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MILNOR NUMBER OF PAIRS AND PENCILS OF PLANE HOLOMORPHIC GERMS

by Adrian Szawlowski

Abstract. — We introduce a Milnor number of pairs of plane holomorphic germs and investigate the relation between the Milnor numbers in a pencil of such functions.

Résumé (*Le nombre de Milnor des paires et des pinceaux de germes holomorphes plans*)

Nous introduisons un nombre de Milnor pour les paires de germes holomorphes plans et nous étudions la relation entre les nombres de Milnor apparaissant dans un tel pinceau.

1. Introduction

The study of germs of holomorphic functions $f\colon(\mathbb{C}^n,\mathbf{0})\to(\mathbb{C},0)$ has very much progressed since the work of Milnor ([13]). Among other things he proved a result which is nowadays called Milnor's fibration theorem and has relations with knot theory. There is another fibration, which is also referred very often to as Milnor fibration (or Milnor-Lê fibration) and is equivalent to the previous one. Some decades later, the interest in meromorphic germs has raised. For example both fibration theorems have extensions to the meromorphic case (compare [1] for the first fibration and [16] resp. [8] for the second fibration

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and [2] for their comparison in the semitame case). It seems however that the investigation of meromorphic germs is still not complete at all. Another way to perceive them is for example by considering a pencil spanned by the two given germs. What we do here is to look at the Milnor numbers that occur in such a pencil. The basic invariant of a holomorphic germ $f\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ with an isolated critical point at the origin is its Milnor number

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{\langle \partial_x f, \partial_y f\rangle}.$$

Given another holomorphic germ $g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ we form the pencil generated by f and g which associates to each $(s : t) \in \mathbb{P}^1$ the function germ $sf + tg$. In this paper we address the question whether there is a relation between all the Milnor numbers of the pencil members. Apart from some minor propositions, the main results in this paper are Theorem 2.6, Theorem 3.3, Conjecture 4.1 and its special case Proposition 4.4.

As is standard we denote by $\mathbb{C}\{x, y\}$ the ring of germs of holomorphic functions at the origin, or equivalently the complex power series ring at the origin and by $\mathfrak{m}$ its maximal ideal. For two germs $f, g \in \mathbb{C}\{x, y\}$ we denote by $i(f, g)$ their intersection number at the origin of $\mathbb{C}^2$. For an ideal I in $\mathbb{C}\{x, y\}$, we denote by $V(I)$ its vanishing locus, the germ of an analytic set at the origin in $\mathbb{C}^2$. Finally we denote by ∂ the partial derivative after x and/or y.

I like to thank the referee for useful comments, especially regarding the last paragraph.

2. Milnor Number of Pairs

For any $f, g \in \mathbb{C}\{x, y\}$ we introduce the notation $\omega(f, g)$ for the ideal in $\mathbb{C}\{x, y\}$ generated by the elements $f_x g - f g_x$ and $f_y g - f g_y$. It is clear that $\omega(f, g)$ is a subideal of $\langle f, g\rangle$. Sometimes if f and g are clear from the context, we simply write ω instead of $\omega(f, g)$. Let us now define the following possibly infinite numbers

$$\mu(f, g) := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{\omega(f, g)}, \tag{2.1}$$

$$\nu(f, g) := \dim_{\mathbb{C}} \frac{\langle f, g\rangle}{\omega(f, g)}. \tag{2.2}$$

Clearly we have $\mu(f, g) = \nu(f, g) + i(f, g)$. At an arbitrary point $p \in \mathbb{C}^2$ we define $\mu_p(f, g)$ by using the local power series ring $\mathcal{O}_p$ and the ideal $\omega_p(f, g)$ at p in the obvious way:

$$\mu_p(f, g) := \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{\omega_p(f, g)}. \tag{2.3}$$

As will follow from the propositions below, $\mu(f,g)$ is the analogue to the classical Milnor number $\mu(f)$ of a holomorphic germ and will be referred to as Milnor number of pairs or meromorphic Milnor number in the following.

The next propositions give first information on these numbers.

PROPOSITION 2.1. — *Let $f,g \in \mathbb{C}\{x,y\}$. Then for any coordinate transformation $\Phi \in \mathrm{Aut}(\mathbb{C}^2,\mathbf{0})$ and any unit $u \in \mathbb{C}\{x,y\}$ we have $\mu(u\cdot f\circ\Phi, u\cdot g\circ\Phi) = \mu(f,g)$ and the analogous property holds for ν.*

This has an elementary proof using derivatives. Easy as well is the following

PROPOSITION 2.2. — *Let $f,g \in \mathbb{C}\{x,y\}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\,\mathbb{C}$. Then*

$$\mu(af+bg, cf+dg) = \mu(f,g)$$

and the same property holds for ν.

The next proposition shows that our generalization of the Milnor number includes as a special case the classical Milnor number.

PROPOSITION 2.3. — *Let $f,g \in \mathbb{C}\{x,y\}$. If $f(\mathbf{0}) \neq 0$ but $g(\mathbf{0}) = 0$, then $\mu(f,g) = \nu(f,g) = \mu(g)$ (maybe infinite).*

Proof. — It is clear that if $f(\mathbf{0}) \neq 0$, we have $\mu(f,g) = \nu(f,g)$. If we assume $f(\mathbf{0}) \neq 0$ then the ideals $\omega, f^{-2}\omega$ are the same and the latter can be written as $\langle \partial(g/f)\rangle$. Hence $\mu(f,g) = \mu(g/f)$. By the invariance of the Milnor number under the action of the contact group we obtain $\mu(f,g) = \mu(g)$. □

It is well-known that the classical Milnor number of a holomorphic germ $f \in \mathfrak{m}$ is equal to one if and only if f is critical and Morse at the origin. The following proposition shows that the situation with the new Milnor number is also very pleasant.

