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LOWER BOUNDS FOR RANKS OF MUMFORD-TATE GROUPS

BY MARTIN ORR

ABSTRACT. — Let A be a complex abelian variety and G its Mumford-Tate group. Supposing that the simple abelian subvarieties of A are pairwise non-isogenous, we find a lower bound for the rank $\operatorname{rk} G$ of G, which is a little less than $\log_2 \dim A$. If we suppose furthermore that $\operatorname{End} A$ is commutative, then we can improve this lower bound to $\operatorname{rk} G \geq \log_2 \dim A + 2$ and prove that this is sharp. We also obtain the same results for the rank of the ℓ-adic monodromy group of an abelian variety defined over a number field.

RÉSUMÉ (*Minoration des rangs de groupes de Mumford-Tate*). — Soit A une variété abélienne complexe et G son groupe de Mumford-Tate. En supposant que les sous variétés abéliennes simples de A sont deux à deux non-isogènes, on trouve une minoration du rang $\operatorname{rk} G$ de G, légèrement inférieure à $\log_2 \dim A$. Si de plus on suppose que $\operatorname{End} A$ est commutatif, alors on peut améliorer cette borne en $\operatorname{rk} G \geq \log_2 \dim A + 2$, et montrer que cette borne-ci est optimale. On obtient les mêmes resultats pour le rang du groupe de monodromie ℓ-adique d'une variété abélienne définie sur un corps de nombres.

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1. Introduction

Let A be a complex abelian variety of dimension g, whose simple abelian subvarieties are pairwise non-isogenous. In this paper we will establish a lower bound for the rank of the Mumford-Tate group of A. The Mumford-Tate group is an algebraic group over $\mathbb{Q}$ defined via the Hodge theory of A (see Section 2 below for the definition). The same argument will also establish a lower bound for the rank of the ℓ-adic monodromy groups G_ℓ, in the case where A is defined over a number field. The ℓ-adic monodromy group is the Zariski closure of the image of the Galois representation on the ℓ-adic Tate module of A. Our main theorems are the following:

THEOREM 1.1. — *Let A be an abelian variety of dimension g such that* $\operatorname{End} A$ *is commutative. Let G be the Mumford-Tate group or the ℓ-adic monodromy group of A. Then* $\operatorname{rk} G \geq \log_2 g + 2$.

THEOREM 1.2. — *Let A be an abelian variety of dimension g whose simple abelian subvarieties are pairwise non-isogenous. Let G be the Mumford-Tate group or the ℓ-adic monodromy group of A. If $n = \operatorname{rk} G$, then*

$$n + \alpha(n)\sqrt{n \log n} \geq \log_2 g + 2$$

for a function $\alpha : \mathbb{N}_{\geq 2} \to \mathbb{R}$ satisfying $\alpha(n) < 2$ for all n and $\alpha(n) \to 1/\log 2 = 1.44...$ as $n \to \infty$.

Each of these theorems is an instance of a more general bound for weak Mumford-Tate triples, which are defined in Section 2. These more general bounds are Theorems 4.1 and 4.4 respectively. These would apply also for example to the analogue of the Mumford-Tate group for a Hodge-Tate module of weights 0 and 1.

Theorem 1.1 was proved by Ribet in the case of an abelian variety with complex multiplication [13]. Our proof is a generalisation of his, relying on the fact that the defining representation of the Mumford-Tate group or ℓ-adic monodromy group has minuscule weights.

The condition on simple subvarieties in Theorem 1.2 is necessary: taking products of copies of the same simple abelian variety increases the dimension without changing the rank of the Mumford-Tate group. Indeed, if A is isogenous to $\prod_i A_i^{m_i}$ where the A_i are simple and pairwise non-isogenous, then according to [4] Lemme 2.2,

$$\operatorname{MT}(A) \cong \operatorname{MT}(\prod_i A_i).$$

Hence Theorem 1.2 implies that for a general abelian variety A, if n denotes the rank of either the Mumford-Tate group or the ℓ-adic monodromy group of A, then

$$n + \alpha(n)\sqrt{n \log n} \geq \log_2 \left(\sum_i \dim A_i \right) + 2$$

where the A_i are one representative of each isogeny class of simple abelian subvarieties of A.

The condition of having pairwise non-isogenous simple abelian subvarieties can be interpreted via the endomorphism algebra like the condition in Theorem 1.1: it is equivalent to $\operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Q}$ being a product of division algebras. Note also that $\operatorname{End} A$ being commutative implies the condition of Theorem 1.2. (Throughout this paper, $\operatorname{End} A$ means the endomorphisms of A after extension of scalars to an algebraically closed field.)

Let G be either the Mumford-Tate group or the ℓ-adic monodromy group of A. It is well known that the rank of G is at most $g+1$, and that this upper bound is achieved for a generic abelian variety. Indeed, if g is odd and $\operatorname{End} A = \mathbb{Z}$, then $\operatorname{rk} G$ is always $g+1$ [16]. So in this case the bound in Theorem 1.1 is far from sharp.

On the other hand if g is a power of 2, then there are abelian varieties for which the bound in Theorem 1.1 is achieved (even with $\operatorname{End} A = \mathbb{Z}$). We construct such examples in Section 5. The exact bound for a given g is very sensitive to the prime factors of g. Equality can happen only when g is a power of 2 (for the trivial reason that otherwise $\log_2 g \notin \mathbb{Z}$) but even near-equality can only occur when g has many small prime factors. This was made precise by Dodson in the complex multiplication case [2], and it is possible that something similar could be proved in general.

Theorem 1.2 is not sharp. The function $\alpha(n)$ is specified exactly in Section 4, but it is likely that this could be improved on, perhaps to something which goes to 0 as $n \to \infty$. In Section 5, we construct a family of examples showing that Theorem 1.2 cannot be improved to $n + k \geq \log_2 g$ for any constant k.

We can deduce a lower bound for the growth of the degrees of the division fields $K(A[\ell^n])$ (for ℓ a fixed prime number) as a straightforward consequence of Theorem 1.1.

