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## WANDERING FATOU COMPONENTS AND ALGEBRAIC JULIA SETS

BY EUGENIO TRUCCO

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**ABSTRACT.** — We study the dynamics of polynomials with coefficients in a non-Archimedean field  $K$ , where  $K$  is a field containing a dense subset of algebraic elements over a discrete valued field  $k$ . We prove that every wandering Fatou component is contained in the basin of a periodic orbit. We obtain a complete description of the new Julia set points that appear when passing from  $K$  to the Berkovich affine line over  $K$ . We give a dynamical characterization of polynomials having algebraic Julia sets. More precisely, we establish that a polynomial with algebraic coefficients has algebraic Julia set if every critical element is nonrecurrent.

**RÉSUMÉ** (*Composantes de Fatou errantes et ensembles de Julia algébriques*)

Nous étudions la dynamique des polynômes à coefficients dans un corps  $K$  non-archimédien, où  $K$  contient un sous-ensemble dense d'éléments algébriques sur un corps  $k$  à valeurs discrètes. Nous montrons que toute composante de Fatou errante est contenue dans le bassin d'une orbite périodique. Nous obtenons une description complète des nouveaux points d'ensemble de Julia qui apparaissent quand on passe de  $K$  à la ligne de Berkovich affine sur  $K$ . Nous donnons une caractérisation dynamique des polynômes ayant des ensembles de Julia algébriques. Plus précisément, nous établissons qu'un polynôme à coefficients algébriques a un ensemble de Julia algébrique si tout élément critique est non-récurrent.

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## 1. Introduction

In this paper we study the dynamics of polynomials  $P: K \rightarrow K$  where  $K$  is an algebraically closed field of characteristic 0 which is complete with respect to a non-Archimedean absolute value. Moreover, we will assume that there exists a discrete valued field  $k \subseteq K$  such that

$$k^a = \{z \in K \mid [k(z) : k] < +\infty\}$$

form a dense subset of  $K$ . Examples of such fields are the field  $\mathbb{C}_p$  of  $p$ -adic numbers and the field, which we will denote by  $\mathbb{L}$ , which is the completion of an algebraic closure of the field of formal Laurent series with coefficients in  $\mathbb{C}$ . Dynamics over  $\mathbb{C}_p$  naturally arises in number theory and dynamics over  $\mathbb{L}$  naturally appears in the study of parameter spaces of complex rational maps [17].

For complex rational maps acting on the Riemann sphere, Sullivan [25] proved, with the aid of quasi-conformal techniques, that every connected component of the Fatou set of a rational map  $R \in \mathbb{C}(z)$  of degree  $\geq 2$  is eventually periodic (Sullivan's No Wandering Domains Theorem). This is no longer true for general non-Archimedean fields. In fact, Benedetto [4] established the existence of  $p$ -adic polynomials having wandering (analytic) domains which are not attracted to a periodic orbit. This result heavily relies on the fact that over  $p$ -adic fields, whose residual characteristic is  $p > 0$ , there exists a phenomenon called *wild ramification*.

The aim of this paper is to study the interplay between algebraic and dynamical properties of points in the Julia set of a polynomial. As a consequence, we establish that for *tame* polynomials (see Definition 2.10), that is, for polynomials such that wild ramification does not occur, the dynamics is free of nontrivial wandering domains (see Corollary B below).

Recent developments on the theory of iteration of rational maps over non-Archimedean fields put in evidence that the correct space to study the action of rational maps is the *Berkovich space* (e.g., [1, 2, 10, 13, 19, 20, 21]). The action of a polynomial  $P \in K[z]$  extends to the *Berkovich affine line*  $\mathbb{A}_K^{1,\text{an}}$  associated to  $K$ . Moreover, the notions of Julia set (chaotic dynamics) and Fatou set (regular dynamics) also extend to  $\mathbb{A}_K^{1,\text{an}}$ . Our first main result is a complete description of the new Julia set points that appear when passing from  $K$  to  $\mathbb{A}_K^{1,\text{an}}$ . We will denote by  $\mathcal{J}_P$  the Julia set of  $P$ . A polynomial is *simple* if its Julia set is a singleton.

**THEOREM A.** — *Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $d \geq 2$ . Then  $\mathcal{J}_P \setminus K$  is empty or, there exist finitely many repelling periodic orbits  $\mathcal{O}_1, \dots, \mathcal{O}_m \subseteq \mathbb{A}_K^{1,\text{an}} \setminus K$  such that*

$$\mathcal{J}_P \setminus K = \text{GO}(\mathcal{O}_1) \sqcup \dots \sqcup \text{GO}(\mathcal{O}_m),$$

where  $\text{GO}(\mathcal{O}_j)$  denotes the grand orbit of  $\mathcal{O}_j$  and  $1 \leq m \leq d - 2$ .

The previous theorem is first proven for polynomials in  $K[z]$  with algebraic coefficients over the field  $k$ . Here, we rely on our study of the interplay between the geometry of the Julia set and the underlying algebraic structure (Section 6). For a general tame polynomial with coefficients in  $K$ , we use a perturbation technique furnished by a key proposition (Proposition 7.1) inspired by complex polynomial dynamics (e.g., [18]).

Standard techniques (see Proposition 2.16) allow us to deduce the above mentioned nonwandering result from Theorem A. We say that  $x$  is in the basin of the periodic orbit  $\mathcal{O}$  if  $\mathcal{O}$  is the set of limits points of  $\{P^n(x) \mid n \in \mathbb{N}\}$ .

**COROLLARY B.** — *Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$ . Then, every wandering Fatou component is in the basin of a periodic orbit.*

Benedetto [3] proved a similar result to Corollary B for rational maps with algebraic coefficients over the field of  $p$ -adic numbers  $\mathbb{Q}_p$  with some slightly different hypothesis.

In terms of  $k, K$  and in our language, Theorem B in [5] says that every wandering Fatou component of a rational map with algebraic coefficients over  $k$  is in the basin of a periodic orbit. Benedetto asks (question (2) at the end of the introduction of [5]) if this is true for rational maps with coefficients in  $K$ , assuming that the characteristic of the residual field of  $K$  is zero. Corollary B above gives an affirmative answer to the question posed by Benedetto in the case of polynomials.

It is not known if for any polynomial  $P \in \mathbb{Q}_p[z]$  such that  $\mathcal{J}_P \cap \mathbb{Q}_p \neq \emptyset$ , there exists a classical repelling periodic point of  $P$  in  $\mathcal{J}_P \cap \mathbb{Q}_p$ . In the case of tame polynomials we have the following result.

**COROLLARY C.** — *Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $d \geq 2$ . Then the classical Julia set of  $P$  contains a repelling periodic point.*

In the case of  $p$ -adic polynomials, Bézivin [7, Proposition A] proved that if there exists a repelling periodic point in  $\mathcal{J}_P \cap \mathbb{Q}_p$  and  $\mathcal{J}_P \cap \mathbb{Q}_p$  is a compact set, then  $\mathcal{J}_P = \mathcal{J}_P \cap \mathbb{Q}_p$ . The following corollary is an analog of that result, nevertheless, we do not need to assume the existence of a repelling periodic point.

**COROLLARY D.** — *Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$ . Then the following statements are equivalent:*

1.  $\mathcal{J}_P \cap K$  is a compact subset of  $\mathbb{A}_K^{1, \text{an}}$ .
2. There is no critical periodic element in  $\mathcal{J}_P$ .
3.  $\mathcal{J}_P = \mathcal{J}_P \cap K$ .

4. All the periodic points of  $P$  in  $K$  are repelling.

After extending the notion of *algebraic degree* of  $x \in K$  over  $k$  for arbitrary points  $x \in \mathbb{A}_K^{1,\text{an}}$  (see Section 5) we obtain the algebraic counterpart of the previous topological dynamics results. We relate the algebraic degree with the recurrent critical elements (a point  $x$  is recurrent if it is a limit point for its iterates)

**THEOREM E.** — *Let  $P(z) \in K[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$  and with algebraic coefficients over  $k$ . If the algebraic degree of every element in  $\mathcal{J}_P$  is finite then the critical elements contained in  $\mathcal{J}_P$  are not recurrent. In that case the algebraic degrees of the elements of  $\mathcal{J}_P$  are uniformly bounded.*

**REMARK 1.1.** — *A posteriori*, using Theorem A we can also obtain that  $\mathcal{J}_P \subseteq k^a$ .

**REMARK 1.2.** — Suppose that  $k$  is a field with the property that for any positive integer  $N$  there are only finitely many extensions of  $k$  of degree less than  $N$ . For example, this is true if  $k$  is a finite extension of  $\mathbb{Q}_p$  or of the field  $\mathbb{C}((\tau))$  of formal Laurent series. Then under the hypotheses of Theorem E, it follows that there exists a finite extension  $k'/k$  such that  $\mathcal{J}_P$  is contained in  $k'$ .

In a special type of fields, which will denote by  $\mathbb{L}_F$ , we obtain the converse of the previous theorem. Here  $F$  is an algebraically closed field of characteristic 0 and  $\mathbb{L}_F$  is the completion of an algebraic closure of the field  $F((\tau))$  of formal Laurent series with coefficients in  $F$  with respect to an appropriate non-Archimedean absolute value. See Subsection 6.6 for definitions.

**THEOREM F.** — *Let  $P(z) \in \mathbb{L}_F[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$  with algebraic coefficients over  $F((\tau))$ . Then the algebraic degree of every element in  $\mathcal{J}_P$  is finite if and only if the critical elements contained in  $\mathcal{J}_P$  are not recurrent. In that case,  $\mathcal{J}_P$  is contained in a finite extension of  $F((\tau))$ .*

**1.1. Outline of the paper.** — Section 2 consists of basic definitions and facts about the Berkovich affine line and the action of polynomials on  $\mathbb{A}_K^{1,\text{an}}$ .

In Sections 3 and 4 we introduce the *dynamical sequence* and the *geometric sequence* of a polynomial. We employ these objects throughout the paper since they organize our topological and algebraic study of the convex hull of the Julia set.

In Section 5 we extend the notion of algebraic degree to the Berkovich line and explore its basic properties.

In Section 6, for polynomials with algebraic coefficients, we describe the behavior of the algebraic degree along geometric sequences. Then we prove Corollary B, in the case of polynomials with algebraic coefficients. In Subsection 6.3 we will explore the relation between the algebraic degree and the *equilibrium measure* (e.g., [2, 10, 13, 12]). We finish the section with the proofs of Theorem E and of Theorem F.

In Section 7 we establish Proposition 7.1 which is the key to perturb polynomials (with transcendental coefficients), preserving the dynamics along an orbit. Then we prove Theorem A in full generality and, as a consequence, we obtain Corollary B, Corollary C and Corollary D.

## 2. Background

In this section we first recall some basic facts about non-Archimedean fields. Then we construct the *Berkovich affine line* and discuss the behavior of the polynomial dynamics on it. See [9, 14, 26] in the case of non-Archimedean fields and [1, 6, 12, 13, 19, 23] for the dynamics on the Berkovich affine line.

**2.1. Non-Archimedean fields.** — Let  $K$  be a field with characteristic zero endowed with a non-Archimedean absolute value  $|\cdot|$ . That is, an absolute value satisfying the *strong triangle inequality*

$$|z_1 + z_2| \leq \max\{|z_1|, |z_2|\}$$

for all  $z_1, z_2 \in K$ . Examples of such fields are the field of  $p$ -adic numbers  $\mathbb{Q}_p$  and the field  $\mathbb{L}_F$  of Puiseux series with coefficients in  $F$  that will discuss in detail in Subsection 6.6. For more about non-Archimedean fields see [9, 14].

The set  $|K^\times| := \{|z| \mid z \in K^\times\}$  of nonzero values of  $|\cdot|$  is a multiplicative subgroup of the positive real numbers called the *value group of  $K$* . We say that the absolute value  $|\cdot|$  is *discrete* if  $|K^\times|$  is discrete as a subset of  $\mathbb{R}$ .

We denote by  $\mathfrak{o}_K := \{z \in K \mid |z| \leq 1\}$  the *ring of integers of  $K$*  and by  $\mathfrak{m}_K$  its unique maximal ideal, i.e.,  $\mathfrak{m}_K := \{z \in K \mid |z| < 1\}$ . The *residual field of  $K$*  is the quotient field  $\tilde{K} := \mathfrak{o}_K / \mathfrak{m}_K$ . As we will see later, there exists a substantial difference according to whether the residue field has characteristic 0 or  $p > 0$ .

For  $z_0 \in K$  and  $r > 0$  we define the sets

$$B_r^+(z_0) := \{z \in K \mid |z - z_0| \leq r\} \quad \text{and} \quad B_r(z_0) := \{z \in K \mid |z - z_0| < r\}.$$

If  $r$  belongs to the value group of  $K$ , then the sets defined above are different and we say that  $B_r^+(z_0)$  (resp.  $B_r(z_0)$ ) is a *closed ball* (resp. *open ball*). If  $r$  is not in the value group of  $K$ , then the sets  $B_r(z_0)$  and  $B_r^+(z_0)$  coincide and we say that  $B_r(z_0) = B_r^+(z_0)$  is an *irrational ball*. Despite these names, every

ball is open and closed in the metric topology induced in  $K$  by the absolute value  $|\cdot|$ .

**2.2. Balls and polynomial.** — Given a nonconstant polynomial  $P$  with coefficients in  $K$ , define the *local degree* of  $P$  at  $z_0 \in K$  as the largest integer  $\deg_{z_0}(P) \geq 1$  such that  $(z - z_0)^{\deg_{z_0}(P)}$  divides  $P(z) - P(z_0)$  in the ring  $K[z]$ . If  $\deg_{z_0}(P) > 1$  we say that  $z_0$  is a *critical point* of  $P$  with *multiplicity*  $\deg_{z_0}(P) - 1$ . We denote by  $\text{Crit}^I(P)$  the subset of  $K$  formed by the critical points of  $P$ .

The image of a ball  $B \subseteq K$ , under the action of  $P$ , is a ball, of the same type as  $B$ , and there exists a positive integer, denoted by  $\deg_B(P)$ , and called *degree of  $P$  at  $B$* , such that

$$\deg_B(P) = \sum_{\{z \in B \mid P(z) = z'\}} \deg_z(P)$$

for all  $z' \in B$  (e.g., see Section 2 in [20]).

Moreover, the preimage of a ball  $B$  is a finite union of pairwise disjoint balls  $B_1, \dots, B_m$  of the same type as  $B$  and

$$\sum_{j=1}^m \deg_{B_j}(P) = \deg(P).$$

**2.3. Berkovich affine line.** — We will need only basic facts about the structure of the Berkovich affine line and its topology. For more details see [11, 20, 21, 23], for the original construction of V. G. Berkovich, see [6].

We identify the Berkovich line with an appropriate quotient of the set  $\mathcal{S}_K$  of all the strictly decreasing sequences of closed balls of  $K$ . This construction is a slight modification of the given one in [23].

On the set  $\mathcal{S}_K$  we define the equivalence relation  $\sim$  given by:  $(B_j) \sim (B'_j)$  if for all  $n \in \mathbb{N}$ , the sequence  $(B_j)$  (resp.  $(B'_j)$ ) is eventually contained in  $B'_n$  (resp.  $B_n$ ).

The *Berkovich analytic space associated to the affine line over  $K$*  (for short, the Berkovich line) denoted by  $\mathbb{A}_K^{1,\text{an}}$ , is (as a set) the quotient  $\mathcal{S}_K / \sim$ .

If the sequence  $(B_j)$  is equivalent to  $(B'_j)$ , then  $\cap B_j = \cap B'_j$ . Note that the field  $K$  is not spherically complete, that is, there exist decreasing sequences of closed balls having empty intersection. However, consider  $(B_j) \in \mathcal{S}_K$  such that  $B = \cap B_j$  is not empty. Then  $B$  is a closed ball, an irrational ball or a point of  $K$ . Moreover, the intersection  $B$  determines completely the equivalence class of  $(B_j)$ . In this case, we denote the equivalence class of  $(B_j)$  by  $x_B$  and we will say that  $x_B$  in  $\mathbb{A}_K^{1,\text{an}}$ , is *the point associated to  $B$*  and that  $B$  is *the ball associated to  $x_B$* .



The elements of the Berkovich line  $\mathbb{A}_K^{1,\text{an}}$  are classified in the following four types:

1. *Type I* or *classical points*, corresponding to the equivalence classes of sequences whose intersection is a point in  $K$ . We identify  $K$  with these elements of  $\mathbb{A}_K^{1,\text{an}}$ .
2. *Type II* or *rational points*, corresponding to elements  $x_B$  where  $B$  is a closed ball.
3. *Type III* or *irrational points* that is, the points of the form  $x_B$  where  $B$  an irrational ball of  $K$ .
4. *Type IV* or *singular points*, corresponding to the equivalence classes of decreasing sequences of closed balls with empty intersection.

The inclusion between the balls of  $K$  induces a partial order, denoted by  $\preceq$ , in  $\mathbb{A}_K^{1,\text{an}}$ . If  $x \in \mathbb{A}_K^{1,\text{an}}$  (resp  $x' \in \mathbb{A}_K^{1,\text{an}}$ ) is the equivalence class of  $(B_j)$  (resp  $(B'_j)$ ) we say that  $x \preceq x'$  if, for each  $n \in \mathbb{N}$ , the sequence  $(B_j)$  is eventually contained in  $B'_n$ . We say that  $x \prec y$  if  $x \preceq y$  and  $x \neq y$ . In the case that  $x_B$  and  $x_{B'}$  are nonsingular elements, we have that  $x_B \preceq x_{B'}$  if and only if  $B$  is contained in  $B'$ .

For all  $x \in \mathbb{A}_K^{1,\text{an}}$  denote the set of elements larger than  $x$  by

$$[x, \infty[ := \{w \in \mathbb{A}_K^{1,\text{an}} \mid x \preceq w\}.$$

Observe that  $[x, \infty[$  is isomorphic, as an ordered set, to  $[0, +\infty[ \subseteq \mathbb{R}$ .

Given two points  $x, y$  in the Berkovich line  $\mathbb{A}_K^{1,\text{an}}$  we have that

$$[x, \infty[ \cap [y, \infty[ = [x \vee y, \infty[$$

where  $x \vee y$  is the smallest element larger than  $x$  and  $y$ . If  $x$  is not related to  $y$  then the element  $x \vee y$  is a type II point.

Given two elements  $x, y \in \mathbb{A}_K^{1,\text{an}}$  let

$$[x, y] := \{w \in \mathbb{A}_K^{1,\text{an}} \mid x \preceq w \preceq x \vee y\} \cup \{w \in \mathbb{A}_K^{1,\text{an}} \mid y \preceq w \preceq x \vee y\}.$$

The sets  $]x, y]$ ,  $[x, y[$  and  $]x, y[$  are defined in the obvious way.

For  $x$  in the Berkovich line, the *diameter* of  $x$  is

$$\text{diam}(x) := \lim_{j \rightarrow \infty} \text{diam}(B_j),$$

where  $(B_j)$  is a representative of  $x$ . For  $x_B$ , a nonsingular element, the diameter of  $x_B$  coincides with the diameter (radius) of the ball  $B$ .

In order to endow the Berkovich affine line with a topology, we define an *open ball* of  $\mathbb{A}_K^{1,\text{an}}$  and a *closed ball* of  $\mathbb{A}_K^{1,\text{an}}$  by

$$\mathcal{B}(a, r) = \{x \in \mathbb{A}_K^{1,\text{an}} \mid \text{diam}(a \vee x) < r\}$$

$$\mathcal{B}^+(a, r) = \{x \in \mathbb{A}_K^{1,\text{an}} \mid \text{diam}(a \vee x) \leq r\}$$

respectively, where  $a \in K$  and  $r > 0$ .

The *weak topology* on the Berkovich line is the smallest topology containing all the open balls and the complements of closed balls of  $\mathbb{A}_K^{1,\text{an}}$ . Note that for all  $a \in K$  and  $r > 0$  the closed ball  $\mathcal{B}^+(a, r)$  is a compact set with respect to the weak topology.

If  $B = B_r^+(a) \subseteq K$  we have that the closure  $\overline{B}$  of  $B$  in  $\mathbb{A}_K^{1,\text{an}}$  is  $\mathcal{B}^+(a, r)$ . The boundary of  $\mathcal{B}^+(a, r)$  is  $\{x_B\}$ , although we will often abuse notation and write simply  $\partial \mathcal{B}^+(a, r) = x_B$ .

For all  $x \in \mathbb{A}_K^{1,\text{an}}$  there exists an order preserving bijection between  $[x, \infty[$  and an interval of  $\mathbb{R}$ . Moreover, for all  $x \neq y \in \mathbb{A}_K^{1,\text{an}}$  there exists an isomorphism between  $[x, x \vee y]$  and a closed interval of  $\mathbb{R}$ . Hence, following Definition 3.5 in [11]  $\mathbb{A}_K^{1,\text{an}}$  is an *unrooted nonmetric tree*.

Let  $B = B_r^+(a)$  be a closed ball of  $K$  and consider  $x_B \in \mathbb{A}_K^{1,\text{an}}$  the type II point associated to  $B$ . We say that two elements  $x, y \prec x_B$  are in the same *direction at  $x_B$*  if  $x \vee y \prec x_B$ . Given  $x \prec x_B$ , the set of elements in the same direction as  $x$  at  $x_B$  is the open ball  $\mathcal{B}(z, r)$  of the Berkovich line, where  $z \in B$  is such that  $z \prec x$ .