PROPOSITION 2.4. — *Let $f,g \in \mathbb{C}\{x,y\}$ with $f(\mathbf{0})g(\mathbf{0}) = 0$. Then $\mu(f,g) = 1$ if and only if we are in one of the following cases:*

- *We have $f(\mathbf{0}) \neq 0$ and g/f vanishes at the origin, is critical and Morse there.*
- *We have $g(\mathbf{0}) \neq 0$ and f/g vanishes at the origin, is critical and Morse there.*
- *The pair (f,g) is right equivalent to (x,y).*

Proof. — Let $\mu(f,g) = 1$. Since $\mu(f,g) = \nu(f,g) + i(f,g)$ two cases are possible:

- $\nu(f,g) = 0, i(f,g) = 1$ or
- $\nu(f,g) = 1, i(f,g) = 0$.

Now $i(f,g)=1$ holds if and only if (f,g) is right equivalent to (x,y). The converse is easily seen. And $i(f,g)=0$ holds if and only if at least one of f,g does not vanish at the origin. By hypothesis, the other function must then vanish at the origin. In this case we can use Proposition 2.3 and what was just said before to conclude the proof. □

Proposition 2.5 (Deformation property of $\mu(-,-)$). — *Let $f=f(x,y)$, $g=g(x,y)\in\mathfrak{m}$ with $\mu(f,g)<\infty$. Let $f(x,y,t),g(x,y,t)$ be unfoldings of $f(x,y),g(x,y)$. Then for any sufficiently small neighbourhood U of the origin in $\mathbb{C}^2$ we have for each sufficiently small $t\in\mathbb{C}, t\approx 0$*

$$\mu(f,g)=\sum_{p\in U(t)}\mu_p(f(x,y,t),g(x,y,t))$$

where

$$U(t):=\left\{(x,y)\in U\;\middle|\; \begin{array}{l} f_x(x,y,t)g(x,y,t)-f(x,y,t)g_x(x,y,t)=0,\\ f_y(x,y,t)g(x,y,t)-f(x,y,t)g_y(x,y,t)=0 \end{array}\right\}.$$

Proof. — The proof of such statements is nowadays standard. First, a sufficiently small representative of the map

$$\begin{aligned}(\mathbb{C}^3,0)\to(\mathbb{C}^3,0),(x,y,t)\to(&f_x(x,y,t)g(x,y,t)-f(x,y,t)g_x(x,y,t),\\ &f_y(x,y,t)g(x,y,t)-f(x,y,t)g_y(x,y,t),t)\end{aligned}$$

is flat and finite. (Since $\mu(f,g)<\infty$, the zero fibre of the last map is just the origin, so finiteness follows from the Weierstraß finiteness theorem. Due to the same reason, the map is flat, compare [7], Proposition 2.7.) From [7], Theorem 1.81 (principle of conservation of numbers) applied to the sheaf $\mathcal{O}_{\mathbb{C}^3}$ restricted to a small neighbourhood of $\mathbf{0}$ the assertion follows. □

The next theorem is one of our main results. It replaces the a priori infinitely many conditions in item (3) by just a single condition in item (1) or (2).

Theorem 2.6. — *Let $f,g\in\mathbb{C}\{x,y\}$ be not identically vanishing germs with $f(\mathbf{0})=g(\mathbf{0})=0$. Then the following statements are equivalent:*

1. *The number $\mu(f,g)$ is finite.*
2. *The number $\nu(f,g)$ is finite.*
3. *The germs f,g are coprime and every linear combination $sf+tg$ (with $(s:t)\in\mathbb{P}^1$) has an isolated singularity at the origin.*

Proof. — Recall the well-known fact that a nontrivial germ $f\in\mathfrak{m}$ is a reduced element of the ring $\mathbb{C}\{x,y\}$ (i.e. in its prime factor decomposition no multiple factor occurs) if and only if f has an isolated singularity at the origin. (In higher dimensions this does not hold anymore.) It is also known that for an ideal $I\subset\mathbb{C}\{x,y\}$ we have $\dim(\mathbb{C}\{x,y\}/I)<\infty$ if and only if $V(I)\subset\{0\}$. This follows e.g. by using Cor. 1.74 of [7] for the coherent sheaf $\mathcal{O}/\mathcal{J}$. We are going to prove the following lemma. With it, the equivalence between *(1)* and *(3)* follows, as one can easily see.

LEMMA 2.7. — *Let $f, g \in \mathfrak{m}$ be not identically zero. Then with $\omega = \omega(f, g)$ we have*

i) *If f is not reduced, then $V(\omega)$ is not just the origin, more precisely $V(\omega) \supset V(f_1)$ where f_1 is any prime factor of f which divides f at least of order two.*
ii) *If f and g have a common divisor h, then $V(\omega(f, g)) = V(h) \cup V(\omega(f/h, g/h))$.*
iii) *If $\gcd(f, g) = 1$ and f, g are reduced, then the following implication is valid:*

$$V(\omega) \subset V(f) \cup V(g) \;\Rightarrow\; V(\omega) \subset V(f) \cap V(g) = \{\mathbf{0}\}.$$

iv) *If $\gcd(f, g) = 1$, then the following statements are equivalent:*
   (a) $V(\omega) \subset V(f) \cup V(g)$
   (b) For every $c \in \mathbb{C}^*$, the germ $f - cg$ is reduced.