Corollary 1.3. — *Let A be an abelian variety of dimension g over a number field K, and ℓ a prime number. If* $\operatorname{End} A$ *is commutative, then there is a constant $C(A, K, \ell)$ such that*

$$[K(A[\ell^n]) : K] \geq C(A, K, \ell) \cdot \ell^{n(\log_2 g + 2)}.$$

Theorem 1.2 implies a similar bound for the degree of $K(A[\ell^n])$ whenever A is an abelian variety whose simple abelian subvarieties are pairwise non-isogenous. One would like to extend these results to lower bounds on the degrees of $K(A[N])$ for any N, but this cannot be done without knowing how $C(A, K, \ell)$ varies with ℓ. The primary obstacle here is the index of the image of $\mathrm{Gal}(\bar{K}/K)$ in $G_\ell(\mathbb{Z}_\ell)$, which is conjectured to be bounded by a constant $C_1(A, K)$ independent of ℓ.

In Section 2 we recall the definitions of Mumford-Tate group, ℓ-adic monodromy groups and weak Mumford-Tate triples; the latter are an axiomatisation of the properties of the groups and representations we will consider. In Section 3 we bound the number of distinct characters of a maximal torus which can appear in such a representation. In Section 4 we bound the multiplicity of absolutely irreducible components of this representation. This is straightforward for the Mumford-Tate group but more difficult for the ℓ-adic monodromy group. Combining these two bounds gives Theorems 1.1 and Theorem 1.2. Finally in Section 5 we give some examples to show that Theorem 1.1 is sharp and to place a limit on the possible improvements of Theorem 1.2.

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2. Mumford-Tate triples: Definitions

We recall the definition of a weak Mumford-Tate triple, which abstracts the key properties of a Mumford-Tate group which we will use. We recall also the definitions of the two examples of Mumford-Tate triple we will consider, namely the Mumford-Tate group and the ℓ-adic monodromy group of an abelian variety.

The following definition is a slight modification of those used by Serre [14] and Wintenberger [21].

DEFINITION. — Let F be a field of characteristic zero and E an algebraically closed field containing F.

A *weak Mumford-Tate triple* is a triple (G, ρ, Ψ) where G is an algebraic group over F, ρ is a rational representation of G and Ψ is a set of cocharacters of $G \times_F E$ satisfying the following conditions:

(i) G is a connected reductive group;
(ii) ρ is faithful;
(iii) the images of all $G(E)$-conjugates of elements of Ψ generate G_E.

The *weights* of a Mumford-Tate triple (G, ρ, Ψ) are the integers which appear as weights of $\rho \circ \nu$ (a representation of $\mathbb{G}_m$) for some $\nu \in \Psi$.

A weak Mumford-Tate triple (G, ρ, Ψ) is called *pure* if $\rho(G)$ contains the torus $\mathbb{G}_m.\mathrm{id}$ of homotheties.

The Mumford-Tate group. — Let A be an abelian variety over $\mathbb{C}$, of dimension g. The singular cohomology group $H^1(A(\mathbb{C}), \mathbb{Q})$ is a vector space of dimension $2g$ over $\mathbb{Q}$. Hodge theory gives a decomposition of $\mathbb{C}$-vector spaces

$$H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^{1,0}(A) \oplus H^{0,1}(A)$$

with $H^{1,0}(A)$ and $H^{0,1}(A)$ being mapped onto each other by complex conjugation (so each has dimension g).

We define a cocharacter $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}_{2g,\mathbb{C}}$ by:

$$\mu(z) \text{ acts as multiplication by } z \text{ on } H^{1,0}(A)$$
$$\text{and as the identity on } H^{0,1}(A).$$

The *Mumford-Tate group* of A is defined to be the smallest algebraic subgroup M of GL_{2g} defined over $\mathbb{Q}$ and such that $M_{\mathbb{C}}$ contains the image of μ.

The triple consisting of the Mumford-Tate group, its defining representation $\rho : M \to \mathrm{GL}_{2g}$, and the set of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-conjugates of the cocharacter μ form a pure weak Mumford-Tate triple of weights $\{0, 1\}$. This is immediate from the definitions.

The functor $A \mapsto H^1(A(\mathbb{C}), \mathbb{Z})$ is an equivalence of categories between complex abelian varieties and polarisable $\mathbb{Z}$-Hodge structures of type $\{(-1, 0), (0, -1)\}$. Furthermore the endomorphism ring of ρ as a representation of the Mumford-Tate group is equal to the endomorphism ring of $H^1(A(\mathbb{C}), \mathbb{Q})$ as a $\mathbb{Q}$-Hodge structure, so

$$\mathrm{End}\,\rho = \mathrm{End}\,A \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The ℓ-adic algebraic monodromy group. — Now suppose that the abelian variety A is defined over a number field K. Its first ℓ-adic cohomology group is a $\mathbb{Q}_\ell$-vector space of dimension $2g$, isomorphic to the dual of the ℓ-adic Tate module:

$$H^1(A_{\bar{K}}, \mathbb{Q}_\ell) \cong (T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^\vee .$$

The Galois group $\mathrm{Gal}(\bar{K}/K)$ acts on the torsion points of $A(\bar{K})$, and this induces an action on $H^1(A_{\bar{K}}, \mathbb{Q}_\ell)$, or in other words a continuous representation

$$\rho_\ell : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_{2g}(\mathbb{Q}_\ell).$$

The *ℓ-adic algebraic monodromy group* of A is the smallest algebraic subgroup G_ℓ of $\mathrm{GL}_{2g,\mathbb{Q}_\ell}$ whose $\mathbb{Q}_\ell$-points contain the image of ρ_ℓ. By working with the ℓ-adic monodromy group instead of the image of ρ_ℓ directly, we gain the ability to use the structure theory of algebraic groups. On the other hand, we do not lose very much because $\mathrm{Im}\rho_\ell$ is known [1] to be an open (and hence finite-index) subgroup of $G_\ell(\mathbb{Q}_\ell) \cap \mathrm{GL}_{2g}(\mathbb{Z}_\ell)$.

Pink [12] has proved that the identity component G_ℓ° together with the representation ρ_ℓ and a certain set Ψ of cocharacters form a pure weak Mumford-Tate triple of weights $\{0,1\}$.

By Faltings' Theorem [3],

$$\operatorname{End}\rho_\ell = \operatorname{End}A \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

3. Bound for the number of characters

Let (G,ρ,Ψ) be a pure weak Mumford-Tate triple of weights $\{0,1\}$, and let T be a maximal torus of G. In this section we will give an upper bound for the number of distinct characters in $\rho_{|T}$ as a function of $\operatorname{rk}G$.

If A has complex multiplication (in other words if G is a torus) then this bound was obtained by Ribet [13]. Our method of proving the bound is inspired by applying Ribet's method to a maximal torus of G, but it is convenient to arrange it differently.