The *tangent space at  $x_B$* , denoted by  $T_{x_B}$ , is the set of all the directions at  $x_B$ . Since we will work only with polynomials, our notion of tangent space is really the *affine tangent space at  $x_B$* .

After an affine change of coordinates  $h$ , such that  $h(B_1^+(0)) = B$  we can identify the directions in  $T_{x_B}$  with the directions at the point associated to the ball  $B_1^+(0)$  and these directions can be naturally identified with the residue field of  $K$ .

We say that a set  $X$  of  $\mathbb{A}_K^{1,\text{an}}$  is *convex*, if for all  $x, y \in X$  we have that  $[x, y]$  is contained in  $X$ . For  $X$  a subset of  $\mathbb{A}_K^{1,\text{an}}$  we define the *convex hull of  $X$*  to be the set

$$\text{conv}(X) = \bigcup_{x, y \in X} [x, y].$$

A convex subset of  $\mathbb{A}_K^{1,\text{an}}$  is always a connected set.

**2.4. The action of a polynomial over the Berkovich line.** — The action of a non-constant polynomial  $P$  with coefficients in  $K$  has a unique continuous extension to  $\mathbb{A}_K^{1,\text{an}}$ , which we also denote by  $P$ . More precisely, if  $x$  is the equivalence class of  $(B_j)$ , then  $P(x)$  is defined as the equivalence class of the sequence  $(P(B_j))$ . If  $x = x_B$  is a non singular element, we have that  $P(x_B) = x_{P(B)}$ .

The map  $P: \mathbb{A}_K^{1,\text{an}} \rightarrow \mathbb{A}_K^{1,\text{an}}$  is increasing, open and preserves the type of the points. For all  $x \in \mathbb{A}_K^{1,\text{an}}$  the set of preimages of  $x$  under  $P$  is finite. The image of a ball  $\mathcal{B}$  of  $\mathbb{A}_K^{1,\text{an}}$  is a ball of the same type, and its preimage is a finite union of pairwise disjoint balls of the same type as  $\mathcal{B}$ .

To extend the notion of local degree of  $P$  to  $\mathbb{A}_K^{1,\text{an}}$  let

$$\deg_x(P) := \lim_{j \rightarrow \infty} \deg_{B_j}(P),$$

where  $(B_j)$  is a representative of the class  $x$ . We have that  $\deg_{x_B}(P) = \deg_B(P)$ , for all non singular elements  $x_B \in \mathbb{A}_K^{1,\text{an}}$ .

REMARK 2.1. — The map  $x \mapsto \deg_x(P)$  is a nondecreasing function with respect to  $\preceq$  and it is upper semi-continuous. See [1, Proposition 9.28] or [13, Section 2]

REMARK 2.2. — Given  $x \in \mathbb{A}_K^{1,\text{an}}$  with preimages  $x_1, \dots, x_m$ , we have that

$$\sum_{j=1}^m \deg_{x_j}(P) = \deg(P).$$

We say that  $x \in \mathbb{A}_K^{1,\text{an}}$  is a *critical element* of  $P$  if  $\deg_x(P) \geq 2$ . The structure of the critical set

$$\text{Crit}(P) := \{x \in \mathbb{A}_K^{1,\text{an}} \mid \deg_x(P) \geq 2\}$$

depends strongly on the characteristic of the residue field  $\tilde{K}$ , as we will see in Subsection 2.6.

Let  $x \in \mathbb{A}_K^{1,\text{an}}$  be a type II point. Given a direction  $\mathcal{D}$  in  $T_x$ , that is, an open ball  $\mathcal{D}$  of  $\mathbb{A}_K^{1,\text{an}}$  such that  $\partial \mathcal{D} = x$ , we have that  $P(\mathcal{D})$  is a direction in  $T_{P(x)}$ . Hence, the action of  $P$  in the Berkovich line induces a map  $T_x P: T_x \rightarrow T_{P(x)}$  between the tangent spaces at  $x$  and  $P(x)$ . After affine changes of coordinates  $h_1, h_2$  such that  $h_2(P(B)) = \mathfrak{o}$  and  $h_1(\mathfrak{o}) = B$  the map  $T_x P$  coincides with the reduction of  $P$  to the residue field  $\tilde{K}$ . Hence,  $T_x P$  is a nonconstant polynomial map in  $\tilde{K}[z]$  of degree lower than or equal to  $\deg(P)$ .

For further reference we establish a relation between the local degree of  $P$  at a type II point  $x \in \mathbb{A}_K^{1,\text{an}}$  and the degree of  $T_x P$ .

REMARK 2.3. — Let  $P \in K[z]$  be polynomial of degree  $\geq 2$  and let  $x \in \mathbb{A}_K^{1,\text{an}}$  be a type II point. If  $\zeta' \in \tilde{K}$  then

$$\deg_x(P) = \deg(T_x P) = \sum_{\{\zeta \in \tilde{K} \mid T_x P(\zeta) = \zeta'\}} \deg_\zeta(T_x P).$$

**2.5. The Hyperbolic Space.** — We denote by  $\mathbb{H}_K$  the *hyperbolic space of  $K$* , that is, the set of nonclassical elements in the Berkovich Line. This set has a tree structure induced by the structure of  $\mathbb{A}_K^{1,\text{an}}$ .

Over  $\mathbb{H}_K$  we can define the *hyperbolic distance*,

$$d_{\mathbb{H}}(x, y) = 2 \log \text{diam}(x \vee y) - \log \text{diam}(x) - \log \text{diam}(y),$$

which is compatible with the tree structure of  $\mathbb{H}_K$ . More precisely, the set  $\mathbb{H}_K$  with the hyperbolic distance is a  $\mathbb{R}$ -tree. That is, for all  $x, y \in \mathbb{H}_K$  the length of the segment  $[x, y]$ , which is a geodesic segment, coincides with the hyperbolic distance between  $x$  and  $y$ .

For further reference we state, without proof, the following straightforward fact

LEMMA 2.4. — *Let  $w, y, x$  in  $\mathbb{A}_K^{1,\text{an}}$ . We have that  $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(x, w) + d_{\mathbb{H}}(w, y)$  if and only if  $w$  belongs to  $[x, y]$ .*

The hyperbolic distance behaves nicely under the action of a polynomial. More precisely we have the following lemma which is a restatement of Corollary 4.8 of [21].

LEMMA 2.5. — *Let  $P \in K[z]$  and consider  $x \prec x' \in \mathbb{A}_K^{1,\text{an}}$ . Suppose that  $\deg_y(P) = \lambda$  for all  $y \in ]x, x'[$ . Then*

$$d_{\mathbb{H}}(P(x), P(x')) = \lambda \cdot d_{\mathbb{H}}(x, x').$$

The metric topology induced in  $\mathbb{H}_K$  by the hyperbolic distance is called the *strong topology*. Every open set  $X \subseteq \mathbb{H}_K$  for the topology induced in  $\mathbb{H}_K$  by the weak topology is an open set for the strong topology. Moreover,  $(\mathbb{H}_K, d_{\mathbb{H}})$  is a complete metric space.

Our default topology will always be the weak topology in  $\mathbb{A}_K^{1,\text{an}}$  and  $\mathbb{H}_K$ .

**2.6. The Critical Set of  $P$ .** — Let  $P \in K[z]$  be a polynomial of degree  $\geq 2$ . The structure of the critical set of  $P$  depends strongly on the characteristic of the residual field of  $K$ , as we will see in the following propositions. We will first assume that  $\text{char}(\tilde{K}) = 0$ .

PROPOSITION 2.6. — *Let  $P \in K[z]$  be a polynomial of degree  $\geq 2$  and let  $B \subseteq K$  be a ball. If the characteristic of  $\tilde{K}$  is zero, then*

$$\deg_B(P) = 1 + \deg_B(P')$$

*Proof.* — After an affine change of coordinates we can suppose that  $B$  and  $P(B)$  contain 0. Since  $\text{char}(\tilde{K}) = 0$  we have that  $|n| = 1$  for all  $n \in \mathbb{N}$ . Therefore, the Newton polygon of  $P'$  is a translation of the Newton polygon of  $P - P(0)$ . Hence, the number of zeros of  $P'$  in  $B$  is the number of zeros of  $P$  in  $B$  minus 1.  $\square$

REMARK 2.7. — Note that if  $\text{char}(\tilde{K}) = 0$  and  $B \subseteq K$  is a ball, then the following holds

$$\begin{aligned} \deg_B(P) - 1 &= \sum_{z \in B} (\deg_z(P) - 1) \\ &= \sum_{w \in B \cap \text{Crit}^I(P)} (\deg_w(P) - 1). \end{aligned}$$

That is, the degree of  $P$  at the ball  $B$  is determined by the critical points of  $P$  contained in  $B$ . We will refer to the identity above as the *Riemann-Hurwitz formula*.

From above, we have that in the case of  $\text{char}(\tilde{K}) = 0$ , the set  $\text{Crit}(P)$  coincides with the set

$$\bigcup_{w \in \text{Crit}^I(P)} [w, \infty).$$

Therefore  $\text{Crit}(P)$  is a finite subtree of  $\mathbb{A}_K^{1, \text{an}}$ . That is,  $\text{Crit}(P)$  has finitely many vertices and finitely many edges. There is one distinguished edge of the form  $[x, \infty[$ . The other edges are closed segments. In particular, we have that the local degree at a singular element is always 1.

The situation in the case of  $\text{char}(\tilde{K}) = p > 0$  is, in general, completely different.

PROPOSITION 2.8. — *Let  $P \in K[z]$  be a polynomial with degree  $\geq 2$  and let  $B \subseteq K$  be a ball such that  $m = \deg_B(P)$ . If the characteristic of  $\tilde{K}$  is  $p > 0$  and  $(p, m) = 1$  then*

$$\deg_B(P) = 1 + \deg_B(P')$$

*Proof.* — After an affine change of coordinates we can suppose that 0 belongs to  $B$  and  $P(B)$ . Since the local degree is  $m$ , we have that  $(m, \log(|a_m|))$  is a vertex of the Newton polygon of  $P$ . Since  $(p, m) = 1$  it follows that

$$(m - 1, \log(|ma_m|)) = (m - 1, \log(|a_m|))$$

is a vertex of the Newton polygon of  $P'$ . Indeed, in the Newton polygon of  $P'$  the slope before  $m - 1$  can only increase and the slope after  $m - 1$  can only decrease with respect to the slopes before and after  $m$  in the Newton polygon

of  $P$ . Hence, the number of zeros of  $P'$  in  $B$  is the number of zeros of  $P$  in  $B$  minus 1.  $\square$

**PROPOSITION 2.9.** — *Let  $P \in K[z]$  be a polynomial with degree  $\geq 2$  and let  $B \subseteq K$  be a closed or irrational ball such that  $m = \deg_B(P)$ . If the characteristic of  $\tilde{K}$  is  $p > 0$  and  $m = p^r n$  with  $(p, n) = 1$  and  $r \geq 1$ , then  $\text{Crit}(P) \cap \mathbb{H}_K$  has nonempty interior with respect to the strong topology. In particular,  $\text{Crit}(P)$  is not a finite tree.*

*Proof.* — Let  $x_B$  the point associated to the ball  $B$ . Since  $\deg_{x_B}(P) = p^r m$  and the local degree is the limit of a nonincreasing sequence of positive integers, there exists a type II point  $x_B \prec x$  such that  $\deg_x(P)$  is also  $p^r n$  and  $\deg_{\mathcal{D}}(P) = p^r n$ , where  $\mathcal{D}$  is the direction at  $x$  that contains  $x_B$ . If we consider the action between  $T_x$  and  $T_{P(x)}$ , we have that  $p^r n = \deg(T_x P) = \deg_{\mathcal{D}}(T_x P)$ . After affine changes of coordinates we can suppose that  $x$  and  $P(x)$  are the point associated to the ball  $B_1^+(0)$  and  $0 \in \mathcal{D} = P(\mathcal{D})$ . It follows that  $T_x P$ , which is a polynomial of degree  $p^r n$ , has a fixed point with local degree  $p^r n$ . Hence

$$T_x P(\zeta) = \zeta^{p^r n} = (\zeta^p)^{p^{r-1} n}.$$

By Lemma 10.1 in [22] we have that  $P$  coincide with  $z^{p^r n}$  in a strong neighborhood  $\mathcal{U}$  of  $x$ . Since  $x$  is a *inseparable fixed point* (Definition 5.4 in [22]) for  $z^{p^r n}$  we can use Proposition 10.2 in [22] to obtain the existence of a strong neighborhood  $\mathcal{V}$  of  $x$  such that  $\deg_y(z^{p^r n}) \geq p$  for all  $y \in \mathcal{V}$ . Then

$$\deg_y(P) = \deg_y(z^{p^r n}) \geq p$$

for all  $y \in \mathcal{U} \cap \mathcal{V}$ . Therefore  $\text{Crit}(P)$  has nonempty interior with respect to the strong topology. In this case  $\text{Crit}(P)$  is not a finite tree.  $\square$

The following definition is motivated by the previous propositions

**DEFINITION 2.10.** — We say that a polynomial  $P \in K[z]$  is *tame* if the critical set of  $P$  is a finite tree.

For instance, if  $\text{char}(\tilde{K}) = 0$  then any polynomial is tame. If the residual characteristic of  $K$  is  $p > 0$ , then any polynomial with degree  $d < p$  is tame.

If the residual characteristic of  $K$  is  $p > 0$  we have, by Proposition 2.9, that  $P$  is a tame polynomial if and only if the set  $\{\deg_x(P) \mid x \in \mathbb{A}_K^{1, \text{an}}\}$  does not contains multiples of  $p$ . Moreover, in this case, the Riemann-Hurwitz formula (see Remark 2.7) is valid. Our main results will be on tame polynomials.

We finish this subsection with a proposition about the structure of  $\text{Crit}(P)$ .

PROPOSITION 2.11. — *Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$ . Then*

$$\text{Crit}(P) = \bigcup_{w \in \text{Crit}^I(P)} [w, \infty[.$$

*In particular  $\deg_x(P) = 1$  for any  $x$  of type IV.*

*Proof.* — Let  $\mathcal{C} = \bigcup_{w \in \text{Crit}^I(P)} [w, \infty[$ . It is clear that  $\mathcal{C}$  is contained in  $\text{Crit}(P)$ .

Let  $x \in \text{Crit}(P)$ . If  $x$  is a nonsingular point then, following Remark 2.7, we have that  $x$  is a classical critical point or  $x = x_B$  and  $B \cap \text{Crit}^I(P) \neq \emptyset$ , therefore  $x \in \mathcal{C}$ .

If  $x$  is a type IV point, then there exists a decreasing sequence  $(x_n)$  of nonsingular critical elements such that  $\lim_{n \rightarrow +\infty} x_n = x$ , see Subsection 2.4. Since the points  $x_n$  are nonsingular and the classical critical points are finitely many, there exists  $w \in \text{Crit}^I(P)$  such that  $x_n$  belongs to  $[w, \infty[$  for all  $n \in \mathbb{N}$ . Hence,  $\lim_{n \rightarrow +\infty} x_n = x$  belongs to  $[w, \infty[$ , but this is impossible because  $x$  is a type IV point. Therefore  $\deg_x(P) = 1$  for any  $x$  of type IV and  $\mathcal{C} = \text{Crit}(P)$ .  $\square$

**2.7. Julia and Fatou sets in the Berkovich line.** — By analogy with complex polynomial dynamics the *filled Julia set* of  $P$  is defined by

$$\mathcal{K}_P := \{x \in \mathbb{A}_K^{1,\text{an}} \mid (P^n(x)) \text{ is precompact}\}.$$

The filled Julia set of  $P$  is always nonempty, since it contains the classical periodic points of  $P$ .

We define the *Julia set* of  $P$ , denoted by  $\mathcal{J}_P$ , as the boundary of the filled Julia set of  $P$ , that is,  $\mathcal{J}_P = \partial \mathcal{K}_P$ . An equivalent definition, which will be useful, is the following: a point  $x \in \mathbb{A}_K^{1,\text{an}}$  belongs to  $\mathcal{J}_P$  if for every open neighborhood  $V$  of  $x$ , we have that

$$\mathbb{A}_K^{1,\text{an}} \setminus \bigcup_{n \geq 0} P^n(V),$$

has at most one element.

The Julia set is a compact, totally invariant (i.e.,  $P(\mathcal{J}_P) = \mathcal{J}_P = P^{-1}(\mathcal{J}_P)$ ) and nonempty set. Moreover, for all  $n \in \mathbb{N}$  we have  $\mathcal{J}_P = \mathcal{J}_{P^n}$ . Furthermore, it can be characterized as the smallest compact set totally invariant by the action of  $P$  and not containing any classical exceptional points (a classical point is exceptional if its grand orbit is finite, the classical exceptional set contains at most 2 elements, see [13]).

The *Fatou set* of  $P$ , denoted by  $\mathcal{F}_P$ , is defined as the complement of the Julia set of  $P$ . This is a nonempty open set. We say that  $\mathbb{A}_K^{1,\text{an}} \setminus \mathcal{K}_P$  is the *basin of attraction of  $\infty$* . Note that the basin of attraction of  $\infty$  is a convex set, and therefore a connected set. Moreover, it is a Fatou component.

The *classical filled Julia set* of  $P$ , denoted by  $\mathcal{K}_P^I$ , is defined as  $\mathcal{K}_P \cap K$ . We define the classical Julia set of  $P$  as  $\mathcal{J}_P^I := \mathcal{J}_P \cap K$ . The *classical Fatou set*  $\mathcal{F}_P^I$  is the intersection of  $\mathcal{J}_P$  and  $K$ . These definitions, of classical Fatou and Julia sets, agree with the ones given by Hsia [15, 16].

Consider  $x$  in  $\mathbb{A}_K^{1,\text{an}}$  a periodic element of period  $q$ . In the case that  $x$  belongs to  $K$ , we say that  $x$  is *attracting*, *neutral* or *repelling* according  $|(P^q)'(x)| < 1$ ,  $|(P^q)'(x)| = 1$  or  $|(P^q)'(x)| > 1$ , respectively. If  $x$  belong to  $\mathbb{A}_K^{1,\text{an}} \setminus K$  we say that  $x$  is *neutral* or *repelling* if  $\deg_x(P^q) = 1$  or  $\deg_x(P^q) \geq 2$ .

A periodic point  $x \in \mathbb{A}_K^{1,\text{an}}$  of  $P$  belongs to the Julia set of  $P$  if and only if it is a repelling periodic point, see [19, 24].

We will use the following proposition which is proved in Section 5 of [21].

**PROPOSITION 2.12.** — *Let  $P \in K[z]$  be a polynomial of degree  $\geq 2$  and let  $x$  be in the Julia set of  $P$ . If  $x$  is a periodic critical element then  $x$  is a type II point.*

**DEFINITION 2.13.** — We say that a polynomial  $P \in K[z]$  of degree  $\geq 2$  is *simple* if there exists a fixed point  $x \in \mathbb{H}_K$  with  $\deg_x(P) = \deg(P)$ .

The simplest Julia set consists of a unique type II point in  $\mathbb{H}_K$  which is fixed under  $P$ . In fact, the polynomials with a unique type II point as Julia set are precisely the simple polynomials. Moreover, a tame polynomial  $P$  is simple, if and only if all the classical critical points of  $P$  belong to  $\mathcal{K}_P^I$  (see Corollary 2.11 in [17], the proof of that corollary is valid for tame polynomials).

From Subsection 2.6 we have that the critical set of  $P$  always contains infinitely many elements. Nevertheless, if  $P$  is a tame polynomial, there are only finitely many critical elements of  $P$  contained in its Julia set, that is one of the important properties of tame polynomials.

**PROPOSITION 2.14.** — *Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $d \geq 2$ . Then  $\mathcal{J}_P$  contains at most  $d - 2$  critical elements of  $P$ , counted with multiplicity, where the multiplicity of  $x$  is  $\deg_x(P) - 1$*

In order to give the proof of the previous proposition we need the following lemma.

**LEMMA 2.15.** — *Let  $P \in K[z]$  be a polynomial of degree  $\geq 2$  and consider  $x \in \mathcal{J}_P$ . Then  $A(x) = \{y \in \mathbb{A}_K^{1,\text{an}} \mid y \prec x\} \subseteq \mathcal{J}_P \cap \mathcal{K}_P$  and  $]x, \infty[$  is contained in the basin of  $\infty$ .*



*Proof.* — If  $x$  is a classical or a singular point then  $\emptyset = A(x) \subseteq \mathcal{F}_P$ . Suppose that  $x = x_B \in \mathcal{J}_P \cap \mathbb{H}_K$  is a nonsingular point and consider  $y \in A(x)$ . Then, for an open ball  $\mathcal{B}$  such that  $y \in \mathcal{B} \subseteq \overline{\mathcal{B}}$  and given  $\mathcal{D}$  a ball of the Berkovich line such that  $\mathcal{J}_P \subseteq \mathcal{D}$  we have that

$$P^n(\mathcal{B}) \subseteq P^n(\overline{\mathcal{B}}) \subseteq \mathcal{D}$$

for all  $n \geq 1$ . That is  $y$  belongs to  $\mathcal{J}_P \cap \mathcal{K}_P$ . Therefore  $A(x) \subseteq \mathcal{F}_P$ .