Proof. — i) and ii) are elementary and are left out.

iii) Let $V(\omega) \subset V(f) \cup V(g)$. Then

$$\begin{aligned} V(\omega) &= V(\omega) \cap (V(f) \cup V(g)) \\ &= V(\omega) \cap V(\langle f \rangle \cap \langle g \rangle) \\ &= V(\langle \omega \rangle + \langle f \rangle \cap \langle g \rangle) \\ &= V(\langle \omega \rangle + \langle fg \rangle), \end{aligned}$$

the latter since $\gcd(f, g) = 1$. From the Nullstellensatz we deduce

$$\sqrt{\langle \omega \rangle} = \sqrt{\langle \omega \rangle + \langle fg \rangle}. \qquad (*)$$

If $V(\omega) = \{\mathbf{0}\}$, then there is nothing to show for *iii)*. So we assume $V(\omega) \underset{\neq}{\supset} \{0\}$ which means that there is an irreducible nonunit h with $\sqrt{\langle \omega \rangle} \subset \langle h \rangle$ ($V(h)$ can be taken as one of the branches of $V(\omega)$). Then by (*) we have not only $h|\, f_x g - f g_x$ and $h|\, f_y g - f g_y$ but also $h|\, fg$. Since $h|\, fg$, the irreducible h must be an irreducible factor of f or g up to a unit, say $h|f$. Then $h|\, f_x g - f g_x, f_y g - f g_y$ implies $h|f_x g, f_y g$ and from $\gcd(f, g) = 1$ we get $h|f_x, f_y$. But then f would not have an isolated singularity at the origin, hence would not be reduced. A contradiction!

iv) We show the implication *(a)* $\Rightarrow$ *(b)* by contraposition. Assume that there is $c \in \mathbb{C}^*$ with $f - cg =: h$ nonreduced. The germ h can't be identically zero since $\gcd(f,g)=1$. We compute

$$\begin{aligned} f\partial g - g\partial f &= (cg+h)\partial g - g\partial(cg+h) \\ &= h\partial g - g\partial h. \end{aligned}$$

Since h is nonreduced it follows already from the item *i)* that $V(\omega) \supset V(h_1)$ where h_1 is an irreducible nonunit and h_1^2 divides h. In particular $V(w) \subset V(f) \cup V(g)$ can't be true since otherwise for example $V(h_1) \subset V(f)$, so h_1 divides f, but since $h = f - cg$, f and g would not be coprime.

Finally, we show *(b)* $\Rightarrow$ *(a)*, again by contraposition. So we assume that it is not true that $V(\omega) \subset V(f) \cup V(g)$. If $V(\omega)$ is two-dimensional, then $f_x g - f g_x$ and $f_y g - f g_y$ are identically zero in a neighbourhood of the origin. Then f/g would be a locally constant, hence constant function on the connected $(\mathbb{C}^2,0)\backslash V(g)$, say equal to $c \in \mathbb{C}$. Hence $f - cg = 0$ on $(\mathbb{C}^2,0)\backslash V(g)$ and therefore on all of $(\mathbb{C}^2,0)$ by continuity (recall $g \not\equiv 0$), which contradicts $\gcd(f,g)=1$ or (if $c=0$) $f \not\equiv 0$.

In the remaining case, $V(\omega)$ is one-dimensional and by the contraposition hypothesis it is not true that $V(\omega) \subset V(f) \cup V(g)$. Then there must be a one-dimensional irreducible component V of $V(w)$ with the property $V \cap V(f) = V \cap V(g) = \{\mathbf{0}\}$. First of all note, since V is irreducible, $V^* := V \backslash \{\mathbf{0}\}$ is smooth and connected. Since $V \subset V(\omega)$ we have $d(f/g) = 0$ on V^*. Therefore f/g is a locally constant function, hence constant on V^*, so equal to some $c \in \mathbb{C}^*$. Let $h = f - cg$. On V^* we have

$$\partial h = \partial f - c\,\partial g = \partial f - \frac{f}{g}\partial g = \frac{g\,\partial f - f\,\partial g}{g^2}.$$

Hence $V(\partial h) \cap V^* = V(\omega) \cap V^* = V^*$ which implies $V(\partial h)$ cannot be $\varnothing$ or $\{\mathbf{0}\}$, hence h could not be an isolated singularity, i.e. is not reduced. This completes the proof of Lemma 2.7. □

It remains to show the equivalence between item *(1)* and *(2)* in Theorem 2.6, that is the equivalence between $\mu(f,g) < \infty$ and $\nu(f,g) < \infty$. Since $\mu(f,g) = i(f,g) + \nu(f,g)$ we assume for a contradiction that $\nu(f,g) < \infty$ but $\mu(f,g) = \infty$. In this case we obviously must have $i(f,g) = \infty$. Now the intersection number of f and g is infinite if and only if their greatest common divisor h is a nonunit. Writing $f = f_1 h, g = g_1 h$, we have $\omega(f,g) = h^2 \omega(f_1,g_1) = h^2\omega_1$. Hence we obtain an isomorphism of vector spaces (by dividing by h):

$$\frac{\langle f,g\rangle}{\omega} \cong \frac{\langle f_1,g_1\rangle}{h\omega_1}.$$

Since $\nu(f,g) < \infty$, both vector spaces are thus finite-dimensional. Furthermore it is clear that we have an exact sequence

$$0 \to \frac{\omega_1}{h\omega_1} \to \frac{\langle f_1, g_1\rangle}{h\omega_1} \to \frac{\langle f_1, g_1\rangle}{\omega_1} \to 0.$$

So since the middle-term vector space is finite-dimensional, so are the other two. Since f_1 and g_1 are coprime we have $i(f_1,g_1) < \infty$. And because the right vector space is finite-dimensional, we deduce $\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\omega_1$. But since the vector space to the left is also finite-dimensional, so would be $\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/h\omega_1$. This can't be true since h is a nonunit! Hence we get a contradiction showing that $\mu(f,g) = \infty$ and $\nu(f,g) < \infty$ cannot hold simultaneously. This completes the proof of Theorem 2.6. □

As a corollary to the proof we get the well-known result

PROPOSITION 2.8. — *Given coprime $f, g \in \mathfrak{m}$, the set $\mathscr{B}_{ni}(f,g) := \{(s:t) \in \mathbb{P}^1 \mid sf + tg$ does not have an isolated singularity at the origin$\}$ is finite.*

The subscript *ni* stands for nonisolated.

