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FLIPS OF MODULI OF STABLE TORSION FREE SHEAVES WITH $c_1 = 1$ ON $\mathbb{P}^2$

by Ryo Ohkawa

Dedicated to Takao Fujita on the occasion of his 60th birthday

Abstract. — We study flips of moduli schemes of stable torsion free sheaves E with $c_1(E) = 1$ on $\mathbb{P}^2$ via wall-crossing phenomena of Bridgeland stability conditions. They are described as stratified Grassmann bundles by a variation of stability of modules over certain finite dimensional algebra.

Résumé (*Flips de modules de faisceaux stables et sans torsion avec $c_1 = 1$ sur $\mathbb{P}^2$*)
Nous étudions des flips de schémas de modules de faisceaux stables et sans torsion E avec $c_1(E) = 1$ sur $\mathbb{P}^2$ à travers des phénomènes de traversée de mur des conditions de stabilité de Bridgeland. Ils sont décrits en tant que fibrés grassmanniens par une variation de stabilité de modules au-dessus d'une certaine algèbre de dimension finie.

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1. Introduction

1.1. Background. — We denote by $M_{\mathbb{P}^2}(r, c_1, n)$ the moduli scheme of semistable torsion free sheaves E on $\mathbb{P}^2$ with the Chern class $c(E) = (r, c_1, n) \in H^{2*}(\mathbb{P}^2, \mathbb{Z})$. In this paper we treat the case where $c_1 = 1$. Then semistability and stability for E coincide. When $n \geq r \geq 2$, or $r = 1$ and $n \geq 2$, the Picard number of $M_{\mathbb{P}^2}(r, 1, n)$ is equal to 2 and we have two birational morphisms from $M_{\mathbb{P}^2}(r, 1, n)$, which are described below.

One is defined by J. Li [12] for general cases. We denote by $M_{\mathbb{P}^2}(r, 1, n)_0$ the open subset of $M_{\mathbb{P}^2}(r, 1, n)$ consisting of stable *vector bundles*. The Uhlenbeck compactification $\overline{M}_{\mathbb{P}^2}(r, 1, n)$ of $M_{\mathbb{P}^2}(r, 1, n)_0$ is described set-theoretically by

$$\overline{M}_{\mathbb{P}^2}(r, 1, n) = \bigsqcup_{i \geq 0} \left(M_{\mathbb{P}^2}(r, 1, n-i)_0 \times S^i(\mathbb{P}^2) \right).$$

The map $\pi\colon M_{\mathbb{P}^2}(r, 1, n) \to \overline{M}_{\mathbb{P}^2}(r, 1, n), E \mapsto \pi(E)$ is defined by

$$\pi(E) = (E^{**}, \operatorname{Supp}(E^{**}/E)) \in M_{\mathbb{P}^2}(r, 1, n-i)_0 \times S^i(\mathbb{P}^2),$$

where E^{**} is the double dual of E and i is the length of E^{**}/E.

In the case where $r = 1$, this morphism is called the Hilbert-Chow morphism $\pi\colon (\mathbb{P}^2)^{[n]} \to S^n(\mathbb{P}^2)$. In the case where $r \geq 2$, this map is also birational since it is an isomorphism on $M_{\mathbb{P}^2}(r, 1, n)_0$ to its image. It is shown that the codimension of the complement of $M_{\mathbb{P}^2}(r, 1, n)_0$ is equal to 1 when we have $M_{\mathbb{P}^2}(r, 1, n-1) \neq \emptyset$ (cf. [13, Proposition 3.23]). Hence this map is a divisorial contraction.

The other birational morphism is defined by Yoshioka. In his paper [17] on moduli of torsion free sheaves on rational surfaces, he studied the following morphism:

$$\psi\colon M_{\mathbb{P}^2}(r, 1, n) \to M_{\mathbb{P}^2}(n+1, 1, n).$$

For any $E \in M_{\mathbb{P}^2}(r, 1, n)$, $\psi(E)$ is defined by the exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2})^* \otimes \mathcal{O}_{\mathbb{P}^2} \to \psi(E) \to E \to 0, \tag{1}$$

which is called the universal extension, where $\operatorname{Ext}^1_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2})^*$ is the dual vector space of $\operatorname{Ext}^1_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2})$. Here we have $\operatorname{Hom}_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2}) = \operatorname{Ext}^2_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2}) = 0$ and $(n+1, 1, n) \in H^{2*}(\mathbb{P}^2, \mathbb{Z})$ is the Chern class of $[E] - \chi(E, \mathcal{O}_{\mathbb{P}^2})[\mathcal{O}_{\mathbb{P}^2}] = [E] + \dim \operatorname{Ext}^1_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2})[\mathcal{O}_{\mathbb{P}^2}] \in K(\mathbb{P}^2)$, where

$$\chi(E, \mathcal{O}_{\mathbb{P}^2}) = \sum_i (-1)^i \dim_{\mathbb{C}} \operatorname{Ext}^i_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2}).$$

Furthermore, the moduli space $M_{\mathbb{P}^2}(r, 1, n)$ has a stratification

$$M_{\mathbb{P}^2}(r, 1, n) = \bigsqcup_{i=0}^{r} M^i_{\mathbb{P}^2}(r, 1, n),$$

where $M^i_{\mathbb{P}^2}(r,1,n) := \{E \in M_{\mathbb{P}^2}(r,1,n) \mid \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}, E) = i\}$ is called the *Brill-Noether locus*. The following theorem is shown in [17].

THEOREM 1.1 ([17, Theorem 5.8]). — (i) *There exists an isomorphism*

$$M^i_{\mathbb{P}^2}(r,1,n) \cong \psi^{-1}\left(M^{n-r+i+1}_{\mathbb{P}^2}(n+1,1,n)\right).$$

(ii) *The restriction of* ψ *to each stratum* $M^i_{\mathbb{P}^2}(r,1,n)$ *is a* $\operatorname{Gr}(n-r+i+1, i)$*-bundle over the stratum* $M^{n-r+i+1}_{\mathbb{P}^2}(n+1,1,n)$.

By the above theorem, if $n > r+2$, ψ is a birational morphism to the image $\operatorname{im}\psi$, and it is a flipping contraction. By the theory of birational geometry [3], we have the diagram called flip:

$$\begin{array}{ccc} M_+(r,1,n) & \dashleftarrow & M_{\mathbb{P}^2}(r,1,n) \\ & {\scriptstyle\psi_+}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \operatorname{im}\psi & \end{array} \tag{2}$$

The purpose of this note is to describe spaces $M_+(r,1,n)$, $\operatorname{im}\psi$ and the morphism ψ_+ in the above diagram in terms of moduli spaces. We follow ideas in [15]. We consider $M_{\mathbb{P}^2}(r,1,n)$ as a moduli scheme of semistable modules over the finite dimensional algebra

$$B := \operatorname{End}_{\mathbb{P}^2}\left(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \Omega_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}(2)\right)$$

via Bridgeland stability conditions on $D^b(\mathbb{P}^2)$. This enables us to study the wall-crossing phenomena of the moduli scheme as the stability changes by using the result of [15].

1.2. Main results. —

We introduce the exceptional collection

$$\mathfrak{E} := \left(\mathcal{O}_{\mathbb{P}^2}(1), \Omega^1_{\mathbb{P}^2}(3), \mathcal{O}_{\mathbb{P}^2}(2)\right)$$

on $\mathbb{P}^2$ and put $\mathcal{E} := \mathcal{O}_{\mathbb{P}^2}(1) \oplus \Omega^1_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ and $B := \operatorname{End}_{\mathbb{P}^2}(\mathcal{E})$. We denote abelian categories of coherent sheaves on $\mathbb{P}^2$ and finitely generated right B-modules by $\operatorname{Coh}(\mathbb{P}^2)$ and mod-B, respectively. Then by Bondal's Theorem [4], the functor $\Phi := \mathbf{R}\operatorname{Hom}_{\mathbb{P}^2}(\mathcal{E}, -)$ gives an equivalence

$$\Phi\colon D^b(\mathbb{P}^2) \cong D^b(B),$$

where $D^b(\mathbb{P}^2)$ and $D^b(B)$ are bounded derived categories of $\operatorname{Coh}(\mathbb{P}^2)$ and mod-B, respectively. The equivalence Φ also induces an isomorphism $\varphi\colon K(\mathbb{P}^2) \cong K(B)$ between Grothendieck groups of $\operatorname{Coh}(\mathbb{P}^2)$ and mod-B.

For a class α in $K(B)$, we put

$$\alpha^\perp := \{\theta \in \operatorname{Hom}_{\mathbb{Z}}(K(B), \mathbb{R}) \mid \theta(\alpha) = 0\}.$$

Any $\theta \in \alpha^{\perp}$ defines a *stability condition* of B-modules E with $[E] = \alpha$ (cf. §2.2). We denote by $M_B(\alpha, \theta)$ the moduli space of θ-semistable B-modules E with $[E] = \alpha$. In particular, we take a class

$$\alpha_r = \alpha_{r,n} := \varphi\left(n\mathcal{O}_{\mathbb{P}^2}(-1)[2] + (2n + r - 1)\mathcal{O}_{\mathbb{P}^2}[1] + (n - 1)\mathcal{O}_{\mathbb{P}^2}(1)\right)$$

in $K(B)$ so that the Chern class $c\left(\varphi^{-1}(\alpha_r)\right) = -(r, 1, n) \in H^{2*}(\mathbb{P}^2, \mathbb{Z})$. Here we omit the subscript "n" (although α_r depends on n), since we almost always fix n in this paper.

There exists a *wall-and-chamber structure* on $\alpha_r^{\perp}$ (cf. §2.2). When n is large enough, in §3 we find two chambers $\mathcal{C}_-$, $\mathcal{C}_+$ and a wall $W_0 \subset \alpha_r^{\perp}$ between them such that the following propositions hold. We put

$$M_-(\alpha_r) := M_B(\alpha_r, \theta_-), \quad M_+(\alpha_r) := M_B(\alpha_r, \theta_+), \quad M_0(\alpha_r) := M_B(\alpha_r, \theta_0)$$

for any $\theta_- \in \mathcal{C}_-$, $\theta_+ \in \mathcal{C}_+$ and $\theta_0 \in C_0$. By using the Bridgeland stability, we get the following theorem as a variation of Le Potier's result [10].

Proposition 1.2 ([15, Main Theorem 1.3 (iii)]). — *We have an isomorphism*

$$M_{\mathbb{P}^2}(r, 1, n) \cong M_-(\alpha_r), \ E \mapsto \Phi(E[1]).$$

We automatically get the following diagram.

$$\begin{array}{ccccc} M_+(\alpha_r) & & & & M_-(\alpha_r) \\ & \searrow^{f_+} & & \swarrow^{f_-} & \\ & & M_0(\alpha_r) & & \end{array} \tag{3}$$

By analyzing this diagram, we see that diagrams (2) and (3) coincide up to isomorphism. In particular, we get the following proposition.

Proposition 1.3. — *We have isomorphisms.*

(i) $M_0(\alpha_r) \cong \operatorname{im} \psi$.

(ii) $M_+(\alpha_r) \cong M_+(r, 1, n)$ *if* $n > r + 2$.

Proofs of Proposition 1.3 (i) and (ii) are given in §3.2 and §3.5, respectively. Using the B-module $S_0 := \Phi(\mathcal{O}_{\mathbb{P}^2}[1])$, we define the *Brill-Noether loci*

$$M_-^i(\alpha_r) = \{E \in M_-(\alpha_r) \mid \dim_{\mathbb{C}} \operatorname{Hom}_B(S_0, E) = i\},$$

$$M_+^i(\alpha_r) = \{E \in M_+(\alpha_r) \mid \dim_{\mathbb{C}} \operatorname{Hom}_B(E, S_0) = i\}$$

similar to the ones in Yoshioka's theory. Our situation is analogous to [14] and we have our main theorem.