Proposition 3.1. — *Let (G,ρ,Ψ) be a pure weak Mumford-Tate triple of weights $\{0,1\}$. The number of distinct characters in $\rho_{|T}$ is at most $2^{\operatorname{rk}G-1}$.*

Proof. — Let $Y = \mathrm{Hom}(\mathbb{G}_{m,E}, T_E) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the quasi-cocharacter space of T, where E is an algebraically closed field of definition for (G,ρ,Ψ).

Let Ψ' be the set of all cocharacters of T_E which are $G(E)$-conjugate to an element of Ψ. Every cocharacter of G has a $G(E)$-conjugate whose image is contained in T_E, so Ψ' still satisfies condition (iii) in the definition of a weak Mumford-Tate triple. Replacing Ψ by Ψ' does not change the weights of our Mumford-Tate triple.

Furthermore Ψ' is closed under the action of the Weyl group of G_E on Y. So condition (iii) implies that Ψ' spans Y as a $\mathbb{Q}$-vector space.

Let Θ be a basis of Y contained in Ψ'. The character space of T is dual to Y, so any character ω is determined by its inner products $\langle\omega,\mu\rangle$ for $\mu\in\Theta$.

Because our Mumford-Tate triple has weights $\{0,1\}$, if μ is a character in $\rho_{|T}$ then these inner products can only have the values 0 or 1. So there are at most $2^{|\Theta|}$ distinct characters in $\rho_{|T}$, and $|\Theta| = \operatorname{rk}G$.

We can use the fact that our Mumford-Tate triple is pure to improve the exponent to $\operatorname{rk}G-1$. We know that $\rho(G)$ contains the homotheties. Since ρ is faithful, there is a unique cocharacter $\mu_0 : \mathbb{G}_m \to G$ such that $\rho\circ\mu_0(z) = z.\mathrm{id}$. We take Θ' to be a subset of Ψ such that $\Theta' \cup \{\mu_0\}$ is a basis of Y. Now $\langle\omega,\mu_0\rangle = 1$ for all characters ω in $\rho_{|T}$, so ω is determined by the values $\langle\omega,\mu\rangle$ for $\mu\in\Theta'$. We may repeat the previous argument with Θ replaced by Θ'. □

COROLLARY 3.2. — *Let* (G, ρ, Ψ) *be a pure weak Mumford-Tate triple of weights* $\{0, 1\}$. *Let* M *be the maximum of the multiplicities of the irreducible components of* ρ *(working over an algebraically closed base field). Then*

$$\dim \rho \leq M \cdot 2^{\operatorname{rk} G - 1}.$$

Proof. — Serre [14] showed that each irreducible component σ in a weak Mumford-Tate triple of weights $\{0, 1\}$ is *minuscule*, that is, the characters in $\sigma_{|T}$ form a single orbit under the action of the Weyl group. Serre only treated strong Mumford-Tate triples, i.e., weak Mumford-Tate triples satisfying the additional condition that all the cocharacters in Ψ are contained in a single $\operatorname{Aut}(E/F)$-orbit. However this extra condition is not used in his argument (see also [12] Section 4 and [22]).

The characters of T in a minuscule representation have multiplicity 1, and non-isomorphic minuscule representations contain disjoint characters. So the multiplicity of any character in $\rho_{|T}$ is equal to the multiplicity of the unique irreducible component which contains that character, and so

$$\dim \rho \leq M \cdot \big(\text{the number of distinct characters in } \rho_{|T}\big).$$

The corollary now follows from Proposition 3.1. □

4. Bound for the multiplicities

Let (G, ρ, Ψ) be a pure weak Mumford-Tate triple of weights $\{0, 1\}$. In this section we will bound the multiplicities of the absolutely irreducible components of $\rho \otimes_F \bar{F}$. If $\operatorname{End} \rho$ is commutative, then it is immediate that all absolutely irreducible components of $\rho \otimes_F \bar{F}$ have multiplicity 1.

Most of the section concerns the case in which the irreducible components of ρ are pairwise non-isomorphic. Because we use a result on division algebras coming from class field theory, we must assume that the field of definition of ρ is a local field or a number field. If $n = \operatorname{rk} G$, then each absolutely irreducible component has multiplicity at most $\alpha(n)\sqrt{n \log n}$ for a function $\alpha(n)$ satisfying the conditions of Theorem 1.2.

To establish this bound, we introduce an invariant $u(G)$ for a reductive group G such that for any F-irreducible representation of G, the multiplicity of its irreducible components over $\bar{F}$ is at most $u(G)$. Then we use Landau's function (the maximum LCM of a set of positive integers with given sum) to obtain a bound for $u(G)$.

The above bounds together with Corollary 3.2 suffice to prove Theorem 1.1 for both the Mumford-Tate group and ℓ-adic monodromy groups, and Theorem 1.2 for the Mumford-Tate group. Proving Theorem 1.2 for the ℓ-adic monodromy group requires additional work because even when an abelian variety satisfies the condition of Theorem 1.2, its associated ℓ-adic representations might not satisfy the corresponding condition of Theorem 4.4.

4.1. The commutative endomorphism case

THEOREM 4.1. — *Let (G, ρ, Ψ) be a pure weak Mumford-Tate triple of weights $\{0, 1\}$. If $\operatorname{End}\rho$ is commutative, then $\operatorname{rk} G \geq \log_2 \dim \rho + 1$.*

Proof. — Let F be the field of definition of ρ. Since $\operatorname{End}\rho$ is commutative, each irreducible component of $\rho \otimes_F \bar{F}$ has multiplicity 1. So the theorem follows immediately from Corollary 3.2. □

Let A be an abelian variety, G its Mumford-Tate group or ℓ-adic monodromy group, and ρ the associated representation. We have observed that $\operatorname{End}\rho = \operatorname{End} A \otimes_{\mathbb{Z}} F$ where $F = \mathbb{Q}$ or $\mathbb{Q}_\ell$ as appropriate, so that if $\operatorname{End} A$ is commutative the same is true of $\operatorname{End}\rho$. Hence Theorem 1.1 follows from Theorem 4.1. The $\log_2 \dim \rho + 1$ becomes $\log_2 \dim A + 2$ because $\dim \rho = 2 \dim A$.