Proof. — Let the c_i's be pairwise different complex numbers such that $f - c_i g$ is not isolated. From the proof above, part (iv), (a) $\Rightarrow$ (b), we infer that a branch of $V(f - c_i g)$ must be contained in $V(\omega)$. Since the case $V(\omega) = (\mathbb{C}^2, 0)$ does not occur (see above proof), $V(\omega)$ is a curve or lower-dimensional. As a curve it can have only finitely many branches. However the $f - c_i g$'s are pairwise coprime. Hence the number of such c_i's is finite. □

The above set $\mathscr{B}_{ni}(f,g)$ is contained in the set $\mathscr{B}(f,g)$ which will be discussed more closely in the next section.

3. The Bifurcation Set

We start this section by recalling several equivalent characterizations of the bifurcation set $\mathscr{B}(f,g)$ associated to two germs $f, g \in \mathbb{C}\{x,y\}$ with $V(f) \cap V(g) = \{\mathbf{0}\}$.

We assume that $f, g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are given nontrivial, coprime germs. It is well-known that the function $\mu\colon \mathbb{P}^1 \to \mathbb{N}_{\geq 0} \cup \{\infty\}, (s:t) \mapsto \mu(sf + tg)$ is upper-semicontinuous in the Zariski topology, so that there is a finite set in projective space whose associated pencil members have a strictly larger Milnor number than the remaining pencil member who all have the same Milnor number. This defines the generic Milnor number and the bifurcation set of the pencil:

$$\mu_{gen}(f,g) := \min_{(s:t)\in\mathbb{P}^1} \mu(sf+tg) \tag{3.1}$$

$$\mathcal{B}_\mu(f,g) := \{(s:t)\in\mathbb{P}^1 \mid \infty \geq \mu(sf+tg) > \mu_{gen}(f,g)\}. \tag{3.2}$$

For a more topological characterization of $\mathcal{B}(f,g)$ we recall that that there is a uniquely determined set $\mathcal{B}_{top}(f,g)\subset\mathbb{P}^1$ ("top" stands for topological) such that the following properties hold for all $(s_1:t_1),(s_2:t_2)\in\mathbb{P}^1$:

a) $\mathcal{B}_{top}(f,g)$ is a finite set
b) if s_1f+t_1g has a nonisolated singularity at the origin, then $(s_1:t_1)\in\mathcal{B}(f,g)_{top}$,
c) if s_1f+t_1g and s_2f+t_2g both have an isolated singularity at the origin, then
   i) $(\{0\},V(s_1f+t_1g),\mathbb{C}^2)\approx(\{0\},V(s_2f+t_2g),\mathbb{C}^2)$ if $(s_1:t_1),(s_2:t_2)$ are both *not* in $\mathcal{B}_{top}(f,g)$,
   ii) $(\{0\},V(s_1f+t_1g),\mathbb{C}^2)\not\approx(\{0\},V(s_2f+t_2g),\mathbb{C}^2)$ if $(s_1:t_1)\in\mathcal{B}_{top}(f,g)$ but $(s_2:t_2)\notin\mathcal{B}_{top}(f,g)$

Above we have used $\approx$ as an abbreviation for homeomorphism of triples of germs of sets at the origin. Recall here that the Milnor number is a topological invariant. From the famous result of Lê and Ramanujam ([10]) it follows the equality $\mathcal{B}_{top}(f,g)=\mathcal{B}_\mu(f,g)$. A third characterization of the bifurcation set in terms of minimal resolutions can be found in [11]. For the next characterization we need a formula originally due to Lê (for generalizations of this formula cf. [9] or [6], for a simple proof cf. [4], Chapter 7). It is an interesting exercise to show that the left-hand side is finite if and only if the right-hand side is finite, by using the Brieskorn module (cf. [18] for details).

Proposition 3.1 (Lê). — *For $p,q\in\mathfrak{m}$ we have the following equality in $\mathbb{N}_{\geq 0}\cup\{\infty\}$*

$$i(p,p_xq_y-p_yq_x)=\mu(p)+i(p,q)-1. \tag{3.3}$$

Now let $h=f_xg_y-f_yg_x$ and $h=\prod_i h_i^{a_i}$ be its prime factor decomposition. Define $C:=V(h)$ and $C_i:=V(h_i)$. Since $\Phi:=(f,g)\colon(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ is a finite map, the image of C_i under Φ is a curve germ Δ_i and we let d_i be the degree of the branched covering $C_i\to\Delta_i$. Finally let $\Delta=\Phi(C)$. Then the Δ_i's and Δ are analytic sets (considered with reduced structure). The first part of the following formula is well-known (see e.g. [14], the second should be known but we know of no reference. Płoski has given also another formula for μ_{gen}.