Theorem 1.4. — *Assume* $n \geq r+2$. *Then for each* i, *the following statements hold.*

(i) *The images* $f_+(M^i_+(\alpha_r))$ *and* $f_-(M^i_-(\alpha_r))$ *coincide in* $M_0(\alpha_r)$. *We put*

$$M^i_0(\alpha_r) := f_+(M^i_+(\alpha_r)) = f_-(M^i_-(\alpha_r)).$$

(ii) *We have isomorphisms* $M^i_0(\alpha_r) \cong M^0_-(\alpha_{r-i}) \cong M^0_+(\alpha_{r-i})$.
(iii) *We have isomorphisms* $M^i_+(\alpha_r) \cong f_+^{-1}(M^i_0(\alpha_r))$.
(iv) *The restriction of* f_+ *to each stratum* $M^i_+(\alpha_r) \to M^i_0(\alpha_r)$ *is a* $\mathrm{Gr}(n-r+i-2, i)$*-bundle over* $M^i_0(\alpha_r)$.

Note that $M_+(\alpha_r) \neq \emptyset$ if and only if $n \geq r+2$. When $n = r+2$, we have

$$M^0_-(\alpha_r) = M^0_+(\alpha_r) = M^0_0(\alpha_r) = \emptyset. \tag{4}$$

By Theorem 1.4 (iv), we see that $M_+(\alpha_r) = M_0(\alpha_r)$, and that $f_-\colon M_-(\alpha_r) \to M_0(\alpha_r)$ is a morphism to the lower dimensional moduli space

$$M_0(\alpha_r) = \bigsqcup_{i>0} M^i_0(\alpha_r) \cong \bigsqcup_{i>0} M^0_-(\alpha_{r-i}).$$

A proof of Theorem 1.4 (i) is given in §3.4 and the rest of the theorem is proven in §3.7. We also show a new proof of Theorem 1.1 in terms of B-modules via the isomorphism $M_{\mathbb{P}^2}(r,1,n) \cong M_-(\alpha_r)$ in §3.7. The equalities in (4) are proven in Proposition 3.6 (ii).

By these descriptions, we see that $M_+(r,1,n)$ is smooth and in §3.8 we get the recursion formula (21) for Hodge polynomials of moduli spaces

$$e\left(M_{\mathbb{P}^2}(r,1,n)\right) - e\left(M_+(r,1,n)\right) =$$
$$\sum_{i=1}^{r} \Big(e\left(\mathrm{Gr}(n-r+i+1,i)\right) - e\left(\mathrm{Gr}(n-r+i-2,i)\right)\Big) e\left(M^0_{\mathbb{P}^2}(r-i,1,n)\right).$$

Here $e(Y) := \sum_{p,q} e^{p,q}(Y) x^p y^q$ denotes the virtual Hodge polynomial for any variety Y (cf. [6]). Using this formula we can compute the Hodge polynomials of $M_+(r,1,n)$ from those of $M_{\mathbb{P}^2}(r,1,n)$ when $r \leq 2$.

We further expect that by repeating this procedure we get a minimal model program of $M_{\mathbb{P}^2}(r,1,n)$. In fact for each $i \in \mathbb{Z}$ we take

$$\mathscr{E}_i := \mathscr{O}_{\mathbb{P}^2}(i) \oplus \Omega^1_{\mathbb{P}^2}(i+2) \oplus \mathscr{O}_{\mathbb{P}^2}(i+1),$$

$\Phi_i := \mathbf{R}\mathrm{Hom}_{\mathbb{P}^2}(\mathscr{E}_i, -)\colon D^b(\mathbb{P}^2) \cong D^b(B)$, and the induced isomorphism $\varphi_i\colon K(\mathbb{P}^2) \cong K(B)$. We put $\alpha'_r := \varphi^{-1}(\alpha_r) \in K(\mathbb{P}^2)$. Then by [15, Proposition 5.4], for each $i \in \mathbb{Z}$ there exists a region

$$P_i \subset (\alpha'_r)^\perp := \{\theta' \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathbb{P}^2), \mathbb{R}) \mid \theta'(\alpha'_r) = 0\}$$

such that for any $\theta' \in P_i$ we have an isomorphism

$$\Phi_i \circ \Phi_{i-1}^{-1}\colon M_B(\varphi_{i-1}(\alpha'_r), \theta' \circ \varphi_{i-1}^{-1}) \cong M_B(\varphi_i(\alpha'_r), \theta' \circ \varphi_i^{-1}).$$

In this paper we study a variation of $M_B(\varphi_1(\alpha'_r), \theta' \circ \varphi_1^{-1})$ when θ' changes from P_2 to P_1. Since the *dimension vector* of $\varphi_i(\alpha'_r)$ (cf. §2.1) decreases when i decreases, eventually this procedure terminates for some i. We hope that this minimal model program will allow us to compute the Hodge polynomials of $M_{\mathbb{P}^2}(r,1,n)$ for arbitrary r.

The paper is organized as follows: In §2 we introduce a description of the Picard group of $M_{\mathbb{P}^2}(r,1,n)$ in terms of θ-stability conditions of right B-modules. In §3 we study the wall-crossing phenomena of moduli of θ-semistable right B-modules. This is described by stratified Grassmann bundles, giving a proof of Theorem 1.4. In Appendix A, we give a proof of Proposition 3.11 by using the Bridgeland stability and Yoshioka's results in [17].

Notation. We fix the following notation in the paper:
For a matrix A, we denote by ${}^tA$ the transpose of A. For a $\mathbb{C}$-vector space V, we denote by V^* the dual vector space $\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ of V and we also denote by $\mathrm{Gr}(V,i)$ the Grassmann manifold of i-dimensional subspaces of V. When $V = \mathbb{C}^k$, we put $\mathrm{Gr}(k,i) := \mathrm{Gr}(V,i)$.

We consider the polynomial ring $\mathbb{C}[x_0,x_1,x_2]$ and the tensor product

$$V \otimes \mathbb{C}[x_0,x_1,x_2]$$

with a vector space V. For any monomial $m \in \mathbb{C}[x_0,x_1,x_2]$, we put

$$V \otimes m := \{v \otimes m \in V \otimes \mathbb{C}[x_0,x_1,x_2] \mid v \in V\}.$$

We denote by x_i the ith embedding

$$V \to (V \otimes x_0) \oplus (V \otimes x_1) \oplus (V \otimes x_2)$$

and by x_i^* the ith projection

$$(V \otimes x_0) \oplus (V \otimes x_1) \oplus (V \otimes x_2) \to V$$

for $i = 0,1,2$. We also use the similar notation for vector bundles.

For any path algebra of a quiver with relations, we identify modules over the algebra with representations of the corresponding quiver with relations.

2. Picard group of $M_{\mathbb{P}^2}(r,1,n)$

We introduce an explicit description of the Picard group of $M_{\mathbb{P}^2}(r,1,n)$ in terms of B-modules.

2.1. Finite dimensional algebra B. — The finite dimensional algebra

$$B = \operatorname{End}_{\mathbb{P}^2}\left(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \Omega_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}(2)\right)$$

is written as the path algebra of the following quiver with relations (Q, J), where Q is defined as

$$Q := \overset{v_{-1}}{\bullet} \xleftarrow{\gamma_i} \overset{v_0}{\bullet} \xleftarrow{\delta_j} \overset{v_1}{\bullet} \qquad (i, j = 0, 1, 2)$$

and J is generated by the relations

(5) $$\gamma_i \delta_j + \gamma_j \delta_i = 0 \quad (i, j = 0, 1, 2).$$

We identify derived categories $D^b(\mathbb{P}^2)$ and $D^b(B)$, and groups $K(\mathbb{P}^2)$ and $K(B)$ via the equivalence

$$\Phi = \mathbf{R}\operatorname{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \Omega_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}(2), -) \colon D^b(\mathbb{P}^2) \cong D^b(B)$$

and the induced isomorphism $\varphi \colon K(\mathbb{P}^2) \cong K(B)$, respectively. For example, the same symbol S_i denotes both $\mathcal{O}_{\mathbb{P}^2}(i)[1-i]$ and the simple B-module $\mathbb{C}v_i = \Phi(\mathcal{O}_{\mathbb{P}^2}(i)[1-i])$ for each $i = -1, 0, 1$.

Then we have

$$K(B) = \mathbb{Z}[S_{-1}] \oplus \mathbb{Z}[S_0] \oplus \mathbb{Z}[S_1].$$

For $\alpha_{-1}, \alpha_0, \alpha_1 \in \mathbb{Z}$, by

(6) $$\alpha = \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{pmatrix} \in K(B)$$

we denote

$$\alpha = \alpha_{-1}[S_{-1}] + \alpha_0[S_0] + \alpha_1[S_1] \in K(B).$$

For $\theta^{-1}, \theta^0, \theta^1 \in \mathbb{R}$, by

(7) $$\theta = (\theta^{-1}, \theta^0, \theta^1) \in \operatorname{Hom}_{\mathbb{Z}}(K(B), \mathbb{R}),$$

we denote

$$\theta = \theta^{-1}[S_{-1}]^* + \theta^0[S_0]^* + \theta^1[S_1]^* \in \operatorname{Hom}_{\mathbb{Z}}(K(B), \mathbb{R}),$$

where $\{[S_{-1}]^*, [S_0]^*, [S_1]^*\}$ is the dual basis of $\{[S_{-1}], [S_0], [S_1]\}$. The vector in (6) is called the *dimension vector* of α.

2.2. Moduli of semistable B-modules. — For any $\alpha \in K(B)$ and $\theta \in \alpha^{\perp} \otimes \mathbb{R} \subset \mathrm{Hom}_{\mathbb{Z}}(K(B), \mathbb{R})$, we define θ-stability as follows: Here

$$\alpha^{\perp} = \{\theta \in \mathrm{Hom}_{\mathbb{Z}}(K(B), \mathbb{Z}) \mid \theta(\alpha) = 0\}.$$

DEFINITION 2.1. — A right B-module E with $[E] = \alpha$ in $K(B)$ is said to be *θ-semistable* if for any non-zero proper submodule $F \subset E$, the inequality $\theta(F) \geq \theta(E) = 0$ holds. If the inequality is always strict, then E is said to be *θ-stable*.

By $M_B(\alpha, \theta)$ we denote the moduli scheme of θ-semistable B-modules E with $[E] = \alpha$. We define a *wall-and-chamber structure* on $\alpha^{\perp} \otimes \mathbb{R}$ as follows: A *wall* is a ray $W = \mathbb{R}_{\geq 0}\theta^W$ in $\alpha^{\perp} \otimes \mathbb{R}$ such that there exists a θ^W-semistable B-module E having a proper submodule $F \subset E$ with $[F] \notin \mathbb{Q}_{>0}\alpha$ in $K(B)$ and $\theta^W(F) = 0$. A *chamber* is a connected component of $(\alpha^{\perp} \otimes \mathbb{R}) \setminus \bigcup W$, where W runs over the set of all walls in $\alpha^{\perp} \otimes \mathbb{R}$. For any chamber $\mathcal{C} \subset \alpha^{\perp} \otimes \mathbb{R}$, the moduli space $M_B(\alpha, \theta)$ does not depend on the choice of $\theta \in \mathcal{C}$.