4.2. Multiplicity of irreducible representations and $u(G)$

DEFINITION. — Let G be a reductive group defined over the field F. Let T be a maximal torus of G and $\Lambda = \operatorname{Hom}(T_{\bar{F}}, \mathbb{G}_m)$ the character group of T. Let Λ_0 be the subgroup of Λ generated by the roots of G and characters which vanish on $T \cap G^{\mathrm{der}}$. The roots of G span the quasi-character space of $T_{\bar{F}} \cap G^{\mathrm{der}}_{\bar{F}}$ as a $\mathbb{Q}$-vector space so Λ_0 spans $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. It follows that Λ/Λ_0 is finite. (In fact Λ/Λ_0 is canonically isomorphic to the dual of the centre of $G^{\mathrm{der}}(\bar{F})$, which is a finite abelian group.)

Hence we can define $u(G)$ to be the exponent of Λ/Λ_0.

LEMMA 4.2. — *Let G be a reductive group over a field F and ρ an F-irreducible representation of G. Let D be the endomorphism ring of ρ and E the centre of D. Then the order of $[D]$ in $\operatorname{Br} E$ divides $u(G)$.*

Proof. — Fix a base Δ for the root system of G with respect to T. When we refer to the action of $\operatorname{Gal}(\bar{F}/F)$ on the character group Λ below, this is the natural action twisted by the Weyl group so that it preserves the set Δ (this is the same action used in [20]).

Let σ be an absolutely irreducible component of $\rho \otimes_F \bar{F}$, and $\lambda_\sigma \in \Lambda$ the highest weight of σ. Let Γ be the subgroup of $\operatorname{Gal}(\bar{F}/F)$ fixing λ_σ. Then E is isomorphic to the subfield of $\bar{F}$ fixed by Γ.

Tits defined a map

$$\alpha_{G,E} : \Lambda^\Gamma \to \operatorname{Br} E$$

as follows: if $\lambda \in \Lambda^\Gamma$ is dominant then there is a unique isomorphism class of E-irreducible representations of G with highest weight λ. The endomorphism ring of such a representation is a division algebra with centre E. We define $\alpha_{G,E}(\lambda)$ to be the inverse of the class of this division algebra in $\operatorname{Br} E$. Tits showed that this map on dominant weights is additive so it extends to a homomorphism $\Lambda^\Gamma \to \operatorname{Br} E$. He also showed that $\alpha_{G,E}$ is trivial on Λ_0^Γ ([20] Corollary 3.5).

In our case we have $[D]^{-1} = \alpha_{G,E}(\lambda_\sigma)$. Since $[D]$ is in the image of $\alpha_{G,E}$, it follows that the order of $[D]$ in $\operatorname{Br} E$ divides the exponent of $\Lambda^\Gamma/\Lambda_0^\Gamma$. But the latter is a subgroup of Λ/Λ_0, so its exponent divides $u(G)$. □

Corollary 4.3. — *Let G be a reductive group defined over a number field or a local field F. Let ρ be an F-irreducible representation of G. Then the multiplicity of each absolutely irreducible component of $\rho \otimes_F \bar{F}$ divides $u(G)$.*

Proof. — Let $D = \operatorname{End} \rho$ and let E be the centre of D. Then the multiplicity of any absolutely irreducible component of $\rho \otimes_F \bar{F}$ is $\sqrt{\dim_E D}$.

Since F is a number field or a local field, it follows from class field theory that $\sqrt{\dim_E D}$ is equal to the order of $[D]$ in $\operatorname{Br} E$ (see e.g., [11] Theorem 18.6).

Now apply Lemma 4.2. □

The following theorem is obtained by combining Corollaries 3.2 and 4.3.

Theorem 4.4. — *Let (G, ρ, Ψ) be a pure weak Mumford-Tate triple of weights $\{0, 1\}$ defined over a number field or a local field F. If the F-irreducible components of ρ are pairwise non-isomorphic, then*

$$\operatorname{rk} G + \log_2 u(G) \geq \log_2 \dim \rho + 1.$$

If A is a complex abelian variety, G its Mumford-Tate group and ρ the associated representation, then $\operatorname{End} \rho = \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Q}$ so the hypothesis that the simple abelian subvarieties of A are pairwise non-isogenous implies that the irreducible components of ρ are pairwise non-isomorphic. Hence Theorem 4.4, together with the bounds for $u(G)$ in Section 4.4, implies Theorem 1.2 for the Mumford-Tate group.

4.3. Multiplicities in ℓ-adic representations. — Let A be an abelian variety over a number field whose simple abelian subvarieties are pairwise non-isogenous. Let G_ℓ be the ℓ-adic monodromy group of A and ρ_ℓ the associated ℓ-adic representation. We shall show that the multiplicities of irreducible components of $\rho_\ell \otimes_{\mathbb{Q}_\ell} \bar{\mathbb{Q}}_\ell$ are bounded above by $u(G_\ell^\circ)$ and hence prove Theorem 1.2.

By Faltings' Theorem, if B and B' are non-isogenous simple abelian varieties, then the associated ℓ-adic representations have no common subrepresentations. Hence it will suffice to suppose that A is simple.

By Faltings' Theorem, $\operatorname{End} \rho_\ell = \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. This implies that the multiplicities of absolutely irreducible components of $\rho_\ell \otimes_{\mathbb{Q}_\ell} \bar{\mathbb{Q}}_\ell$ are independent of ℓ. We will use results of Serre and Pink to show that $u(G_\ell^\circ)$ is also independent of ℓ, and then we can consider all ℓ at once to show that the multiplicities are bounded above by $u(G_\ell^\circ)$.

LEMMA 4.5. — *$u(G_\ell^\circ)$ is independent of ℓ.*

Proof. — Let ℓ, ℓ' be any two rational primes. Via ρ_ℓ, we view G_ℓ° as a subgroup of $\mathrm{GL}_{2g,\mathbb{Q}_\ell}$.

For a finite place v of K, let T_v be the Frobenius torus of A in the sense of Serre [15]. Serre showed that we can choose v such that $T_{v,\mathbb{Q}_\ell}$ is $\mathrm{GL}_{2g,\mathbb{Q}_\ell}$-conjugate to a maximal torus of G_ℓ°, and such that the analogous property holds for ℓ'.

Hence we get maximal tori $T_{v,\ell}$ of G_ℓ and $T_{v,\ell'}$ of G_ℓ' together with an isomorphism $\Lambda(T_{v,\ell}) \cong \Lambda(T_v) \cong \Lambda(T_{v,\ell'})$. Furthermore, under this isomorphism, the formal character of ρ_ℓ corresponds to the formal character of $\rho_{\ell'}$.