Proposition 3.2. — *Let $f,g\in\mathfrak{m}$ be coprime. Then*

1. $\mathcal{B}(f,g)$ is the projective tangent cone of Δ and

2. $\mu_{gen}(f,g) = 1 - i(f,g) + \sum_k a_k d_k \operatorname{mult}(\Delta_k)$.

Proof. — Applying Lê's formula to $f + ag$ instead of f for any $a \in \mathbb{P}^1$, we get

$$\mu(f+ag) = 1 - i(f,g) + i(f+ag, f_x g_y - f_y g_x),$$

which shows that $\mu(f+ag)$ is minimal if and only if $i(f+ag, f_x g_y - f_y g_x)$ becomes minimal. Let U be a neighbourhood of the origin in $\mathbb{C}^2$ where f and g are defined and where the map $\Phi = (f,g)\colon U \to V$ is finite. Let the coordinates in the image space be (u,v). According to the projection formula for the intersection multiplicity ([15]) we have

$$\begin{aligned} i(f+ag,h) &= i(\Phi^{-1}(u+av),h) \\ &= \sum_k a_k i(\Phi^{-1}(u+av),h_k) \\ &= \sum_k a_k i(u+av,\Phi_* C_k) \\ &= \sum_k a_k d_k i(u+av,\Delta_k). \end{aligned}$$

It is well-known that for plane curves E, F we have

$$i(E,F) = \operatorname{mult}(E)\operatorname{mult}(F) + \sum_p i_p(\widehat{E},\widehat{F})$$

where the summation is over points in the exceptional divisor $\mathbb{E}$ of the blowup of $(\mathbb{C}^2,0)$ and $\widehat{E},\widehat{F}$ are the respective strict transforms. Now it is also known that $\widehat{E} \cap \mathbb{E}$ consists of the points that corresponds to tangents of E. Hence $i(E,F)$ is minimal if and only if E,F have no common tangents and the minimal value is the product of the multiplicities of E,F. In our situation this amounts to say that each $i(u+av,\Delta_k)$ is minimal if and only if a does not belong to the tangent cone of Δ_k and the minimal value of $\mu(f+ag)$ is $1 - i(f,g) + \sum_k a_k d_k \operatorname{mult}(\Delta_k)$. □

All above characterizations of $\mathscr{B}(f,g)$ do not allow an explicit computation of this finite set except for some trivial input (f,g). There is however an algorithmic description of the bifurcation set due to Maugendre and Michel ([12]) which allows to write a computer program (see [18]) for the easy determination of all special pencil members and their Milnor numbers. There are other characterizations of the bifurcation set as well. As a starting point for their investigation we recommend the paper [5], which gives several descriptions of $\mathscr{B}(f,g)$ and indicates proofs when g is a power of a linear function, a situation which appears in the study of polynomials at infinity. In [1], the authors used a description of the bifurcation set which measures transversality of the fibres of f/g with spheres.

Part (b) of the following theorem states that the bifurcation set can have any arbitrary finite cardinality even under the additional condition of a finite meromorphic Milnor number.

Theorem 3.3. — *Let $n \geq 0$ be a given integer.*

(a) *Then there exists $f, g \in \mathfrak{m}$ such that the pencil generated by f and g has precisely n nonisolated members and all other members have the same Milnor number.*
(b) *Then there exists $p, q \in \mathfrak{m}$ with $\mu(p, q) < \infty$ such that $\mathscr{B}(p, q)$ has cardinality n.*

Proof. — If $n = 0$ we can take $(f, g) = (p, q) = (x, y)$. If $n = 1$, then $(f, g) = (x, x + y^2)$ resp. $(p, q) = (x, x + x^2 + y^2)$ will provide examples for $(a), (b)$ respectively. Assume $n \geq 2$ from now on. For part (a) we let $f(x, y) = (n-1)x^n$ and $g(x, y) = y^n - nyx^{n-1}$. If $(s : t) = (1 : 0)$ we have $sf + tg = (n - 1)x^n$ which is a nonisolated singularity, for $n \geq 2$. Let now $t \neq 0$. Then we may put $\lambda := s/t$ and compute the partial derivatives

$$(\lambda f + g)_x = n(n-1)(\lambda x^{n-1} - yx^{n-2})$$
$$(\lambda f + g)_y = n(y^{n-1} - x^{n-1}).$$

So if both derivatives vanished at (x, y), then from the second equation we would get $\xi x = y$ for some $\xi \in \mathbb{C}$ with $\xi^{n-1} = 1$. Inserting this into the first equation gives

$$\lambda x^{n-1} - yx^{n-2} = 0$$
$$\lambda x^{n-1} - \xi x^{n-1} = 0$$
$$\Rightarrow x^{n-1}(\lambda - \xi) = 0.$$

So if λ is not an $(n - 1)$st root of unity then any common solution (x, y) of $(\lambda f + g)_x = 0$ and $(\lambda f + g)_y = 0$ is only the origin itself. I.e. we have an isolated singularity. Hence the values of $(s : t)$ for which $sf + tg$ is a nonisolated singularity are thus given by $(1 : 0)$ and $(\xi : 1)$ where ξ goes through the $(n-1)$st roots of unity. At all other values of $(s : t)$ we have an isolated singularity given by a homogeneous polynomial of degree n. By a result of Milnor and Orlik any such singularity has Milnor number $(n-1)^2$. In particular the Milnor numbers are equal for those $(s : t)$. This proves part (a) of the assertion.

Now we prove part (b). When $n = 2$ we can use $(p, q) = (x^2 + y^3, x^3 + y^2)$ where obviously the special pencil members are p and q with Milnor numbers both equal to 2 and all other pencil members have Milnor number equal to 1. Finally we have $\mu(p, q) = 11 < \infty$.