Here we assume that $\alpha \in K(B)$ is indivisible and θ is not on any wall in $\alpha^{\perp}$. Then there exists a universal family $\mathcal{U}$ of B-modules on $M_B(\alpha, \theta)$

$$\mathcal{U} := \left(\mathcal{U}_{-1} \xrightarrow{\quad \gamma_i^* \quad} \mathcal{U}_0 \xrightarrow{\quad \delta_j^* \quad} \mathcal{U}_1 \right) \quad (i, j = 0, 1, 2), \tag{8}$$

where $\mathcal{U}_{-1}$, $\mathcal{U}_0$ and $\mathcal{U}_1$ are vector bundles corresponding to vertices v_{-1}, v_0, v_1 and $\gamma_i^* \colon \mathcal{U}_{-1} \to \mathcal{U}_0$, $\delta_j^* \colon \mathcal{U}_0 \to \mathcal{U}_1$ are morphisms corresponding to arrows γ_i, δ_j.

2.3. Deformations of B-modules. — We take $\alpha \in K(B)$ defined by (6). For any B-module E with $[E] = \alpha$, by choosing a basis of Ev_{-1}, Ev_0 and Ev_1, we identify E with a representation of the quiver (Q, J) with relations as follows: We consider E as a collection of $\mathbb{C}$-linear maps

$$E = (\mathbb{C}^{\alpha_{-1}} \overset{C_i}{\to} \mathbb{C}^{\alpha_0} \overset{D_j}{\to} \mathbb{C}^{\alpha_1}) \quad (i, j = 0, 1, 2), \tag{9}$$

such that they satisfy the relations

$$D_j C_i + D_i C_j = 0 \quad (i, j = 0, 1, 2),$$

where $C_i \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0})$ and $D_j \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1})$ correspond to actions of γ_i and δ_j, respectively. The pull back of the heart mod-B of the standard t-structure of $D^b(B)$ by Φ is the full subcategory

$$\mathcal{A} := \langle \mathcal{O}_{\mathbb{P}^2}(-1)[2], \mathcal{O}_{\mathbb{P}^2}[1], \mathcal{O}_{\mathbb{P}^2}(1) \rangle$$

of $D^b(\mathbb{P}^2)$. The complex of coherent sheaves on $\mathbb{P}^2$

$$\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus \alpha_{-1}} \xrightarrow{\sum_i C_i x_i} \mathcal{O}_{\mathbb{P}^2}^{\oplus \alpha_0} \xrightarrow{\sum_i D_j x_j} \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus \alpha_1}$$

corresponds to E in (9) via the equivalence Φ, where x_0, x_1, x_2 are homogeneous coordinates of $\mathbb{P}^2$.

By [15, Lemma 4.6 (1)], $\mathrm{Ext}^2_B(E,E)$ is isomorphic to the cokernel of the map

$$(10)\quad d\colon \left(\bigoplus_i \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0}) \otimes x_i\right) \bigoplus \left(\bigoplus_j \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1}) \otimes x_j\right) \to \bigoplus_{i\le j} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_1}) \otimes x_i x_j,$$

defined by

$$\left(\sum_i \xi_i \otimes x_i, \sum_j \eta_j \otimes x_j\right) \mapsto \sum_{i,j} (D_j \xi_i + \eta_j C_i) \otimes x_i x_j$$

for $\xi_i \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0})$ and $\eta_j \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1})$ for $i, j = 0, 1, 2$.

We study the deformation functor $\mathscr{D}_E\colon (\mathrm{Artin}/\mathbb{C}) \to (\mathrm{Sets})$. For any Artin local $\mathbb{C}$-ring R, the set $\mathscr{D}_E(R)$ consists of right $R \otimes B$-modules E^R such that E^R is flat as a R-module and $E^R \equiv E$ modulo the maximal ideal m_R of R. This means that E^R is a collection of R-linear maps

$$E^R = (R^{\alpha_{-1}} \overset{C_i^R}{\to} R^{\alpha_0} \overset{D_j^R}{\to} R^{\alpha_1}) \quad (i, j = 0, 1, 2)$$

such that they satisfy the relations $D_j^R C_i^R + D_i^R C_j^R = 0$, and

$$C_i^R \equiv C_i, \quad D_j^R \equiv D_j$$

modulo m_R for each $i, j = 0, 1, 2$. We show the following lemma.

Lemma 2.2. — *The deformation functor $\mathscr{D}_E$ has an obstruction theory with values in $\mathrm{Ext}^2_B(E,E)$.*

Proof. — For any small extension

$$0 \to \mathfrak{a} \to R' \to R \to 0$$

with $m_{R'}\mathfrak{a} = 0$ and $E^R = (R^{\alpha_{-1}} \overset{C_i^R}{\to} R^{\alpha_0} \overset{D_j^R}{\to} R^{\alpha_1}) \in \mathscr{D}_E(R)$, we write

$$C_i^R = C_i + \xi_i, \quad D_j^R = D_j + \eta_j$$

for $\xi_i \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0}) \otimes m_R$ and $\eta_j \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1}) \otimes m_R$. By the isomorphism $m_R \cong m_{R'}/\mathfrak{a}$, we take lifts

$$\xi_i' \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0}) \otimes m_{R'}, \quad \eta_j' \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1}) \otimes m_{R'}$$

of ξ_i, η_j, respectively and put

$$C_j^{R'} := C_j + \xi_i', \quad D_j^{R'} := D_j + \eta_j'.$$

Since $D_j^{R'}C_i^{R'} + D_i^{R'}C_j^{R'} \equiv D_j^R C_i^R + D_i^R C_j^R = 0$ modulo $\mathfrak{a}$, we have an element

$$\sum_{i,j} D_j^{R'}C_i^{R'} \otimes x_i x_j \in \bigoplus_{i\leq j} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_1}) \otimes x_i x_j \otimes \mathfrak{a}.$$

By the image of this element in the cokernel of (10) tensored by $\mathfrak{a}$, we define an element $\mathfrak{o}(E^R)$ in $\mathrm{Ext}^2_B(E,E)\otimes\mathfrak{a}$. This gives a well-defined map

$$\mathfrak{o}\colon \mathscr{D}_E(R) \to \mathrm{Ext}^2_B(E,E)\otimes\mathfrak{a},$$

and we easily see that E^R lifts to $\mathscr{D}_E(R')$ if and only if $\mathfrak{o}(E^R)=0$. □

2.4. Chambers $\mathscr{C}_{\mathbb{P}^2}$ and the Picard group of $M_{\mathbb{P}^2}(r,1,n)$. — We take

$$\alpha_r = \begin{pmatrix} n \\ 2n+r-1 \\ n-1 \end{pmatrix} \in K(B) \tag{11}$$

in the notation in (6) so that the Chern class

$$c(\alpha_r) = -(r,1,n) \in H^{2*}(\mathbb{P}^2,\mathbb{Z}).$$

In this subsection, we assume that $r>0$ and $M_{\mathbb{P}^2}(r,1,n)$ is not the empty set. Then there exists a chamber $\mathscr{C}_{\mathbb{P}^2} \subset \alpha_r^\perp \otimes \mathbb{R}$ such that $\Phi(\,\cdot\,[1])$ induces an isomorphism

$$M_{\mathbb{P}^2}(r,1,n) \cong M_B(\alpha_r,\theta) \tag{12}$$

for any $\theta \in \mathscr{C}_{\mathbb{P}^2}$ by [15, Theorem 5.1 and Proposition 5.4].

The chamber $\mathscr{C}_{\mathbb{P}^2}$ is characterized as follows: We put

$$\theta_{\mathbb{P}^2} := (-r-1, 1, -1+r), \quad \theta_0 := (-n+1, 0, n) \in \mathrm{Hom}_{\mathbb{Z}}(K(B),\mathbb{R})$$

in the notation in (7). By [15, Lemma 6.2], we have

$$\mathbb{R}_{>0}\theta_0 + \mathbb{R}_{>0}\theta_{\mathbb{P}^2} \subset \mathscr{C}_{\mathbb{P}^2}. \tag{13}$$

In the case where $r=1$ and $n\geq 2$, by [15, Lemma 6.3 (2)] we see that an equality holds in (13). In the case where $r\geq 2$, we will describe the chamber in §3.5.

Since α_r is indivisible by definition (11), for any $\theta\in\mathscr{C}_{\mathbb{P}^2}$ we have a universal family $\mathscr{U}$ on $M_B(\alpha_r,\theta)$ as in (8). We define a homomorphism from $\alpha_r^\perp$ to $\mathrm{Pic}\,(M_B(\alpha_r,\theta))$ by

$$\rho(\mathbf{m}) = m_{-1}\det(\mathscr{U}_{-1}) + m_0\det(\mathscr{U}_0) + m_1\det(\mathscr{U}_1),$$

for $\mathbf{m} = (m_{-1}, m_0, m_1) \in \alpha_r^\perp$ in the notation in (7). By (12) this gives a homomorphism

$$\rho\colon \alpha_r^\perp \to \mathrm{Pic}\,(M_{\mathbb{P}^2}(r,1,n)).$$

We have the following proposition, which is a particular case of [7, Theorems D,F].

PROPOSITION 2.3. — *We assume that $n \geq r$. Then, the above map $\rho\colon \alpha_r^\perp \to \operatorname{Pic}(M_{\mathbb{P}^2}(r,1,n))$ is an isomorphism. Furthermore $\rho(-3\theta_{\mathbb{P}^2})$ is the class of the canonical bundle of $M_{\mathbb{P}^2}(r,1,n)$.*

3. Proof of Theorem 1.4

In the following we assume that $r \geq 0$ and $n \geq 1$ except §3.5. As in §1.2 and the previous section, we take

$$\alpha_r = \alpha_{r,n} := \begin{pmatrix} n \\ 2n+r-1 \\ n-1 \end{pmatrix} \in K(B)$$

in the notation in (6) so that the Chern class $c(\alpha_r) = -(r,1,n)$ in $H^{2*}(\mathbb{P}^2,\mathbb{Z})$. Similarly in the notation in (7) we put

$$\theta_0 := \left(-n+1, 0, n\right) \in \alpha_r^\perp$$

so that we have $\theta_0(S_0) = 0$, and define a subset $W_0 := \mathbb{R}_{\geq 0}\theta_0$ of $\alpha_r^\perp$. For $\varepsilon > 0$, we define $\theta_-, \theta_+ \in \alpha^\perp$ by

$$\theta_- := \theta_0 - \varepsilon\left(2n-1+r, -n, 0\right), \quad \theta_+ := \theta_0 + \varepsilon\left(2n-1+r, -n, 0\right) \in \alpha^\perp$$

so that we have $\theta_-(S_0) > 0$ and $\theta_+(S_0) < 0$. We take ε small enough such that θ_-, θ_+ lie on no wall, and subsets

$$\mathbb{R}_{>0}\theta_- + \mathbb{R}_{\geq 0}\theta_0, \quad \mathbb{R}_{>0}\theta_+ + \mathbb{R}_{\geq 0}\theta_0$$

are contained in chambers $\mathcal{C}_-$ and $\mathcal{C}_+$, respectively. We put

$$M_-(\alpha_r) := M_B(\alpha_r, \theta_-), \quad M_+(\alpha_r) := M_B(\alpha_r, \theta_+), \quad M_0(\alpha_r) := M_B(\alpha_r, \theta_0).$$

When $r > 0$ and n is large enough, we show in §3.5 that W_0 is a wall and $\mathcal{C}_-$ and $\mathcal{C}_+$ are different chambers. Since $\theta_- \in \mathcal{C}_{\mathbb{P}^2}$ by (13), we have $\mathcal{C}_- = \mathcal{C}_{\mathbb{P}^2}$ and the isomorphism $M_{\mathbb{P}^2}(r,1,n) \cong M_-(\alpha_r)$ in (12).

However when $r = 0$, we show in §A.3 that W_0 is not a wall, and two chambers $\mathcal{C}_-$ and $\mathcal{C}_+$ coincide and contain W_0 for any $n \geq 1$.

