stab}[\mu_8]$. □

LEMMA 9.6. — *For $x \in \mathrm{Stab}[\alpha_1, \alpha_3]$ we have*

$$(*) \quad \psi(h_{e_2})^{-1} \psi_{v_1}(x) \psi(h_{e_2}) = \psi_{v_1}(h_{e_2}^{-1} x h_{e_2}).$$

Proof. — Let $h = h_{e_2} = a_2 a_3 a_1 a_2$. We have $h \in \mathrm{Stab}[\mu_g] \cap \mathrm{Stab}[\xi]$ and $\psi(h) = \psi_{v_2}(h) = \psi_{v_3}(h)$. Therefore for $x \in \mathrm{Stab}[\alpha_1, \alpha_3, \mu_g]$ we have by Lemma 9.1

$$\psi(h)^{-1} \psi_{v_1}(x) \psi(h) = \psi_{v_2}(h^{-1} x h) = \psi_{v_1}(h^{-1} x h).$$

Analogously, if $g \in \{5, 6\}$ then $(*)$ holds for $x \in \mathrm{Stab}[\alpha_1, \alpha_3, \xi]$ by Lemma 9.4. By Lemma 9.5 this finishes the proof for $g \in \{5, 6\}$. For $g \geq 7$ we have

$$\psi(h)^{-1} \psi_{v_1}(a_{g-1}) \psi(h) = (a_2 a_1 a_3 a_2)^{-1} a_{g-1} (a_2 a_1 a_3 a_2) = a_{g-1} = \psi_{v_1}(a_{g-1}).$$

It can be checked that for $g = 8$ in $\mathcal{M}$ we have

$$\begin{aligned}
&h^{-1} b_3 h = b_3, \qquad h^{-1} T_{\gamma_{\{5,6,7,8\}}} h = T_{\gamma_{\{5,6,7,8\}}},\\
&h^{-1} T_{\gamma_{\{1,2,5,6,7,8\}}} h = T_{\gamma_{\{3,4,5,6,7,8\}}} = b^{-1} b_3 a_1 a_3 T_{\gamma_{\{5,6,7,8\}}} T_{\gamma_{\{1,2,5,6,7,8\}}}^{-1},
\end{aligned}$$

and for $w = a_6 a_7 a_5 a_6 a_4 a_5 a_3 a_4$,

$$T_{\gamma_{\{5,6,7,8\}}} = w^{-1} c w, \qquad T_{\gamma_{\{1,2,5,6,7,8\}}} = w^{-1} b_2 w.$$

It follows that for $x \in \{b_3, T_{\gamma_{\{5,6,7,8\}}}, T_{\gamma_{\{1,2,5,6,7,8\}}}\}$, $\psi(h)$, $\psi_{v_1}(x)$ and $\psi_{v_1}(h^{-1}xh)$ are in the subgroup of $\mathscr{G}$ generated by a_i's and b_j's and so $(*)$ is satisfied in $\mathscr{G}$ by Lemma 3.12. □

LEMMA 9.7. — *If $g = 5$ then for $x \in \mathrm{Stab}[\mu_5, \beta]$*

$$(*)\ \psi(h_{e_5})^{-1}\psi_{v_2}(x)\psi(h_{e_5}) = \psi_{v_1}(h_{e_5}^{-1}xh_{e_5}).$$

Proof. — By similar argument as in the proof of Lemma 9.4, $\mathrm{Stab}[\mu_5, \beta]$ is generated by $\mathrm{Stab}^+[\mu_5, \beta]$ together with $h_{e_5}u_1h_{e_5}^{-1}$ (preserves orientation of μ_5, preserves orientation and swaps sides of β) and $r_5\Delta_4$ (reverses orientation of μ_5, reverses orientation and preserves sides of β). By Lemma 9.3 $\mathrm{Stab}^+[\mu_5, \beta]$ is generated by b, a_1, a_2, a_3 and $e = a_4u_4a_3u_4^{-1}a_4^{-1}$. Note that $\{b, a_1, a_2, a_3, r_5\Delta_4\} \subset \mathrm{Stab}[\xi]$ and for $x = b, a_1, a_2, a_3$ we have $\psi_{v_2}(x) = \psi_{v_3}(x)$. Also $\psi_{v_2}(\Delta_4) = \psi_{v_3}(\Delta_4)$ and because $r_5 \in \mathrm{Stab}[\alpha_1, \mu_5] \cap \mathrm{Stab}[\xi]$, thus $\psi_{v_2}(r_5) = \psi_{v_1}(r_5) = \psi_{v_3}(r_5)$ by Lemmas 9.1 and 9.4. Also $h_{e_5} = a_4ba_3a_4a_2a_1a_3a_2 \in \mathrm{Stab}[\xi]$ and $\psi(h_{e_5}) = \psi_{v_3}(h_{e_5})$ and thus $(*)$ holds for $x \in \{b, a_1, a_2, a_3, r_5\Delta_4\}$ by Lemma 9.4. Note that $h_{e_5} = a_4ba_3a_4h_{e_2}$ and $\psi(h_{e_5}) = \psi_{v_1}(a_4ba_3a_4)\psi(h_{e_2})$. Also $\psi_{v_2}(e) = \psi_{v_1}(e)$ and