From now on assume $n \geq 3$. For any $m > (n-1)^2 + 1$ with the previously defined f, g the function pair

$$(p, q) = (f + (n-1)y^m, g) = ((n-1)(x^n + y^m), y^n - nyx^{n-1})$$

is as desired.

First note that $m > n$ since $n \geq 3$. We show that the germs $x^n + y^m$ and $y^n - nyx^{n-1}$ are coprime. In general $x^n + y^m = \prod_\xi (x^{n/d} - \xi y^{m/d})$ where the product is over the dth roots of -1 and $d = \gcd(m, n)$. So for a branch of $x^n + y^m$ we have a parametrization of the form $\gamma(t) = (\sqrt[n/d]{\xi} t^{m/d}, t^{n/d})$ which does not kill $(y^n - nyx^{n-1}) \circ \gamma(t)$ as is easily checked.

The germ p is an isolated singularity with Milnor number $(n-1)(m-1)$. Now define $\lambda = s/t$ and compute the derivatives

$$\begin{aligned}(\lambda p + q)_x &= \lambda(n-1)nx^{n-1} - n(n-1)yx^{n-2} \\ &= n(n-1)x^{n-2}(\lambda x - y), \\ (\lambda p + q)_y &= \lambda(n-1)my^{m-1} + ny^{n-1} - nx^{n-1}.\end{aligned}$$

Let (x, y) be a critical point of $\lambda p + q$. Then we have either $x = 0$ or $\lambda x - y = 0$. In the first case we obtain $\lambda(n-1)my^{m-1} + ny^{n-1} = 0$ which can be rewritten as $y^{n-1}(\lambda(n-1)y^{m-n} + n) = 0$. Since $m > n$, the expression in brackets does not vanish at the origin and so $y = 0$, in which case we would get an isolated singularity of $\lambda p + q$. In the second case we have $\lambda x - y = 0$. Inserting this into the equation for $\partial_y(\lambda p + q)$ we get

$$\begin{aligned}\lambda(n-1)m(\lambda x)^{m-1} + n(\lambda x)^{n-1} - nx^{n-1} &= 0 \\ x^{n-1}\big((n-1)m\lambda^m x^{m-n} + n\lambda^{n-1} - n\big) &= 0.\end{aligned}$$

We again have two cases. Either $\lambda^{n-1} = 1$ or not. If $\lambda^{n-1} \neq 1$, then the expression in brackets is nonvanishing at the origin. So $x = 0$ and therefore $y = \lambda x = 0$, too. In the case $\lambda^{n-1} = 1$, we rewrite the last equation as

$$x^{m-1}\big((n-1)m\lambda^m\big) = 0,$$

which implies $x = 0$, too. So for any $\lambda \in \mathbb{C}$, $\lambda p + q$ has an isolated singularity at the origin. Since this holds also for $\lambda = \infty$ (i.e. for p), we have $\mu(p, q) < \infty$ by Theorem 2.6. What about the bifurcation set? We know by part (a) that the Milnor number of $\lambda f + g$ is equal to $(n-1)^2$ for $\lambda \in \mathbb{C}$ not an $(n-1)$th root of unity. Since $m > \mu(\lambda f + g) + 1$, by Tougeron's finite determinacy result adding $\lambda(n-1)y^m$ to the isolated singularity $\lambda f + g$ will not change the Milnor number. So the generic Milnor number of the pencil generated by p, q is $\mu_{gen}(p, q) = (n-1)^2$ and it remains to compare the Milnor number of p resp.

$\lambda p+q$ for $\lambda^{n-1}=1$ with $(n-1)^2$. For p we have $\mu(p)=(n-1)(m-1)>(n-1)^2$, so $(s:t)=(1:0)$ belongs to $\mathscr{B}(p,q)$. Finally let $\lambda^{n-1}=1$. We have

$$\begin{aligned}\mu(\lambda p+q) &= i(\lambda p_x+q_x, \lambda p_y+q_y)\\ &= i(x^{n-2}(\lambda x-y), \lambda(n-1)my^{m-1}+ny^{n-1}-nx^{n-1})\\ &= (n-2)i(x, \lambda(n-1)my^{m-1}+ny^{n-1}-nx^{n-1})\\ &\quad + i(\lambda x-y, \lambda(n-1)my^{m-1}+ny^{n-1}-nx^{n-1})\\ &= (n-2)\operatorname{ord}_t(-nt^{n-1})+\operatorname{ord}_t(\lambda(n-1)m(\lambda t)^{m-1}+n(\lambda t)^{n-1}-nt^{n-1})\\ &= (n-2)(n-1)+(m-1)\\ &= (n-1)^2+(m-n)\\ &> (n-1)^2=\mu_{gen}(f,g).\end{aligned}$$

So $(\lambda:1)\in\mathscr{B}(p,q)$. Altogether we deduce that $\mathscr{B}(p,q)$ has n elements. □

4. The Bifurcation Formula

The following statement is the conjecture that there is a certain relation between all the Milnor numbers in the pencil. For its formulation we define the reduced bifurcation set $\mathscr{B}^*$ to be $\mathscr{B}^* := \mathscr{B}\setminus\{0,\infty\}$.