In any case, we automatically get the following diagram:

$$(14) \qquad \begin{array}{ccccc} M_+(\alpha_r) & & & & M_-(\alpha_r) \\ & \searrow^{f_+} & & {}^{f_-}\swarrow & \\ & & M_0(\alpha_r) & & \end{array}$$

When $r > 0$ and $n > r+2$, we show that this diagram (14) coincides with the flip diagram (2) in §1.1 and we describe the diagram (14) by stratified Grassmann bundles. This gives a proof of Theorem 1.4.

3.1. Kronecker modules. — We consider the *3-Kronecker quiver*, which has 2 vertices v_{-1}, v_1 and 3 arrows $\beta_0, \beta_1, \beta_2$ from v_1 to v_{-1}

$$\overset{v_{-1}}{\bullet} \xleftarrow{\quad \beta_i \quad} \overset{v_1}{\bullet} \quad (i = 0, 1, 2).$$

and consider the path algebra T. Any right T-module G has a decomposition $G = Gv_{-1} \oplus Gv_1$ and actions of β_i define linear maps $Gv_{-1} \to Gv_1$ for $i = 0, 1, 2$. For $\theta_0 = (-n+1, 0, n) \in \mathbb{R}^3$, by abuse of notation we define $\theta_0(G) \in \mathbb{R}$ by

$$\theta_0(G) := (-n+1) \dim_{\mathbb{C}} Gv_{-1} + n \dim_{\mathbb{C}} Gv_1.$$

We denote by $K(T)$ the Grothendieck group of the abelian category of finitely generated right T-modules, and take $\alpha_T := n[\mathbb{C}v_{-1}] + (n-1)[\mathbb{C}v_1] \in K(T)$.

Definition 3.1. — A right T-module G with $[G] = \alpha_T \in K(T)$ is *stable* if and only if for any non-zero proper submodule G' of G we have $\theta_0(G') > 0$.

We denote by $M_T(\alpha_T)$ the moduli space of stable T-modules G with $[G] = \alpha_T$. For any B-module $E = \left(\mathbb{C}^n \overset{C_i}{\to} \mathbb{C}^{2n-1+r} \overset{D_j}{\to} \mathbb{C}^{n-1}\right)$, we define a T-module E_T by

$$E_T := \left(\mathbb{C}^n \overset{A_i}{\to} \mathbb{C}^{n-1}\right), \quad A_i := D_{i+2}C_{i+1}$$

for each $i \in \mathbb{Z}/3\mathbb{Z}$. By using the map $E \mapsto E_T$ and the simple B-module $S_0 = \mathbb{C}v_0$, we get the following criterion.

Lemma 3.2. — *For any B-module $E = \left(\mathbb{C}^n \overset{C_i}{\to} \mathbb{C}^{2n-1+r} \overset{D_j}{\to} \mathbb{C}^{n-1}\right)$, the following statements hold.*

(i) *E is θ_0-semistable if and only if E_T is stable.*
(ii) *E is θ_0-stable if and only if E_T is stable and*

$$\mathrm{Hom}_B(E, S_0) = \mathrm{Hom}_B(S_0, E) = 0.$$

(iii) *The following conditions are equivalent.*
  (a_-) *E is θ_--stable.*
  (b_-) *E is θ_--semistable.*
  (c_-) *E_T is stable and* $\mathrm{Hom}_B(E, S_0) = 0$.
(iv) *The following conditions are equivalent.*
  (a_+) *E is θ_+-stable.*
  (b_+) *E is θ_+-semistable.*
  (c_+) *E_T is stable and* $\mathrm{Hom}_B(S_0, E) = 0$.

Proof. — (i) For every submodule $F \subset E$, we have the submodule F_T of E_T and $\theta_0(F) = \theta_0(F_T)$. Conversely for any submodule G' of E_T, we define a submodule F of E such that $F_T = G'$ as follows: We put

$$Fv_{-1} := G'v_{-1}, \quad Fv_1 := G'v_1 \quad Fv_0 := \sum_i C_i(Fv_{-1}) \subset Ev_0.$$

By the relations (5), we have a submodule $F := Fv_{-1} \oplus Fv_0 \oplus Fv_1$ of E and $\theta_0(F) = \theta_0(F_T) = \theta_0(G')$. This yields the claim.
(ii) For any non-zero proper submodule $F \subset E$, the equality $\theta_0(F) = 0$ holds if and only if the dimension vector of F is equal to

$$\begin{pmatrix} 0 \\ l \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} n \\ l \\ n-1 \end{pmatrix} \in K(B)$$

for some $0 < l < 2n + r - 1$. Non-existence of such a F is equivalent to the equalities

$$\mathrm{Hom}_B(S_0, E) = \mathrm{Hom}_B(E, S_0) = 0.$$

(iii) ($(a_-) \implies (b_-)$) It is trivial.
($(b_-) \implies (c_-)$) We choose

$$\theta_- = \theta_0 - \varepsilon(2n - 1 + r, -n, 0) \in \alpha_r^\perp$$

for $\varepsilon > 0$ small enough. If E is θ_--semistable, then for any submodule $F \subset E$ we have $\theta_0(F) \geq 0$, since by assumption we have $\theta_-(F) \geq 0$ for arbitrary small $\varepsilon > 0$. This implies that E is θ_0-semistable and hence by (i), E_T is stable. Any non-zero element $\phi \in \mathrm{Hom}_B(E, S_0)$ breaks θ_--semistability of E. Hence we also have $\mathrm{Hom}_B(E, S_0) = 0$.
($(c_-) \implies (a_-)$) We assume that E_T is stable and $\mathrm{Hom}_B(E, S_0) = 0$. Hence for every non-zero proper submodule $F \subset E$, we have $\theta_0(F) = \theta_0(F_T) \geq 0$. If $\theta_0(F) = 0$, then $\mathrm{Hom}_B(E, S_0) = 0$ implies that $F \cong S_0^{\oplus l}$ for some $0 < l \leq 2n + r - 1$. In this case we have $\theta_-(F) = \varepsilon nl > 0$. If $\theta_0(F) > 0$, then we also have $\theta_-(F) > 0$ for ε small enough. Hence E is θ_--stable.
(iv) It is similar to the proof of (iii). □

3.2. Morphisms between moduli spaces.

— By Lemma 3.2, we have morphisms

$$\pi_-^r : M_-(\alpha_r) \to M_T(\alpha_T), \quad \pi_+^r : M_+(\alpha_r) \to M_T(\alpha_T), \quad E \mapsto E_T.$$

For θ_0-stability, we also see that the map $E \mapsto E_T$ is independent of representatives E of any S-equivalence class up to isomorphism of T-modules E_T. Hence we get a morphism

$$\pi_0^r : M_0(\alpha_r) \to M_T(\alpha_T), \quad E \mapsto E_T$$

and this map is set-theoretically injective.

LEMMA 3.3. — *The morphism $\pi_0^r \colon M_0(\alpha_r) \to M_T(\alpha_T)$ is a closed embedding.*

Proof. — The morphism π_0^r is induced from a homomorphism of graded rings of invariant sections, and therefore is an affine morphism. Since both $M_0(\alpha_r)$ and $M_T(\alpha_T)$ are projective, π_0^r is finite. Since π_0^r is set-theoretically injective, the claim holds. □

Furthermore we easily see that when $r = n+1$ and $r = n-2$, the morphisms

$$\pi_-^{n+1} \colon M_-(\alpha_{n+1}) \to M_T(\alpha_T), \quad \pi_+^{n-2} \colon M_+(\alpha_{n-2}) \to M_T(\alpha_T)$$

are isomorphisms. Inverse maps

$$E_T = \left(\mathbb{C}^n \overset{A_i}{\to} \mathbb{C}^{n-1}\right) \mapsto \left(\pi_-^{n+1}\right)^{-1}(E_T) = \left(\mathbb{C}^n \overset{C_i^-}{\to} \mathbb{C}^{3n} \overset{D_j^-}{\to} \mathbb{C}^{n-1}\right)$$

$$E_T = \left(\mathbb{C}^n \overset{A_i}{\to} \mathbb{C}^{n-1}\right) \mapsto \left(\pi_+^{n-2}\right)^{-1}(E_T) = \left(\mathbb{C}^n \overset{C_i^+}{\to} \mathbb{C}^{3n-3} \overset{D_j^+}{\to} \mathbb{C}^{n-1}\right)$$

are defined as follows:

For π_-^{n+1}, using an identification $\mathbb{C}^{3n} = \bigoplus_j (\mathbb{C}^n \otimes x_j)$, we put

$$C_i^- := x_i \colon \mathbb{C}^n \to \mathbb{C}^{3n},$$
$$D_i^- := A_{i+1} \circ x_{i+2}^* - A_{i+2} \circ x_{i+1}^* \colon \mathbb{C}^{3n} \to \mathbb{C}^{n-1},$$

such that we have equalities of matrices

$$(C_0^-, C_1^-, C_2^-) := I_{3n}, \quad \begin{pmatrix} D_0^- \\ D_1^- \\ D_2^- \end{pmatrix} := \begin{pmatrix} 0 & -A_2 & A_1 \\ A_2 & 0 & -A_0 \\ -A_1 & A_0 & 0 \end{pmatrix}.$$

Similarly, for π_+^{n-2}, using an identification $\mathbb{C}^{3n-3} = \bigoplus_j (\mathbb{C}^{n-1} \otimes x_j)$, we put

$$C_i^+ := x_{i+1} \circ A_{i+2} - x_{i+2} \circ A_{i+1} \colon \mathbb{C}^n \to \mathbb{C}^{3n-3},$$
$$D_i^+ := x_i^* \colon \mathbb{C}^{3n-3} \to \mathbb{C}^{n-1},$$

such that we have equalities of matrices

$$(C_0^+, C_1^+, C_2^+) := \begin{pmatrix} 0 & -A_2 & A_1 \\ A_2 & 0 & -A_0 \\ -A_1 & A_0 & 0 \end{pmatrix}, \quad \begin{pmatrix} D_0^+ \\ D_1^+ \\ D_2^+ \end{pmatrix} := I_{3n-3}.$$

Here indices run over $\mathbb{Z}/3\mathbb{Z}$, and I_{3n} and I_{3n-3} are unit matrices with sizes $3n$ and $3n-3$, respectively.

Hence we get the diagram:

$$
\begin{array}{ccccc}
M_+(\alpha_r) & & & & M_-(\alpha_r) \\
& \searrow^{f_+} & & {}^{f_-}\swarrow & \\
\downarrow g_+ & & M_0(\alpha_r) & & \downarrow g_- \\
& & \downarrow \pi_0^r & & \\
M_+(\alpha_{n-2}) & \xrightarrow[\pi_+^{n-2}]{\cong} & M_T(\alpha_T) & \xleftarrow[\pi_-^{n+1}]{\cong} & M_-(\alpha_{n+1})
\end{array}
\tag{15}
$$

where $g_- := (\pi_-^{n+1})^{-1} \circ \pi_0^r \circ f_-$ and $g_+ := (\pi_+^{n-2})^{-1} \circ \pi_0^r \circ f_+$.