As observed by Larsen and Pink [7], the formal character of a faithful irreducible representation of a reductive group determines the root lattice Λ_0. Hence $\Lambda/\Lambda_0(G_\ell^\circ) \cong \Lambda/\Lambda_0(G_{\ell'}^\circ)$ so $u(G_\ell^\circ) = u(G_{\ell'}^\circ)$. □

We will also need the following lemma on pure weak Mumford-Tate triples. Let (G, ρ, Ψ) be a pure weak Mumford-Tate triple. Let H be the identity component of $\ker \det \rho \subset G$. (In the case where G is the Mumford-Tate group of an abelian variety A and ρ is the standard representation on $H^1(A(\mathbb{C}), \mathbb{Q})$, H is called the Hodge group of A). Because (G, ρ, Ψ) is pure, there is a cocharacter μ_0 of G such that $\rho \circ \mu_0(z) = z.\mathrm{id}$. Then the quasi-cocharacter space of a maximal torus T splits as

$$(\mathrm{Hom}(\mathbb{G}_m, T \cap H) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus \mathbb{Q}.\mu_0. \qquad (*)$$

LEMMA 4.6. — *Let (G, ρ, Ψ) be a pure weak Mumford-Tate triple of weights 0 and 1, with multiplicities g_0 and g_1 respectively. Choose $\mu \in \Psi$ and let T*

be a maximal torus of G containing the image of μ. Suppose that μ splits as $\mu_H + r\mu_0$ in the decomposition (*). Then for all characters ω in $\rho_{|T}$,

$$\langle \omega, \mu_H \rangle = \frac{g_0}{g_0 + g_1} \ \text{or} \ \frac{-g_1}{g_0 + g_1}.$$

Proof. — By the definition of μ_0, $\langle \omega, \mu_0 \rangle = 1$ for every character ω in $\rho_{|T}$. Hence

$$\langle \det \rho, \mu_0 \rangle = \dim \rho = g_0 + g_1.$$

Because $\det \rho$ is trivial on H, $\langle \det \rho, \mu_H \rangle = 0$. Therefore

$$\langle \det \rho, \mu \rangle = \langle \det \rho, r\mu_0 \rangle = r(g_0 + g_1).$$

On the other hand,

$$\langle \det \rho, \mu \rangle = g_0.0 + g_1.1 = g_1$$

so $r = g_1/(g_0 + g_1)$. Combining with $\langle \omega, \mu \rangle = 0$ or 1 gives the result. □

PROPOSITION 4.7. — *Let A be a simple abelian variety defined over a number field, and G_ℓ its ℓ-adic monodromy group. The multiplicity of every absolutely irreducible component of $\rho_\ell \otimes_{\mathbb{Q}_\ell} \bar{\mathbb{Q}}_\ell$ divides $u(G_\ell^\circ)$.*

Proof. — Let $D = \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Q}$ be the endomorphism algebra of A, and let E be the centre of D. Let $m^2 = \dim_E D$.

By Faltings' Theorem, $\operatorname{End} \rho_\ell = D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. This is a product of simple algebras, each of dimension m^2 over its centre. So every absolutely irreducible component of $\rho_\ell \otimes_{\mathbb{Q}_\ell} \bar{\mathbb{Q}}_\ell$ has multiplicity m, and it will suffice to show that m divides $u(G_\ell^\circ)$.

According to Albert's classification of endomorphism algebras of simple abelian varieties, there are two cases: E is totally real or E is a CM field.

Case 1. E is totally real. — In this case, the Albert classification implies that $m \leq 2$, so it will suffice to show that 2 divides $u(G_\ell^\circ)$.

Let H_ℓ be the identity component of $\ker \det \rho_\ell \subset G_\ell$ (in other words, the ℓ-adic analogue of the Hodge group). By [19] Lemma 1.4, the condition that E is totally real implies that the Hodge group of A is semisimple and by [18] Theorem 3.2 this implies that H_ℓ is semisimple. Hence H_ℓ is the derived group of G_ℓ°.

Let μ be a weak Hodge cocharacter of G_ℓ in the sense of [12] Definition 3.2 and let T be a maximal torus of G_ℓ containing the image of μ. Then $\rho_\ell \circ \mu$ has weights 0 and 1 each with multiplicity $\dim A$, so by Lemma 4.6,

$$\langle \omega, \mu_H \rangle = \pm\tfrac{1}{2}$$

for all characters ω in $\rho_{\ell|T}$, where μ_H is the component of μ in the quasi-cocharacter space of $T \cap H_\ell$.

Now $\langle -, \mu \rangle$ takes integer values on the roots of G_ℓ°. Since μ_0 is orthogonal to the roots, $\langle -, \mu_H \rangle$ also takes integer values on the roots. We remarked above that H_ℓ is equal to the derived group of G_ℓ°, so μ_H is orthogonal to all characters which vanish on $T \cap G_\ell^{\circ\mathrm{der}}$. Hence $\langle -, \mu_H \rangle$ takes integer values on $\Lambda_0(G_\ell^\circ)$.

So in order for $\langle \omega, \mu_H \rangle$ to have denominator 2, the order of ω in $\Lambda(G_\ell)/\Lambda_0(G_\ell^\circ)$ must be even. Therefore $u(G_\ell^\circ)$ is divisible by 2.

Case 2. E is a CM field. — For each place λ of E, let E_λ denote the completion of E at λ. Then $D_\lambda = D \otimes_E E_\lambda$ is a matrix ring over a division algebra with centre E_λ. Let m_λ be the order of $[D_\lambda]$ in $\mathrm{Br}\, E_\lambda$. By the Albert-Brauer-Hasse-Noether theorem ([11] Theorem 18.5), the map $[D] \mapsto ([D_\lambda])$ is an injection

$$\mathrm{Br}\, E \to \bigoplus_\lambda \mathrm{Br}\, E_\lambda$$

so m is the lowest common multiple of the m_λ. So it suffices to show that m_λ divides $u(G_\ell^\circ)$ for every place λ.

Since E is a CM field, all its archimedean places have trivial Brauer group, so we need only consider non-archimedean places. Let λ be a non-archimedean place of E and ℓ' its residue characteristic. Then

$$\mathrm{End}\, \rho_{\ell'} = D \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell'} = D \otimes_E \left(\prod_{\lambda' | \ell'} E_{\lambda'} \right) = \prod_{\lambda' | \ell'} D_{\lambda'}.$$

Hence $\rho_{\ell'}$ has a $\mathbb{Q}_{\ell'}$-irreducible subrepresentation with endomorphism algebra D_λ.