$$\psi(h_{e_5})^{-1}\psi_{v_2}(e)\psi(h_{e_5}) = \psi(h_{e_2})^{-1}\psi_{v_1}((a_4ba_3a_4)^{-1}e(a_4ba_3a_4))\psi(h_{e_2}).$$

It can be checked that $(a_4ba_3a_4)^{-1}e(a_4ba_3a_4) \in \mathrm{Stab}[\alpha_1, \alpha_3]$ and thus $(*)$ holds for $x = e$ by Lemma 9.6. Similarly, since $u_1 \in \mathrm{Stab}[\alpha_1, \alpha_3]$

$$\psi(h_{e_5})\psi_{v_1}(u_1)\psi(h_{e_5})^{-1} = \psi_{v_1}(h_{e_5}u_1h_{e_5}^{-1}) = \psi_{v_2}(h_{e_5}u_1h_{e_5}^{-1})$$

by Lemmas 9.1 and 9.6. □

LEMMA 9.8. — *If $g = 6$ then for $x \in \mathrm{Stab}[\alpha_1, \gamma_{\{3,4,5,6\}}]$*

$$(*)\ \psi(h_{e_6})^{-1}\psi_{v_1}(x)\psi(h_{e_6}) = \psi_{v_1}(h_{e_6}^{-1}xh_{e_6}).$$

Proof. — Let $\gamma = \gamma_{\{3,4,5,6\}}$ and $d = u_3u_4u_5u_3u_4u_3$. By similar argument as in the proof of Lemma 9.4, $\mathrm{Stab}[\alpha_1, \gamma]$ is generated by $\mathrm{Stab}^+[\alpha_1, \gamma]$ together with u_1a_1, $h_{e_6}u_1a_1h_{e_6}^{-1}$ and u_1r_6d (preserves sides and reverses orientation of α_1, preserves sides and reverses orientation of γ).

By Lemma 9.3 $\mathrm{Stab}^+[\alpha_1, \gamma]$ is generated by c, a_i for $i \in \{1, 3, 4, 5\}$, r_6br_6 and $d^{-1}bd$. Note that the last element can be expressed in terms of the other generators of $\mathrm{Stab}[\alpha_1, \gamma]$. For $x \in \{c, r_6br_6, u_1r_6d\} \cup \{a_i \mid i = 1, 3, 4, 5\}$ we have $x \in \mathrm{Stab}[\xi]$, and since also $h_{e_6} \in \mathrm{Stab}[\xi]$ and $\psi(h_{e_6}) = \psi_{v_3}(h_{e_6})$, thus $(*)$ holds by Lemma 9.4.

We have $u_1a_1 = Y_{\mu_1,\alpha_1}$ and

$$\begin{aligned} h_{e_6}u_1a_1h_{e_6}^{-1} &= Y_{\mu_3,\gamma} = Y_{\mu_3,\gamma_{\{3,6\}}}Y_{\mu_3,\gamma_{\{3,5\}}}Y_{\mu_3,\gamma_{\{3,4\}}} \\ &= (u_3u_4Y_{\mu_5,\alpha_5}u_4^{-1}u_3^{-1})(u_3Y_{\mu_4,\alpha_4}u_3^{-1})Y_{\mu_3,\alpha_3} = u_3u_4u_5a_5a_4a_3. \end{aligned}$$

Hence $\psi_{v_1}(h_{e_6}u_1a_1h_{e_6}^{-1}) = u_3u_4u_5a_5a_4a_3$. Recall that $c = s^2bs^{-2}$ for $s = a_1\cdots a_5$. We have

$$\psi(h_{e_6})\psi_{v_1}(u_1a_1)\psi(h_{e_6})^{-1} = a_2ca_1a_2u_1a_1a_2^{-1}a_1^{-1}c^{-1}a_2^{-1} = a_2cu_2^{-1}a_2c^{-1}a_2^{-1}$$
$$= a_2s^2bs^{-2}u_2^{-1}a_2s^2b^{-1}s^{-2}a_2^{-1} = sa_1sbs^{-1}u_1a_1sb^{-1}s^{-1}a_1^{-1}s^{-1}.$$