Morphisms g_- and g_+ are explicitly defined by the following universal extensions for each $E_- \in M_-(\alpha_r)$ and $E_+ \in M_+(\alpha_r)$:

$$0 \to \operatorname{Ext}^1_B(E_-, S_0)^* \otimes S_0 \to g_-(E_-) \to E_- \to 0, \tag{16}$$

$$0 \to E_+ \to g_+(E_+) \to \operatorname{Ext}^1_B(S_0, E_+) \otimes S_0 \to 0. \tag{17}$$

To see (16), for any B-module $E_- = \left(\mathbb{C}^n \overset{C_i}{\to} \mathbb{C}^{2n-1+r} \overset{D_j}{\to} \mathbb{C}^{n-1}\right)$, we put

$$A = \begin{pmatrix} 0 & -A_2 & A_1 \\ A_2 & 0 & -A_0 \\ -A_1 & A_0 & 0 \end{pmatrix} = DC, \quad C = (C_0, C_1, C_2), \quad D = \begin{pmatrix} D_0 \\ D_1 \\ D_2 \end{pmatrix},$$

i.e, $A_i = D_{i+2}C_{i+1}$ for each $i \in \mathbb{Z}/3\mathbb{Z}$. Since $\operatorname{coker} C \cong \operatorname{Hom}_B(E_-, S_0) = 0$ by Lemma 3.2 (iii), the map C is surjective. Hence the following diagram gives the exact sequence (16)

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \ker C & \longrightarrow & & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \circlearrowleft & & \downarrow \\
\mathbb{C}^n & \xrightarrow{x_i} & \mathbb{C}^{3n} & \xrightarrow{A} & \mathbb{C}^{3n-3} & \xrightarrow{x_j^*} & \mathbb{C}^{n-1} \\
\| & \circlearrowleft & \downarrow C & & \circlearrowleft & & \| \\
\mathbb{C}^n & \xrightarrow{C_i} & \mathbb{C}^{2n+r-1} & & \xrightarrow{D_j} & & \mathbb{C}^{n-1}
\end{array}
\qquad (i, j = 0, 1, 2).
$$

Here the middle raw is $g_-(E_-)$ using identifications $\mathbb{C}^{3n} = \bigoplus_j (\mathbb{C}^n \otimes x_j)$ and $\mathbb{C}^{3n-3} = \bigoplus_j (\mathbb{C}^{n-1} \otimes x_j)$, and the bottom law is E_-. A similar diagram gives the exact sequence (17).

Hence, via the isomorphism (12), the morphism g_- coincides with the Yoshioka's map ψ, which is defined by the exact sequence (1) similar to (16). By Lemma 3.3 and the diagram (15), we have

$$M_0(\alpha_r) \cong \operatorname{im} g_- \cong \operatorname{im} \psi.$$

This gives a proof of (i) in Proposition 1.3.

3.3. Brill-Noether locus. — We introduce the *Brill-Noether loci* $M^i_-(\alpha_r)$ and $M^i_+(\alpha_r)$ by using $S_0 = \Phi(\mathscr{O}_{\mathbb{P}^2}[1])$ as follows:

$$M^i_-(\alpha_r) := \{E_- \in M_-(\alpha_r) \mid \dim_{\mathbb{C}} \operatorname{Hom}_B(S_0, E_-) = i\},$$
$$M^i_+(\alpha_r) := \{E_+ \in M_+(\alpha_r) \mid \dim_{\mathbb{C}} \operatorname{Hom}_B(E_+, S_0) = i\}.$$

When we replace "$= i$" by "$\geq i$" in the right hand sides, the corresponding moduli spaces are denoted by the left hand side with "i" replaced by "$\geq i$".

We consider a universal family $\mathscr{U} = \left(\mathscr{U}_{-1} \xrightarrow{\gamma_i^*} \mathscr{U}_0 \xrightarrow{\delta_j^*} \mathscr{U}_1\right)$ of B-modules on $M_-(\alpha_r)$. If we put

$$\delta^* := \sum_i x_i \circ \delta_i^* \colon \mathscr{U}_0 \to \bigoplus_i (\mathscr{U}_1 \otimes x_i)\,,$$

then the zero locus of $\bigwedge^{\operatorname{rk}\mathscr{U}_0 - i+1} \delta^*$ defines $M^{\geq i}_-(\alpha_r)$ as a closed subscheme of $M_-(\alpha_r)$. This is because we have

$$(\ker \delta^*)|_p \cong \operatorname{Hom}_B(S_0, \mathscr{U}|_p)$$

for any $p \in M_-(\alpha_r)$. Here we consider the fiber $\mathscr{U}|_p$ of the universal family over p as the θ_--semistable B-module corresponding to the point $p \in M_-(\alpha_r)$.

Similarly, $M^{\geq i}_+(\alpha_r)$ is defined as a closed subscheme of $M_+(\alpha_r)$. The loci

$$M^i_-(\alpha_r) = M^{\geq i}_-(\alpha_r) \setminus M^{\geq i+1}_-(\alpha_r), \quad M^i_+(\alpha_r) = M^{\geq i}_+(\alpha_r) \setminus M^{\geq i+1}_+(\alpha_r)$$

are open subsets of $M^{\geq i}_-(\alpha_r)$ and $M^{\geq i}_+(\alpha_r)$, respectively.

3.4. Set-theoretical description of Grassmann bundles. — By Lemma 3.2, we have the following proposition.

Proposition 3.4. — (i) *For any $E_- \in M^i_-(\alpha_r)$, we put*

$$E' := \operatorname{coker}\left(\operatorname{Hom}_B(S_0, E_-) \otimes S_0 \to E_-\right).$$

Then E' is θ_--stable and $\operatorname{Hom}_B(S_0, E') = 0$, *that is, $E' \in M^0_-(\alpha_{r-i})$. Hence E' is also θ_0-stable.*

(ii) *Conversely, for any $E' \in M^0_-(\alpha_{r-i})$ and any i-dimensional vector subspace $V \subset \operatorname{Ext}^1_B(E', S_0)$, we obtain a B-module E_- by the canonical exact sequence*

$$0 \to V^* \otimes S_0 \to E_- \to E' \to 0.$$

Then E_- is θ_--stable and $\operatorname{Hom}_B(S_0, E_-) \cong V$, *that is, $E_- \in M^i_-(\alpha_r)$.*

(iii) *For any $E_+ \in M^i_+(\alpha_r)$, we put*

$$E' := \ker\left(E_+ \to \operatorname{Hom}_B(E, S_0)^* \otimes S_0\right).$$

Then E' is θ_+-stable and $\operatorname{Hom}_B(E', S_0) = 0$, that is, $E' \in M^0_+(\alpha_{r-i})$. Hence E' is also θ_0-stable.

(iv) *Conversely, for any $E' \in M^0_+(\alpha_{r-i})$ and any i-dimensional vector subspace $V \subset \operatorname{Ext}^1_B(S_0, E')$, we obtain a B-module E_+ by the canonical exact sequence*

$$0 \to E' \to E_+ \to V \otimes S_0 \to 0.$$

Then E_+ is θ_+-stable and $\operatorname{Hom}_B(E_+, S_0) \cong V$, that is, $E_+ \in M^i_+(\alpha_r)$.

By Lemma 3.2, $M^0_-(\alpha_{r-i})$ is set-theoretically equal to $M^0_+(\alpha_{r-i})$. For any B-module E, we have $\operatorname{Ext}^2_B(S_0, E) = \operatorname{Ext}^2_B(E, S_0) = 0$ by [15, Lemma 4.6 (1)]. Hence by the Riemann-Roch formula, for any element $E' \in M^0_-(\alpha_{r-i}) = M^0_+(\alpha_{r-i})$, we have $\dim_{\mathbb{C}} \operatorname{Ext}^1_B(E', S_0) = n-r+i+1$ and $\dim_{\mathbb{C}} \operatorname{Ext}^1_B(S_0, E') = n-r+i-2$.

If $n - r - 2 \geq 0$, then by the above lemma we have set-theoretical equalities

$$\begin{aligned} f_-\left(M^i_-(\alpha_r)\right) &= \left\{S_0^{\oplus i} \oplus E' \mid E' \in M^0_-(\alpha_{r-i}) = M^0_+(\alpha_{r-i})\right\} / \equiv_{\mathrm{S}} \\ &= f_+\left(M^i_+(\alpha_r)\right), \end{aligned} \tag{18}$$

where $\equiv_{\mathrm{S}}$ denotes the S-equivalence relation (cf. [15, §4.1]). This gives a proof of Theorem 1.4 (i). We put $M^i_0(\alpha_r) := f_-\left(M^i_-(\alpha_r)\right) = f_+\left(M^i_+(\alpha_r)\right)$ as in Theorem 1.4.

Fibers of the S-equivalence class of $S_0^{\oplus i} \oplus E'$ by f_- and f_+ are parameterized by $\operatorname{Gr}(\operatorname{Ext}^1_B(E', S_0), i)$ and $\operatorname{Gr}(\operatorname{Ext}^1_B(S_0, E'), i)$ for $E' \in M^0_-(\alpha_{r-i}) = M^0_+(\alpha_{r-i})$.

Lemma 3.5. — *For any integer $i > r$, we have*

$$M^i_-(\alpha_r) = M^i_+(\alpha_r) = \emptyset.$$

Proof. — By [17, Lemma 5.7], we have $M^i_-(\alpha_r) = \emptyset$ for any $i > r$. By (18) this implies $M^i_+(\alpha_r) = \emptyset$. □

3.5. Description of the chamber $\mathcal{C}_{\mathbb{P}^2}$. — In this subsection, we assume that $r > 0$ and $n \in \mathbb{Z}$ is arbitrary. In the following proposition, we use the symbol $\alpha_{r,n} = {}^t(n, 2n+r-1, n-1) \in K(B)$ instead of α_r to avoid confusion.

Proposition 3.6. — (i) *$M_{\mathbb{P}^2}(r, 1, n) \neq \emptyset$ if and only if $n \geq r-1$.*

(ii) *$M_+(\alpha_{r,n}) \neq \emptyset$ if and only if $n \geq r+2$. When $n = r+2$, we have $M^0_-(\alpha_{r,n}) = M^0_+(\alpha_{r,n}) = M^0_0(\alpha_{r,n}) = \emptyset$.*

In the following, we assume $r \geq 2$.

(iii) *$W_0 = \mathbb{R}_{\geq 0}\theta_0$ is a wall on $\alpha_{r,n}^{\perp} \otimes \mathbb{R}$ for $n \geq r-1$.*

(iv) $W_{\mathbb{P}^2} = \mathbb{R}_{\geq 0}\theta_{\mathbb{P}^2}$ *is a wall on* $\alpha_{r,n}^{\perp} \otimes \mathbb{R}$ *for* $n \geq r$.

Hence we have $\mathbb{R}_{>0}\theta_0 + \mathbb{R}_{>0}\theta_{\mathbb{P}^2} = \mathcal{C}_{\mathbb{P}^2}$ *if* $n \geq r$.