So by Lemma 4.2, m_λ divides $u(G_{\ell'}^\circ)$, and this is equal to $u(G_\ell^\circ)$ by Lemma 4.5. □

Theorem 1.2 follows from Corollary 3.2, Proposition 4.7 and the bounds for $u(G)$ in Section 4.4.

4.4. Bounds for $u(G)$

Definition. — Let $g(n)$ be the maximum value of $\mathrm{LCM}(a_i)$ where a_i are positive integers satisfying $\sum a_i = n$. (This is Landau's function.)

Let $g_1(n)$ be the maximum value of $\mathrm{LCM}(a_i)$ where a_i are integers greater than 1 satisfying $\sum (a_i - 1) = n$.

For $n \geq 2$, let

$$\alpha(n) = \frac{\log_2 g_1(n)}{\sqrt{n \log n}}.$$

Lemma 4.8. — *For any reductive group G, $u(G) \leq g_1(\mathrm{rk}\, G)$.*

Proof. — Let Φ_i (for $i \in I$) be the simple components of the root system of G.

The group Λ/Λ_0 is a subgroup of the product of the fundamental groups of the Φ_i. So $u(G)$ divides the lowest common multiple of the exponents of these fundamental groups.

Let e_i be the exponent of the fundamental group of Φ_i. Then $e_i \leq \operatorname{rk} \Phi_i + 1$ for all i (by the classification of simple root systems), and so $\sum_i (e_i - 1) \leq \operatorname{rk} G$.

By the definition of g_1,

$$u(G) \leq g_1 \left(\sum_{i \in I} (e_i - 1) \right)$$

and this is less than or equal to $g_1(\operatorname{rk} G)$ because g_1 is nondecreasing. □

The following lemma gives the asymptotic behaviour of $\alpha(n)$ and hence of $g_1(n)$.

LEMMA 4.9. — *$\alpha(n) \to \frac{1}{\log 2}$ as $n \to \infty$ and $\alpha(n) < 2$ for all $n \geq 2$.*

Proof. — We use two results on the size of $g(n)$: Landau's asymptotic result ([6] Section 61)

$$\frac{\log g(n)}{\sqrt{n \log n}} \to 1 \text{ as } n \to \infty$$

and Massias' bound [9]

$$\log g(n) < 1.05314 \sqrt{n \log n} \text{ for all } n \geq 2.$$

We note that $g(n) \leq g_1(n) \leq g(n + \lfloor \sqrt{2n} \rfloor)$ since any set of distinct positive integers satisfying $\sum_i (a_i - 1) = n$ will satisfy $\sum_i a_i \leq n + \lfloor \sqrt{2n} \rfloor$.

Let

$$f(x) = \frac{(x + \sqrt{2x}) \log(x + \sqrt{2x})}{x \log x}.$$

Since $f(x) \to 1$ as $x \to \infty$, we conclude that $\alpha(n) \to \frac{1}{\log 2}$.

Likewise by Massias' bound

$$\alpha(n) \leq \frac{\log g(n + \lfloor \sqrt{2n} \rfloor)}{\log 2 \cdot \sqrt{n \log n}} < \frac{1.05314 \sqrt{f(n)}}{\log 2} \leq \frac{1.05314 \sqrt{f(9)}}{\log 2} < 2$$

for $n \geq 9$ since $f(x)$ is decreasing for $x > 1$.

Manual calculation shows that $\alpha(n) < 2$ for $2 \leq n \leq 8$. □

5. Some examples

In this section, we will give three examples of families of abelian varieties with commutative endomorphism ring for which Theorem 1.1 is sharp. Note that these examples show that the bound is sharp for the ℓ-adic monodromy groups as well as for the Mumford-Tate group, because the ranks of the ℓ-adic monodromy groups are less than or equal to that of the Mumford-Tate group and satisfy the same lower bound. Furthermore these examples satisfy the Mumford-Tate conjecture, because any abelian variety for which the ranks of the Mumford-Tate group and ℓ-adic monodromy groups are equal satisfies the Mumford-Tate conjecture by [8] Theorem 4.3.

We also give one family of simple abelian varieties with noncommutative endomorphism ring for which the Mumford-Tate group has rank n and the dimension g satisfies $\log_2 g = n + \frac{1}{2}\log_2 n + \mathrm{O}(1)$. This shows that the bound in Theorem 1.2 cannot be improved to $n \geq \log_2 g + \mathrm{O}(1)$. Because we have not calculated the exact lower bound in the noncommutative case we cannot deduce that these varieties satisfy the Mumford-Tate conjecture purely from the rank bound. But for the examples constructed here, we can show that they satisfy the Mumford-Tate conjecture by using [12] Proposition 4.3.

5.1. Examples with commutative endomorphism ring

Example 1: Complex multiplication. — Let F be a totally real field such that $[F : \mathbb{Q}] = n-1$. By [17] Theorem 1.10, there is an imaginary quadratic extension K of F such that for every CM type (K, Φ), the reflex type (K', Φ') satisfies $[K' : \mathbb{Q}] = 2^{n-1}$. Such a CM type is primitive.

Let A be a complex abelian variety corresponding to the CM type (K', Φ'). Then the Mumford-Tate group M is a torus, isomorphic to the image of the homomorphism $\mathrm{Res}_{K/\mathbb{Q}}\,\mathbb{G}_m \to \mathrm{Res}_{K'/\mathbb{Q}}\,\mathbb{G}_m$ induced by the reflex norm $K^\times \to K'^\times$.

This image has rank at most $[K : \mathbb{Q}] + 1 = n + 1$. But $\dim A = 2^{n-1}$ so by Theorem 1.1, $\operatorname{rk} M \geq n + 1$. So in fact $\operatorname{rk} M = n + 1 = \log_2 \dim A + 2$.

The endomorphism ring of A is the field K'.

Example 2: Spin group. — This example generalises the Kuga-Satake construction of an abelian variety attached to a polarised $K3$ surface [5].