Thus for $x = h_{e_6}u_1a_1h_{e_6}^{-1}$, $(*)$ is equivalent to

$$a_1sbs^{-1}u_1a_1sb^{-1}s^{-1}a_1^{-1} = s^{-1}u_3u_4u_5a_5a_4a_3s.$$

The last relation holds in $\mathcal{G}$ because its both sides are in $\mathcal{S}(v_2)$. Indeed, we have $sbs^{-1} = a_1a_2a_3a_4ba_4^{-1}a_3^{-1}a_2^{-1}a_1^{-1}$ and $s^{-1}u_3u_4u_5a_5a_4a_3s = u_2^{-1}u_3^{-1}u_4^{-1}a_4a_3a_2$. It can be checked that $h_{e_6}^2 \in \operatorname{Stab}[\alpha_1]\cap\operatorname{Stab}[\xi]$ and by Lemma 9.4 $\psi(h_{e_6})^2 = \psi_{v_3}(h_{e_6}^2) = \psi_{v_1}(h_{e_6}^2)$. It follows that $(*)$ holds for $x = u_1a_1$ because

$$\psi(h_{e_6})^{-1}\psi_{v_1}(u_1a_1)\psi(h_{e_6}) = \psi_{v_1}(h_{e_6}^{-2})\psi(h_{e_6})\psi_{v_1}(u_1a_1)\psi(h_{e_6})^{-1}\psi_{v_1}(h_{e_6}^2)$$

and the right hand side is equal to $\psi_{v_1}(h_{e_6}^{-1}u_1a_1h_{e_6})$ by earlier part of the proof. □

LEMMA 9.9. — *If $g = 6$ then for $x \in \operatorname{Stab}[\mu_6, \gamma_{\{1,2,3,4,5\}}]$*

$$(*)\ \psi(h_{e_7})^{-1}\psi_{v_2}(x)\psi(h_{e_7}) = \psi_{v_2}(h_{e_7}^{-1}xh_{e_7}).$$

Proof. — Let $\gamma = \gamma_{\{1,2,3,4,5\}}$. By similar argument as in the proof of Lemma 9.4, $\operatorname{Stab}[\mu_6,\gamma]$ is generated by $\operatorname{Stab}^+[\mu_6,\gamma]$ and an element reversing orientation of μ_6 and γ, for which we can take

$$w = a_2a_1a_3a_2a_4a_3\Delta_5(a_5u_5)^{-1}.$$

The surface obtained by cutting $N_{6,0}$ along $\mu_6\cup\gamma$ is homeomorphic to $S_{2,2}$ and it can be deduced from Theorem 3.1 that $\operatorname{Stab}^+[\mu_6,\gamma]$ is generated by b, a_1, a_2, a_3, a_4, $u_4^{-1}a_4^{-1}ba_4u_4$. We have $h_{e_7} = b_2^{-1}$ and $\psi(h_{e_7}) = b_2^{-1}$ commutes with $b = \psi_{v_2}(b)$ and $a_i = \psi_{v_2}(a_i)$. It can be checked that for $x \in \{w, u_4^{-1}a_4^{-1}ba_4u_4\}$ we have $x \in \operatorname{Stab}[\alpha_1]$. Since also $h_{e_7} \in \operatorname{Stab}[\alpha_1]$ and $\psi(h_{e_7}) = \psi_{v_1}(h_{e_7})$, thus $(*)$ holds for these x by Lemma 9.1. □

10. Triangles

In this section we finish the proof of Theorem 3.6 by showing that the map ψ defined at the beginning of the previous section respects the relations corresponding to the triangles of X. For $f \in \mathcal{S}_2(X)$ and $i = 1,2,3$ let $\nu_i = \nu_i(f) \in \mathcal{S}_0(X)$, $\varepsilon_i = \varepsilon_i(f) \in \mathcal{S}_1(X)$, $\widetilde{\varepsilon_i} = \widetilde{\varepsilon_i}(f) \in \mathcal{S}_1(\widetilde{X})$ and $x_i = x_i(f) \in \operatorname{Stab} s(\nu_i)$ be as defined in Subsection 5.3. We have to prove that