Proof. — (i) By the criterion for the existence of non-exceptional stable sheaves in [11, §16.4], we have our claim.
(ii) As in the introduction, we have the isomorphism

$$M_+(\alpha_{r,n}) \cong M_B(\varphi_0^{-1}(\alpha_r'), \theta' \circ \varphi_0^{-1})$$

for certain $\theta' \in (\alpha_r')^{\perp}$. Since the dimension vector of $\varphi_0^{-1}(\alpha_r')$ is equal to

$$\begin{pmatrix} n-1 \\ 2n-3 \\ n-r-2 \end{pmatrix}, \tag{19}$$

if $M_+(\alpha_{r,n}) \neq \emptyset$, it must be $n \geq r+2$. On the other hand, if $n \geq r+2$, by (i) and (18) we see that $M_+(\alpha_{r,n}) \neq \emptyset$. When $n = r+2$, we have $\dim_{\mathbb{C}} \operatorname{Hom}_B(S_0, E_-) = 2n-3 > 0$ by the dimension vector (19). Hence by (18) we have $M_-^0(\alpha_{r,n}) = M_+^0(\alpha_{r,n}) = M_0^0(\alpha_{r,n}) = \emptyset$.
(iii) We assume $n \geq r-1$. By (i), there exists an element E of $M_-(\alpha_{r-1,n}) \cong M_{\mathbb{P}^2}(r-1, 1, n)$. By Lemma 3.2 (i), a B-module $E \oplus S_0$ is θ_0-semistable and has a submodule S_0 with $\theta_0(S_0) = 0$. Hence $W_0 = \mathbb{R}_{\geq 0}\theta_0$ is a wall on $\alpha_{r,n}^{\perp} \otimes \mathbb{R}$.
(iv) We assume $n \geq r$ and take an element $\mathcal{F}$ of $M_{\mathbb{P}^2}(r, 1, n-1)$. We consider the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{O}_x \to 0$$

for the skyscraper sheaf $\mathcal{O}_x$ at any point $x \in \mathbb{P}^2$. Then $\mathcal{F}'$ is stable since $\mu(\mathcal{F}') = \mu(\mathcal{F})$. This gives elements $F := \Phi(\mathcal{F}[1])$, $F' := \Phi(\mathcal{F}'[1])$ of $M_-(\alpha_{r,n-1})$, $M_-(\alpha_{r,n})$ respectively, and an exact sequence of B-modules

$$0 \to \Phi(\mathcal{O}_x) \to F' \to F \to 0.$$

Since $\theta_{\mathbb{P}^2}(\mathcal{O}_x) = 0$, we see that $W_{\mathbb{P}^2} = \mathbb{R}_{\geq 0}\theta_{\mathbb{P}^2}$ is a wall on $\alpha_r^{\perp} \otimes \mathbb{R}$. These together with (13) imply the last assertion. □

By this proposition and [15, Lemma 6.3 (2)], we have $\mathbb{R}_{>0}\theta_0 + \mathbb{R}_{>0}\theta_{\mathbb{P}^2} = \mathcal{C}_{\mathbb{P}^2}$ if $r = 1$ and $n \geq 2$, or $r \geq 2$ and $n \geq r$. In these cases, $\mathcal{C}_+$ is different from $\mathcal{C}_- = \mathcal{C}_{\mathbb{P}^2}$ and it is adjacent to $\mathcal{C}_- = \mathcal{C}_{\mathbb{P}^2}$ with the common boundary W_0. By the description of the canonical bundle of $M_-(\alpha_r)$ in Proposition 2.3, we see that if $n > r+2$, then the diagram (14) gives the flip of $M_-(\alpha_r)$. Hence we get an isomorphism $M_+(\alpha_r) \cong M_+(r, 1, n)$ and a proof of (ii) in Proposition 1.3.

Proposition 3.7. — *The moduli schemes* $M_-(\alpha_r)$ *and* $M_+(\alpha_r)$ *are smooth.*

Proof. — By Lemma 2.2, deformation functors of B-modules E have obstruction theories with values in $\mathrm{Ext}^2_B(E, E)$. Since $\mathrm{Ext}^2_B(E_-, E_-) = 0$ for any $E_- \in M_-(\alpha_r) \cong M_{\mathbb{P}^2}(r, 1, n)$, we see that $M_-(\alpha_r)$ is smooth (cf. [9, Corollary 4.5.2]). Furthermore, if E_+ is an element of $M_+(\alpha_r)$, then by Proposition 3.4 we have an exact sequence

$$0 \to E' \to E_+ \to \mathbb{C}^i \otimes S_0 \to 0$$

for some i and $E' \in M^0_-(\alpha_{r-i})$. Since $\mathrm{Ext}^2_B(S_0, E_+) = \mathrm{Ext}^2_B(E', E') = \mathrm{Ext}^2_B(E', S_0) = 0$, we also have $\mathrm{Ext}^2_B(E_+, E_+) = 0$. Thus $M_+(\alpha_r)$ is also smooth. □

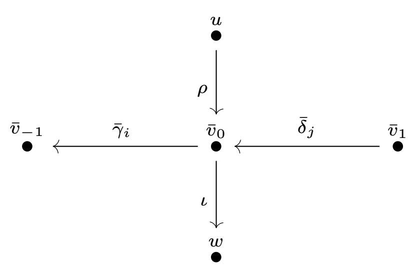
In the rest of this section we show that the diagram (14) is scheme-theoretically described by stratified Grassmann bundles.

3.6. Coherent systems. — For $r \geq i \geq 0$ we define moduli of coherent systems $M_-(\alpha_r, i)$ and $M_+(\alpha_r, i)$:

$$M_-(\alpha_r, i) := \{(E_-, V) \mid E_- \in M_-(\alpha_r), V \subset \mathrm{Hom}_B(S_0, E_-) \text{ with } \dim_{\mathbb{C}} V = i\},$$
$$M_+(\alpha_r, i) := \{(E_+, V) \mid E_+ \in M_+(\alpha_r), V \subset \mathrm{Hom}_B(E_+, S_0) \text{ with } \dim_{\mathbb{C}} V = i\}.$$

These moduli schemes are constructed as follows: We only show the construction of $M_-(\alpha_r, i)$ because the construction of $M_+(\alpha_r, i)$ is similar.

We introduce the quiver with relations $(\bar{Q}, I)$, where

$\bar{Q} :=$  $(i, j = 0, 1, 2)$

and I is generated by the relations

$$\iota\bar{\delta}_j = \bar{\gamma}_i\bar{\delta}_j + \bar{\gamma}_j\bar{\delta}_i = \iota\rho = 0 \quad (i, j = 0, 1, 2).$$

Let $\bar{B}$ be the path algebra $\mathbb{C}\bar{Q}/I$ of the quiver with relations $(\bar{Q}, I)$. We have simple $\bar{B}$-modules $\mathbb{C}\bar{v}_{-1}$, $\mathbb{C}\bar{v}_0$, $\mathbb{C}\bar{v}_1$, $\mathbb{C}u$ and $\mathbb{C}w$. For each $\alpha_r \in K(B)$, we put

$$\bar{\alpha}_r := n[\mathbb{C}\bar{v}_{-1}] + (2n + r - 1)[\mathbb{C}\bar{v}_0] + (n - 1)[\mathbb{C}\bar{v}_1] + (2n + r - i - 1)[\mathbb{C}u] + i[\mathbb{C}w] \in K(\bar{B}).$$

For $\theta_- = (\theta_-^{-1}, \theta_-^0, \theta_-^1) \in \alpha_r^\perp$ and $\varepsilon' > 0$ small enough, we put

$$\bar{\theta}_- := \theta_-^{-1}[\mathbb{C}\bar{v}_{-1}]^* + \theta_-^0[\mathbb{C}\bar{v}_0]^* + \theta_-^1[\mathbb{C}\bar{v}_1]^* + \frac{\varepsilon'}{2n + r - i - 1}[\mathbb{C}u]^* - \frac{\varepsilon'}{i}[\mathbb{C}w]^* \in \bar{\alpha}_r^\perp.$$

For any right $\bar{B}$-module

$$\begin{array}{ccccc} & & \mathbb{C}^{2n+r-i-1} & & \\ & & \uparrow{\scriptstyle A'} & & \\ \bar{E}_- = \mathbb{C}^n & \xrightarrow{\bar{C}_i} & \mathbb{C}^{2n+r-1} & \xrightarrow{\bar{D}_j} & \mathbb{C}^{n-1} \\ & & {\scriptstyle A}\uparrow & \circlearrowleft & \uparrow \\ & & \mathbb{C}^i & \longrightarrow & 0 \end{array}$$

we define a B-module

$$E_- := \left(\mathbb{C}^n \xrightarrow{\bar{C}_i} \mathbb{C}^{2n+r-1} \xrightarrow{\bar{D}_j} \mathbb{C}^{n-1} \right).$$

The following lemma is proved similarly as in Lemma 3.2 (iii).

Lemma 3.8. — *If we take ε' small enough, then $\bar{E}_-$ is $\bar{\theta}_-$-semistable if and only if E_- is θ_--semistable, A is injective and A' is surjective.*

Hence, if we denote by $M_{\bar{B}}(\bar{\alpha}_r, \bar{\theta}_-)$ the moduli of $\bar{\theta}_-$-semistable $\bar{B}$-modules $\bar{E}_-$ with $[\bar{E}_-] = \bar{\alpha}_r$, we get an isomorphism $M_{\bar{B}}(\bar{\alpha}_r, \bar{\theta}_-) \cong M_-(\alpha_r, i)$. We write $\bar{E}_- = (E_-, \mathbb{C}^i) \in M_-(\alpha_r, i)$ by abuse of notation.

We have morphisms

$$q_1 \colon M_-(\alpha_r, i) \to M_-(\alpha_r), \quad \bar{E}_- = (E_-, \mathbb{C}^i) \mapsto E_-$$

and

$$q_2 \colon M_-(\alpha_r, i) \to M_-(\alpha_{r-i}), \quad \bar{E}_- \mapsto q_2(\bar{E}_-),$$

where $q_2(\bar{E}_-)$ is defined by the canonical exact sequence

$$0 \to \mathbb{C}^i \otimes S_0 \to E_- \to q_2(\bar{E}_-) \to 0.$$

Similarly we have morphisms

$$q_1' \colon M_+(\alpha_r, i) \to M_+(\alpha_r), \quad q_2' \colon M_+(\alpha_r, i) \to M_+(\alpha_{r-i}).$$

If we take an element $\bar{E}_+ := (E_+, \mathbb{C}^i) \in M_+(\alpha_r, i)$, then q_1' and q_2' are defined by $q_1'(\bar{E}_+) = E_+$ and $q_2'(\bar{E}_+) := \ker\big(E_+ \to (\mathbb{C}^i)^* \otimes S_0\big)$.

Proposition 3.9. — (i) *The morphism $q_1 \colon M_-(\alpha_r, i) \to M_-(\alpha_r)$ is a $\mathrm{Gr}(j,i)$-bundle over each stratum $M_-^j(\alpha_r)$. In particular, we have an isomorphism*

$$q_1 \colon q_1^{-1}(M_-^i(\alpha_r)) \cong M_-^i(\alpha_r).$$

(ii) *The morphism $q_2 \colon M_-(\alpha_r, i) \to M_-(\alpha_{r-i})$ is a $\mathrm{Gr}(n-r+i+1, i)$-bundle. In particular, we have an isomorphism $q_2 \colon M_-(\alpha_{n+1}, i) \cong M_-(\alpha_{n+1-i})$.*

(iii) *For any $j \geq 0$, we have $q_1^{-1}(M_-^{i+j}(\alpha_r)) \cong q_2^{-1}(M_-^j(\alpha_{r-i}))$.*

(iv) *The morphism* $q_1'\colon M_+(\alpha_r, i) \to M_+(\alpha_r)$ *is a* $\mathrm{Gr}(j,i)$*-bundle over each stratum* $M_+^j(\alpha_r)$*. In particular, we have an isomorphism*

$$(q_1')^{-1}(M_+^j(\alpha_r)) \cong M_+^j(\alpha_r).$$

(v) *The morphism* $q_2'\colon M_+(\alpha_r, i) \to M_+(\alpha_{r-i})$ *is a* $\mathrm{Gr}(n-r+i-2, i)$*-bundle. In particular, we have an isomorphism* $q_2'\colon M_+(\alpha_{n-2}, i) \cong M_+(\alpha_{n-2-i})$.

(vi) *For any* $j \geq 0$*, we have* ${q_1'}^{-1}(M_+^{i+j}(\alpha_r)) \cong {q_2'}^{-1}(M_+^j(\alpha_{r-i}))$.

Proof. — (i) The fiber of q_1 over $E_- \in M_-^j(\alpha_r)$ is parameterized by the Grassmann manifold $\mathrm{Gr}(\mathrm{Hom}_B(S_0, E_-), i)$ for all $j \geq i$. As in §3.3 we take a universal family $\mathcal{U}$ of B-modules on $M_-(\alpha_r)$ and put

$$\delta^* := \sum_i x_i \circ \delta_i^* \colon \mathcal{U}_0 \to \bigoplus_i (\mathcal{U}_1 \otimes x_i)\,.$$

Then for any point $p \in M_-(\alpha_r)$, we have $\mathrm{Hom}_B(S_0, \mathcal{U}|_p) \cong (\ker \delta^*)|_p$. Since $\ker \delta^*$ is locally free of rank j on $M_-^j(\alpha_r)$ (cf. [1, Chapter II]), we have a $\mathrm{Gr}(j,i)$-bundle $\mathrm{Gr}(\ker \delta^*|_{M_-^j(\alpha_r)}, i)$ on $M_-^j(\alpha_r)$.