If there exists  $y \in \mathcal{J}_P$  such that  $x \prec y$ , then  $x \in A(y) \subseteq \mathcal{F}_P$ , which is impossible. It follows that  $]x, \infty[ \subseteq \mathcal{F}_P$ .  $\square$

*Proof of Proposition 2.14.* — Recall that the critical points of  $P$  belong to

$$\bigcup_{w \in \text{Crit}^I(P)} [w, \infty[.$$

For each critical element  $c \in \mathcal{J}_P$ , we can choose  $w_c \in \text{Crit}^I(P)$  such that  $c$  belongs to  $[w_c, \infty[$ . In view of Lemma 2.15 we have that  $[w_c, c[$  and  $]c, \infty[$  are contained in the Fatou set of  $P$ . Then, the map  $c \mapsto w_c$  is injective. It follows that  $\mathcal{J}_P$  contains at most  $d - 1$  critical elements, since  $\text{Crit}^I(P)$  contains at most  $d - 1$  elements.

Seeking a contradiction suppose that  $\mathcal{J}_P \cap \text{Crit}(P)$  contains  $d - 1$  elements, it follows that  $\text{Crit}^I(P)$  is a subset of  $\mathcal{K}_P^I$ . Following Corollary 2.11 in [17] we have that  $P$  is a simple polynomial and the Proposition follows.  $\square$

The following proposition shows that the existence of wandering Fatou components is equivalent with the existence of nonpreperiodic points in  $\mathcal{J}_P \cap \mathbb{H}_K$ .

**PROPOSITION 2.16.** — *Let  $P \in K[z]$  be a nonsimple polynomial of degree  $\geq 2$ . There exists a wandering component of  $\mathcal{F}_P$  which is not in the basin of a periodic orbit if and only if there exists a nonpreperiodic point of type II or III in  $\mathcal{J}_P$ .*

*Proof.* — First note that the Fatou components different from the basin of  $\infty$  are open balls of the Berkovich affine line.

We proceed by contradiction. Let  $\mathcal{B}$  be a wandering Fatou component which is not in the basin of a periodic point and let  $x = \partial\mathcal{B}$ . Since  $x$  belongs to  $\mathcal{J}_P$ , passing to an iterate if necessary, we can suppose that  $P(x) = x$ .

Let  $\mathcal{V}$  be a neighborhood of  $x$ . By the description of the neighborhoods of  $x$  we have that the set  $\{\mathcal{D} \in T_x \mid \mathcal{D} \not\subseteq \mathcal{V}\}$  contains finitely many elements. On the other hand  $\{n \in \mathbb{N} \mid P^n(\mathcal{B}) \in \mathcal{V}\}$  contains infinitely many elements, this implies that  $\mathcal{B}$  belongs to the basin of attraction of the orbit of  $x$ , which is a contradiction.

Conversely, if  $x \in \mathcal{J}_P \cap \mathbb{H}_K$  is a nonsingular point which is nonpreperiodic, we have that for each  $a \in K$  with  $a \prec x$  the open ball  $\mathcal{B}(a, \text{diam}(x)) \subseteq$

$\mathcal{J}_P \cap \mathcal{K}_P$  is a wandering Fatou component which is not in the basin of a periodic orbit.  $\square$

For more results about Julia and Fatou set for rational maps see [13, 21].

**2.8. Measure on the Berkovich affine line.** — Given a polynomial  $P \in K[z]$  of degree  $\geq 2$ , Favre and Rivera [13] construct an ergodic probability measure, defined on the Borel sets of  $\mathbb{A}_K^{1,\text{an}}$ . See also [1, 2, 10, 12]. This measure is denoted by  $\rho_P$  and called *the equilibrium measure of  $P$* . The measure  $\rho_P$  is characterized by the followings properties:

- $\rho_P$  does not charge classical points.
- If  $\mathcal{B}$  is a ball of  $\mathbb{A}_K^{1,\text{an}}$  then

$$\rho_P(\mathcal{B}) = \frac{\deg_{\mathcal{B}}(P)}{\deg(P)} \rho_P(P(\mathcal{B})).$$

The equilibrium measure of  $P$  is supported on the Julia set of  $P$  and is an atom free measure for all  $P$  which are not simple. Moreover, for any open set  $V$  such that  $\mathcal{J}_P \cap V \neq \emptyset$  we have that  $\rho_P(V) > 0$ .

### 3. Dynamical Points.

Consider a nonsimple polynomial  $P \in K[z]$  of degree  $\geq 2$ . To establish properties about  $\mathcal{J}_P$  we study the action of  $P$  in the convex hull of its Julia set, that is

$$\text{conv}(\mathcal{J}_P) = \bigcup_{x,y \in \mathcal{J}_P} [x,y].$$

For each  $x \in \mathcal{J}_P$  we will construct a decreasing sequence  $(L_n(x)) \subseteq \text{conv}(\mathcal{J}_P)$  of type II points having  $x$  as its limit. This sequence is dynamically defined, therefore every dynamical property of  $x$  can be obtained from the properties of the sequence  $(L_n(x))$ . Compare with the *lemniscates* in [7] and the *dynamical ends* in [17].

At the end of the section we will introduce the concept of *good starting level*, which will be useful to compare the distances between the points in  $(L_n(x))$  and  $(L_n(P(x)))$ .

From Proposition 6.7 in [19] we know that

$$r_P = \max\{|z_0 - z_1| \mid z_0, z_1 \in \mathcal{K}_P^{\text{I}}\}$$

belongs to the value group of  $K$ . Thus, the closed ball  $D_0 = B_{r_P}^+(z) \subseteq K$ , where  $z$  is any periodic point of  $P$ , is the smallest ball of  $K$  containing  $\mathcal{K}_P^{\text{I}}$ , and therefore  $\overline{D_0}$  is the smallest ball of  $\mathbb{A}_K^{1,\text{an}}$  containing  $\mathcal{K}_P$ . In particular we have that the Julia set of  $P$  is contained in  $\overline{D_0}$ .

LEMMA 3.1. — *Let  $P \in K[z]$  be a polynomial of degree  $\geq 2$ . Then  $\overline{D_0}$  is the smallest ball of  $\mathbb{A}_K^{1,\text{an}}$  containing  $\mathcal{J}_P$ .*

*Proof.* — We proceed by contradiction. Suppose that there exists a ball  $\mathcal{B}$  such that  $\mathcal{J}_P \subseteq \mathcal{B} \subsetneq \overline{D_0}$ . From Lemma 2.15 we have that

$$\mathcal{K}_P = \bigsqcup_{x \in \mathcal{J}_P} \{y \in \mathbb{A}_K^{1,\text{an}} \mid y \preccurlyeq x\}.$$

Hence,  $\mathcal{K}_P^{\text{I}} = \mathcal{K}_P \cap K \subseteq \mathcal{B} \cap K$ , which contradicts that  $\text{diam}(\mathcal{K}_P^{\text{I}}) = \text{diam}(D_0)$ .  $\square$

DEFINITION 3.2. — The *level 0 dynamical point of  $P$* , denoted by  $L_0$ , is defined as the point associated to the ball  $D_0$ , that is,  $L_0 := x_{D_0} = \partial \overline{D_0}$ .

DEFINITION 3.3. — For each  $n \in \mathbb{N}$  the *level  $n$  dynamical set of  $P$*  is defined as  $\mathcal{L}_n := P^{-n}(L_0)$ . We say that an element  $L_n$  of  $\mathcal{L}_n$  is a *level  $n$  dynamical point of  $P$* .

From the definition we have that  $L_0$  is a type II point and that  $x \preccurlyeq L_0$  for all  $x \in \mathcal{J}_P$ . Moreover,  $L_0$  is the smallest element in  $\mathbb{A}_K^{1,\text{an}}$  with this property.

PROPOSITION 3.4. — *Let  $P \in K[z]$  be a nonsimple polynomial of degree  $d \geq 2$ . Then the following statements hold:*

1.  $\{L_0\} = P^{-1}(P(L_0))$
2.  $L_0 \prec P(L_0)$ .
3.  $\text{diam}(P^n(L_0)) \longrightarrow +\infty$  as  $n \rightarrow +\infty$ .
4.  $P^{-1}(L_0)$  has at least two elements. Moreover, the elements of  $P^{-1}(L_0)$  are pairwise incomparable with respect to  $\preccurlyeq$ .
5.  $P^{-1}(L_0)$  contains points in at least two directions in  $T_{L_0}$ .

*Proof.* — To prove the first statement, note that, the Julia set of  $P$  is forward invariant, therefore

$$\mathcal{J}_P = P(\mathcal{J}_P) \subseteq P(\overline{D_0}) = \overline{P(D_0)}.$$

By definition of  $L_0$ , we have  $L_0 \preccurlyeq P(L_0)$ . Now seeking a contradiction, suppose that there exists  $x_B$  in  $P^{-1}(P(L_0))$  different than  $L_0$ . Since  $\mathcal{J}_P \subseteq P(\overline{D_0})$  we have that  $\overline{B} \cap \mathcal{J}_P \neq \emptyset$ . Hence,  $x_B$  is comparable to  $L_0$ . If  $x_B \prec L_0$  it follows that

$$P(L_0) = P(x_B) \prec P(L_0),$$

which is impossible. Analogous assertions hold if we suppose that  $L_0 \prec x_B$ . Therefore  $x_B = L_0$ , which is a contradiction. Hence, we have proved that  $\{L_0\} = P^{-1}(P(L_0))$ .

To prove (2) note that  $L_0 \preccurlyeq P(L_0)$  and  $\{L_0\} = P^{-1}(P(L_0))$ . Using Remark 2.2 we obtain that  $\deg_{L_0}(P) = d$ , because  $L_0$  is the unique preimage of  $P(L_0)$ . Hence  $L_0 \prec P(L_0)$ , since  $P$  is not a simple polynomial.

To show (3), let  $0 < a = d_{\mathbb{H}}(L_0, P(L_0))$ . We proved that  $\deg_{L_0}(P) = d$ , therefore  $\deg_{P^n(L_0)}(P) = d$  for all  $n \in \mathbb{N}$ . Following Lemma 2.5, we obtain that

$$d_{\mathbb{H}}(P^{n-1}(L_0), P^n(L_0)) = a \cdot d^{n-1}.$$

Hence,  $d_{\mathbb{H}}(L_0, P^n(L_0)) > a \cdot d^{n-1}$ , because all the iterates of  $L_0$  belong to the segment  $[L_0, \infty[$ . Using the definition of the hyperbolic distance we obtain

$$\text{diam}(P^n(L_0)) > \text{diam}(L_0) \exp(a \cdot d^{n-1}).$$

Then,  $\text{diam}(P^n(L_0)) \rightarrow +\infty$ .

To prove the first statement in (4) we proceed by contradiction. Suppose that  $P^{-1}(L_0)$  has exactly one element  $x_B$ . It follows that the Julia set of  $P$  is contained in  $\overline{B}$ , since  $\mathcal{J}_P$  is totally invariant. By the definition of  $L_0$  we have that  $L_0 \preccurlyeq x_B$ . By monotonicity of  $P$ , we obtain that  $P(L_0) \preccurlyeq P(x_B) = L_0$ , which is a contradiction with (2). That is,  $P^{-1}(L_0)$  contains at least two element.

Suppose now that there exist  $x_1, x_2 \in P^{-1}(L_0)$  with  $x_1 \prec x_2$ . It follows that

$$L_0 = P(x_1) \prec P(x_2) = L_0,$$

which is impossible. Therefore the elements in  $P^{-1}(L_0)$  are pairwise incomparable.

To show (5) note that if  $P^{-1}(L_0) = \{y_1, \dots, y_m\}$  we have that  $x \preccurlyeq y_1 \vee \dots \vee y_m$  for all  $x \in \mathcal{J}_P$ , since,

$$\mathcal{J}_P = \bigcup_{1 \leq j \leq m} \{x \in \mathcal{J}_P \mid x \preccurlyeq y_j\}.$$

In particular, if  $P^{-1}(L_0)$  is contained in a direction  $\mathcal{D} \in T_{L_0}$  we have that

$$y_1 \vee \dots \vee y_m \prec L_0,$$

which contradicts the definition of  $L_0$ . Now (5) follows.  $\square$

From (1), (2), (4) of Proposition 3.4 we have that each level  $n \geq 1$  dynamical point is strictly smaller than exactly one level  $n - 1$  dynamical point.

DEFINITION 3.5. — A *dynamical sequence* is a decreasing sequence  $(L_n)_{n \geq 0}$  of dynamical points such that  $L_0$  is the level 0 point and  $L_n \in \mathcal{L}_n$  for all  $n \geq 1$ .

PROPOSITION 3.6. — Let  $P \in K[z]$  be a polynomial of degree  $d \geq 2$ . Then

$$\mathcal{J}_P = \{\lim L_n \mid (L_n) \text{ is a dynamical sequence of } P\}.$$

*Proof.* — If  $P$  is a simple polynomial then  $L_n = L_0$  for all  $n \in \mathbb{N}$  and  $\mathcal{J}_P = \{L_0\}$ .

When  $P$  is not simple, let  $(L_n)$  be a dynamical sequence of  $P$  and  $x = \lim L_n$ . For all  $n \geq 0$  we have that  $x \prec L_n$ , therefore  $P^n(x) \prec L_0$ . Hence  $x$  belongs to  $\mathcal{R}_P$ . The dynamical points do not belong to  $\mathcal{R}_P$  (see Proposition 3.4 (3)), hence  $x \in \mathcal{J}_P = \partial \mathcal{R}_P$ .

Let  $x$  be in the Julia set of  $P$  and let  $n \in \mathbb{N}$ . From the definition of  $L_0$  and Proposition 3.4 (2) we have that  $L_0$  belongs to  $]P^n(x), P^n(L_0[$ . Hence, the intersection  $[x, L_0] \cap \mathcal{L}_n$  contains exactly one element, denoted by  $L_n(x)$ .

Suppose that  $\lim L_n(x) = y \neq x$ . It follows that  $x \prec y$ , because  $x \preceq L_n(x)$  for all  $n \geq 0$ . From the above we conclude that  $y$  is a Julia point. Following Lemma 2.15 we have that  $x \in A(y) \subseteq \mathcal{J}_P$ , which is impossible. Hence we have  $\lim L_n(x) = x$ .  $\square$

DEFINITION 3.7. — We will refer to the sequence  $(L_n(x))_{n \geq 0}$  constructed in the proof of Proposition 3.6 as *the dynamical sequence of  $x$* .

The dynamical sequences of  $x$  and  $P(x)$  are related according the following identity

$$L_n(P(x)) = P(L_{n+1}(x)),$$

for all  $n \geq 1$ .

As an immediate consequence of Proposition 3.6 we have the following Corollary.

COROLLARY 3.8. — Let  $P \in K[z]$  be a polynomial of degree  $\geq 2$ . Denote by  $\mathcal{D}(P)$  the set of dynamical sequences of  $P$  endowed with the topology induced by the following distance

$$d((L_n), (L'_n)) = \frac{1}{m}$$

where  $m = \min\{j \geq 0 \mid L_j \neq L'_j\}$  and  $d((L_n), (L_n)) = 0$ .

Let  $\hat{P}: \mathcal{D}(P) \rightarrow \mathcal{D}(P)$  be the map defined by  $\hat{P}((L_n)) = (P(L_{n+1}))$ . Then  $P: \mathcal{J}_P \rightarrow \mathcal{J}_P$  is topologically conjugate to  $\hat{P}: \mathcal{D}(P) \rightarrow \mathcal{D}(P)$ . The topological conjugacy is given by  $\sigma: \mathcal{J}_P \rightarrow \mathcal{D}(P)$  where  $\sigma(x) = (L_n(x))$ .

To distinguish whether a Julia point  $x$  is classical (i.e  $x \in K$ ) or not (i.e  $x \in \mathbb{H}_K$ ) we consider the hyperbolic distance between the level 0 dynamical point  $L_0$  and  $x$ . In view of Lemma 2.4 we have that

$$\begin{aligned} d_{\mathbb{H}}(L_n(x), L_0) &= \log(\text{diam}(L_0)) - \log(\text{diam}(L_n(x))) \\ &= \sum_{j=0}^{n-1} d_{\mathbb{H}}(L_j(x), L_{j+1}(x)). \end{aligned}$$

Hence  $x$  is a classical point if and only if the sum of the right hand side of the expression above is divergent. The convergence of the sum does not allow us to decide whether the point  $x$  is of type II, III or IV. It only determines if  $\log(\text{diam}(x))$  is a positive rational or irrational number.

The following corollaries are applications of Proposition 2.14 to dynamical sequences.

**COROLLARY 3.9.** — *Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$ . Then there exists  $\mathcal{M}(P) \in \mathbb{N}$ , only depending on  $P$ , such that*

1. *if  $n \geq \mathcal{M}(P)$  and  $L_n$  is a level  $n$  point which is critical, then  $L_n = L_n(c)$  for some  $c \in \text{Crit}(P) \cap \mathcal{J}_P$ .*
2.  *$\deg_{L_n(c)}(P) = \deg_{L_{\mathcal{M}(P)}(c)}(P)$  for all  $n \geq \mathcal{M}(P)$  and all  $c \in \text{Crit}(P) \cap \mathcal{J}_P$ .*

*Proof.* — Note that  $w \in \text{Crit}^I(P)$  belongs to  $\mathcal{X}_P$  if and only if  $[w, \infty[ \cap \mathcal{L}_n \neq \emptyset$  for all  $n \geq 1$ . It follows that there exists a smallest integer  $M_1$  such that if  $w \in \text{Crit}^I(P)$  and  $[w, \infty[ \cap \mathcal{L}_{M_1} \neq \emptyset$ , then  $w \in \mathcal{X}_P$ .

From the definition of the local degree, we have that for each  $c \in \text{Crit}(P) \cap \mathcal{J}_P$  there exist a smallest integer  $M_c$  such that  $\deg_c(P) = \deg_{L_{M_c}(c)}(P)$ . Consider  $M_2 = \max\{M_c \mid c \in \text{Crit}(P) \cap \mathcal{J}_P\}$ , we can consider  $\max$  instead  $\sup$  by Proposition 2.14. Then  $\mathcal{M}(P) = \max\{M_1, M_2\}$  only depends on  $P$  and is the smallest integer satisfying (1) and (2).  $\square$

To state and prove the following corollary we need two definitions.

**DEFINITION 3.10.** — Let  $P \in K[z]$  be a polynomial of degree  $\geq 2$  and consider  $x \in \mathbb{A}_K^{1,\text{an}}$ . The *forward orbit* of  $x$  is the set

$$\mathcal{O}^+(x) := \{P^j(x) \mid j \in \mathbb{N}\}.$$

**DEFINITION 3.11.** — Let  $P \in K[z]$  be a polynomial of degree  $\geq 2$  and consider  $x \in \mathbb{A}_K^{1,\text{an}}$ . The  *$\omega$ -limit* of  $x$  is the set

$$\omega(x) := \left\{ y \in \mathbb{A}_K^{1,\text{an}} \mid \text{there exists } (n_j) \subseteq \mathbb{N}, n_j < n_{j+1} \text{ and } \lim_{j \rightarrow +\infty} P^{n_j}(x) = y \right\}$$

**COROLLARY 3.12.** — *Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$  and consider  $x$  in the Julia set of  $P$ . Then there exists an integer  $N \geq \mathcal{M}(P)$ , depending on  $x$ , such that*

1.  *$\deg_x(P) = \deg_{L_n(x)}(P)$  for all  $n \geq N$ .*
2. *if  $n \geq N, j \geq 1$  and  $L_n(P^j(x))$  is critical, then  $L_n(P^j(x)) = L_n(c)$  for some  $c \in \mathcal{O}^+(x) \cap \text{Crit}(P)$ .*

*Proof.* — From the definition of local degree, there exists  $N_1 \geq \mathcal{M}(P)$  such that

$$\deg_x(P) = \deg_{L_{N_1}(x)}(P).$$

A critical element  $c$  belongs to  $\omega(x) \setminus \mathcal{O}^+(x)$  if and only if there exists a increasing sequence  $(n_j)$  of integers such that the dynamical sequence of  $P^{n_j}(x)$  coincides with the dynamical sequence of  $c$  at least up to the level  $j$ . If  $c \notin \omega(x) \setminus \mathcal{O}^+(x)$  there exists an integer  $N_c$  such that  $L_{N_c}(P^j(x)) \neq L_{N_c}(c)$  for all  $j \geq 1$ . By Proposition 2.14 we can consider

$$N_2 = \max\{N_c \mid c \in (\text{Crit}(P) \cap \mathcal{J}_P) \setminus \omega(x)\}.$$

It follows that every  $N \geq \max\{N_1, N_2\}$  satisfies (1) and (2).  $\square$

DEFINITION 3.13. — Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$  and consider  $x \in \mathcal{J}_P$ . We define the *good starting level* of  $x$ , denoted by  $\mathcal{N}(x)$ , as the smallest integer satisfying the two properties in Corollary 3.12.

In general, for  $x \in \mathcal{J}_P$  we want to estimate the distance  $d_{\mathbb{H}}(x, L_0)$ . But in practice we estimate the distance  $d_{\mathbb{H}}(x, L_{\mathcal{N}(x)})$ , where  $\mathcal{N}$  is the good starting level of  $x$ , since it is easier to control.