$$(**)\quad \psi(h_{\varepsilon_1})\psi_{\nu_2}(x_2)\psi(h_{\varepsilon_2})\psi_{\nu_3}(x_3)\psi(h_{\varepsilon_3})^{-1} = \psi_{\nu_1}(x_1)$$

holds in $\mathcal{G}$. Note that we have not yet chosen the elements x_i. Once any two of them are chosen, the third one is determined by the relation $h_{\varepsilon_1}x_2h_{\varepsilon_2}x_3h_{\varepsilon_3}^{-1} = x_1$. As explained in Subsection 5.3, for $j \in \{1,\dots,10\}$ and for each permutation $\sigma \in \mathrm{Sym}_3$, $x_i(f_j^\sigma)$ are determined by $x_i(f_j)$. Moreover, it is easy to check that if $(**)$ holds for f_j then it also holds for f_j^σ. Therefore it suffices to prove the following.

Lemma 10.1. — *For $j \in \{1,\dots,10\}$ and $i \in \{1,2,3\}$ the elements $x_i(f_j)$ can be chosen in such a way that $(**)$ is satisfied.*

Proof. — Fix $f = f_j$ and let $[\gamma_1,\gamma_2,\gamma_3] = s(f)$.

Case 1. — Suppose that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$ and $h_{\varepsilon_1}(\gamma_3) = \gamma_3$. Then $s(\varepsilon_1) = [\gamma_1,\gamma_2]$ and $h_{\varepsilon_1}[\gamma_1,\gamma_3] = [\gamma_2,\gamma_3]$. Once x_1 is chosen such that $x_1[\gamma_1,\gamma_2] = [\gamma_1,\gamma_3]$, then we can take $x_2 = x_1$ because $h_{\varepsilon_1}x_1[\gamma_1,\gamma_2] = [\gamma_2,\gamma_3]$, and x_3 is determined. The assumption of this case is satisfied for $j \in \{1,4,7\}$. Indeed, for $j = 1$ we have $s(f) = [\alpha_1,\alpha_3,\alpha_5]$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = e_2$, $h_{e_2} = a_2a_3a_1a_2$ and we take $x_1 = a_4a_5a_3a_4$. For $j \in \{4,7\}$ we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = e_3$, $h_{e_3} = a_{g-1}^{-1}$ and $s(f_4) = [\mu_g,\mu_{g-1},\mu_{g-2}]$, $s(f_7) = [\mu_5,\mu_4,\gamma_{\{1,2,3\}}]$ $(g = 5)$. We take $x_1(f_4) = a_{g-2}^{-1}$ and $x_1(f_7) = b^{-1}$. It is easy to check that in each case we have $x_3 = x_1^{-1}$ and $(**)$ is a consequence of the relations (A1, A2, A4).

Case 2. — Suppose that $\varepsilon_2 = \varepsilon_3$, $h_{\varepsilon_3} = 1$, $s(\varepsilon_3) = [\gamma_1,\gamma_3]$ and $h_{\varepsilon_1}(\gamma_3) = \gamma_3$. Then $h_{\varepsilon_1}[\gamma_1,\gamma_3] = [\gamma_2,\gamma_3]$ and we can take $x_1 = x_2 = 1$, $x_3 = h_{\varepsilon_1}^{-1}$. In this case $(**)$ is $\psi(h_{\varepsilon_1}) = \psi_{\nu_3}(h_{\varepsilon_1})$. The assumption of this case is satisfied for $j \in \{2,5\}$. Indeed, for $j = 2$ we have $\varepsilon_1 = e_2$, $\varepsilon_2 = \varepsilon_3 = e_1$, $s(f) = [\alpha_1,\alpha_3,\mu_g]$, $h_{e_2} = a_2a_3a_1a_2$. Since $\psi(h_{e_2}) = \psi_{v_2}(h_{e_2})$ thus $(**)$ holds. For $j = 5$ we have $\varepsilon_1 = e_2$, $\varepsilon_2 = \varepsilon_3 = e_4$, $s(f) = [\alpha_1,\alpha_3,\xi]$ and $(**)$ holds because $\psi(h_{e_2}) = \psi_{v_3}(h_{e_2})$.