On the other hand, by the definition of $M_-^j(\alpha_r)$ (cf §3.3), we easily see that $M_-^j(\alpha_r)$ represents the moduli functor parameterizing families of θ_--semistable B-modules E_- with $[E_-] = \alpha_r$ and $\dim_{\mathbb{C}} \mathrm{Hom}_B(S_0, E_-) = j$. Hence $q_1^{-1}(M_-^j(\alpha_r))$ has the same universal property as $\mathrm{Gr}(\ker \delta^*|_{M_-^j(\alpha_r)}, i)$, and we have $q_1^{-1}(M_-^j(\alpha_r)) \cong \mathrm{Gr}(\ker \delta^*|_{M_-^j(\alpha_r)}, i)$.

(ii) The fiber of q_2 over $E' = q_2(\bar{E}_-)$ is parameterized by $\mathrm{Gr}\left(\mathrm{Ext}_B^1(E', S_0), i\right)$. For the universal family $\mathcal{U}' = \left(\mathcal{U}_{-1}' \overset{\gamma_i'^*}{\to} \mathcal{U}_0' \overset{\delta_j'^*}{\to} \mathcal{U}_1'\right)$ of B-modules on $M_-(\alpha_{r-i})$, we put

$$\gamma'^* := \sum_i \gamma_i'^* \circ x_i^* \colon \bigoplus_i (\mathcal{U}_{-1} \otimes x_i) \to \mathcal{U}_0.$$

Since we have $(\ker \gamma'^*|_{p'})^* \cong \mathrm{Ext}_B^1(\mathcal{U}'|_{p'}, S_0)$ for any $p' \in M_-(\alpha_{r-i})$, similarly as in (i) we get

$$M_-(\alpha_r, i) \cong \mathrm{Gr}\left((\ker \gamma'^*)^*, i\right).$$

(iii) Since spaces of both sides have the same universal property, our claim holds.

(iv), (v) and (vi) are proved similarly as in (i), (ii) and (iii). □

Corollary 3.10. — *$M_-^i(\alpha_r)$ and $M_+^i(\alpha_r)$ are smooth for any $i, r \geq 0$.*

Proof. — The restriction of the morphism $q_1\colon M_-(\alpha_r, i) \to M_-(\alpha_r)$ gives an isomorphism

$$q_1^{-1}(M_-^i(\alpha_r)) \cong M_-^i(\alpha_r).$$

By Proposition 3.9 (iii) we have an isomorphism $M_-^i(\alpha_r) \cong q_2^{-1}(M_-^0(\alpha_{r-i}))$. Hence by Proposition 3.9 (ii), $M_-^i(\alpha_r)$ is isomorphic to a Grassmann-bundle over $M_-^0(\alpha_{r-i})$. Since $M_-^0(\alpha_{r-i})$ is smooth by Proposition 3.7, we see that $M_-^i(\alpha_r)$ is also smooth. Similarly $M_+^i(\alpha_r)$ is shown to be smooth. □

3.7. Stratified Grassmann bundle. — In this section, we show that morphisms $f_-: M_-(\alpha_r) \to M_0(\alpha_r)$ and $f_+: M_+(\alpha_r) \to M_0(\alpha_r)$ are described by stratified Grassmann bundles by using Proposition 3.9.

We consider the diagram:

$$\begin{array}{ccc} & M_-(\alpha_{n+1}, n+1-r) & \\ {\scriptstyle \cong}\swarrow {\scriptstyle q_2} & & \searrow {\scriptstyle q_1} \\ M_-(\alpha_r) & & M_-(\alpha_{n+1}). \end{array}$$

By Proposition 3.9 (ii), q_2 is an isomorphism and we have a map

$$q_1 \circ q_2^{-1}: M_-(\alpha_r) \cong M_-(\alpha_{n+1}, n+1-r) \to M_-(\alpha_{n+1}),$$

which coincides with g_- by (16). This gives another proof of Theorem 1.1. Similarly the map

$$q_1' \circ {q_2'}^{-1}: M_+(\alpha_r) \cong M_+(\alpha_{n-2}, n-2-r) \to M_+(\alpha_{n-2})$$

coincides with the map g_+ by (17). For any $r \geq 0$, we have isomorphisms

$$M_-^0(\alpha_r) \cong M_-^{n+1-r}(\alpha_{n+1}), \quad M_+^0(\alpha_r) \cong M_+^{n-2-r}(\alpha_{n-2})$$

via g_- and g_+ respectively. In particular, we have isomorphisms

$$M_-^0(\alpha_{r-i}) \cong M_-^{n-r+i+1}(\alpha_{n+1}), \quad M_+^0(\alpha_{r-i}) \cong M_+^{n-r+i-2}(\alpha_{n-2}).$$

Since $\dim \mathrm{Ext}_B^1(E_-, S_0) = n - r + 1$ and $\dim \mathrm{Ext}_B^1(S_0, E_+) = n - r - 2$ for $E_- \in M_-(\alpha_r)$ and $E_+ \in M_+(\alpha_r)$, the strata $M_-^{n-r+i+1}(\alpha_{n+1})$ and $M_+^{n-r+i-2}(\alpha_{n-2})$ coincide with the images $f_-(M_-^i(\alpha_r)) \cong f_+(M_+^i(\alpha_r))$ via the diagram (15). This gives a proof of (ii) in Theorem 1.4.

By Proposition 3.9 and the diagram (15), we also have proofs of (iii) and (iv) in Theorem 1.4.

3.8. Hodge polynomials of flips. — We assume that $r > 0, n \geq r+2$, and study the difference between the Hodge polynomials of $M_-(\alpha_r)$ and $M_+(\alpha_r)$. To do this we use the virtual Hodge polynomial $e(Y) := \sum_{p,q} e^{p,q}(Y)x^p y^q$ for any variety Y (cf. [6]).

By Theorem 1.4 we get the diagram

$$\begin{array}{ccc} \coprod M_+^i(\alpha_r) & & \coprod M_-^i(\alpha_r) \\ & \searrow^{f_+} \quad {}^{f_-}\swarrow & \\ & \coprod M_0^i(\alpha_r) & \end{array} \tag{20}$$

where restrictions of f_- and f_+ to $M_-^i(\alpha_r)$ and $M_+^i(\alpha_r)$ are a $\mathrm{Gr}(n-r+i+1,i)$-bundle and a $\mathrm{Gr}(n-r+i-2,i)$-bundle over $M_0^i(\alpha_r) \cong M_-^0(\alpha_{r-i}) \cong M_+^0(\alpha_{r-i})$, respectively. Hence we get the following equality.

$$\begin{aligned} &e\left(M_-(\alpha_r)\right) - e\left(M_+(\alpha_r)\right) = \\ &\quad \sum_{i=1}^{r} \Big(e\left(\mathrm{Gr}(n-r+i+1,i)\right) - e\left(\mathrm{Gr}(n-r+i-2,i)\right) \Big) e\left(M_-^0(\alpha_{r-i})\right). \end{aligned} \tag{21}$$

In the following we compute the Hodge polynomial of $M_+(\alpha_r)$ from that of $M_-(\alpha_r)$ in the case where $r = 1, 2$. In this case, we know the Hodge polynomial of $M_-(\alpha_r) \cong M_{\mathbb{P}^2}(r,1,n)$ from [8] and [16]. We need the following proposition.

Proposition 3.11. — *We have the following isomorphisms:*

$$M_-(\alpha_0) \cong M_+(\alpha_0) \cong \mathbb{P}^2.$$

A proof of this proposition is given in Appendix A. From this proposition and (21), we get the following:

$$\cdots = M_+(1,1,1) = M_+(1,1,2) = \emptyset,$$

$$\begin{aligned} e\left(M_+(1,1,3)\right) &= t^{12} + t^{10} + 3t^8 + 3t^6 + 3t^4 + t^2 + 1, \\ e\left(M_+(1,1,4)\right) &= t^{16} + 2t^{14} + 5t^{12} + 8t^{10} + 10t^8 + 8t^6 + 5t^4 + 2t^2 + 1, \\ e\left(M_+(1,1,5)\right) &= \cdots + 21t^{10} + 19t^8 + 11t^6 + 6t^4 + 2t^2 + 1, \\ e\left(M_+(1,1,n)\right) &= e\left(M_{\mathbb{P}^2}(1,1,n)\right) - (t^{2n+4} + 2t^{2n+2} + 3t^{2n} + 2t^{2n-2} + t^{2n-4}), \end{aligned}$$

and

$$\cdots = M_+(2,1,1) = M_+(2,1,2) = M_+(2,1,3) = \emptyset,$$

$$\begin{aligned} e\left(M_+(2,1,4)\right) &= \cdots + 12t^{12} + 10t^{10} + 8t^8 + 5t^6 + 3t^4 + t^2 + 1, \\ e\left(M_+(2,1,5)\right) &= \cdots + 67t^{16} + 60t^{14} + 48t^{12} + 32t^{10} + 20t^8 + 10t^6 \\ &\qquad + 5t^4 + 2t^2 + 1, \end{aligned}$$

where $t = xy$.

Appendix A
Proof of Proposition 3.11

We give a proof of Proposition 3.11 by using the Bridgeland stability and Yoshioka's results in [17].

A.1. Bridgeland stability. — We briefly introduce the concept of the Bridgeland stability. For details the reader can consult [5]. Let $\mathscr{A}$ be an abelian category, $K(\mathscr{A})$ the Grothendieck group of $\mathscr{A}$.

Definition A.1. — A *stability function* Z on $\mathscr{A}$ is a group homomorphism from $K(\mathscr{A})$ to $\mathbb{C}$ such that for any object $E \in \mathscr{A}$, if E is not equal to zero we have $Z(E) \in \mathbb{R}_{>0} \exp(\sqrt{-1}\pi\phi(E))$ with $0 < \phi(E) \leq 1$.

The real number $\phi(E)$ is called the phase of E.

Definition A.2. — A non-zero object $E \in \mathscr{A}$ is *semistable* with respect to Z if and only if for any proper subobject $0 \neq F \subsetneq E$ we have $\phi(F) \leq \phi(E)$. If the inequality is always strict we call E to be *stable* with respect to Z.

Let $\mathscr{T}$ be a triangulated category and $K(\mathscr{T})$ the Grothendieck group of $\mathscr{T}$.

Definition A.3. — A *stability condition* σ on $\mathscr{T}$ is a pair $\sigma = (\mathscr{A}, Z)$, which consists of a full subcategory $\mathscr{A}$ of $\mathscr{T}$ and a group homomorphism $Z\colon K(\mathscr{T}) \to \mathbb{C}$ satisfying the following conditions:

- $\mathscr{A}$ is the heart of a bounded t-structure of $\mathscr{T}$, which implies that $\mathscr{A}$ is an abelian category and $K(\mathscr{A})$ is isomorphic to $K(\mathscr{T})$ by the inclusion $\mathscr{A} \subset \mathscr{T}$. Hence we always identify them.

- Z is a stability function on $\mathscr{A}$ via the above identification $K(\mathscr{A}) = K(\mathscr{T})$.

- Z has the Harder-Narasimhan property.

We omit the definition of "the heart of a bounded t-structure" and "Harder-Narasimhan property" (see [5, §2 and §3]). We denote by $\mathrm{Stab}(\mathscr{T})$ the set of all stability conditions satisfying the technical condition called "local finiteness" (see [5, §5]).