PROPOSITION 3.14. — Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$  and consider  $x$  in the Julia set of  $P$ . If  $x$  belongs to  $\mathbb{H}_K$ , then the  $\omega$ -limit of  $x$  contains at least one critical point of  $P$

*Proof.* — Consider  $x \in \mathcal{J}_P$ . Suppose that  $\omega(x) \cap \text{Crit}(P)$  is empty, it is enough to show that  $x \in K$ . Passing to an iterate if necessary we can suppose that  $x$  is noncritical and has no critical iterates. Since  $\omega(x) \cap \text{Crit}(P) = \emptyset$  and  $x$  has no critical iterates, we have that  $L_n(P^j(x))$  is noncritical for all  $n \geq \mathcal{N}$  and all  $j \geq 1$ , where  $\mathcal{N} = \mathcal{N}(x)$  is the good starting level of  $x$ . Equivalently,

$$L_{n-j}(P^j(x)) = P^j(L_n(x))$$

is noncritical provided that  $n - j \geq \mathcal{N}$  (see Definition 3.13).

Since, the dynamical level sets  $\mathcal{L}_n$  are finite, for all  $n \geq 1$ , there exist finitely many intervals of the form  $[L_{\mathcal{N}+1}(y), L_{\mathcal{N}}(y)]$  with  $y \in \mathcal{J}_P$ . Hence, there exist a point  $y_0$  in  $\mathcal{J}_P$  and a strictly increasing sequence  $(n_j)$  of dynamical levels larger than  $\mathcal{N}$  such that

$$P^{n_j - \mathcal{N}}([L_{n_j+1}(x), L_{n_j}(x)]) = [L_{\mathcal{N}+1}(y_0), L_{\mathcal{N}}(y_0)].$$

We have that

$$\deg_{L_{n_j}(x)}(P^{n_j - \mathcal{N}}) = \prod_{i=0}^{n_j - \mathcal{N} - 1} \deg_{L_{n_j-i}(P^i(x))}(P),$$

since the levels  $n_j$  are larger than  $\mathcal{N}$ , we obtain that  $\deg_{L_{n_j-i}(P^i(x))}(P) = 1$  for all  $0 \leq i \leq n_j - \mathcal{N} - 1$ . Therefore  $\deg_{L_{n_j}(x)}(P^{n_j-\mathcal{N}}) = 1$ . Applying Lemma 2.5 it follows that

$$\begin{aligned} d_{\mathbb{H}}(x, L_{\mathcal{N}}(x)) &= \lim_{m \rightarrow +\infty} \sum_{j=\mathcal{N}}^m d_{\mathbb{H}}(L_{j+1}(x), L_j(x)) \\ &\geq \lim_{m \rightarrow +\infty} \sum_{j=0}^m d_{\mathbb{H}}(L_{n_j+1}(x), L_{n_j}(x)) \\ &= \lim_{m \rightarrow +\infty} \sum_{j=0}^m d_{\mathbb{H}}(L_{\mathcal{N}+1}(y_0), L_{\mathcal{N}}(y_0)) \\ &= +\infty. \end{aligned}$$

Thus,  $x$  belongs to  $K$ . □

#### 4. The Geometric Sequence

Let  $P \in K[z]$  be a tame polynomial of degree  $d \geq 2$ , consider  $x \in \mathcal{J}_P$  and let  $L_n = L_n(x)$  for all  $n \geq 1$ .

In order to compute the hyperbolic distance between  $x$  and  $L_0$ , we want to estimate the distance between two consecutive levels of the dynamical sequence of  $x$ . However, Lemma 2.5 does not always applies to relate  $d_{\mathbb{H}}(L_{n+1}, L_n)$  to the distance  $d_{\mathbb{H}}(P(L_{n+1}), P(L_n))$  because the local degree of  $P$  is not necessarily constant in the segment  $]L_{n+1}, L_n[$ .

In view of this, to have a better control of the distance between consecutive dynamical levels, we need a finer subdivision of the segment joining  $x$  to the level 0 dynamical point. This subdivision, that will be called *the geometric sequence of  $x$* , is motivated by the following propositions about *branch points* of  $\text{conv}(\mathcal{J}_P)$ . First we need a formal definition.

**DEFINITION 4.1.** — We say that a type II point  $x \in \text{conv}(\mathcal{J}_P)$  is a *branch point of  $\text{conv}(\mathcal{J}_P)$*  if  $\mathcal{J}_P$  intersects at least two directions in  $T_x$ .

**PROPOSITION 4.2.** — *Let  $P \in K[z]$  be a nonsimple polynomial of degree  $\geq 2$ . Then  $L_0$  is a branch point of  $\text{conv}(\mathcal{J}_P)$ .*

*Proof.* — From Proposition 3.4 (2) it follows that the elements of  $P^{-1}(L_0)$  are strictly smaller than  $L_0$ . Following Proposition 3.4 (5) there exist  $x_1, x_2 \in P^{-1}(L_0)$  such that  $x_1, x_2$  belong to different directions in  $T_{L_0}$ . Hence,  $L_0 = x_1 \vee x_2$  is a branch point of  $\text{conv}(\mathcal{J}_P)$ . □



PROPOSITION 4.3. — *Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$  and consider  $x_B$  a branch point of  $\text{conv}(\mathcal{J}_P)$ . Then there exists a direction  $\mathcal{D} \in T_{x_B}$  and a critical point  $w$  of  $P$  not contained in  $\mathcal{K}_P^I$  such that  $\mathcal{D} \cap \mathcal{J}_P = \emptyset$  and  $\text{GO}(w) \cap \mathcal{D} \neq \emptyset$ .*

*Proof.* — In view of Proposition 3.6 there exists  $n$  such that  $L_n \prec x_B \preccurlyeq L_{n-1}$  for some dynamical points  $L_n$  and  $L_{n-1}$ , it follows that  $P^n(x_B)$  belongs to the segment  $]L_0, P(L_0)]$ .

Since the dynamical points of a given dynamical level are finitely many, we have that there exist only finitely many directions in  $T_{x_B}$  having nonempty intersection with  $\mathcal{J}_P$ . Let  $\mathcal{D}_1, \dots, \mathcal{D}_m$  be such directions.

We have that  $P^n(\mathcal{D}_j) = \mathcal{D}_0$  for all  $1 \leq j \leq m$ , where  $\mathcal{D}_0$  is the direction at  $P^n(x_B)$  containing  $L_0$ . From the Riemann-Hurwitz formula (see Remark 2.7), we obtain that

$$\begin{aligned} \deg_{x_B}(P^n) &= 1 + \sum_{z \in B \cap \text{Crit}^I(P^n)} (\deg_z(P^n) - 1) \\ &= \deg(T_{x_B} P^n) \\ &= \deg_{\mathcal{D}_1}(P^n) + \dots + \deg_{\mathcal{D}_m}(P^n) \\ &= m + \sum_{z \in (\mathcal{D}_1 \sqcup \dots \sqcup \mathcal{D}_m) \cap \text{Crit}^I(P^n)} (\deg_z(P^n) - 1). \end{aligned}$$

Since  $2 \leq m$ , it follows that

$$\sum_{z \in (\mathcal{D}_1 \sqcup \dots \sqcup \mathcal{D}_m) \cap \text{Crit}^I(P^n)} (\deg_z(P^n) - 1) < \sum_{z \in B \cap \text{Crit}^I(P^n)} (\deg_z(P^n) - 1).$$

That is, there exists direction  $\mathcal{D} \in T_{x_B} \setminus (\mathcal{D}_1 \sqcup \dots \sqcup \mathcal{D}_m)$  and critical point  $u$  of  $P^n$  contained in  $\mathcal{D}$ . Since  $\mathcal{D} \cap \mathcal{J}_P = \emptyset$  and  $u \in \text{GO}(w)$  for some  $w \in \text{Crit}^I(P) \setminus \mathcal{K}_P^I$ , the proposition follows.  $\square$

REMARK 4.4. — Taking  $x_B = L_0$ , the previous proof shows that if  $P$  is tame and nonsimple, then there exists  $w \in \text{Crit}^I(P) \setminus \mathcal{K}_P^I$ .

Let  $P \in K[z]$  be a tame polynomial. Let  $\{w_1, \dots, w_{q'}\}$  be the set of classical critical points of  $P$  which are not contained in  $\mathcal{K}_P^I$ . Note that  $1 \leq q' \leq d - 1$ . For each  $1 \leq j \leq q'$ , let  $x_{w_j}$  be defined by

$$(\text{conv}(\mathcal{J}_P) \cup [L_0, \infty]) \cap [w_j, \infty[ = [x_{w_j}, \infty[.$$

By Proposition 4.3 all the branch points of  $\text{conv}(\mathcal{J}_P)$  are contained in the grand orbit of the set  $\{x_{w_1}, \dots, x_{w_{q'}}\}$ .

DEFINITION 4.5. — Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $d \geq 2$ . The geometric sequence of  $x \in \mathcal{J}_P$ , denoted by  $(G_n(x))_{n \geq 0}$ , is the decreasing sequence enumerating the elements in

$$[x, L_0] \cap \text{GO}(\{x_{w_1}, \dots, x_{w_{q'}}\}),$$

where GO denotes the grand orbit.

Note that  $(G_n)$  is a constant sequence when  $P$  is simple.

REMARK 4.6. — Consider  $v_{q-1}, \dots, v_0 \in ]L_0, P(L_0)]$  satisfying

$$\text{GO}(\{x_{w_1}, \dots, x_{w_{q'}}\}) \cap ]L_0, P(L_0)] = \{v_{q-1}, v_{q-2}, \dots, v_0\},$$

and  $v_{q-1} \prec v_{q-2} \prec \dots \prec v_0 = P(L_0)$ . Note that  $1 \leq q \leq d$  and that every element of

$$\text{GO}(\{x_{w_1}, \dots, x_{w_{q'}}\}) \cap \text{conv}(\mathcal{J}_P)$$

is eventually mapped to  $v_j$ , for some  $j$ .

Consider  $n \geq 0$  and put  $n = q \cdot \lfloor n/q \rfloor + j$  for some  $0 \leq j \leq q-1$ , where  $\lfloor \cdot \rfloor$  denotes the floor function, that is  $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$  is defined by

$$\lfloor a \rfloor := \sup\{n \in \mathbb{Z} \mid n \leq a\}.$$

It follows that

$$P^{\lfloor n/q \rfloor + 1}(G_n(x)) = v_j.$$

Moreover

$$P^{\lfloor n/q \rfloor + 1}([G_{n+1}(x), G_n(x)]) = I_j,$$

where  $I_j = [v_{j+1}, v_j]$  for  $0 \leq j \leq q-2$  and  $I_{q-1} = [L_0, v_{q-1}]$ .

DEFINITION 4.7. — We say that  $v_{q-1}, \dots, v_0$  are the *generators of the geometric sequences of  $P$* .

REMARK 4.8. — If  $q$  is the number of generators of the geometric sequences of  $P$ , and if  $q'$  is the number of classical critical points of  $P$  that escape to infinity, then  $q \leq q'$ . Indeed, first note that for tame polynomials, we have  $\text{Crit}^1(P) \subseteq D_0$ . So  $x_{w_j} \in \overline{D_0}$ . Also, the segment  $]P^i(L_0), P^{i+1}(L_0)]$  maps  $d$ -to-1 onto  $]P^{i+1}(L_0), P^{i+2}(L_0)]$  for each  $i \geq 0$ , which shows that for each  $j = 1, \dots, q'$ , there is exactly one positive integer  $n$  such that  $P^n(x_{w_j}) \in ]L_0, P(L_0)]$ . Thus we have a well-defined surjection of sets  $\{x_{w_1}, \dots, x_{w_{q'}}\} \longrightarrow \{v_0, \dots, v_{q-1}\}$ , and hence  $q \leq q'$ .

The next lemma states the basic properties of the geometric sequence. The proof is straightforward and we omit it.

PROPOSITION 4.9. — Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$  with  $q$  generators for its geometric sequences and consider  $x \in \mathcal{J}_P$ . Then, the following statements hold:

1.  $L_n(x) = G_{nq}(x)$  for all  $n \geq 0$ .
2.  $\lim_{n \rightarrow +\infty} G_n(x) = x$ .
3. The geometric sequences of  $x$  and  $P(x)$  are related by

$$P(G_n(x)) = G_{n-q}(P(x))$$

for all  $n \geq q$ .

Note that

$$P([G_{n+1}(x), G_n(x)]) = [G_{n-q+1}(P(x)), G_{n-q}(P(x))]$$

for all  $n \geq q+1$ . Moreover, for all  $n \geq 0$  it follows that

$$P^{-1}([G_{n+1}(x), G_n(x)]) = \bigcup_{y \in P^{-1}(x)} [G_{n+q+1}(y), G_{n+q}(y)]$$

The main result in this section is the following proposition. It will allow us to use Lemma 2.5 to relate  $d_{\mathbb{H}}(G_{n+1}, G_n)$  with  $d_{\mathbb{H}}(P(G_{n+1}), P(G_n))$ .

**PROPOSITION 4.10.** — *Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$  and consider  $x \in \mathcal{J}_P$ . Let  $(G_n)$  be the geometric sequence of  $x$ . Then for all  $n \geq 0$  the local degree of  $P$  is constant in the segment  $[G_{n+1}, G_n]$ . In fact*

$$\deg_y(P) = \deg_{G_{n+1}}(P)$$

for all  $y \in [G_{n+1}, G_n]$ .

*Proof.* — Let  $n \geq 0$  and define

$$\mathcal{J}_n = \{y \in [G_{n+1}, G_n] \mid \deg_w(P) = \deg_{G_{n+1}}(P) \text{ for all } w \in [G_{n+1}, y]\}.$$

To show that  $\mathcal{J}_n \neq \emptyset$ , consider  $a \in K$  and  $r > 0$  such that  $B_r^+(a)$  is the ball associated to  $G_{n+1}$ . Consider

$$R = \min\{|a - w| \mid w \in \text{Crit}^I(P) \setminus B_r^+(a)\}.$$

In view of the Riemann-Hurwitz formula, see Remark 2.7, the local degree of  $P$  at the point associated to the ball  $B_{r+\varepsilon}^+(a)$  coincides with  $\deg_{G_{n+1}}(P)$  for all  $0 \leq \varepsilon < R$ . In particular, we obtain that  $\mathcal{J}_n \neq \emptyset$ . Moreover, the previous argument shows that largest element contained in  $\mathcal{J}_n$  is a type II point.

Denote by  $y_B = \max \mathcal{J}_n$ . We will show that  $y_B = G_n$ .

We proceed by contradiction. Suppose that  $y_B \prec G_n$ . It follows that the degree of the map  $P$  in  $\overline{B}$  is larger than the degree of  $P$  in the direction  $\mathcal{D}$  at  $y_B$  that contains  $G_{n+1}$ . In particular, there exists a critical point  $w$  in  $\overline{B} \setminus \mathcal{D}$ .

We have two cases:

If  $w \notin \mathcal{K}_P^I$ , then  $y_B$  belongs to  $\{x_{w_1}, \dots, x_{w_{q'}}\}$  (see Definition 4.5). In particular,  $G_n$  and  $G_{n+1}$  are not consecutive elements of the sequence  $(G_n)$ .

If  $w \in \mathcal{K}_P^I$  we have that  $y_B$  is a branch point of  $\mathcal{J}_P$ , which is a contradiction because  $]G_{n+1}, G_n[$  is branch point free, see Proposition 4.3.

It follows that  $\max \mathcal{J}_n = G_n$ . Therefore  $\deg_y(P) = \deg_{G_{n+1}}(P)$  for all  $y$  in  $]G_{n+1}, G_n[$ .  $\square$

From Lemma 2.5 it follows that for each  $n \geq 0$  the lengths of the segments  $]G_{n+1}, G_n[$  and  $P^{\lfloor n/q \rfloor + 1}([G_{n+1}, G_n]) = I_j$  (see Remark 4.6) are related according the following identity

$$(1) \quad d_{\mathbb{H}}(G_{n+1}, G_n) = [\deg_{G_{n+1}}(P^{\lfloor n/q \rfloor + 1})]^{-1} \cdot |I_j|,$$

where  $|I_j|$  denotes the length of the segment  $I_j$  for all  $0 \leq j \leq q-1$ . Therefore

$$(2) \quad \sum_{n=0}^{+\infty} d_{\mathbb{H}}(G_{n+1}, G_n) = \sum_{n=0}^{+\infty} [\deg_{G_{n+1}}(P^{\lfloor n/q \rfloor + 1})]^{-1} \cdot |I_{j_n}|,$$

where,  $0 \leq j_n \leq q-1$  is such that  $P^{\lfloor n/q \rfloor + 1}([G_{n+1}, G_n]) = I_{j_n}$ .

In order to write the previous sum in a more convenient manner we need to introduce the following notation.

DEFINITION 4.11. — Consider  $x \in \mathcal{J}_P$ . For all  $n \geq 0$  we define, the *dynamical degree of level  $n$  around  $x$*  as

$$d_n(x) := \deg_{G_{n+1}(x)}(P^{\lfloor n/q \rfloor + 1}).$$

DEFINITION 4.12. — The *range of dynamical degrees of  $x$*  is defined by

$$\mathcal{D}(x) := \{d_n(x) \mid n \geq 0\}.$$

The previous definitions are inspired in complex polynomials dynamics (see [8])

Rephrasing equation (1) and the sum (2) in this notation gives the following lemma.

LEMMA 4.13. — Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$  with  $q$  generators for its geometric sequences. Consider  $x \in \mathcal{J}_P$  and denote by  $(G_n)$  the geometric sequence of  $x$ . Then

$$d_{\mathbb{H}}(G_{n+1}, L_0) = \sum_{\ell=0}^n d_{\mathbb{H}}(G_{\ell+1}, G_{\ell}) = \sum_{j=0}^{q-1} \left( |I_j| \sum_{\substack{\ell \equiv j \pmod q \\ 0 \leq \ell \leq n}} [d_{\ell}^{-1}] \right)$$

for all  $n \geq 0$ , where  $|I_j|$  is the length of the interval  $I_j$ ,  $0 \leq j \leq q-1$

The above sum gives us a relation between the sum of the length of the intervals defined by the generators of the geometric sequences of  $P$  and the range of dynamical degrees around  $x$ .

Now we present two direct applications of the geometric sequence. Recall that, for  $a \in \mathbb{N}$  and  $X$  a subset of  $\mathbb{N}$ , the set  $a \cdot X$  is defined by

$$a \cdot X := \{ab \mid b \in X\}.$$

LEMMA 4.14. — *Let  $P \in K[z]$  be a tame polynomial of degree  $d \geq 2$  and let  $x \in \mathcal{J}_P$ . Then the following statements hold:*

1. *If  $\omega(x) \cap \text{Crit}(P) = \emptyset$ , then  $\mathcal{D}(x)$  is finite.*
2. *If  $\omega(x) \cap \text{Crit}(P) \neq \emptyset$ , then the range of dynamical degrees  $\mathcal{D}(x)$  is contained in*

$$\{1, 2, \dots, d^{\mathcal{N}(x)+2}\} \cup \left( \deg_x(P) \cdot \bigcup_{c \in \mathcal{C}(x)} \mathcal{D}(c) \right),$$

where  $\mathcal{C}(x) = \text{Crit}(P) \cap \overline{\mathcal{O}^+(x)}$ .

*Proof.* — Suppose that  $P$  has  $q$  generators for its geometric sequences and let  $\mathcal{N} = \mathcal{N}(x)$  be the good starting level of  $x$  (see Definition 3.13).

Note that if  $0 \leq n \leq q\mathcal{N}$ , then

$$d_n = \deg_{G_{n+1}}(P^{\lfloor n/q \rfloor + 1}) \leq \deg_{G_{n+1}}(P^{\lfloor q\mathcal{N}/q \rfloor + 1}) \leq d^{\mathcal{N}+1}.$$

In order to prove the first statement observe that if  $\omega(x) \cap \text{Crit}(P) = \emptyset$  then  $x$  has at most  $d - 2$  critical images. Consider  $n > q\mathcal{N}$  and let  $\ell \geq 1$  be the smallest integer such that  $(n + 1) - \ell q \leq q\mathcal{N}$ . It follows that,

$$\begin{aligned} d_n &= \deg_{G_{n+1}}(P^{\lfloor n/q \rfloor + 1}) \\ &= \deg_{G_{n+1}}(P^\ell) \cdot \deg_{P^\ell(G_{n+1})}(P^{\lfloor n/q \rfloor + 1 - \ell}) \\ &= \deg_{G_{n+1}}(P^\ell) \cdot \deg_{G_{n-q\ell+1}(P^\ell(x))}(P^{\lfloor n/q \rfloor + 1 - \ell}) \\ &= \deg_{G_{n+1}}(P^\ell) \cdot \deg_{G_{n-q\ell+1}(P^\ell(x))}(P^{\lfloor (n-q\ell)/q \rfloor + 1}) \\ &\leq d^{d-1} \cdot d^{\mathcal{N}+1} \\ &\leq d^{d+\mathcal{N}}. \end{aligned}$$

Therefore, the range of dynamical degrees of  $x$  is contained in the finite set

$$\{1, 2, \dots, d^{d+\mathcal{N}}\}.$$

To prove (2), consider  $n > q\mathcal{N}$ . Then there exists a minimal  $1 \leq j_n \leq \lfloor n/q \rfloor + 1$  such that  $P^{j_n}(G_{n+1})$  is critical element.

We have two cases:

If  $n - qj_n \leq q\mathcal{N}$  we have that

$$\begin{aligned} d_n &= \deg_{G_{n+1}}(P^{\lfloor n/q \rfloor + 1}) \\ &= \deg_{G_{n+1}}(P^{j_n}) \cdot \deg_{P^{j_n}(G_{n+1})}(P^{\lfloor n/q \rfloor + 1 - j_n}) \\ &= \deg_x(P) \cdot \deg_{G_{n-qj_n+1}}(P^{\lfloor (n-qj_n)/q \rfloor + 1}) \\ &\leq \deg_x(P) \cdot d^{c\mathcal{N}+1} \\ &\leq d^{c\mathcal{N}+2}. \end{aligned}$$

Therefore,  $d_n$  belongs to  $\{1, 2, \dots, d^{c\mathcal{N}+2}\}$ .