Case 3. — Suppose that $\varepsilon_1 = \varepsilon_3$, $h_{\varepsilon_1} = 1$, $s(\varepsilon_2) = [\gamma_2,\gamma_3]$ and $h_{\varepsilon_2}(\gamma_1) = \gamma_1$. Analogously as in Case 2 we can take $x_2 = x_3 = 1$, $x_1 = h_{\varepsilon_2}$ and $(**)$ becomes $\psi(h_{\varepsilon_2}) = \psi_{\nu_1}(h_{\varepsilon_1})$. The assumption of this case is satisfied for $j \in \{3,8\}$. Indeed, for $j = 3$ we have $\varepsilon_1 = \varepsilon_3 = e_1$, $\varepsilon_2 = e_3$, $s(f) = [\alpha_1,\mu_g,\mu_{g-1}]$, $h_{e_3} = a_{g-1}^{-1}$ and $(**)$ holds because $\psi(h_{e_3}) = \psi_{v_1}(h_{e_3})$. For $j = 8$ we have $\varepsilon_1 = \varepsilon_3 = e_1$, $\varepsilon_2 = e_7$, $s(f) = [\alpha_1,\mu_g,\gamma_{\{1,\dots,5\}}]$, $h_{e_7} = b_2^{-1}$ and $(**)$ holds because $\psi(h_{e_7}) = \psi_{v_1}(h_{e_7})$.

Case 4. — Suppose that $j \in \{6,9,10\}$. We have $h_{\varepsilon_i} \in \mathrm{Stab}[\xi]$ and $\psi(h_{\varepsilon_i}) = \psi_{v_3}(h_{\varepsilon_i})$ for $i = 1,2,3$. If we can choose $x_i \in \mathrm{Stab}[\xi] \cap \mathrm{Stab}\, s(\nu_i)$ such that $\psi_{\nu_i}(x_i) = \psi_{v_3}(x_i)$ then $(**)$ will follow from the fact that ψ_{v_3} is a homomorphism.

For $j = 6$ we have $s(f) = [\mu_6,\mu_5,\beta]$ and $h_{\varepsilon_1} = a_5^{-1}$. It can be checked that for $x_1 = a_4ba_3a_4a_2a_3a_1a_2$ we have $x_1[\mu_6,\alpha_1] = [\mu_6,\beta]$. Since $a_5^{-1}[\mu_6,\beta] = [\mu_5,\beta]$ we can take $x_2 = x_1$ and x_3 is determined. Clearly $x_1 \in \mathrm{Stab}[\xi] \cap \mathrm{Stab}[\mu_6]$

and $\psi_{v_2}(x_1) = \psi_{v_3}(x_1)$. It follows that $x_3 \in \mathrm{Stab}[\xi] \cap \mathrm{Stab}[\alpha_1]$ and $\psi_{v_1}(x_3) = \psi_{v_3}(x_3)$ by Lemma 9.4.

For $j = 9$ we have $s(f) = [\alpha_1, \mu_5, \beta]$. It can be checked that for $x_1 = a_4 b a_3 a_4$ we have $x_1[\alpha_1, \alpha_3] = [\alpha_1, \beta]$. Since $s(\varepsilon_2) = [\mu_5, \beta]$ and $h_{\varepsilon_1} = 1$, we can take $x_2 = 1$ and x_3 is determined. We have $x_3 \in \mathrm{Stab}[\xi]$ and $\psi_{v_1}(x_3) = \psi_{v_3}(x_3)$ by Lemma 9.4.

For $j = 10$ we have $s(f) = [\alpha_1, \alpha_3, \gamma_{\{3,4,5,6\}}]$, $\varepsilon_1 = \varepsilon_2 = e_2$, $\varepsilon_3 = e_6$ and $h_{e_2} = a_2 a_3 a_1 a_2$, $h_{e_6} = a_2 c a_1 a_2$. We take $x_1 = 1$ and $x_2 = a_4 a_5 a_3 a_4^2 b a_3 a_4$. It can be checked that $h_{e_2} x_2 [\alpha_1, \alpha_3] = [\alpha_3, \gamma_{\{3,4,5,6\}}]$. We have $\psi_{v_1}(x_2) = \psi_{v_3}(x_2)$, $x_3 \in \mathrm{Stab}[\xi]$ and $\psi_{v_1}(x_3) = \psi_{v_3}(x_3)$ by Lemma 9.4. □

Appendix A
The presentation of $\mathcal{M}(S_{\rho,r})$

In this section we prove Theorem 3.1. For $r = 1$ the presentation given in Theorem 3.1 is essentially the same as the one given in [19], with additional generators and relations. In the case $r = 2$ we follow the idea of [16].