CONJECTURE 4.1 (Bifurcation Formula). — *Assume that $f,g\in\mathbb{C}\{x,y\}$ both vanish at the origin and are coprime. Then the following formula holds as an equality in $\mathbb{N}_{\geq 0}\cup\{\infty\}$:*

$$\mu(f,g)=\mu(fg)+\sum_{(s:t)\in\mathscr{B}(f,g)^*}\big(\mu(sf+tg)-\mu_{gen}(f,g)\big). \tag{4.1}$$

Note that Theorem 2.6 is already a part of the bifurcation formula since it states that the left-hand side of this formula is finite if and only if the right-hand side is finite. As a finite example consider $f(x,y)=2(x^3+y^5)$ and $g(x,y)=y^3-3yx^2$. Then we have $\mathscr{B}(f,g)=\{1,-1,0\}$ with corresponding Milnor numbers $\mu=6,6,8$ (i.e. $\mu(f+1\cdot g)=6$, $\mu(f-1\cdot g)=6$ and $\mu(f+0\cdot g)=8$). The generic Milnor number of the pencil is $\mu_{\text{gen}}(f,g)=4$ and we have $\mu(fg)=29$. Altogether the bifurcation formula becomes $33=29+(6-4)+(6-4)$.

Recalling that a meromorphic germ $(f:g)\colon(\mathbb{C}^2,0)\dashrightarrow\mathbb{P}^1$ is called semitame in [1] if $\mathscr{B}(f,g)\subset\{0,\infty\}$, this conjecture has as a corollary the following

CONJECTURE 4.2. — *Under the above hypothesis $\mu(f,g)\geq\mu(fg)$ with equality if and only if the meromorphic germ $(f:g)$ is semitame.*

In the special case that one of the functions involved is noncritical we can indeed prove the bifurcation formula. For this we use the following lemma.

Lemma 4.3. — *The right-hand side of the bifurcation formula is invariant against the change* $(f,g)\mapsto(af+bg,cf+dg)$ *for any* $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2\,\mathbb{C}$.

Proof. — We rewrite the right-hand side of the formula as

$$\begin{aligned}
&\mu(f\cdot g)+\sum_{(s:t)\in\mathscr{B}(f,g)^*}\big(\mu(sf+tg)-\mu_{\mathrm{gen}}(f,g)\big)\\
&=\mu(f\cdot g)+\sum_{(s:t)\in\mathbb{P}^1\setminus\{0,\infty\}}\big(\mu(sf+tg)-\mu_{\mathrm{gen}}(f,g)\big)\\
&=\mu(f)+\mu(g)+2i(f,g)-1+\sum_{(s:t)\in\mathbb{P}^1\setminus\{0,\infty\}}\big(\mu(sf+tg)-\mu_{\mathrm{gen}}(f,g)\big)\\
&=2i(f,g)-1+2\mu_{\mathrm{gen}}(f,g)+\sum_{(s:t)\in\mathbb{P}^1}\big(\mu(sf+tg)-\mu_{\mathrm{gen}}(f,g)\big)
\end{aligned}$$

from which the assertion is immediate since $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ induces a bijection of $\mathbb{P}^1$ and $i(f,g)=i(af+bg,cf+dg)$ as well as obviously $\mu_{\mathrm{gen}}(f,g)=\mu_{\mathrm{gen}}(af+bg,cf+dg)$. □

Proposition 4.4. — *The bifurcation formula holds when f or g is a smooth germ.*

Proof. — Let f be smooth. We have three cases:

i) g is smooth and transversal to f.
ii) g is smooth but not transversal to f.
iii) g is not smooth, i.e. $g\in\mathfrak{m}^2$.

In case *i)*, all pencil members are smooth and hence $\mathscr{B}(f,g)=\varnothing$. The product fg is Morse-critical and hence $\mu(fg)=1$. By 2.4 we have $\mu(f,g)=1$. So the bifurcation formula is true in this case.

For the cases *ii)*, *iii)* first note that the asserted formula is invariant against a coordinate transformation. So we may assume $f(x,y)=x$ and $g(x,y)\in x+\mathfrak{m}^2$ in case *ii)* resp. $g\in\mathfrak{m}^2$ in case *iii)*.

Due to Proposition 2.2 and Lemma 4.3, the bifurcation formulae for the cases *ii)* and *iii)* are equivalent, so it suffices to prove it in the case *iii)*. Here we have $f(x,y)=x$ and $g\in\mathfrak{m}^2$ and we like to show the formula

$$\mu(f,g)=\mu(fg)+\sum_{(s:t)\in\mathscr{B}(f,g)^*}(\mu(sf+tg)-\mu_{\mathrm{gen}}(f,g))$$

whenever $\mu(f,g) = \mu(x,g)$ is finite. By Theorem 2.6 this is equivalent to the requirement that g must have an isolated singularity at the origin and x must not divide g. Since all germs $sf+tg = sx+tg$ are smooth except the one with $s=0$, we have $\mathscr{B}(x,g) = \{\infty\}$, so that $\mathscr{B}^*(x,g) = \mathscr{B}(x,g) \setminus \{0,\infty\} = \varnothing$. By the general formula $\mu(xg) = \mu(x) + \mu(g) + 2i(x,g) - 1$, the asserted formula is

$$\mu(x,g) = \mu(g) + 2i(x,g) - 1 + 0.$$

Now it is easily checked that $\mu(x,g) = i(g - xg_x, xg_y)$ so that the formula to be shown is equivalent to

$$i(g - xg_x, xg_y) = \mu(g) + 2i(x,g) - 1.$$

We now apply a special case of Lê's formula $\mu(g) + i(x,g) - 1 = i(g, g_y)$ to get the equivalent statements

$$i(g - xg_x, g_y) + i(g - xg_y, x) = i(g, g_y) + i(x,g).$$
$$i(g - xg_x, g_y) = i(g, g_y)$$

Finally, this last equation follows from (the proof of) Lemma 6.5.7 in [19] which we recall.