Suppose that  $n - qj_n > q\mathcal{N}$ . We have that  $P^{j_n}(G_{n+1}) = G_{n-qj_n+1}(c)$  for some critical element  $c \in \text{Crit}(P) \cap \overline{\mathcal{O}^+(x)}$  (see Corollary 3.12). It follows that

$$\begin{aligned} d_n &= \deg_{G_{n+1}}(P^{\lfloor n/q \rfloor + 1}) \\ &= \deg_{G_{n+1}}(P^{j_n}) \cdot \deg_{P^{j_n}(G_{n+1})}(P^{\lfloor n/q \rfloor + 1 - j_n}) \\ &= \deg_x(P) \cdot \deg_{G_{n-qj_n+1}(c)}(P^{\lfloor (n-qj_n)/q \rfloor + 1}) \\ &= \deg_x(P) \cdot d_{n-qj_n}(c), \end{aligned}$$

that is,  $d_n$  belongs to

$$\deg_x(P) \cdot \mathcal{D}(c) \subseteq \deg_x(P) \cdot \bigcup_{c \in \mathcal{C}(x)} \mathcal{D}(c).$$

Now (2) follows. □

LEMMA 4.15. — *Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $d \geq 2$  and  $x \in \mathcal{J}_P$ . The range of dynamical degrees  $\mathcal{D}(x)$  is unbounded if and only if there exists a recurrent critical element  $c$  contained in the  $\omega$ -limit of  $x$ .*

To prove the previous lemma we need the following definition.

DEFINITION 4.16. — For  $x_1, x_2 \in \mathcal{J}_P$  the *greatest common geometric level between  $x_1$  and  $x_2$*  is defined by

$$\text{gcg}(x_1, x_2) := \begin{cases} \max\{j \geq 0 \mid G_j(x_1) = G_j(x_2)\} & x_1 \neq x_2 \\ +\infty & x_1 = x_2 \end{cases}.$$

*Proof of Lemma 4.15.* — Suppose that  $\mathcal{D}(x)$  unbounded. By Lemma 4.14) it follows that there exists a critical element  $c_1 \in \overline{\mathcal{O}^+(P(x))}$  such that  $\mathcal{D}(c_1)$  is also unbounded. Hence, we can find a critical element  $c_2 \in \overline{\mathcal{O}^+(P(c_1))}$  with  $\mathcal{D}(c_2)$  also unbounded. Recursively we obtain a sequence  $(c_n) \subseteq \overline{\mathcal{O}^+(P(x))}$  of critical elements with  $\mathcal{D}(c_n)$  unbounded and  $c_{n+1} \in \overline{\mathcal{O}^+(P(c_n))}$ , that is  $\overline{\mathcal{O}^+(P(c_{n+1}))} \subseteq \overline{\mathcal{O}^+(P(c_n))}$ . By Proposition 2.14 there are at most  $d-2$  critical elements of  $P$  contained in  $\mathcal{J}_P$ . It follows that there exists  $N \in \mathbb{N}$  such that

$\overline{\mathcal{O}^+(P(c_N))}$  coincides with  $\overline{\mathcal{O}^+(P(c_{N+1}))}$ . That is,  $c_{N+1} \in \overline{\mathcal{O}^+(P(c_{N+1}))}$ . This means that  $c_{N+1}$  is a recurrent critical element in  $\overline{\mathcal{O}^+(P(x))}$ . Hence,  $c_{N+1}$  belongs to  $\omega(x)$ .

Conversely, suppose that there exists a recurrent critical element  $c$  in the  $\omega$ -limit of  $x$ . We split the proof in two parts. First we show that  $\mathcal{D}(c)$  is unbounded and then we prove that the range of dynamical degrees  $\mathcal{D}(x)$  is unbounded.

Suppose that  $P$  has  $q$  generators for its geometric sequences and denote by  $(G_n)$  the geometric sequence of  $c$  and by  $d_n$  the dynamical degree of level  $n$  around  $c$ . The element  $c$  is recurrent, thus we can choose  $n_1$  such that the greatest common geometric level between  $P^{n_1}(c)$  and  $c$  is larger than 1, that is,  $\text{gcg}(P^{n_1}(c), c) > 1$  (see Definition 4.16). Hence, we have

$$\begin{aligned} d_{qn_1} &= \deg_{G_{1+qn_1}}(P^{\lfloor qn_1/q \rfloor + 1}) \\ &= \deg_{G_{1+qn_1}}(P^{1+n_1}) \\ &= \deg_{G_{1+qn_1}}(P^{n_1}) \cdot \deg_{P^{n_1}(G_{1+qn_1})}(P) \\ &= \deg_{G_{1+qn_1}}(P^{n_1}) \cdot \deg_{G_1}(P) \\ &\geq 2^2. \end{aligned}$$

Now we can pick  $n_2 > n_1$  such that  $\text{gcg}(P^{n_2}(c), c) > 1 + qn_1$ . Therefore

$$\begin{aligned} d_{qn_1+qn_2} &= \deg_{G_{1+qn_1+qn_2}}(P^{\lfloor (qn_1+qn_2)/q \rfloor + 1}) \\ &= \deg_{G_{1+qn_1+qn_2}}(P^{n_1+n_2+1}) \\ &= \deg_{G_{1+qn_1+qn_2}}(P^{n_2}) \cdot \deg_{G_{1+qn_1}}(P^{1+n_1}) \\ &\geq \deg_{G_{1+qn_1+qn_2}}(P^{n_2}) \cdot 2^2 \\ &\geq 2^3. \end{aligned}$$

Recursively, we can find an increasing sequence  $(n_j)$  of natural numbers such that, for  $\sigma_j = qn_1 + \dots + qn_j$

$$d_{\sigma_j} = \deg_{G_{\sigma_j+1}}(P^{\lfloor \sigma_j/q \rfloor + 1}) \geq 2^{j+1}.$$

It follows that  $\mathcal{D}(c)$  is an unbounded set.

Since  $c$  belongs to  $\omega(x)$ , for each  $n \geq 0$  there exists at least one  $j_n \in \mathbb{N}$  such that  $P^{j_n}(G_{n+1+qj_n}(x)) = G_{n+1}$ . Then

$$\begin{aligned} d_{n+qj_n}(x) &= \deg_{G_{n+qj_n+1}(x)}(P^{\lfloor (n+qj_n)/q \rfloor + 1}) \\ &= \deg_{G_{n+qj_n+1}(x)}(P^{\lfloor n/q \rfloor + j_n + 1}) \\ &= \deg_{G_{n+qj_n+1}(x)}(P^{j_n}) \cdot \deg_{P^{j_n}(G_{n+qj_n+1}(x))}(P^{\lfloor n/q \rfloor + j_n + 1 - j_n}) \\ &= \deg_{G_{n+qj_n}(x)}(P^{j_n}) \cdot \deg_{G_{n+1}}(P^{\lfloor n/q \rfloor + 1}) \\ &\geq \deg_x(P) \cdot \deg_{G_{n+1}}(P^{\lfloor n/q \rfloor + 1}) \\ &= \deg_x(P) \cdot d_n. \end{aligned}$$

Hence, the range of dynamical degrees  $\mathcal{D}(x)$  is unbounded.  $\square$

## 5. Algebraic Degree on the Affine Line

From now, we consider that  $K$  is an algebraically closed field which is complete with respect to a non-Archimedean absolute value. We will assume that there exists a complete field  $k \subseteq K$  so that  $|\cdot|$  restricted to  $k$  is a discrete absolute value and such that the elements of  $K$  which are algebraic over  $k$  are dense in  $K$ , that is

$$\{z \in K \mid [k(z) : k] < +\infty\}$$

is a dense subset of  $K$ . Moreover, changing  $|\cdot|$  for  $|\cdot|^\lambda$  for some  $\lambda > 1$  we can always suppose that

$$\log(|k^\times|) = \mathbb{Z} \quad \text{and} \quad \log(|K^\times|) = \mathbb{Q}.$$

Since  $K$  is algebraically closed and complete, it is not difficult to see that  $K$  coincides with the completion of an algebraic closure of  $k$ .

The algebraic degree of an element  $z \in K$  over  $k$  is the number  $[k(z) : k]$ , that is, the degree of the smallest extension of  $k$  containing  $z$ . To extend this notion to the Berkovich line, note that the ball associated to every nonsingular element in  $\mathbb{H}_K$  contains points in  $k^a$ , where  $k^a$  is the algebraic closure of  $k$  inside  $K$ . It follows that, if  $x_B$  is a type II or III point in  $\mathbb{H}_K$ , then

$$\{[k(a) : k] \mid a \in B \cap K\},$$

contains its minimum.

**DEFINITION 5.1.** — For  $z \in K$  we define  $\delta(z) := [k(z) : k]$ . For  $x_B \in \mathbb{A}_K^{1,\text{an}}$  nonsingular, we define the *algebraic degree of  $x_B$  over  $k$*  as

$$\delta(x_B) = \min\{[k(a) : k] \mid a \in B \cap k^a\}.$$

If  $x \in \mathbb{A}_K^{1,\text{an}}$  singular, we define the algebraic degree of  $x$  as  $+\infty$ .



DEFINITION 5.2. — We say that a nonempty subset  $X$  of  $\mathbb{A}_K^{1,\text{an}}$  is an *algebraic set over  $k$*  if  $\delta(x) < +\infty$  for all  $x \in X$ .

Note that for  $x_B \in \mathbb{H}_K$  nonsingular and  $F$  a finite extension of  $k$ , we have that  $x_B$  belongs to  $\text{conv}(F)$ , see Subsection 2.3, if and only if  $B \cap F \neq \emptyset$ . In fact, if  $x_B$  belongs to  $\text{conv}(F)$ , by definition there exist two points  $z_0, z_1 \in F$  such that  $x_B$  belongs to the segment  $]z_0, z_1[$ . In particular,  $z_0 \prec x_B$  or  $z_1 \prec x_B$ , that is  $z_0 \in B \cap F$  or  $z_1 \in B \cap F$ . Conversely, if  $B = B_r^+(z_0)$  with  $z_0 \in F$  then  $z_0 \preceq x_B$ . It follows that  $x_B$  belongs to  $\text{conv}(F)$ . Hence, we have proved

LEMMA 5.3. — *Let  $x_B$  be a nonsingular element in  $\mathbb{A}_K^{1,\text{an}}$ . Then*

$$\delta(x_B) = \min\{[F : k] \mid x \in \text{conv}(F), F \text{ is a finite extension of } k\}.$$

The next lemma states the basic properties of the algebraic degree.

LEMMA 5.4. — *Let  $P$  be a polynomial with coefficients in  $k$  of degree  $\geq 2$ . Then the following statements hold:*

1.  $\delta(P(x)) \leq \delta(x)$  for all  $x \in \mathbb{A}_K^{1,\text{an}}$ .
2. If  $x \preceq y$ , then  $\delta(y) \leq \delta(x)$ .
3. If  $x$  belongs to  $\mathbb{H}_K$  then  $\delta(x) = \lim_{j \rightarrow +\infty} \delta(x_j)$  for each decreasing sequence  $(x_j)$  such that  $x_j \rightarrow x$ .

*Proof.* — 1 and 2 follow directly from the definition of the algebraic degree.

To prove 3 note that for any  $n \in \mathbb{N}$  the diameters of the branch points of  $\mathcal{D}_n = \text{conv}\left(\bigcup_{[F:k] \leq n} F\right)$  form a discrete subset of the real interval  $[0, +\infty[$ , because  $|\cdot|$  restricted to  $k$  is a discrete absolute value. Moreover, the algebraic degree is constant between two consecutive branch points of  $\mathcal{D}_n$ .

Let  $x \in \mathbb{H}_K$  and consider  $(x_j)$  such that  $x_j \rightarrow x$ . We have two cases:

If  $x$  is nonsingular then there exists a smallest  $j$  such that  $x$  belongs to  $\mathcal{D}_n$ , in particular  $x$  belongs to a segment between two consecutive branch points of  $\mathcal{D}_n$ . Therefore  $\delta(x) = \lim_{j \rightarrow +\infty} \delta(x_j)$ .

If  $x$  is a type IV point then there exist a strictly increasing subsequence  $(m_n)$  of integers and a subsequence  $(x_{j_n})$  of  $(x_j)$  such that  $x_{j_m} \in \mathcal{D}_{m_n}$  and  $x_{j_n} \notin \mathcal{D}_\ell$  for  $\ell < m_n$ . Hence  $\delta(x) = \lim_{j \rightarrow +\infty} \delta(x_{m_j}) = \lim_{j \rightarrow +\infty} \delta(x_j)$ .  $\square$

In the case of the Julia set, the fact of being an algebraic set over  $k$  is a local property:

PROPOSITION 5.5. — *Let  $P \in k[z]$  be a nonsimple polynomial of degree  $\geq 2$ . The Julia set of  $P$  is algebraic over  $k$  if and only if there exists a point  $x \in \mathcal{J}_P$  and a neighborhood  $V$  of  $x$  such that  $\mathcal{J}_P \cap V$  is an algebraic set over  $k$ .*

*Proof.* — If  $\mathcal{J}_P$  is algebraic over  $k$ , then taking  $V$  as an open ball of the Berkovich line which contains  $\mathcal{J}_P$ , we have that  $V$  is a neighborhood of all  $x \in \mathcal{J}_P$  and  $\mathcal{J}_P = V \cap \mathcal{J}_P$  is an algebraic set over  $k$ .

Conversely, if there exists a element  $x \in \mathcal{J}_P$  and a neighborhood  $V$  containing  $x$  we have that

$$\mathcal{J}_P = \bigcup_{n \geq 0} P^n(\mathcal{J}_P \cap V) \subseteq \bigcup_{n \geq 0} P^n(V).$$

From Lemma 5.4 (1) it follows that  $\delta(P^{n+1}(x)) \leq \delta(P^n(x))$  for all  $x \in \mathcal{J}_P \cap V$  and all  $n \geq 0$ . That is, the Julia set of  $P$  is an algebraic set over  $k$ .  $\square$

## 6. Polynomials with Algebraic Coefficients

In this section we will fix a polynomial  $P$  with algebraic coefficients over  $k$  and we will study how the algebraic degree behaves along the geometric sequence of  $x$  in  $\mathcal{J}_P$ . To do this, we need the following *dynamical version* of the well known Krasner's Lemma which is adapted for our applications. See Corollary 3 in chapter seven of [9] for the standard version of the lemma.

**LEMMA 6.1** (Krasner's Lemma). — *Let  $k$  be a field of characteristic 0 and complete with respect to a non-Archimedean absolute value. Let  $k^a$  be an algebraic closure of the field  $k$ . Consider  $P \in k[z]$  and let  $\alpha \in k^a$  such that  $P(\alpha) = 0$ . If  $B \subseteq k^a$  is a ball containing  $\alpha$  such that  $\deg_B(P) = 1$ , then  $k(\alpha) \subseteq k(\beta)$ , for all  $\beta \in B$ .*

**6.1. Algebraic degree along a geometric sequence.** — The first statement of Lemma 5.4 shows that the algebraic degree behaves nicely under the action of polynomials with algebraic coefficients.

Through this subsection, let  $P$  be a nonsimple and tame polynomial with algebraic coefficients over  $k$ . Passing to a finite extension if necessary, we can suppose that the coefficients and the critical points belong to  $k$ .

To state the proposition that allow us to estimate the algebraic degree along a geometric sequence we need two definitions.

**DEFINITION 6.2.** — Consider  $x$  in the Julia set of  $P$ , and let  $(G_n)$  be its geometric sequence. Let  $t_{-1} = -1$  and for each  $n \geq 0$  the *injectivity time* of  $G_n$  is the largest integer  $0 \leq t_n$  such that  $\deg_{G_n}(P^{t_n}) = 1$ . The *critical pullbacks* of  $P$  around  $G_n$  are the elements in  $D_n \cap P^{-t_n}(\text{Crit}^I(P))$ , where  $x_{D_n} = G_n$ .

Note that if  $P$  has  $q$  generators for its geometric sequences then the injectivity time  $t_n$  is smaller or equal than  $\lfloor n/q \rfloor + 1$ . In the case that  $G_n$  is a critical element we have that  $t_n = 0$  and that the critical pullbacks around  $G_n$  are the critical points of  $P$  contained in  $D_n$ .

If  $t_n > 0$  then the injectivity time of  $G_n$  is the smallest integer such that

$$P^{t_n}(G_n) \in \text{Crit}(P) \quad \text{and} \quad P^{t_n-1}(G_n) \notin \text{Crit}(P).$$

DEFINITION 6.3. — Let  $P \in k[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$  and consider  $x \in \mathcal{J}_P$ . Let  $s_{-1} = 1$ . For each  $n \geq 0$  define  $s_n(x)$  as the index of  $|k^\times|$  in the group generated by  $|k^\times|$  and  $\text{diam}(G_n(x))$ , that is

$$s_n(x) := \left[ |k^\times| (\text{diam}(G_n(x))) : |k^\times| \right]$$

Note that  $s_n(x) < +\infty$  since  $G_n(x)$  is a type II point and  $|K^\times| = \sqrt{|k^\times|}$ .

PROPOSITION 6.4. — Let  $P \in k[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$ . Consider  $x$  in the Julia set of  $P$  and denote by  $(G_n)$  the geometric sequence of  $x$ . For each  $n \geq -1$  let  $s_n = s_n(x)$ . Then

$$(3) \quad \max\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n\} \leq \delta(G_{n+1}).$$

The reader may find the proof of this proposition is at the end of this subsection.

LEMMA 6.5. — Let  $P \in k[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$ . Consider  $x \in \mathcal{J}_P$  and denote by  $(G_n)$  its geometric sequence. Consider critical pullbacks  $u_n, u_{n+1}$  around  $G_n$  and  $G_{n+1}$  respectively. Then, for all  $n \geq 0$  the followings statements hold:

1.  $\delta(G_n) = \delta(u_n) = \delta(u'_n)$  for all critical pullbacks  $u'_n$  around  $G_n$ .
2.  $k(u_n)$  is contained in  $k(u_{n+1})$ .
3.  $u_n \notin \mathcal{D}_n$  and  $\delta(G_{n+1}) = \delta(\mathcal{D}_n \cap K)$ , where  $\mathcal{D}_n$  is the direction at  $G_n$  that contains  $G_{n+1}$ .

*Proof.* — Note that if  $G_n$  is a critical element we have that  $u_n$  is a critical point of  $P$ . Therefore (1) and (2) follow directly in this case, since we are assuming that  $\text{Crit}^I(P)$  is contained in  $k$ .

If  $G_n$  is noncritical, then  $t_n > 1$  and

$$\deg_{D_n} [P^{t_n}(z) - P^{t_n}(u_n)] = 1.$$

By Krasner's lemma (see Lemma 6.1), it follows that  $k(u_n) \subseteq k(v)$  for all  $v$  in  $D_n \cap k^a$ .

Taking  $v \in D_n$  such that  $\delta(v) = \delta(G_n)$  we have that  $\delta(v) \leq \delta(u_n) \leq \delta(v)$ , that is  $\delta(u_n) = \delta(G_n)$ . If  $u'_n$  is another critical pullback around  $G_n$  then interchanging  $u_n$  with  $u'_n$  we obtain the second equality in (1).

Taking  $v = u_{n+1}$  we obtain (2).

In order to prove (3), recall that according to Proposition 4.10 we have that

$$\deg_{G_{n+1}}(P^j) = \deg_y(P^j)$$

for all  $y \in [G_{n+1}, G_n[$  and all  $j \geq 1$ . In particular  $t_{n+1}$  is the smallest integer such that  $P^{t_{n+1}}(\mathcal{D}_n) \cap \text{Crit}(P) \neq \emptyset$  and  $P^{t_{n+1}-1}(\mathcal{D}_n) \cap \text{Crit}(P) = \emptyset$ . Then, as in the proof of the first part of the lemma, we have that

$$\deg_{\mathcal{D}_n \cap K} [P^{t_{n+1}}(z) - P^{t_{n+1}}(u_{n+1})] = 1.$$

Therefore  $\delta(G_{n+1}) = \delta(\mathcal{D}_n \cap K)$ . □

**COROLLARY 6.6.** — *Let  $P \in k[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$ . Let  $x$  be a noncritical Julia point and denote by  $(G_n)$  its geometric sequence.*

1. *If  $t_n = t_{n+1}$  then  $\delta(G_n) = \delta(G_{n+1})$ .*
2. *If  $\delta(G_n) < \delta(G_{n+1})$  then  $t_n < t_{n+1}$ .*

Now we can give the proof of Proposition 6.4. This proposition is a key ingredient in order to prove our main results.

*Proof of Proposition 6.4.* — We proceed by induction in  $n$ . For  $n = -1$  we have that  $-1 = t_{-1} \neq t_0 = 0$ , since  $\deg_{L_0}(P) = \deg(P)$ , therefore

$$\max\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq -1\} = \max\{s_{-1}\} = 1 \leq \delta(G_0) = 1.$$

That is, the first step of the induction is valid.

Suppose that (3) is valid for  $n - 1$ . If  $n$  is such that  $t_n = t_{n+1}$  we have that,  $\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n - 1\} = \{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n\}$ . It follows that  $\max\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n\} \leq \delta(G_n) \leq \delta(G_{n+1})$ .