We are going to use some basic facts about geometric representations of Artin groups (see [16] for details). Let Γ_1, Γ_2 be the Artin groups of types A_5 and D_6 respectively. For $i \geq 1$ and $j = 1, 2$ we have homomorphisms $\theta_{i,j} \colon \Gamma_j \to \mathcal{M}(S_{\rho,r})$, such that $\theta_{i,1}$ maps the standard generators of Γ_1 on $b_{i-1}, a_{2i}, a_{2i+1}, a_{2i+2}, a_{2i+3}$ and $\theta_{i,2}$ maps the standard generators of Γ_2 on $b_{i-1}, a_{2i}, a_{2i+1}, a_{2i+2}, a_{2i+3}, b_i$. Let $\Delta(\Gamma_j)$ be the fundamental element of Γ_j and $C_{i,j} = \theta_{i,j}(\Delta(\Gamma_j))$. Then we have

$$C_{i,1} = b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_{i-1} a_{2i} a_{2i+1} a_{2i+2} b_{i-1} a_{2i} a_{2i+1} b_{i-1} a_{2i} b_{i-1},$$
$$C_{i,1}^2 = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3})^6, \quad C_{i,2} = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_i)^5.$$

Proof of Theorem 3.1. Let $S_1 = S_{\rho,1}$ and $S_2 = S_{\rho,2}$ for a fixed $\rho \geq 1$. We assume that S_1 is obtained from S_2 by gluing a disc along β_ρ (see Figure 5) and let P be the center of the glued disc. We have the following short exact sequences, which are special cases of (2.1) and (2.2).

$$1 \to \pi_1(S_1, P) \to \mathcal{M}(S_1, P) \to \mathcal{M}(S_1) \to 1 \tag{A.1}$$

$$1 \to \langle b_\rho \rangle \to \mathcal{M}(S_2) \to \mathcal{M}(S_1, P) \to 1 \tag{A.2}$$

By [19] $\mathcal{M}(S_1)$ admits a presentation with generators b_1, a_i for $i = 1, \ldots, 2\rho$ and relations (A1–A6). We add to this presentation the generators b_j for $j = 0, 2, \ldots, \rho - 1$ and relations (A7,A8). We need to show that (A8) are satisfied in $\mathcal{M}(S_1)$. Fix $i \geq 1$ and let M_1 and M_2 be regular neighborhoods of $\beta_i \cup \beta_{i-1} \cup \alpha_{2i} \cup \cdots \cup \alpha_{2i+3}$ and $\beta_{i-1} \cup \alpha_{2i} \cup \cdots \cup \alpha_{2i+3}$ respectively. The boundary of M_1 consists of three connected components, one of which is isotopic to β_{i+1} and one bounds a disc. Let δ be the third component and note that the boundary of M_2 consists of two connected components isotopic to β_{i+1} and δ. By [16, Proposition 2.12] we have $C_{i,2} = T_\delta b_{i+1}^2$ and $C_{i,1}^2 = T_\delta b_{i+1}$. The relation (A8) follows and Theorem 3.1 is proved for $r = 1$.

We are going to show that $\mathcal{M}(S_1, P)$ admits a presentation with generators a_i for $i = 1, \dots, 2\rho+1$ and b_j for $j = 0, 1, \dots, \rho-1$ and relations (A1–A8) and if $\rho \geq 2$

$$\text{(A8a)}\quad (b_{\rho-2}a_{2\rho-2}a_{2\rho-1}a_{2\rho}a_{2\rho+1}b_{\rho-1})^5 = (b_{\rho-2}a_{2\rho-2}a_{2\rho-1}a_{2\rho}a_{2\rho+1})^6.$$

To prove this we apply Lemma 2.1 to the sequence (A.1). By gluing a punctured annulus along the boundary of S_1 we obtain an induced embedding $\mathcal{M}(S_1) \to \mathcal{M}(S_1, P)$, which is a splitting of (A.1). Through this embedding we will identify the generators of $\mathcal{M}(S_1)$ with elements of $\mathcal{M}(S_1, P)$ which will be the cokernel generators, and the defining relations (A1–A8) of $\mathcal{M}(S_1)$ will be the cokernel relations in our presentation of $\mathcal{M}(S_1, P)$. For notational convenience we set

$$c_0 = b_{\rho-1},\ c_i = a_{2\rho+2-i} \text{ for } i = 1, \dots, 2\rho+1, \quad d = b_{\rho-2} \text{ if } \rho \geq 2.$$

The kernel is freely generated by

$$x_1 = c_1 c_0^{-1}, \quad x_i = c_i x_{i-1} c_i^{-1} \text{ for } i = 2, \dots, 2\rho.$$

These will be our kernel generators. We also add c_1 to the set of generators together with the relation

$$(0)\quad c_1 = x_1 c_0.$$

The following relations are consequences of (A1–A8).