Let γ be a parametrization of a branch of g_y. Then it suffices to show

$$i(g - xg_x, \gamma) = i(g, \gamma).$$

Without loss of generality we can assume $x \circ \gamma(t) \not\equiv 0$, for otherwise the last equation would be trivially true.

We can furthermore assume that g is an isolated singularity (otherwise $\mu(f,g) = \infty$ by Theorem 2.6 and thus, Proposition 4.4 holds as the equality $\infty = \infty$). Thus, $\gcd(g_x, g_y) = 1$.

Therefore it is possible to write

$$x \circ \gamma(t) = at^{\alpha} + o(t^{\alpha}),\ 0 \neq a \in \mathbb{C}, 1 \leq \alpha \in \mathbb{N}$$
$$g_x \circ \gamma(t) = bt^{\beta} + o(t^{\beta}),\ 0 \neq b \in \mathbb{C}, 1 \leq \beta \in \mathbb{N}$$

and we can compute

$$\begin{aligned} g \circ \gamma(t) &= \int_0^t \frac{\partial(g \circ \gamma(t))}{\partial t} dt \\ &= \int_0^t (g_x \circ \gamma(t)(x \circ \gamma(t))' + g_y \circ \gamma(t)(y \circ \gamma(t))')dt \\ &= \int_0^t g_x \circ \gamma(t)(x \circ \gamma(t))' dt. \end{aligned}$$

By inserting the above series expansions and integrating we obtain

$$g \circ \gamma(t) = \frac{ab\alpha}{\alpha+\beta} t^{\alpha+\beta} + o(t^{\alpha+\beta})$$
$$\Rightarrow (g - xg_x) \circ \gamma(t) = -\frac{ab\beta}{\alpha+\beta} t^{\alpha+\beta} + o(t^{\alpha+\beta})$$

Both have the same order in t, showing $i(g - xg_x, \gamma) = i(g, \gamma)$, as desired. □

5. Final Notes

On the bifurcation formula. — It is well known that the classical Milnor number of a holomorphic germ f with an isolated singularity can alternatively be characterized as the dimension of the parameter space of the semi-universal unfolding of f. Already in 1983, Suwa ([17]) has shown that for meromorphic germs $(f : g)\colon (\mathbb{C}^2, \mathbf{0}) \dashrightarrow \mathbb{P}^1$ universal unfoldings exist, under an appropriate finiteness condition. This condition is nothing but $\nu(f, g) < \infty$ and if true, this number is also the dimension of the parameter space of the semi-universal unfolding in the meromorphic setting. Due to this circumstance the author eventually came to the conjecture that there must be a relation between the Milnor numbers of the pencil members and $\nu(f, g)$. In [18] we have sketched some ideas which might lead to a proof of of the bifurcation formula for semitame germs. In the general case, maybe a proof resembling the one of Brieskorn in the appendix of [3] showing that the covering degree of $\nabla f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ equals $\operatorname{rk} H^{n-1}(\mathrm{Mil}_{f,0})$ may be feasible. Propositions 2.4 and 2.5 could be helpful in combination with [17].

A generalization of the Milnor number of pairs to n-tuples. — Let $f_1, \ldots, f_n : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be holomorphic germs defined in a neighbourhood of the origin in $\mathbb{C}^n$. It is suggested to look at the following generalization

$$(5.1) \qquad \nu(f_1, \ldots, f_n) = \dim_{\mathbb{C}} \frac{\langle f_1 \cdots \widehat{f_i} \cdots f_n \rangle_{i=1}^n}{\langle \sum_{i=1}^n (-1)^i \, \partial f_i \cdot f_1 \cdots \widehat{f_i} \cdots f_n \rangle_{\partial=\partial_1}^{\partial_n}}$$

for ν, where a hat above a function means leaving out this function. If the common vanishing locus of the f_i's is presicely the origin, their intersection number is defined and it may be a good idea to put $\mu(f_1, \ldots, f_n) = \nu(f_1, \ldots, f_n) + i(f_1, \ldots, f_n)$.

Singularities in the pencil. — For a nontrivial holomorphic germ $f \in \mathfrak{m}$ it is known that there is a neighbourhood U of the origin such that $f\colon U \to \mathbb{C}$ has all its critical points only in the zero fibre. If we have not just a single function f but instead a bunch of them, e.g. all members of a pencil $sf + tg$, $(s : t) \in \mathbb{P}^1$, we may ask whether such a neighbourhood U exists independently of the choice of $(s : t)$. In general, the other is no, even under the additional hypothesis that $\mu(f, g) < \infty$ as can be checked with the example $f(x, y) = 2(x^3 + y^5)$ and $g(x, y) = y^3 - 3yx^2$. However the following problem seems to be still open.

PROBLEM 5.1. — *Given coprime $f, g \in \mathfrak{m}$. Is it true that there exists a neighbourhood U of the origin in $\mathbb{C}^2$ such that for each $(s : t) \in \mathbb{P}^1$, there is no singular point of the variety $\{(x, y) \in U |\ sf(x, y) + tg(x, y) = 0\}$ (it is not a germ!) other than the origin?*

One might also ask whether there is a neighbourhood U of the origin, such that if a sequence of points $\mathbf{p}_k \neq \mathbf{0}$ converges to the origin $\mathbf{0} \in \mathbb{C}^2$ and if there are $(s_k : t_k) \in \mathbb{P}^1$ such that $\mathbf{p}_k$ is a critical point of $s_k f + t_k g \colon U \to \mathbb{C}$, then $(s_k : t_k)$ has as accumulation points only values in $\mathscr{B}(f, g)$? This question has a positive answer by the additivity of the Milnor number ([7], Theorem 2.6 (3)).

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