Suppose that  $n$  is such that  $t_n < t_{n+1}$ . In this case, the corresponding critical pullbacks  $u_n$  and  $u_{n+1}$  are necessarily different. By Lemma 6.5 (2), we have that  $k(u_{n+1})$  is a finite extension of  $k(u_n)$ , and therefore  $\delta(u_n)$  divides  $\delta(u_{n+1})$ .

Following Lemma 6.5 (3) we have that the distance between  $u_n$  and  $u_{n+1}$  coincides with the diameter of  $G_n$ . Therefore

$$\begin{aligned} s_n &= \left[ |k^\times|(\text{diam}(G_n)) : |k^\times| \right] \\ &= \left[ |k^\times|(|u_n - u_{n+1}|) : |k^\times| \right] \\ &\leq [k(u_n - u_{n+1}) : k] \\ &\leq \max\{\delta(u_n), \delta(u_{n+1})\}. \end{aligned}$$

Hence, we obtain that

$$\max\{s_n, \delta(u_n)\} \leq \delta(u_{n+1}).$$

Applying the inductive hypothesis it follows that

$$\begin{aligned}\delta(u_{n+1}) &\geq \max\{s_n, \delta(u_n)\} \\ &\geq \max\{s_n, \max\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n-1\}\} \\ &= \max\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n\}.\end{aligned}$$

Thus, we have proven Proposition 6.4.  $\square$

## 6.2. No wandering components for polynomials with algebraic coefficients. —

The following proposition is the key to prove Corollary B. In fact, combining the proposition below with Proposition 2.16 we obtain Corollary B in the case of polynomials with algebraic coefficients over  $k$ .

**PROPOSITION 6.7.** — *Let  $P \in k[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$ . If  $x$  is a nonpreperiodic algebraic element in  $\mathcal{J}_P$ , then  $x \in K$ .*

*Proof.* — Let  $x$  be a noncritical and nonpreperiodic algebraic element of the Julia set of  $P$ . Since  $x \in \mathcal{J}_P$  if and only if  $P^j(x) \in \mathcal{J}_P$  for all  $j \geq 1$  and there are only finitely many critical elements in  $\mathcal{J}_P$ , we may assume that the forward orbit of  $x$  is free of critical elements. Denote by  $(G_n)$  the geometric sequence of  $x$ .

Since  $x$  is not the preimage of a critical element it follows that

$$\{n \in \mathbb{N} \mid t_n \neq t_{n+1}\}$$

contains infinitely many elements. For each  $n \geq 1$  let  $m_n$  be the  $n$ -th nonnegative integer such that  $t_{m_n} \neq t_{m_n+1}$ .

In view of Proposition 6.4, the set  $\{s_{m_j}(x) \mid j \in \mathbb{N}\}$  is bounded by  $\delta(x)$ . Hence, the denominators of  $\log(\text{diam}(G_{m_n}))$  are bounded.

Let  $D \in \mathbb{N}$  be the maximum among the denominators of  $\log(\text{diam}(G_{m_n}))$ ,  $n \in \mathbb{N}$ , and  $\log(\text{diam}(L_0))$ . It follows that

$$d_{\mathbb{H}}(G_{m_j}, G_{m_{j-1}}) = \log(\text{diam}(G_{m_{j-1}})) - \log(\text{diam}(G_{m_j})) \geq \frac{1}{D^2}$$

for all  $j \geq 1$ . Hence

$$d_{\mathbb{H}}(G_{m_n}, L_0) = d_{\mathbb{H}}(G_{m_0}, L_0) + \sum_{j=1}^n d_{\mathbb{H}}(G_{m_j}, G_{m_{j-1}}) \geq d_{\mathbb{H}}(G_{m_0}, L_0) + \frac{n}{D^2}.$$

Thus,

$$\begin{aligned} d(x, L_0) &= \lim_{n \rightarrow +\infty} d_{\mathbb{H}}(G_n, L_0) \\ &= \lim_{n \rightarrow +\infty} d_{\mathbb{H}}(G_{m_n}, L_0) \\ &\geq \lim_{n \rightarrow +\infty} \left( d_{\mathbb{H}}(G_{m_0}, L_0) + \frac{n}{D^2} \right) \\ &= +\infty. \end{aligned}$$

Therefore,  $x$  belongs to  $\mathcal{J}_P^I$ . □

**6.3. Equilibrium measure and algebraic degree.** — Let  $P \in k[z]$  be a nonsimple polynomial of degree  $\geq 2$ . In this subsection, we will show that if there exists an element  $y \in \mathcal{J}_P$  with  $\delta(y) = +\infty$  then  $\delta(x) = +\infty$  for  $\rho_P$ -almost all  $x \in \mathcal{J}_P$ .

First we need the following lemma, which is a stronger version of Lemma 5.4 (3) in the case of Julia points.

**LEMMA 6.8.** — *Let  $P \in k[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$ . If  $x \in \mathcal{J}_P$  then  $\delta(x) = \lim_{n \rightarrow +\infty} \delta(x_n)$ , for each decreasing sequence  $(x_n)$  such that  $x_n \rightarrow x$ .*

*Proof.* — By Lemma 5.4 (3) we only need to prove the lemma for the classical points in  $\mathcal{J}_P$ . We prove the lemma for a classical point in  $x \in \mathcal{J}_P$  and for the geometric sequence  $(G_n)$  of  $x$ . Applying Lemma 5.4 (2) we obtain the result for any decreasing sequence  $(x_n)$  satisfying the hypothesis.

We proceed by contradiction. Suppose that  $\delta(x) \neq \lim_{n \rightarrow +\infty} \delta(G_n)$ , in view of Lemma 5.4 (2) we have that  $\delta(x) > \lim_{n \rightarrow +\infty} \delta(G_n)$ . Since  $\delta(G_n)$  is a positive integer and the sequence  $\delta(G_n)$  is increasing and bounded, there exists  $N \in \mathbb{N}$  such that  $\delta(G_n) = \delta(G_N)$  for all  $n \geq N$ . By Lemma 6.5 (2) it follows that  $k(u_N) = k(u_n)$  for all  $n \geq N$ , where  $u_n$  is a critical pullback around  $G_n$  for all  $n \geq N$ . Since  $(u_n)_{n \geq N} \subseteq k(u_N)$  is a sequence such that  $u_n \rightarrow x$  and  $k(u_N)$  is a finite extension of  $k$  we have that  $x$  belongs to  $k(u_N)$ . Hence,  $\delta(x) = [k(x) : k] \leq [k(u_N) : k]$ , which is a contradiction. □

To relate the algebraic degree with the equilibrium measure we need to note that  $\delta : \mathbb{A}_K^{1, \text{an}} \rightarrow \mathbb{R}$  is Borel measurable. In fact,  $\text{conv}(F)$  is closed in the weak topology for all  $F/k$  finite, then

$$\bigcup_{[F:k] \leq n} \text{conv}(F)$$

is a measurable set for all  $n \in \mathbb{N}$ . Using Lemma 5.3 we have that

$$\{x \in \mathbb{A}_K^{1, \text{an}} \mid \delta(x) \leq n\}$$

is measurable for all  $n \in \mathbb{N}$ .

LEMMA 6.9. — *Let  $P \in k[z]$  be a polynomial of degree  $\geq 2$  with  $\mathcal{J}_P$  algebraic over  $k$ . Let  $\mathcal{A}_n = \{x \in \mathcal{J}_P \mid \delta(x) \geq n\}$ . If  $\mathcal{A}_n \neq \emptyset$  then  $\rho_P(\mathcal{A}_n) = 1$  where  $\rho_P$  is the equilibrium measure of  $P$ .*

*Proof.* — If there exists  $x \in \mathcal{J}_P$  with  $\delta(x) \geq n$  then there exists  $m_n \in \mathbb{N}$  such that  $\delta(L_{m_n}(x)) \geq n$ , see Lemma 6.8. Let  $B_n$  be the closed ball associated to  $L_{m_n}$ . We have that  $\rho_P(\overline{B_n}) > 0$ , since  $\overline{B_n}$  contains an open set that contains a Julia point. Now we will use the ergodicity of  $\rho_P$  (see [13]). Let

$$U = \{x \in \mathcal{J}_P \mid P^\ell(x) \in \overline{B_n} \text{ for some } \ell \geq 0\}.$$

Then  $U$  is a backward invariant set, by the ergodicity  $\rho_P(U) = 0$  or  $\rho_P(U) = 1$ . But  $\overline{B_n} \cap U$  is contained in  $U$ , then  $\rho_P(U) = 1$ . It follows that  $\rho_P$ -almost every  $x \in \mathbb{A}_K^{1,\text{an}}$  have infinitely many iterates in  $\overline{B_n}$ . From Lemma 5.4 (1) and (2) we conclude that  $\rho_P(\mathcal{A}_n) = 1$ .  $\square$

PROPOSITION 6.10. — *Let  $P \in k[z]$  be a polynomial of degree  $\geq 2$  with  $\mathcal{J}_P$  algebraic over  $k$ . Then the set  $\{\delta(x) \mid x \in \mathcal{J}_P\}$  is bounded. Moreover,*

$$\rho_P(x \in \mathcal{J}_P \mid \delta(x) = \Delta) = 1$$

where  $\Delta = \max\{\delta(x) \mid x \in \mathcal{J}_P\}$  and  $\rho_P$  is the equilibrium measure of  $P$ .

*Proof.* — For each  $n \in \mathbb{N}$  let  $\mathcal{A}_n$  as in Lemma 6.9.

Suppose that  $\{\delta(x) \mid x \in \mathcal{J}_P\}$  is an unbounded set. Then there exists a sequence  $(x_n)$  of Julia points such that  $\delta(x_n) \geq n$  and therefore, using Lemma 6.9, it follows that  $\rho_P(\mathcal{A}_n) = 1$  for all  $n \in \mathbb{N}$ . Since  $\mathcal{A}_{n+1} \subseteq \mathcal{A}_n$  and

$$\rho_P(\cap \mathcal{A}_n) = \lim_{n \rightarrow +\infty} \rho_P(\mathcal{A}_n) = 1,$$

we have that the intersection  $\cap \mathcal{A}_n$  is nonempty. It follows that there exists  $y \in \mathcal{J}_P$  with  $\delta(y) = +\infty$ , which contradicts our assumption that  $\mathcal{J}_P$  is an algebraic set over  $k$ . Hence, the set  $\{\delta(x) \mid x \in \mathcal{J}_P\}$  is bounded.

Consider  $\Delta = \max\{\delta(x) \mid x \in \mathcal{J}_P\}$ . By Lemma 6.9 we have that

$$\{x \in \mathcal{J}_P \mid \delta(x) = \Delta\} = \mathcal{A}_\Delta \text{ a.e.}$$

Then  $\rho_P(\{x \in \mathcal{J}_P \mid \delta(x) = \Delta\}) = 1$ .  $\square$

The following corollary will be useful in the proofs of Theorem E and of Theorem F.

COROLLARY 6.11. — *Let  $P \in k[z]$  be a tame polynomial of degree  $\geq 2$  and denote by  $\rho_P$  the equilibrium measure of  $P$ . Then the following statements are equivalent:*

1. *There exists  $y \in \mathcal{J}_P$  with  $\delta(y) = +\infty$ .*

2. For each  $n \geq 1$  there exists  $y_n \in \mathcal{J}_P$  such that  $\delta(y_n) \geq n$ .
3.  $\rho_P(x \in \mathcal{J}_P \mid \delta(x) = +\infty) = 1$ .

**6.4. Algebraic Julia sets and recurrent critical elements.** — Through this subsection let  $P \in k[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$  with  $q$  generators for its geometric sequences. Recall that we are assuming that  $k$  contains the critical points of  $P$ .

Consider  $x \in \mathcal{J}_P$  and let  $G_n = G_n(x)$  for all  $n \geq 1$ . From Proposition 6.4 it follows that there exists a relation between  $\text{diam}(G_n)$  and  $\delta(G_n)$ . Moreover, note that we can obtain  $\text{diam}(G_n)$  from  $d_{\mathbb{H}}(G_n, L_0)$ .

**6.5. Proof of Theorem E.** — In order to prove Theorem E and Theorem F we need to establish some relations between the range of dynamical degrees and the existence of recurrent critical points.

First we need to recall some notation. Let  $P \in K[z]$  be a nonsimple and tame polynomial, see Definition 2.10, and consider  $x$  in the Julia set of  $P$ . We denote by  $\omega(x)$  and  $\mathcal{O}^+(x)$  the  $\omega$ -limit and the forward orbit of  $x$  respectively. We denote by  $G_n = G_n(x)$  the geometric sequence of  $x$ , see Definition 4.5. The fundamental property of the geometric sequence is the fact that

$$\deg_y(P) = \deg_{G_n}(P)$$

for all  $y \in [G_n, G_{n+1}[$ .

**LEMMA 6.12.** — *Let  $P \in k[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$ . Suppose that there exists  $x \in \mathcal{J}_P$  such that  $\{s_n(x) \mid t_n \neq t_{n+1}\}$  is unbounded. Then the range of dynamical degrees around  $x$  is unbounded.*

*Proof.* — If  $\{s_n(x) \mid t_n \neq t_{n+1}, n \in \mathbb{N}\}$  is unbounded we have that

$$\log(\text{diam}(G_j)) = \log(\text{diam}(L_0)) - d_{\mathbb{H}}(G_j, L_0)$$

is at least  $\frac{1}{k_j}$  for an unbounded sequence  $(k_j)$ , then, in view of Lemma 4.13,

$$\liminf_{n \rightarrow +\infty} [d_n(x)]^{-1} = 0.$$

Hence, the range of dynamical degrees  $\mathcal{D}(x)$  is unbounded. □

**LEMMA 6.13.** — *Let  $P \in k[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$ . Suppose that there exists a recurrent critical element  $c \in \mathcal{J}_P$ . Then, there exists a sequence  $(y_n) \subseteq \mathcal{J}_P$  such that  $\lim_{n \rightarrow +\infty} y_n = c$  and  $\lim_{n \rightarrow +\infty} \delta(y_n) = +\infty$ .*



To prove the previous lemma it is convenient to use  $p$ -adic absolute value notation. Recall that we are supposing that

$$\log(|k^\times|) = \mathbb{Z} \quad \text{and} \quad \log(|K^\times|) = \mathbb{Q}.$$

Given a prime number  $p$ , the absolute value  $|\cdot|_p$  is defined as follows. For  $a \in \mathbb{Z}$  we put  $|a|_p = p^{-n}$ , where,  $a = p^n m$  and  $p$  does not divide  $m$ . For a rational number  $\frac{a}{b}$  we put  $\left|\frac{a}{b}\right|_p = \frac{|a|_p}{|b|_p}$ . Recall that  $|\cdot|_p$  is a non-Archimedean absolute value.

For convenience we record the following general fact about non-Archimedean absolute values as a remark.

REMARK 6.14. — Let  $r_1, r_2 \in \mathbb{Q}$  such that  $|r_1|_p < |r_2|_p$ , then  $|r_1 + r_2|_p = |r_2|_p$ .

*Proof of Lemma 6.13.* — In view of Lemma 4.15 we have that the range of dynamical degrees  $\mathcal{D}(c)$  is an unbounded set. For each  $n \geq 0$  the dynamical degree  $d_n = d_n(c)$  around  $c$  is a product of  $\lfloor n/q \rfloor + 1$  numbers bounded by  $d$ . Hence, there exists a prime number  $p \leq d$  such that arbitrarily large powers of  $p$  divide elements in the range of dynamical degrees around  $c$ , that is,  $(|d_n^{-1}|_p)$  is an unbounded sequence.

If  $I_{j_n}$  denotes the segment such that  $P^{\lfloor n/q \rfloor + 1}(]G_n, G_{n-1}[) = I_{j_n}$  and

$$\alpha_n = d_n^{-1} \cdot |I_{j_n}|,$$

it follows that the sequence  $(|\alpha_n|_p)$  is also unbounded.

Let  $m_1 \geq 1$  be the smallest integer such that

$$\max\{1, |\log(\text{diam}(L_0))|_p\} < |\alpha_{m_1}|_p.$$

Let  $1 < e_1$  such that  $|\alpha_{m_1}|_p = p^{e_1}$ .

Recursively, we define  $m_n$  as the smallest integer larger than  $m_{n-1}$  such that  $p^{e_{n-1}} < |\alpha_{m_n}|_p$ . We define  $e_n$  as the integer satisfying  $|\alpha_{m_n}|_p = p^{e_n}$ . Note that both  $(m_n)$  and  $(e_n)$  are increasing sequences of integers.

To construct the sequence  $(y_n)$  we need to prove first that  $\{s_{m_n}(c) \mid n \in \mathbb{N}\}$  is an unbounded set. We have that

$$\begin{aligned} -\log(\text{diam}(G_{m_n})) &= -\log(\text{diam}(L_0)) + \sum_{\ell=1}^{m_n} d_\ell^{-1} \cdot |I_{j_\ell}| \\ &= -\log(\text{diam}(L_0)) + \sum_{\ell=1}^{m_n} \alpha_\ell. \end{aligned}$$

Using Remark 6.14 and the choice of  $m_n$  we have that

$$\begin{aligned} |\log(\text{diam}(G_{m_n}))|_p &= \left| -\log(\text{diam}(L_0)) + \sum_{\ell=1}^{m_n} \alpha_\ell \right|_p \\ &= |\alpha_{m_n}|_p = p^{e_n}. \end{aligned}$$

Hence, the largest power of  $p$  dividing the denominator, say  $b_n$ , of  $\log(\text{diam}(G_{m_n}))$  is  $p^{e_n}$ . Then

$$s_{m_n} = \frac{\text{lcm}(m, b_n)}{m} \geq \frac{\text{lcm}(m, p^{e_n})}{m}$$

by definition of  $s_n$  (see Definition 6.3), where  $m$  is the least common multiple of the denominators of  $\log(|\beta|)$  for  $\beta$  running in the coefficients and the critical points of  $P$ . In particular,  $\lim s_{m_n} = +\infty$ , since  $(e_n)$  is an increasing sequence of integers.

For all  $n \geq q\mathcal{N}(c) + q$  there exists a noncritical element  $y_n \in \mathcal{J}_P$  such that  $G_{m_n}(y_n) = G_{m_n}$  and  $G_{m_n+1}(y_n) \neq G_{m_n+1}$ , see Definition 3.13. Note that,  $\text{gcg}(y_n, c) = m_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Then the sequence  $(y_n)$  converges to  $c$ . In view of this, we only need prove that  $\lim \delta(y_n) = +\infty$

The injectivity time of  $G_{m_n+1}(y_n)$  is larger than the injectivity time of  $G_{m_n}(y_n)$ , since  $G_{m_n}(y_n)$  is a critical element and  $G_{m_n+1}$  is not. Hence, applying Proposition 6.4 and using Lemma 5.4 (2) we obtain a bound, from below, for the algebraic degree of  $y_n$ . More precisely,

$$\delta(y_n) \geq \delta(G_{m_n+1}(y_n)) \geq \max\{1, s_{m_n}(c)\} = s_{m_n}.$$

That is,  $\lim \delta(y_n) = +\infty$ . □

Now we can give the proof of Theorem E.

*Proof of the Theorem E.* — We prove that in the presence of a recurrent critical element,  $\mathcal{J}_P$  is not an algebraic set over  $k$ . Suppose that there exists a recurrent critical element  $c$  contained in  $\mathcal{J}_P$ . Following Lemma 6.13, there exists a sequence  $(y_n) \subseteq \mathcal{J}_P$  such that  $\lim \delta(y_n) = +\infty$ . Applying Corollary 6.11 we obtain  $x$  in the Julia set of  $P$  with  $\delta(x) = +\infty$ .

In particular, if  $\mathcal{J}_P$  is algebraic over  $k$ , then there are not critical periodic elements in  $\mathcal{J}_P$ . Following Proposition 6.7 we obtain that  $\mathcal{J}_P$  is contained in  $K$ . In Proposition 6.10 we showed the existence of (the smallest)  $\Delta \in \mathbb{N}$  with  $\delta(x) \leq \Delta$  for all  $x \in \mathcal{J}_P$ . □

**6.6. The completion of the field of formal Puiseux series.** — Let  $F$  be an algebraically closed field of characteristic 0. We denote by  $F((\tau))$  the *field of formal Laurent series* with coefficients in  $F$ . For a nonzero element

$$z = \sum_{j \geq j_0} a_j \tau^j \in F((\tau))$$

we define  $\text{ord}(z) = \min\{j \mid a_j \neq 0\}$  and  $|z| = e^{-\text{ord}(z)}$ . Observe that  $|\cdot|$  is a non-Archimedean absolute value and that  $F((\tau))$  is complete with respect to  $|\cdot|$  but not algebraically closed. An algebraic closure of  $F((\tau))$  is the *field of formal Puiseux series*  $F\langle\langle\tau\rangle\rangle$  with coefficients in  $F$  (e.g., Chapter IV, Theorem 3.1 in [26]). More precisely,  $F\langle\langle\tau\rangle\rangle$  is the direct limit of the fields  $F((\tau))(\tau^{1/m}) = F((\tau^{1/m}))$  for  $m \in \mathbb{N}$ , with the obvious inclusions, that is,  $F((\tau^{1/m_1})) \subseteq F((\tau^{1/m_2}))$  if and only if  $m_1$  divides  $m_2$ .