(H1) $c_i c_j = c_j c_i$ for $1 \leq i < j-1 \leq 2\rho$,

(H2) $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$ for $1 \leq i \leq 2\rho$,

(H3) $c_0 c_i = c_i c_0$ for $i \neq 2$, (H4) $c_0 c_2 c_0 = c_2 c_0 c_2$,

(H5) $d c_i = c_i d$ for $i \neq 4$, (H6) $d c_4 d = c_4 d c_4$.

Indeed, (H1,H2) are simply (A1,A2) rewritten in the symbols c_i, (H4, H6) involve only cokernel generators, and so they are consequences of the cokernel relations, because of the splitting, and so are (H3, H5) for $i \neq 1$. If $i = 1$ then the last relations follow from (A1, A8).

Since $\mathcal{M}(S_1)$ is generated by d and c_i for $i = 0, 2, \dots, 2\rho$, we can use these cokernel generators to produce the conjugation relations. We write $c_i(x_j)$ instead of $c_i x_j c_i^{-1}$.

(1) $c_i(x_{i-1}) = x_i$, (2) $c_i(x_i) = x_i x_{i-1}^{-1} x_i$,

(3) $c_i(x_j) = x_j$, for $i \geq 2, j \notin \{i-1, i\}$,

(4) $c_0(x_1) = x_1$, (5) $c_0(x_2) = x_1^{-1} x_2$,

(6) $c_0(x_i) = c_i c_0(x_{i-1})$ for $i > 2$,

(7) $d(x_i) = x_i$ for $i = 1, 2, 3$, (8) $d(x_4) = x_1^{-1} x_2 x_3^{-1} x_4$,
(9) $d(x_j) = c_j d(x_{j-1})$ for $j > 4$.

We are going to show that (2–7,9) are consequences of (0,1) and (H1–H6). (6) follows from (1) and (H3):

$$c_0(x_i) = c_0 c_i(x_{i-1}) = c_i c_0(x_{i-1}).$$

Analogously (9) follows from (1) and (H5). (7) follows from (0, 1, H5), and (3) follows from (0, 1, H1) if $j < i - 1$, for $j = i + 1$ we have

$$c_i(x_{i+1}) = c_i c_{i+1} c_i(x_{i-1}) = c_{i+1} c_i c_{i+1}(x_{i-1}) = x_{i+1}$$

and for $j = i + 1 + k$ by induction

$$c_i(x_{i+1+k}) = c_i c_{i+1+k}(x_{i+k}) = c_{i+1+k} c_i(x_{i+k}) = x_{i+1+k}.$$

For $i = 2$ (2) is equivalent to

$$\begin{aligned} c_2^2 c_1 c_0^{-1} c_2^{-2} &= c_2 c_1 \underline{c_0^{-1} c_2^{-1} c_0 c_1^{-1} c_2 c_1} c_0^{-1} c_2^{-1} \\ \iff c_2 c_1 c_0^{-1} c_2^{-1} &= c_1 c_2 \underline{c_0^{-1} c_1} c_2^{-1} c_0^{-1} \\ \iff c_2 c_1 c_0^{-1} c_2^{-1} &= \underline{c_1 c_2 c_1} c_0^{-1} c_2^{-1} c_0^{-1} \\ \iff c_0^{-1} c_2^{-1} &= c_2 c_0^{-1} c_2^{-1} c_0^{-1} \\ \iff c_2 c_0 c_2 &= c_0 c_2 c_0 \end{aligned}$$