Let n be a positive integer congruent to 1 or 2 mod 4. Let W be a $\mathbb{Q}$-vector space of dimension $2n + 1$, and let Q be the quadratic form

$$Q(x) = x_1^2 + x_2^2 - x_3^2 - \cdots - x_{2n+1}^2$$

of signature $(2, 2n - 1)$. The even Clifford algebra $C^+(W, Q)$ is isomorphic to $\mathrm{M}_{2^n}(\mathbb{Q})$, and so it has a unique faithful irreducible $\mathbb{Q}$-representation of dimension 2^n, called the spin representation.

Let M be the Clifford group

$$\mathrm{GSpin}(W, Q) = \{x \in C^{+}(W, Q) \mid xWx^{-1} \subseteq W\}.$$

This is a reductive group of rank $n+1$, with root system B_n and centre $\mathbb{G}_m$. Let $\rho : M \to \mathrm{GL}(V)$ be the spin representation of M. This is an absolutely irreducible representation of dimension 2^n.

Let $\{e_1, e_2\}$ be an orthonormal basis for the positive definite subspace of W. The homomorphism $\varphi : \mathbb{C}^\times \to M(\mathbb{R})$ given by

$$\varphi(a + ib) = a + be_1e_2$$

defines a Hodge structure on V of type $\{(0,-1),(-1,0)\}$. The conditions on $n \bmod 4$ and on the signature of W ensure that this Hodge structure is polarisable.

Because M^{der} is almost simple, replacing φ by a generic $M(\mathbb{R})$-conjugate gives a Hodge structure whose Mumford-Tate group is M. Let A be a complex abelian variety corresponding to such a Hodge structure. It has dimension 2^{n-1} and endomorphism algebra $\mathbb{Q}$, and its Mumford-Tate group has rank $n+1$.

Example 3: Product of copies of SL_2. — This example generalises the example of Mumford [10] of a family of abelian varieties of dimension 4 with Mumford-Tate group M such that $M_\mathbb{C}$ is isogenous to $\mathbb{G}_m \times (\mathrm{SL}_2)^3$.

Let n be an odd positive integer, and F a totally real number field of degree n. Let D be a quaternion algebra over F such that:

(i) $\mathrm{Cor}_{F/\mathbb{Q}} D$ is split over $\mathbb{Q}$, i.e., is isomorphic to $\mathrm{M}_{2^n}(\mathbb{Q})$.
(ii) D is split at exactly one real place of F.

Let M be the $\mathbb{Q}$-algebraic group $M(A) = \{x \in (D \otimes A)^\times | x\bar{x} \in A^\times\}$ (where $\bar{x}$ is the standard involution of D). By condition (ii), $M_\mathbb{R}$ is isomorphic to

$$\left(\mathbb{G}_{m,\mathbb{R}} \times \mathrm{SL}_{2,\mathbb{R}} \times \mathrm{SU}_2^{n-1}\right) / \{(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) \mid \varepsilon_i \in \{\pm 1\}, \varepsilon_0\varepsilon_1 \cdots \varepsilon_n = 1\}.$$

By condition (i), M has a faithful irreducible $\mathbb{Q}$-representation ρ of dimension 2^n. Then $\rho \otimes_\mathbb{Q} \mathbb{C}$ is isomorphic to the tensor product of the standard 1-dimensional representation of $\mathbb{G}_{m,\mathbb{C}}$ with the standard 2-dimensional representation of each factor $\mathrm{SL}_{2,\mathbb{C}}$.

Let $\varphi : \mathbb{C}^\times \to M(\mathbb{R})$ be the homomorphism

$$\varphi(a+ib) = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \text{ in } \mathrm{GL}_2 \cong (\mathbb{G}_m \times \mathrm{SL}_2)/\{\pm 1\}$$
$$\text{and trivial in the } \mathrm{SU}_2 \text{ factors.}$$

Then $\rho \circ \phi$ defines a Hodge structure of type $\{(0,-1),(-1,0)\}$. By condition (ii), this Hodge structure is polarisable.

Again M^{der} is almost $\mathbb{Q}$-simple, so replacing φ by a generic element of its $M(\mathbb{R})$-conjugacy class gives a Hodge structure with Mumford-Tate group equal to M. An abelian variety corresponding to such a Hodge structure will have dimension 2^{n-1}, endomorphism algebra $\mathbb{Q}$ and Mumford-Tate group of rank $n+1$.

5.2. An example with large multiplicity. — Let n be an odd integer and $r = (n-1)/2$. We will construct a simple abelian variety of dimension $g(n) = n\binom{n}{r}$ whose Mumford-Tate group is a $\mathbb{Q}$-form of GL_n. The Mumford-Tate representation is isomorphic over $\mathbb{C}$ to the sum of $2n$ copies of the r-th exterior power of the standard representation. By Stirling's formula $\log_2 g(n) = n + \frac{1}{2}\log_2 n + \mathrm{O}(1)$.

Let K be an imaginary quadratic field, and D a central division algebra over K of dimension n^2 with an involution $*$ of the second kind. The $\mathbb{Q}$-algebraic groups

$$H(A) = \{d \in (D \otimes_{\mathbb{Q}} A)^\times \mid dd^* = 1\},$$
$$G(A) = \{d \in (D \otimes_{\mathbb{Q}} A)^\times \mid dd^* \in A^\times\}$$

are $\mathbb{Q}$-forms of SL_n and GL_n. By choosing $*$ appropriately, we may suppose that $H_{\mathbb{R}}$ is the unitary group of a Hermitian form of signature $(1, n-1)$.

We can view D as a K-irreducible representation of H_K. Over $\mathbb{C}$, $D_{\mathbb{C}}$ is isomorphic to the sum of n copies of the standard representation of SL_n, so its highest weight is ϖ_1. The endomorphism ring of this representation is D^{op}, so

$$\alpha_{H,K}(\varpi_1) = [D]$$

for Tits' homomorphism $\alpha_{H,K} : \Lambda^\Gamma \to \mathrm{Br}\, K$.

Let $r = (n-1)/2$ and let $\tilde{D}$ be the central division algebra over K such that $[\tilde{D}] = [D]^r$ in $\mathrm{Br}\, K$. Now $[D]$ has order n in $\mathrm{Br}\, K$. Since r and n are coprime, $[\tilde{D}]$ also has order n and $\tilde{D} \otimes_K \mathbb{C} \cong M_n(\mathbb{C})$.