Therefore an element in  $F\langle\langle\tau\rangle\rangle$  has the form

$$z = \sum_{j \geq j_0} a_j \tau^{j/m}$$

for some  $m \in \mathbb{N}$ . The unique extension of  $|\cdot|$  to  $F\langle\langle\tau\rangle\rangle$  (also denoted by  $|\cdot|$ ) is completely determined by  $|\tau^{1/m}| = e^{-1/m}$ .

Note that the degree of the field  $F((\tau^{1/m}))$  over  $F((\tau))$  is precisely  $m$ . Moreover, the field  $F((\tau^{1/m}))$  is the unique field extension of  $F((\tau))$  of degree  $m$ .

We denote by  $\mathbb{L}_F$  the completion of  $F\langle\langle\tau\rangle\rangle$  with respect to  $|\cdot|$ . Every  $z \in \mathbb{L}_F$  can be represented as

$$\sum_{j \geq 0} a_j \tau^{\lambda_j},$$

where  $(\lambda_j)$  is an increasing sequence of rational numbers tending to  $+\infty$ . In this case  $|z| = e^{-\text{ord}(z)}$  where  $\text{ord}(z) = \min\{\lambda_j \mid a_j \neq 0\}$ . The field  $\mathbb{L}_F$  is also algebraically closed since it is the completion of an algebraically closed non-Archimedean field.

The following Proposition is a complement of Proposition 6.4. The result is not true in general, it depends on the structure of the finite extensions of  $F((\tau))$ .

**PROPOSITION 6.15.** — *Let  $P \in F\langle\langle\tau\rangle\rangle[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$ . Consider  $x$  in the Julia set of  $P$  and denote by  $(G_n)$  the geometric sequence of  $x$ . For each  $n \geq -1$  let  $s_n = s_n(x)$ . Then*

$$(4) \quad \delta(G_{n+1}) \leq \text{lcm}\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n\},$$

where  $\text{lcm}$  denotes the least common multiple.

*Proof.* — We proceed by induction. If  $n = -1$  we have that

$$\delta(G_0) = 1 \leq \text{lcm}\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq -1\}.$$

Suppose that (4) holds for  $n - 1$ . If  $t_n = t_{n+1}$  then

$$\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n - 1\} = \{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n\}.$$

By Corollary 6.6 we have that  $\delta(G_n) = \delta(G_{n+1})$ , therefore (4) holds for  $n$ .

Suppose that  $t_n \neq t_{n+1}$  and let  $\mathcal{D}_n$  be the direction in  $T_{G_n}$  containing  $G_{n+1}$ . We have that, there exists  $0 \neq a \in F$  such that  $u_n + a\tau^{b_n/a_n}$  belongs to  $\mathcal{D}_n \cap \mathbb{L}_F$  where  $b_n/a_n = |\log \text{diam}(G_n)|_{\mathbb{R}}$  with  $(a_n, b_n) = 1$ .

In view of the structure of the algebraic extensions of the field of formal Laurent series we have that  $\delta(u_n + a\tau^{b_n/a_n}) \leq \text{lcm}\{\delta(\tau^{b_n/a_n}), \delta(u_n)\}$  and

$$\delta(\tau^{b_n/a_n}) = \text{lcm}\{m, a_n\} \cdot m^{-1} = s_n,$$

where  $m = [k : F(\langle\tau\rangle)]$ .

It follows, by Lemma 6.5, that

$$\begin{aligned} \delta(u_{n+1}) &\leq \delta(u_n + a\tau^{b_n/a_n}) \\ &\leq \text{lcm}\{\delta(\tau^{b_n/a_n}), \delta(u_n)\} \\ &= \text{lcm}\{\delta(u_n), s_n\}. \end{aligned}$$

Hence, we have that  $\delta(G_{n+1}) \leq \text{lcm}\{s_n, \delta(G_n)\}$ .

Applying the inductive hypothesis it follows that

$$\begin{aligned} \delta(G_{n+1}) &\leq \text{lcm}\{s_n, \delta(G_n)\} \\ &\leq \text{lcm}\{s_n, \text{lcm}\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n - 1\}\} \\ &\leq \text{lcm}\{s_j \mid t_j \neq t_{j+1}, -1 \leq j \leq n\}. \end{aligned}$$

This proves the proposition.  $\square$

*Proof of Theorem F.* — Assume that the algebraic degree of every element in  $\mathcal{J}_P$  is finite. From Theorem E we have that there is no recurrent critical element in  $\mathcal{J}_P$  and that there exists a smallest  $\Delta \in \mathbb{N}$  such that  $\delta(x) \leq \Delta$  for all  $x \in \mathcal{J}_P$ . In particular, in view of the structure of the subfields of  $F(\langle\tau\rangle)$ , it follows that  $\mathcal{J}_P$  is contained in the unique extension of  $F(\langle\tau\rangle)$  with degree  $\Delta$ .

Conversely, suppose that there exists  $x \in \mathcal{J}_P$  with  $\delta(x) = +\infty$ . By Proposition 6.15 it follows that

$$\{\text{lcm}\{s_n(x) \mid t_n \neq t_{n+1}, n \leq m\} \mid m \in \mathbb{N}\}$$

is an unbounded set, hence  $\{s_n(x) \mid t_n \neq t_{n+1}, n \in \mathbb{N}\}$  is also an unbounded set and therefore the range of dynamical degrees  $\mathcal{D}(x)$  is also unbounded (see Lemma 6.12). By Lemma 4.15, there exists a critical element  $c \in \mathcal{J}_P$  with its range of dynamical degrees  $\mathcal{D}(c)$  unbounded.  $\square$

## 7. Polynomials with Coefficients in $K$

In the previous section we proved Corollary B for polynomials with algebraic coefficients. To prove Theorem A for polynomials with coefficients in  $K$  we use a perturbation argument.

The key to perturb the coefficients of  $P$  while preserving a suitable orbit is the following proposition.

**PROPOSITION 7.1.** — *Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $d \geq 2$  and consider  $x \in \mathcal{J}_P$ . Then*

$$d_{\mathbb{H}}(y, L_0) \leq d^{d-1} \cdot d_{\mathbb{H}}(x, L_0),$$

for all  $y$  in  $\omega(x)$ .

**7.1. Key Lemma and Proof of Proposition 7.1.** — To prove Proposition 7.1 we need to compare the distance from  $d_{\mathbb{H}}(x, L_0)$  with  $d_{\mathbb{H}}(y, L_0)$ . To do this, we need to introduce the concepts of *level* and *time sequences*. Recall that  $\text{gcg}(x, y)$  denotes the greatest common geometric level between  $x$  and  $y$ , see Definition 4.16.

Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $\geq 2$  with  $q$  generators for its geometric sequences. Let  $x \in \mathcal{J}_P$  a nonpreperiodic point and consider  $y$  in the  $\omega$ -limit of  $x$ .

We define  $k_0 = 0$  and  $\ell_0 = \text{gcg}(x, y)$ . The point  $y$  belongs to  $\omega(x)$ , hence

$$\limsup_{j \rightarrow +\infty} \text{gcg}(P^j(x), y) = +\infty.$$

Let

$$k_1 = \min\{j \in \mathbb{N} \mid \text{gcg}(P^j(x), y) > \ell_0\}$$

and let  $\ell_1 = \text{gcg}(P^{k_1}(x), y)$ .

Recursively, we define  $k_n = \min\{j \in \mathbb{N} \mid \text{gcg}(P^j(x), y) > \ell_{n-1}\}$  and

$$\ell_n = \text{gcg}(P^{k_n}(x), y)$$

for all  $n \geq 2$ .

**DEFINITION 7.2.** — We say that  $(\ell_n)_{n \geq 0}$  is the *level sequence from  $x$  to  $y$*  and that  $(k_n)_{n \geq 0}$  is the *time sequence from  $x$  to  $y$* .

In Figure 1 we see a representation of the level and time sequences. In this picture we see the segments  $[x, L_0]$ ,  $[P^{k_{n-1}}(x), L_0]$  and  $[P^{k_n}(x), L_0]$ . The serpentine line represent the intersection with the segment  $[y, L_0]$ . The dashed lines represent the action of an iterate of  $P$ .

Note that

$$P^{k_n}([G_{\ell_n + qk_n}(x), G_{\ell_{n-1} + qk_n}(x)]) = [G_{\ell_n}(y), G_{\ell_{n-1}}(y)].$$

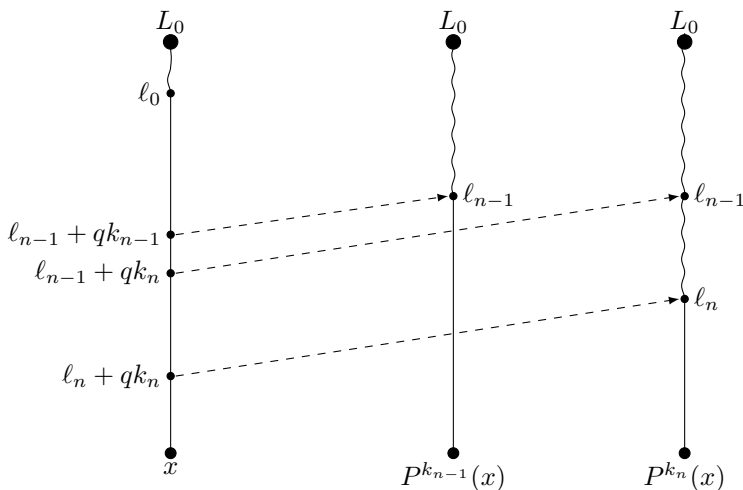


FIGURE 1. Level and time sequences

REMARK 7.3. — Observe that  $k_n$  is the smallest integer such that

$$P^{k_n}([x, L_0]) \cap [y, G_{\ell_{n-1}}(y)] \neq \emptyset.$$

LEMMA 7.4. — Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $d \geq 2$  with  $q$  generators for its geometric sequences. Let  $x \in \mathcal{T}_P$  and consider  $y$  in the  $\omega$ -limit of  $x$ , we denote by  $(\ell_n)$  (resp.  $(k_n)$ ) the level (resp. time) sequence from  $x$  to  $y$ . Then, for any  $n \geq 1$ ,

$$\deg_{G_j(x)}(P^{k_n}) \leq d^{d-1}$$

for all the elements  $G_j(x)$  contained in the segment  $[x, G_{\ell_{n-1}+qk_n}(x)]$ .

*Proof.* — Let  $n > 1$  and let  $G_j(x) \in [x, G_{\ell_{n-1}+qk_n}(x)[$  be a geometric point. Note that the elements in the set

$$O_n(G_j(x)) = \{G_j(x), P(G_j(x)), \dots, P^{k_n-1}(G_j(x))\}$$

are pairwise incomparable. In fact, if we suppose that there exist  $0 \leq i$  and  $1 \leq \ell$  such that  $1 \leq i + \ell < k_n$  and  $P^i(G_j(x)) \preceq P^{i+\ell}(G_j(x))$ , it follows that

$$G_{j-qi}(P^i(x)) \preceq G_{j-qi-q\ell}(P^{i+\ell}(x)).$$

Hence

$$G_{j-qi-q\ell}(P^i(x)) = G_{j-qi-q\ell}(P^{i+\ell}(x)).$$

Then we have that

$$\begin{aligned}
 P^{k_n-\ell}(G_{j-q\ell}(x)) &= P^{k_n-i-\ell}(P^i(G_{j-q\ell}(x))) \\
 &= P^{k_n-i-\ell}(G_{j-qi-q\ell}(P^i(x))) \\
 &= P^{k_n-i-\ell}(G_{j-qi-q\ell}(P^{i+\ell}(x))) \\
 &= G_{j-qi-q\ell-q(k_n-i-\ell)}(P^{i+\ell+k_n-i-\ell}(x)) \\
 &= G_{j-qk_n}(P^{k_n}(x)) \\
 &= P^{k_n}(G_j(x)).
 \end{aligned}$$

Since  $P^{k_n}(G_j(x)) \in [y, G_{\ell_{n-1}}(y)[$ , we have  $P^{k_n-\ell}([x, L_0]) \cap [y, G_{\ell_{n-1}}(y)[ \neq \emptyset$ , which contradicts Remark 7.3.

Hence, we obtain that the set  $O_n(G_j(x))$  contains at most  $d-1$  critical elements. In consequence, the local degree of  $P^{k_n}$  at  $G_j(x)$  is bounded, from above, by  $d^{d-1}$ .  $\square$

*Proof of Proposition 7.1.* — Let  $n \geq 1$  and suppose that  $P$  has  $q$  generators for its geometric sequences. Let  $x$  be in the Julia set of  $P$  and consider  $y \in \omega(x)$ . Denote by  $(\ell_n)$  (resp.  $(k_n)$ ) the level (resp. time) sequence from  $x$  to  $y$ . From the previous Lemma and Proposition 4.10, the local degree of  $P^{k_n}$  is constant and smaller than  $d^{d-1}$  in each segment of the form  $]G_{j+1}(x), G_j(x)[$  contained in  $[G_{\ell_n+qk_n}(x), G_{\ell_{n-1}+qk_n}(x)[$ . By Lemma 2.5 it follows that

$$d_{\mathbb{H}}(P^{k_n}(G_{j+1}(x)), P^{k_n}(G_j(x))) \leq d^{d-1} \cdot d_{\mathbb{H}}(G_{j+1}(x), G_j(x)).$$

Applying this to all the segments  $]G_{j+1}(x), G_j(x)[$  contained in

$$[G_{\ell_n+qk_n}(x), G_{\ell_{n-1}+qk_n}(x)[$$

we obtain that

$$d_{\mathbb{H}}(G_{\ell_n}(y), G_{\ell_{n-1}}(y)) \leq d^{d-1} \cdot d_{\mathbb{H}}(G_{\ell_n+qk_n}(x), G_{\ell_{n-1}+qk_n}(x)).$$

Therefore, if we put  $a = d_{\mathbb{H}}(G_{\ell_0}(x), L_0) = d_{\mathbb{H}}(G_{\ell_0}(y), L_0)$ , we have

$$\begin{aligned}
 d_{\mathbb{H}}(y, L_0) &= a + \sum_{n=1}^{+\infty} d_{\mathbb{H}}(G_{\ell_n}(y), G_{\ell_{n-1}}(y)) \\
 &\leq a + d^{d-1} \cdot \sum_{n=1}^{+\infty} d_{\mathbb{H}}(G_{\ell_n+qk_n}(x), G_{\ell_{n-1}+qk_n}(x)) \\
 &< a + d^{d-1} \cdot \sum_{n=\ell_0+qk_1}^{+\infty} d_{\mathbb{H}}(G_{n+1}(x), G_n(x)) \\
 &< a + d_{\mathbb{H}}(G_{\ell_0+qk_1}(x), G_{\ell_0}(x)) + d^{d-1} \cdot d_{\mathbb{H}}(x, G_{\ell_0+qk_1}(x)) \\
 &\leq d^{d-1} \cdot d_{\mathbb{H}}(x, L_0).
 \end{aligned}$$

$\square$

**7.2. Recurrent Orbits in the Hyperbolic Space.** — Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$  and consider  $x \in \mathcal{J}_P \setminus K$ . From Proposition 3.14 we know that  $\omega(x)$  contains a critical element. Nevertheless, using Proposition 7.1 we can be more precise.

**COROLLARY 7.5.** — *Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$  and consider  $x \in \mathcal{J}_P \setminus K$ . Then,  $\omega(x) \cap \mathbb{H}_K$  contains a recurrent critical element.*

*Proof.* — Let  $x \in \mathcal{J}_P \cap \mathbb{H}_K$ . From Proposition 3.14 there exists a critical point  $c_1$  in  $\omega(x)$ . As  $\mathcal{J}_P$  is a closed set,  $c_1 \in \mathcal{J}_P$ . By Proposition 7.1 it follows that

$$d_{\mathbb{H}}(c_1, L_0) \leq d^{d-1} \cdot d_{\mathbb{H}}(x, L_0) < +\infty.$$

Hence,  $c_1$  belongs to  $\mathbb{H}_K$ . Recursively, we can find a sequence of critical elements  $(c_n) \subseteq \mathcal{J}_P \cap \mathbb{H}_K$  such that  $c_{n+1} \in \omega(c_n)$ . By Proposition 2.14 there exists  $N$  such that  $c_N \in \omega(c_N)$ . That is,  $c_N$  is a recurrent critical element in  $\omega(x) \cap \mathbb{H}_K$ .  $\square$

By the corollary above, we need to study the recurrent critical elements in  $\mathcal{J}_P \cap \mathbb{H}_K$ .

In the case that  $\omega(x)$  contains a periodic critical orbit we will show, Proposition 7.6, that  $x$  is a preperiodic point. In Proposition 7.7, we will prove that the recurrent critical orbits in  $\mathcal{J}_P \cap \mathbb{H}_K$  are the periodic critical orbits.

**PROPOSITION 7.6.** — *Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$  and consider  $x \in \mathcal{J}_P$ . If the  $\omega$ -limit of  $x$  contains a periodic critical element, then  $x$  is preperiodic or  $x$  is a classical point.*

**PROPOSITION 7.7.** — *Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$ . Then the critical elements of  $P$  contained in  $\mathcal{J}_P \cap \mathbb{H}_K$  are preperiodic critical elements of type II.*

To show Proposition 7.7 we use a perturbation argument. To prove Proposition 7.6 we need the following lemma about geometric sequences.

**LEMMA 7.8.** — *Let  $P \in K[z]$  be a tame polynomial of degree  $\geq 2$  with  $q$  generators for its geometric sequences. Consider  $c \in \text{Crit}(P) \cap \mathcal{J}_P$  a critical element which is fixed by  $P$ . Let  $x$  be a Julia point such that  $G_n(x) = G_n(c)$  and  $G_{n+1}(x)$  is noncritical for some  $n > q + q\mathcal{N}(c)$ . Then  $P(G_{n+1}(x))$  is not critical.*

*Proof.* — We first show that if  $P(G_{n+1}(x))$  is critical it coincides with  $G_{n+1-q}(c)$ , then we will show that this is impossible.

By the properties of  $\mathcal{N}(c)$  (see Proposition 3.12) we have that if a geometric point  $G_\ell$  is critical and  $G_\ell \prec G_{n-q}(c)$  then  $G_\ell = G_\ell(c)$ . We know that  $P(G_{n+1}(x)) \prec G_{n-q}(c)$ . Then if we suppose that  $P(G_{n+1}(x))$  is critical, it follows that it coincides with  $G_{n-q+1}(c)$ .



Now suppose that  $P(G_{n+1}(x)) = G_{n+1-q}(c)$ . Since  $G_{n+1}(x)$  is different than  $G_{n+1}(c)$  it follows, using the Riemann-Hurwitz formula (see 2.7), that

$$\deg_{G_n(c)}(P) \geq 1 + \deg_{G_{n+1}(c)}(P),$$

which contradicts the definition of  $\mathcal{N}(c)$ , since the local degrees of  $P$  at  $G_n(c)$  and at  $G_{n+1}(c)$  coincide. Therefore  $P(G_{n+1}(x))$  is not critical.  $\square$

*Proof of Proposition 7.6.* — We assume that  $P$  is a tame polynomial with  $q$  generators for its geometric sequences.

Suppose that there exists a nonpreperiodic element  $x \in \mathcal{J}_P$  such that the  $\omega$ -limit of  $x$  contains a periodic critical element  $c$ . Passing to an iterate we can suppose that  $c$  is a fixed point.

Let  $(\ell_n)$  be the level sequence from  $x$  to  $c$  and  $(k_n)$  be the time sequence from  $x$  to  $c$ .

Since  $1 + \ell_n + qk_n > \ell_{n-1} + qk_n$  we can use Lemma 7.4 to conclude that

$$(5) \quad \deg_{G_{1+\ell_n+qk_n}(x)}(P^{k_n}) \leq d^{d-1}$$

for all  $n \geq 1$ .