and for $i > 2$

$$\begin{aligned} c_i x_i c_i^{-1} &= c_i c_{i-1} x_{i-2} \underline{c_{i-1}^{-1} c_i^{-1} c_{i-1}} x_{i-2}^{-1} \underline{c_{i-1}^{-1} c_i c_{i-1}} x_{i-2} c_{i-1}^{-1} c_i^{-1} \\ \iff c_i c_{i-1} x_{i-2} c_{i-1}^{-1} c_i^{-1} &= c_{i-1} x_{i-2} c_i c_{i-1}^{-1} \underline{c_i^{-1} x_{i-2}^{-1}} c_i c_{i-1} c_i^{-1} x_{i-2} c_{i-1}^{-1} \\ \iff \underline{c_{i-1}^{-1} c_i c_{i-1}} x_{i-2} \underline{c_{i-1}^{-1} c_i^{-1} c_{i-1}} &= \underline{x_{i-2} c_i} c_{i-1}^{-1} x_{i-2}^{-1} c_{i-1} \underline{c_i^{-1} x_{i-2}} \\ \iff c_i \underline{c_{i-1} c_i^{-1} x_{i-2} c_i c_{i-1}^{-1}} c_i^{-1} &= c_i x_{i-2} c_{i-1}^{-1} x_{i-2}^{-1} c_{i-1} x_{i-2} c_i^{-1} \\ \iff x_{i-1} &= x_{i-2} c_{i-1}^{-1} x_{i-2}^{-1} c_{i-1} x_{i-2} \\ \iff c_{i-1} x_{i-1} c_{i-1}^{-1} &= c_{i-1} x_{i-2} c_{i-1}^{-1} x_{i-2}^{-1} c_{i-1} x_{i-2} c_{i-1}^{-1} = x_{i-1} x_{i-2}^{-1} x_{i-1} \end{aligned}$$

and we are done by induction. (4) is equivalent to $c_0 c_1 = c_1 c_0$ and (5) to

$$c_0 c_2 c_1 \underline{c_0^{-1} c_2^{-1} c_0^{-1}} = c_0 c_1^{-1} c_2 c_1 c_0^{-1} c_2^{-1} \iff c_2 c_1 c_2^{-1} = c_1^{-1} c_2 c_1.$$

Finally we are going to show that (8) follows from (0,1), (H1–H6) and (A8a). Let $C_j = C_{\rho-1,j}$ for $j = 1, 2$ so that we have

$$C_1 = d c_4 c_3 c_2 c_1 d c_4 c_3 c_2 d c_4 c_3 d c_4 d, \quad C_1^2 = (d c_4 c_3 c_2 c_1)^6, \quad C_2 = (d c_4 c_3 c_2 c_1 c_0)^5.$$

We leave it as an exercise for the reader to check that by using (0,1) and (H1–H6) the relation (8) can be rewritten as

$$C_1 = c_0 c_2 c_1 c_3 c_2 c_0 c_4 c_3 c_2 c_1 d c_4 c_3 c_2 c_0,$$

and by (H1–H6) we have

$$C_2 = C_1 c_0 c_2 c_1 c_3 c_2 c_0 c_4 c_3 c_2 c_1 d c_4 c_3 c_2 c_0.$$

Thus we have obtained the relation $C_2 = C_1^2$, which is exactly (A8a).

We can drop the generators x_i and relations (0–9) to obtain a presentation of $\mathcal{M}(S_1, P)$ with generators a_i for $i = 1, \dots, 2\rho+1$ and b_j for $j = 0, 1, \dots, \rho-1$ and relations (A1–A8,A8a).

Now we will obtain a presentation of $\mathcal{M}(S_2)$ by applying Lemma 2.1 to the sequence (A.2). We take the generators of $\mathcal{M}(S_1, P)$ as cokernel generators and b_ρ as kernel generator. The relations (A1–A8) are satisfied in $\mathcal{M}(S_2)$ and the cokernel relation corresponding to (A8a) is $C_2 C_1^{-2} = b_\rho$ which gives (A8) for $i + 1 = \rho$. The conjugation relations are

$$(*) \quad b_\rho y = y b_\rho$$

for every cokernel generator y. It suffices to consider $y = a_i$ for $i = 1, \dots, 2\rho+1$ and $y = b_1$ if $\rho \geq 2$. If $\rho = 1$ then $(*)$ follows from (A3), so we suppose that $\rho \geq 2$. Since $b_\rho = C_2 C_1^{-2}$ and C_2, C_1^2 are central in $\theta_{\rho-1,2}(\Gamma_2)$ and $\theta_{\rho-1,1}(\Gamma_1)$ respectively, $(*)$ is a consequence of (H1–H6) for $y = c_i = a_{2\rho+2-i}$, $i = 1, 2, 3, 4$ and $y = d = b_{\rho-2}$. In particular b_ρ commutes with $a_1 = b_0$ if $\rho = 2$ and with b_1 if $\rho = 3$. If $\rho \geq 3$ then it follows from (A1–A8) that b_ρ commutes with a_i for $i \leq 2\rho - 4$ and b_1 if $\rho \geq 4$. Finally (A9a, A9b) imply that it also commutes with $a_{2\rho-3}$ if $\rho \geq 3$ and b_1 if $\rho = 2$. Since all conjugation relations are consequences of (A1–A9), $\mathcal{M}(S_2)$ admits the presentation from Theorem 3.1. □

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