Let $\tilde{\rho}$ be the K-irreducible representation of H_K with highest weight ϖ_r. We know that $\varpi_r \equiv r\varpi_1$ modulo the roots of H_K, so $\alpha_{H,K}(\varpi_r) = [D]^r = [\tilde{D}]$. Hence $\tilde{\rho}$ has endomorphism ring $\tilde{D}^{\mathrm{op}}$, so $\tilde{\rho}_{\mathbb{C}}$ is the sum of n copies of an irreducible representation of SL_n. This irreducible representation is the r-th exterior power of the standard representation, so $\dim_K \tilde{\rho} = n\binom{n}{r}$.

If λI is a scalar matrix in $H(\mathbb{C})$, then $\tilde{\rho}_{\mathbb{C}}(\lambda I)$ is multiplication by λ^r. So we can extend $\tilde{\rho}$ to a representation of G_K by letting each scalar matrix λI act as multiplication by λ^r.

Let $\rho = \mathrm{Res}_{K/\mathbb{Q}}\, \tilde{\rho}$. This is a $\mathbb{Q}$-irreducible representation of G of dimension $2n\binom{n}{r}$. We have $\ker \rho = \mu_r$ so ρ factorises through $M = G/\mu_r$, and the resulting representation of M is faithful.

In order to specify the Hodge structure, we will first define $\varphi' : \mathbb{C}^\times \to G(\mathbb{R})$ as follows: recall that $H_{\mathbb{R}}$ is the unitary group of a Hermitian form Ψ of signature $(1, n-1)$. Then let $\phi'(z)$ act as $z^r/\bar{z}^{r-1}$ on the 1-dimensional space where h is positive definite and as $\bar{z}$ on the $(n-1)$-dimensional space where h is negative definite.

Then $\rho \circ \varphi'$ has weights z^r and $\bar{z}^r$. Because ρ is faithful as a representation of M, it follows that there is a homomorphism $\varphi : \mathbb{C}^\times \to M(\mathbb{R})$ whose r-th power is φ'. Then (M, ρ, φ) defines a $\mathbb{Q}$-Hodge structure of type $\{(-1, 0), (0, -1)\}$. The Hermitian form Ψ induces a polarisation of this Hodge structure.

Once again, M^{der} is almost simple, so replacing φ by a generic $M(\mathbb{R})$-conjugate gives a Hodge structure with Mumford-Tate group M. A corresponding abelian variety will have endomorphism algebra $\tilde{D}^{\mathrm{op}}$ and dimension $g = n\binom{n}{r}$.

We shall confirm that this variety satisfies the Mumford-Tate conjecture. Let σ be an absolutely irreducible component of $\rho \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$. Then $(G \times_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell, \sigma)$, with a suitable set of cocharacters, form a weak Mumford-Tate triple of weights $\{0, 1\}$. By Faltings' theorem, the restriction of σ to $G_{\ell, \bar{\mathbb{Q}}_\ell}$ must remain irreducible, where G_ℓ is the ℓ-adic monodromy group. It also is part of a weak Mumford-Tate triple of weights $\{0, 1\}$. But our $(G \times_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell, \sigma)$ is in the fourth column of [12] Table 4.2: type A with σ not the standard representation. Hence according to Pink's Proposition 4.3, $G_\ell = G \times_{\mathbb{Q}} \mathbb{Q}_\ell$.

BIBLIOGRAPHY

[1] F. A. BOGOMOLOV – "Points of finite order on an abelian variety", *Mathematics of the USSR-Izvestiya* **17** (1981), p. 55–72.

[2] B. DODSON – "On the Mumford-Tate group of an abelian variety with complex multiplication", *J. Algebra* **111** (1987), p. 49–73.

[3] G. FALTINGS – "Endlichkeitssätze für abelsche Varietäten über Zahlkörpern", *Invent. Math.* **73** (1983), p. 349–366.

[4] M. HINDRY & N. RATAZZI – "Torsion dans un produit de courbes elliptiques", *J. Ramanujan Math. Soc.* **25** (2010), p. 81–111.

[5] M. KUGA & I. SATAKE – "Abelian varieties attached to polarized K_3-surfaces", *Math. Ann.* **169** (1967), p. 239–242.

[6] E. LANDAU – *Handbuch der Lehre von der Verteilung der Primzahlen, I*, Teubner, Leipzig, 1909.

[7] M. LARSEN & R. PINK – "Determining representations from invariant dimensions", *Invent. Math.* **102** (1990), p. 377–398.

[8] ———, "Abelian varieties, l-adic representations, and l-independence", *Math. Ann.* **302** (1995), p. 561–579.

[9] J. MASSIAS – "Majoration explicite de l'ordre maximum d'un élément du groupe symétrique", *Annales Faculté des Sciences Toulouse Math.* **6** (1984), p. 269–291.

[10] D. MUMFORD – "A note of Shimura's paper "Discontinuous groups and abelian varieties"", *Math. Ann.* **181** (1969), p. 345–351.

[11] R. S. Pierce – *Associative algebras*, Graduate Texts in Math., vol. 88, Springer, New York-Berlin, 1982.

[12] R. Pink – "l-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture", *J. reine angew. Math.* **495** (1998), p. 187–237.

[13] K. A. Ribet – "Division fields of abelian varieties with complex multiplication", *Mém. Soc. Math. France* **2** (1980/81), p. 75–94.

[14] J-P. Serre – "Groupes algébriques associés aux modules de Hodge-Tate", in *Journées de Géométrie Algébrique de Rennes. (Rennes, 1978), Vol. III*, Astérisque, vol. 65, Soc. Math. France, Paris, 1979, p. 155–188.

[15] ———, "Lettre à Ken Ribet du 1er janvier 1981", 1981.

[16] ———, *Résumés des cours au Collège de France*, 1984–85.

[17] G. Shimura – "On canonical models of arithmetic quotients of bounded symmetric domains", *Ann. of Math.* **91** (1970), p. 144–222.

[18] A. Silverberg & Y. G. Zarhin – "Hodge groups of abelian varieties with purely multiplicative reduction", *Izv. Ross. Akad. Nauk Ser. Mat.* **60** (1996), p. 149–158.

[19] S. G. Tankeev – "Algebraic cycles on abelian varieties. II", *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), p. 418–429.

[20] J. Tits – "Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque", *J. reine angew. Math.* **247** (1971), p. 196–220.

[21] J.-P. Wintenberger – "Groupes algébriques associés à certaines representations p-adiques", *Amer. J. Math.* **108** (1986), p. 1425–1466.

[22] Y. G. Zarkhin – "Weights of simple Lie algebras in the cohomology of algebraic varieties", *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), p. 264–304.