Consider  $\mathcal{N} = \max\{\mathcal{N}(x), \mathcal{N}(c)\}$ . If  $\ell_n > q + q\mathcal{N}$  the points

$$G_{1+\ell_n}(P^{k_n}(x)) = P^{k_n}(G_{1+\ell_n+qk_n}(x)) \text{ and } G_{\ell_n}(P^{k_n}(x)) = P^{k_n}(G_{\ell_n+qk_n}(x))$$

satisfy the hypothesis of Lemma 7.8, since  $G_{\ell_n}(P^{k_n}(x)) = G_{\ell_n}(c)$  and

$$G_{1+\ell_n}(P^{k_n}(x)) = P^{k_n}(G_{1+\ell_n+qk_n}(x))$$

is not a critical element, by definition of  $(\ell_n)$  and  $(k_n)$  (see Definition 7.2). Hence

$$P(G_{1+\ell_n}(P^{k_n}(x)))$$

is not a critical element, that is  $\deg_{G_{1+\ell_n}(P^{k_n}(x))}(P^2) = 1$ . If  $\ell_n - q > q + q\mathcal{N}$  the points

$$G_{1+\ell_n-q}(P^{k_n+1}(x)) = P(G_{1+\ell_n}(P^{k_n}(x)))$$

and

$$G_{\ell_n-q}(P^{k_n+1}(x)) = P(G_{\ell_n}(P^{k_n}(x))) = P(G_{\ell_n}(c)) = G_{\ell_n-q}(c)$$

satisfy the hypothesis of Lemma 7.8 by the definition of the level and time sequences. Hence

$$P(G_{1+\ell_n-q}(P^{k_n+1}(x))) = P^2(G_{1+\ell_n}(P^{k_n}(x)))$$

is not a critical element, that is  $\deg_{G_{1+\ell_n-q}(P^{k_n+1}(x))}(P^3) = 1$ . Applying Lemma 7.8 recursively we obtain that the elements in the set

$$\{P(G_{1+\ell_n}(P^{k_n}(x))), P^2(G_{1+\ell_n}(P^{k_n}(x))), \dots, P^{e_n-1}(G_{1+\ell_n}(P^{k_n}(x)))\}$$

are not critical, where  $e_n$  is the smallest positive integer such that

$$\ell_n - q(e_n - 2) \leq q + q\mathcal{N}.$$

In particular, it follows that

$$(6) \quad \deg_{G_{1+\ell}(P^{k_n}(x))}(P^{e_n}) = \deg_{G_{1+\ell_n+qk_n}(x)}(P^{k_n+e_n}) = 1.$$

If we have that  $\ell_n - q(e_n - 2) \leq q + q\mathcal{N}$  it follows that

$$\ell_n - qe_n \leq -q + q\mathcal{N} \leq q\mathcal{N},$$

since  $1 \leq q$ . Then

$$\begin{aligned} \deg_{G_{1+\ell_n-qe_n}}(P^{\lfloor \ell_n/q \rfloor - e_n + 1}) &= \deg_{G_{1+\ell_n-qe_n}}(P^{\lfloor (\ell_n - qe_n)/q \rfloor + 1}) \\ &\leq \deg_{G_{1+\ell_n-qe_n}}(P^{\lfloor q\mathcal{N}/q \rfloor + 1}) \\ &= \deg_{G_{1+\ell_n-qe_n}}(P^{\mathcal{N}+1}) \\ &\leq d^{\mathcal{N}+1} \end{aligned}$$

that is

$$(7) \quad \deg_{G_{1+\ell_n-qe_n}}(P^{\lfloor \ell_n+qe_n/q \rfloor + 1}) = \deg_{G_{\ell_n-qe_n}}(P^{\lfloor \ell_n/q \rfloor + e_n + 1}) \leq d^{\mathcal{N}+1}$$

for any geometric point of level  $1 + \ell_n - qe_n$ .

If

$$\Delta_n = \deg_{G_{1+\ell_n+qk_n}(x)}(P^{\lfloor (\ell_n+qk_n)/q \rfloor + 1})$$

we have that

$$\begin{aligned} \Delta_n &= \deg_{G_{1+\ell_n+qk_n}(x)}(P^{\lfloor \ell_n/q \rfloor + k_n + 1}) \\ &= \deg_{G_{1+\ell_n+qk_n}(x)}(P^{k_n}) \cdot \deg_{P^{k_n}(G_{1+\ell_n+qk_n}(x))}(P^{e_n}) \\ &\quad \cdot \deg_{P^{k_n+e_n}(G_{1+\ell_n+qk_n}(x))}(P^{\lfloor \ell_n/q \rfloor + e_n + 1}) \\ &= \deg_{G_{1+\ell_n+qk_n}(x)}(P^{k_n}) \cdot \deg_{G_{1+\ell_n}(P^{k_n}(x))}(P^{e_n}) \\ &\quad \cdot \deg_{G_{1+\ell_n-qe_n}(P^{k_n+e_n}(x))}(P^{\lfloor \ell_n/q \rfloor + e_n + 1}) \end{aligned}$$

by (5), (6) and (7) we conclude that

$$(8) \quad \deg_{G_{1+\ell_n+qk_n}(x)}(P^{\lfloor (1+\ell_n+qk_n)/q \rfloor + 1}) \leq d^{d-1} \cdot 1 \cdot d^{\mathcal{N}+1} = d^{d+\mathcal{N}}$$

for all  $n \geq M$  where  $M \in \mathbb{N}$  is such that  $\ell_M > q + q\mathcal{N}$ .

Following Proposition 4.10 and (8) above, we have that  $\deg_y(P^{\lfloor (\ell_n+qk_n)/q \rfloor + 1})$  is constant and bounded by  $d^{d+\mathcal{N}}$  for any point  $y$  in the segment

$$[G_{1+\ell_n+qk_n}(x), G_{\ell_n+qk_n}(x)].$$

Since  $P^{\lfloor (\ell_n+qk_n)/q \rfloor + 1}([G_{1+\ell_n+qk_n}(x), G_{\ell_n+qk_n}(x)])$  belong to  $\{I_0, I_1, \dots, I_{q-1}\}$  (see discussion below Definition 4.5) there exists a segment  $I \in \{I_0, I_1, \dots, I_{q-1}\}$  and an increasing sequence  $(n_j)$  of integers larger than  $M$  such that

$$P^{\lfloor (\ell_{n_j}+qk_{n_j})/q \rfloor + 1}([G_{1+\ell_{n_j}+qk_{n_j}}(x), G_{\ell_{n_j}+qk_{n_j}}(x)]) = I,$$

for all  $j \geq 1$ .

Therefore, following Lemma 2.5 and using (8), we have that

$$\begin{aligned} d_{\mathbb{H}}(x, L_0) &\geq \sum_{j=1}^{+\infty} d_{\mathbb{H}}(G_{1+\ell_{n_j}+qk_{n_j}}(x), G_{\ell_{n_j}+qk_{n_j}}(x)) \\ &\geq \sum_{j=1}^{+\infty} \frac{|I|}{d^{d+\mathcal{N}}} \\ &= +\infty, \end{aligned}$$

where  $|I|$  denotes the length of the segment  $I$ . It follows,  $x$  is a classical point.  $\square$

Recall that we are assuming that  $k \subseteq K$  is a complete field such that the restriction of  $|\cdot|$  to  $k$  is a discrete absolute value and that

$$\{z \in K \mid [k(z) : k] < +\infty\}$$

is a dense subset of  $K$ .

In order to prove Proposition 7.7 we need a proposition to relate the dynamics of a polynomial  $P \in K[z]$  with the dynamics of a small perturbation  $Q$  of  $P$ , of degree  $d$  and with coefficients in a finite extension of  $k$ .

**PROPOSITION 7.9.** — *Let  $P \in K[z]$  be a tame polynomial of degree  $d \geq 2$  and let  $0 < \varepsilon < \text{diam}(L_0)$ . Then, there exists a tame polynomial  $Q \in k^a[z]$  such that  $P(x) = Q(x)$  and  $\deg_x(P) = \deg_x(Q)$  for all  $x$  in  $\overline{D_0} \cap \{\text{diam}(x) > \varepsilon\}$ .*

To prove this Proposition we need to establish some notation. Let

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \in K[z]$$

be a tame polynomial of degree  $d \geq 2$ . Since  $K$  is an algebraically closed field, we can write  $P'(z) = a_d \cdot d \cdot (z - w_1)^{n_1} \cdot (z - w_2)^{n_2} \cdots (z - w_\ell)^{n_\ell}$ , where  $w_i \neq w_j$  if  $i \neq j$ . For any  $0 < \delta < \min_{i \neq j} |w_j - w_i|$  we consider

$$V_\delta = \bigcup_{j=1}^{\ell} \mathcal{B}(w_j, \delta) \quad \text{and} \quad W_\delta = P^{-1}(P(V_\delta)).$$

Note that  $W_\delta$  is a finite union of open balls and that  $\max\{\text{diam}(x) \mid x \in W_\delta\}$  goes to zero when  $\delta$  goes to zero.

For convenience, we establish the following straightforward facts about  $V_\delta$  and  $W_\delta$  as a lemma.

LEMMA 7.10. — Let  $P \in K[z]$  be a tame polynomial with degree  $d > 2$ . Let  $\mathcal{D} = \mathcal{B}^+(0, R)$  be a closed ball of the Berkovich line containing  $D_0$  and the roots of  $P$ . Let  $m_\delta = \min\{|P|(x) \mid x \in \mathcal{D} \setminus V_\delta\}$ . Then the following statements hold:

1. For any  $a \in \mathbb{A}_K^{1,\text{an}} \setminus W_\delta$  the polynomial  $P(z) - P(a)$  is separable and its roots belong to  $\mathbb{A}_K^{1,\text{an}} \setminus V_\delta$ .
2. If  $\alpha, \beta$  are distinct roots of  $P(z) - P(a)$  we have that

$$|\alpha - \beta| \geq |a_d|^{-1} \cdot R^{2-d} \cdot m_\delta.$$

3. Let  $r_\delta = |a_d|^{-1} \cdot R^{2-d} \cdot m_\delta$ . Denote by  $U_\delta(a)$  the union of  $d$  open balls of radius  $r_\delta$  centered in the roots of  $P(z) - P(a)$  it follows that

$$|P - P(a)| \geq m_\delta \cdot r_\delta$$

$$\text{on } \mathbb{A}_K^{1,\text{an}} \setminus U_\delta(a)$$

To prove Proposition 7.9 we use the notation of the previous Lemma.

Recall that if  $P$  is polynomial in  $K[z]$  and  $x_B \in \mathbb{A}_K^{1,\text{an}}$  is a type II point, then  $|P|(x_B) = \sup\{|P(z)| \mid z \in B\}$ . Using continuity we extend definition of  $|P|(x)$  to other types of points.

*Proof of Proposition 7.9.* — Let

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \in K[z]$$

be a tame polynomial of degree  $d \geq 2$ . Let  $\mathcal{D} = \mathcal{B}^+(0, R)$  be a closed ball of the Berkovich line containing  $D_0$  and the roots of  $P$ .

Since  $K$  is an algebraically closed field, we can write

$$P'(z) = a_d \cdot d \cdot (z - w_1)^{n_1} \cdot (z - w_2)^{n_2} \cdots (z - w_\ell)^{n_\ell},$$

where  $w_i \neq w_j$  if  $i \neq j$  and  $n_j$  is the multiplicity of  $w_j$ .

Let  $b \in K$  such that  $0 < |P'(b)| \leq 1$  and consider

$$0 < \delta < \min \left\{ \min_{i \neq j} |w_j - w_i|, \min_j |b - w_j| \right\}.$$

Note that by this choice on  $\delta$  we have that  $m_\delta \leq 1$ .

Let  $0 < \eta < \delta$ . Since  $k^a$  is dense in  $K$ , we can pick  $w'_1, \dots, w'_\ell \in k^a$  such that  $|w_j - w'_j| < \eta$ , and  $|w_j| = |w'_j|$  for all  $1 \leq j \leq \ell$ . Moreover, pick  $b_d, b_0 \in k^a$  such that  $|b_j - a_j| < \eta$  and  $|a_j| = |b_j|$  for  $j = 0, d$ . Let  $Q(z) \in k^a[z]$  be the formal primitive of

$$b_d \cdot d \cdot (z - w'_1)^{n_1} (z - w'_2)^{n_2} \cdots (z - w'_\ell)^{n_\ell}$$

such that  $Q(0) = b_0$ . Note that by the choice of  $w_j, b_d$  and  $b_0$  we have that  $P$  is a tame polynomial and  $\deg_x(P) = \deg_x(Q)$  for all  $x \in \mathbb{A}_K^{1,\text{an}}$  such that  $\text{diam}(x) \geq \eta$ .

The functions  $|P'|$  and  $|Q'|$  coincide on  $\mathbb{A}_K^{1,\text{an}} \setminus V_\delta$ . In particular, if  $W'_\delta = Q^{-1}(Q(V_\delta))$  then Lemma 7.10 holds for  $Q$  and  $a \in \mathbb{A}_K^{1,\text{an}} \setminus W'_\delta$ . We denote by  $U'_\delta(a)$  the union of  $d$  open balls of radius  $r_\delta$  centered in the roots of  $Q(z) - Q(a)$ .

If we choose  $\eta$  small enough, we can assume that  $|P - Q| < m_\delta \cdot r_\delta$  on the ball  $\mathcal{D}$ . For any  $a \in \mathcal{D}$  it follows that

$$|(P - P(a)) - (Q - Q(a))| < m_\delta \cdot r_\delta$$

on  $\mathcal{D}$ .

Note that if  $a \notin W_\delta \cup W'_\delta$  then  $U_\delta(a) = U'_\delta(a)$ . In fact, if  $\alpha$  is a root of  $P(z) - P(a)$ , we have that  $|P(\alpha) - P(a)| < m_\delta \cdot r_\delta$ , that is,  $\alpha$  belongs to  $U'_\delta(a)$ . By symmetry we can conclude that  $U_\delta(a) = U'_\delta(a)$ .

In particular if  $a \notin W_\delta \cup W'_\delta$ , the functions  $|P - P(a)|$  and  $|Q - Q(a)|$  coincide on the boundary of  $U_\delta(a)$ . Moreover, their also coincide on  $\partial\mathcal{D}$ . It follows from the maximum principle that  $|P - P(a)| = |Q - Q(a)|$  on  $\mathcal{D} \setminus U_\delta(a)$ .

Let  $a \notin W_\delta \cup W'_\delta$  and consider  $r_\delta < r$ . Let  $x = x_B$ , where  $B = B_r^+(a)$ . If  $\hat{r} = |P - P(a)|(x) = |Q - Q(a)|(x)$ , it follows that

$$P(B) = B_{\hat{r}}^+(P(a)) \quad \text{and} \quad Q(B) = B_{\hat{r}}^+(Q(a)),$$

as well as  $|P(a) - Q(a)| < m_\delta \cdot r_\delta \leq \hat{r}$ , hence  $P(B) = B$  and therefore  $P(x) = Q(x)$ .

Let  $\delta' = \sup\{\text{diam}(x) \mid x \in W_\delta \cup W'_\delta\}$ . Note that any type II point  $x \in \mathcal{D}$  with diameter  $r > \max\{\delta, \delta'\}$  can be written  $x = x_B$ , where  $B = B_r(a)$  with  $a \notin W_\delta \cup W'_\delta$ , that is  $P$  coincides with  $Q$  on the type II points in  $\mathcal{D} \cap \{\text{diam}(x) > \max\{\delta, \delta'\}\}$ . Using continuity we have that  $P = Q$  on  $\mathcal{D} \cap \{\text{diam}(x) > \max\{\delta, \delta'\}\}$ . Since  $\delta'$  goes to 0 when  $\delta$  goes to 0, it follows that  $P(x) = Q(x)$  for all  $x$  in  $\overline{D_0} \cap \{\text{diam}(y) > \varepsilon\}$   $\square$

LEMMA 7.11. — *Let  $P \in K[z]$  be a tame of degree  $d \geq 2$ . Suppose that there exists a nonperiodic recurrent critical element  $c \in \mathcal{J}_P$ . Then  $c$  belongs to  $K$ .*

*Proof.* — Seeking a contradiction, suppose that there exists a recurrent and nonperiodic critical element  $c \in \mathcal{J}_P \cap \mathbb{H}_K$ . In view of the recurrence of  $c$  we have that  $P^n(c)$  belongs to the  $\omega$ -limit of  $c$  for all  $n \in \mathbb{N}$ . Following Proposition 7.1 we obtain that

$$d_{\mathbb{H}}(L_0, P^n(c)) \leq d^{d-1} \cdot d_{\mathbb{H}}(L_0, c),$$

for all  $n \in \mathbb{N}$ . In particular,  $0 < \inf\{\text{diam}(P^n(c)) \mid n \in \mathbb{N} \cup \{0\}\}$ .

Let  $0 < R < \inf\{\text{diam}(P^n(c)) \mid n \in \mathbb{N} \cup \{0\}\}$ . By Proposition 7.9 we have that there exists a tame polynomial  $Q \in k^a[z]$  of degree  $d$  such that  $Q$  coincides with  $P$  in

$$\overline{D_0} \cap \{\text{diam}(y) > R\} \cap P^{-1}(\{\text{diam}(y) > R\}).$$

In particular, the dynamical sequence  $(L_n(c))$  of  $P$  is also a dynamical sequence of  $Q$ , it follows that  $c$  belongs to the Julia set of  $Q$ . Then,  $c \in \mathcal{J}_Q \setminus K$  is a nonperiodic and recurrent algebraic element (since it is critical). Moreover,  $c$  is not strictly preperiodic, since it is recurrent. Which contradicts Proposition 6.7, therefore  $c$  belongs to  $\mathcal{J}_P^I$ .  $\square$

*Proof of Proposition 7.7.* — We proceed by contradiction. Suppose that there exists a critical element  $c \in \mathcal{J}_P \cap \mathbb{H}_K$  that is not preperiodic. According to Corollary 7.5 there exists a recurrent critical element  $c_1$  in  $\omega(c) \cap \mathbb{H}_K$  which, by Proposition 7.6, is not a periodic point. Applying Lemma 7.11 we have that  $c_1 \in K$ , which is impossible.

Therefore  $c$  is a preperiodic critical element. The proposition follows since periodic critical elements are of type II by Proposition 2.12.  $\square$

**7.3. Proof of Theorem A and Corollary B.** — In this subsection we prove a slightly different version of Theorem A and we obtain some corollaries.

**THEOREM 7.12.** — *Let  $P$  be a nonsimple and tame polynomial with coefficients in  $K$  of degree  $d \geq 2$ . Then  $\mathcal{J}_P \setminus K$  is empty or*

$$\mathcal{J}_P \setminus K = \text{GO}(x_1) \sqcup \cdots \sqcup \text{GO}(x_m),$$

where  $1 \leq m \leq d - 2$  and  $x_1, \dots, x_m \in \mathbb{H}_K$  are periodic critical elements.

*Proof.* — Consider  $x \in \mathcal{J}_P \cap \mathbb{H}_K$ . From Corollary 7.5 we have that  $\omega(x)$  contains a nonclassical recurrent critical element. Using Proposition 7.7 and Proposition 7.6 we have that  $x$  is in the backward orbit of a periodic critical element. Following Proposition 2.14 there exist, at most,  $d - 2$  critical elements contained in the Julia set of  $P$ . Now the theorem follows.  $\square$

Corollary B follows directly from Theorem 7.12 applying Proposition 2.16.

**REMARK 7.13.** — If we consider a nonsimple and tame polynomial  $P \in K[z]$  but we study its action in the spherical completion of  $K$ , we obtain again Corollary B. This is not true for non-tame polynomials. The Example 6.3 in [20] shows that for the non-tame polynomial

$$f(z) = \frac{1}{p}(z^p - z^{p^2}) \in \mathbb{C}_p[z],$$

we have that

$$J_f \subseteq \{x \in \mathbb{A}_{\mathbb{C}_p}^{1,\text{an}} \mid \text{diam}(x) = p^{-\frac{1}{p-1}}\},$$

moreover, the type II points in  $J_f$  are preperiodic and the type IV points in  $J_f$  are not preperiodic. Hence, if we consider the action of  $f$  on the affine line in the sense of Berkovich associated with the spherical completion of  $\mathbb{C}_p$  we have

that there exist wandering domains which are not attracted to an attracting cycle.

The following corollary is about the equilibrium measure of the classical Julia set and the relation between the *topological entropy* ( $h_{\text{top}}$ ) and the *metric entropy* ( $h_{\rho_P}$ ) of  $P$ , see [13] for the definitions of topological and metric entropy. The first statement follows from the countability of  $\mathcal{J}_P \cap \mathbb{H}_K$ . The second statement is a direct consequence of Theorem D in [13].

**COROLLARY 7.14.** — *Let  $P \in K[z]$  be a nonsimple and tame polynomial of degree  $d \geq 2$ . Then, the following statements hold:*

1. *The classical Julia set has full measure, that is,  $\rho_P(\mathcal{J}_P^I) = 1$ .*
2. *The equilibrium measure  $\rho_P$  is a measure of maximal entropy and*

$$h_{\rho_P} = h_{\text{top}} = \log(d).$$

Theorem 7.12 and Corollary 7.14 are not valid for rational maps, see examples in [13].

*Proof of Corollary C.* — By [1, Theorem 10.88] or [24] we know that  $\mathcal{J}_P$  is the closure of the repelling periodic points. Following Theorem 7.12 there are only finitely many such points in  $\mathcal{J}_P \setminus K$ , while the Julia set is infinite.  $\square$

*Proof of Corollary D.* — (1)  $\Rightarrow$  (2) Since  $\mathcal{J}_P^I$  is a compact and totally invariant set, disjoint of the classical exceptional set of  $P$ , we have that  $\mathcal{J}_P = \mathcal{J}_P^I \subseteq K$ . From Theorem 7.12, it follows that there are no critical periodic elements.

(2)  $\Rightarrow$  (3) There is no critical periodic elements in  $\mathcal{J}_P$ , then in view of Theorem 7.12 we have that  $\mathcal{J}_P \subseteq K$ . Therefore  $\mathcal{J}_P = \mathcal{J}_P^I$ .

(3)  $\Rightarrow$  (4) If  $\mathcal{J}_P = \mathcal{J}_P \cap K$  we have that  $\mathcal{J}_P = \mathcal{K}_P^I$ . Since any classical periodic point belongs to  $\mathcal{K}_P^I$ , it follows that them are repelling.

(4)  $\Rightarrow$  (1) We will show that  $\mathcal{J}_P^I$  is a compact set showing that  $\mathcal{J}_P^I = \mathcal{J}_P$ . According Theorem 7.12 if  $\mathcal{J}_P \neq \mathcal{J}_P^I$  then there exists a period  $n$  critical element  $x_B \in \mathcal{J}_P \setminus \mathcal{J}_P^I$ . It follows that  $P^n(B) = B$  and  $\deg_B(P^n) \geq 2$ , therefore following [17, Lemma 2.4], there exists a periodic point  $z_0$  of  $P$  in  $B$ . Since the classical periodic points are repelling we have that  $z_0$  belongs to  $\mathcal{J}_P$ . But  $z_0 \prec x_B$ , which is a contradiction with Lemma 2.15.  $\square$

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