Definition A.4. — For a stability condition $\sigma = (\mathscr{A}, Z) \in \mathrm{Stab}(\mathscr{T})$, an object $E \in \mathscr{T}$ is called *σ-(semi)stable* if and only if E belongs to $\mathscr{A}$ up to shift functors $[n]\colon \mathscr{T} \to \mathscr{T}$ for $n \in \mathbb{Z}$, and it is (semi)stable with respect to Z.

In the following we only consider the case where $\mathscr{T} = D^b(\mathbb{P}^2)$ and we put $\mathrm{Stab}(\mathbb{P}^2) := \mathrm{Stab}(\mathscr{T})$. For $\alpha \in K(\mathbb{P}^2)$ and $\sigma = (\mathscr{A}, Z) \in \mathrm{Stab}(\mathbb{P}^2)$, we define the *moduli functor* $\mathscr{M}_{D^b(\mathbb{P}^2)}(\alpha, \sigma)$ of σ-semistable objects $E \in \mathscr{A}$ with $[E] = \alpha \in K(\mathbb{P}^2)$ as follows: The moduli functor $\mathscr{M}_{D^b(\mathbb{P}^2)}(\alpha, \sigma)$ is a functor from $(\mathrm{Sch}/\mathbb{C})$ to (Set). For a scheme S over $\mathbb{C}$ it sends S to the set $\mathscr{M}_{D^b(\mathbb{P}^2)}(\alpha, \sigma)(S)$ of isomorphism classes of *families* $\mathscr{J} \in D^b(\mathbb{P}^2 \times S)$ of σ-semistable objects with the class α in $K(\mathbb{P}^2)$. This means that for any $\mathbb{C}$-valued point $s \in S$, the fiber $\mathbf{L}\iota_s^* \mathscr{J} \in D^-(\mathbb{P}^2)$ belongs to the full subcategory $\mathscr{A} \subset D^b(\mathbb{P}^2)$ and is σ-semistable with $[\mathbf{L}\iota_s^* \mathscr{J}] = \alpha \in K(\mathbb{P}^2)$.

There exists a right action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on $\mathrm{Stab}(\mathbb{P}^2)$ and this action does not change semistable objects. Hence for any $\alpha \in K(\mathbb{P}^2)$, $\sigma \in \mathrm{Stab}(\mathbb{P}^2)$ and $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, there exists $n \in \mathbb{Z}$ such that the shift functor $[n]\colon D^b(\mathbb{P}^2) \cong D^b(\mathbb{P}^2)$ induces an isomorphism of functors

$$\mathscr{M}_{D^b(\mathbb{P}^2)}(\alpha, \sigma) \cong \mathscr{M}_{D^b(\mathbb{P}^2)}((-1)^n \alpha, \sigma g), \quad E \mapsto E[n].$$

A.2. Geometric stability. — Let H be the ample generator of $\mathrm{Pic}(\mathbb{P}^2)$ and $s, t \in \mathbb{R}$ with $t > 0$. For any torsion free sheaf E on $\mathbb{P}^2$, the slope of E is defined by $\mu_H(E) := \frac{c_1(E)}{\mathrm{rk}(E)}$ and defines μ_H*-semistability* (cf. [9, 1.6.9]). E has the Harder-Narasimhan filtration with μ_H-semistable factors. We denote the maximal value and the minimal value of slopes of μ_H-semistable factors of E by $\mu_{H-\max}(E)$ and $\mu_{H-\min}(E)$, respectively. Then we define a pair $\sigma_{(sH,tH)} = (\mathscr{A}_{(sH,tH)}, Z_{(sH,tH)})$ as follows:

Definition A.5. — [2, §2]

- An object $E \in D^b(\mathbb{P}^2)$ belongs to the full subcategory $\mathscr{A}_{(sH,tH)}$ if and only if
  - $\mathscr{H}^i(E) = 0$ for all $i \neq 0, -1$,
  - $\mathscr{H}^0(E)$ is torsion or $\mu_{H-\min}(\mathscr{H}^0(E)_{\mathrm{fr}}) > st$, where $\mathscr{H}^0(E)_{\mathrm{fr}}$ is the free part of $\mathscr{H}^0(E)$ and
  - $\mathscr{H}^{-1}(E)$ is torsion free and $\mu_{H-\max}(\mathscr{H}^{-1}(E)) \leq st$.
- The group homomorphism $Z_{(sH,tH)}$ is defined by

$$Z_{(sH,tH)}(E) := -\int_{\mathbb{P}^2} \mathrm{ch}(E) \exp(-sH - \sqrt{-1}tH),$$

  where $\mathrm{ch}(E) = (r(E), c_1(E), \mathrm{ch}_2(E)) \in H^{2*}(\mathbb{P}^2, \mathbb{Q}) \cong \mathbb{Q}^3$ is the Chern character of a object E in $D^b(\mathbb{P}^2)$.

We have the following criterion due to Bridgeland.

Proposition A.6. — **cf.** [15, Proposition 3.6] *For $\sigma \in \mathrm{Stab}(\mathbb{P}^2)$, there exist $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and $s, t \in \mathbb{R}$ with $t > 0$ such that $\sigma = \sigma_{(sH,tH)} g$ if and only if the following conditions* (i) *and* (ii) *hold:*

(i) *For any closed point $x \in \mathbb{P}^2$, the skyscraper sheaf $\mathcal{O}_x$ is σ-stable.*
(ii) *For any object E of $D^b(\mathbb{P}^2)$, if $Z(E) = 0$ then we have an inequality $c_1(E)^2 - 2r(E)\,\mathrm{ch}_2(E) < 0$.*

A.3. Proof of Proposition 3.11. — Here we identify derived categories $D^b(\mathbb{P}^2)$ and $D^b(B)$, and groups $K(\mathbb{P}^2)$ and $K(B)$ via the equivalence

$$\Phi := \mathbf{R}\,\mathrm{Hom}_{\mathbb{P}^2}(\mathscr{E}, -)\colon D^b(\mathbb{P}^2) \cong D^b(B)$$

and the induced group homomorphism $\varphi\colon K(\mathbb{P}^2) \cong K(B)$, where

$$B := \mathrm{End}_{\mathbb{P}^2}(\mathscr{E}), \quad \mathscr{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \Omega^1_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}(2).$$

In the notation in (6) we take

$$\alpha_0 = \begin{pmatrix} n \\ 2n-1 \\ n-1 \end{pmatrix} \in K(B),$$

so that the Chern class $c(\alpha_0) = -(0, 1, n)$ in $H^{2*}(\mathbb{P}^2, \mathbb{Z})$.

We define a family of stability conditions $\sigma^s = (\mathscr{A}, Z^s) \in \mathrm{Stab}(\mathbb{P}^2)$ for $s \in \mathbb{R}$ with $-1 < s < 1$ as follows: We take the heart of a bounded t-structure

$$\mathscr{A} = \langle \mathcal{O}_{\mathbb{P}^2}(-1)[2], \mathcal{O}_{\mathbb{P}^2}[1], \mathcal{O}_{\mathbb{P}^2}(1) \rangle$$

of $D^b(\mathbb{P}^2)$. Then, by the equivalence Φ, we see that $\mathscr{A}$ is equivalent to the abelian category mod-B of finitely generated B-modules. We define a group homomorphism $Z^s\colon K(\mathbb{P}^2) \to \mathbb{C}$ by

$$Z^s(\mathcal{O}_{\mathbb{P}^2}(-1)[2]) = \frac{-s-1}{2}, \quad Z^s(\mathcal{O}_{\mathbb{P}^2}[1]) = 1 + \sqrt{-1}, \quad Z^s(\mathcal{O}_{\mathbb{P}^2}(1)) = \frac{-s+1}{2}.$$

Then, by Proposition A.6, we see that there exists a family of elements $g^s \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that

$$\sigma_{(sH,tH)} g^s = \sigma^s, \tag{22}$$

where $t = \sqrt{1-s^2}$.

Fixing $\alpha_0 \in K(B)$, we define a group homomorphism $\tilde{\theta}^s\colon K(B) \to \mathbb{C}$ by

$$\tilde{\theta}^s(\beta) = \det \begin{pmatrix} \mathrm{Re}\, Z^s(\beta) & \mathrm{Re}\, Z^s(\alpha_0) \\ \mathrm{Im}\, Z^s(\beta) & \mathrm{Im}\, Z^s(\alpha_0) \end{pmatrix}$$

for each $\beta \in K(B)$. Then $M_B(\alpha_0, \tilde{\theta}^s)$ corepresents the moduli functor $\mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha_0, \sigma^s)$ by [15, Proposition 1.2]. Furthermore, by (22), we have an isomorphism

$$\mathcal{M}_{D^b(\mathbb{P}^2)}(-\alpha_0, \sigma_{(sH,tH)}) \cong \mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha_0, \sigma^s), \quad E \mapsto E[1] \tag{23}$$

of moduli functors. In the following lemma, we consider the case where

$$s = 0, \quad t = \sqrt{1 - s^2} = 1.$$

Lemma A.7. — *For an object $E \in \mathcal{A}_{(0,H)}$ with $[E] = -\alpha_0$, the following conditions are equivalent:*

(i) *E is $\sigma_{(0,H)}$-stable.*
(ii) *E is $\sigma_{(0,H)}$-semistable.*
(iii) *E is isomorphic to $\mathcal{O}_L(1-n)$ for a line L on $\mathbb{P}^2$.*

Proof. — ((i) $\Longrightarrow$ (ii)) It is trivial.
((ii) $\Longrightarrow$ (iii)) We assume that $E \in \mathcal{A}_{(0,H)}$ is $\sigma_{(0,H)}$-semistable and $\mathcal{H}^{-1}(E) \neq 0$. We put $F := \mathcal{H}^{-1}(E)[1]$ and $G := \mathcal{H}^0(E)$. Then an exact sequence in $\mathcal{A}_{(0,H)}$

$$0 \to F \to E \to G \to 0$$

implies that $0 \leq \operatorname{Im} Z_{(0,H)}(F) < \operatorname{Im} Z_{(0,H)}(E) = 1$. If we put $\operatorname{ch}(F) = -(r, c_1, \operatorname{ch}_2)$ with $r > 0$, then $\operatorname{Im} Z_{(0,H)}(F) = -c_1$ is an integer and must be equal to 0. Hence we see that $\phi(E) < \phi(F) = 1$, contradicting $\sigma_{(0,H)}$-semistability of E. Thus E is a sheaf with $\operatorname{ch}(E) = (0, 1, \frac{1}{2} - n)$. Any subsheaf of E with the support dimension 0 breaks the $\sigma_{(0,H)}$-semistability of E. Hence we see that E is a pure sheaf. This shows that $E \cong \mathcal{O}_L(1-n)$ for a line L on $\mathbb{P}^2$.
((iii) $\Longrightarrow$ (i)) We show that $\mathcal{O}_L(1-n) \in \mathcal{A}_{(0,H)}$ is $\sigma_{(0,H)}$-stable for any line $L \subset \mathbb{P}^2$. We take an exact sequence in $\mathcal{A}_{(0,H)}$

$$0 \to F \to \mathcal{O}_L(1-n) \to G \to 0.$$

Then we have a long exact sequence

$$0 \to \mathcal{H}^{-1}(G) \to F \to \mathcal{O}_L(1-n) \to \mathcal{H}^0(G) \to 0.$$

If the dimension of the support of $\mathcal{H}^0(G)$ is equal to 1, we have

$$\operatorname{rk}(F) = \operatorname{rk}(\mathcal{H}^{-1}(G)), \quad c_1(F) = c_1(\mathcal{H}^{-1}(G)).$$

If $\operatorname{rk}(F) \neq 0$, these equalities contradict the fact that $F, G \in \mathcal{A}_{(0,H)}$ implies inequalities $\mu_H(\mathcal{H}^{-1}(G)) \leq 0 < \mu_H(F)$. Hence F is a torsion sheaf and $\mathcal{H}^{-1}(G) = 0$. This implies $F = 0$ and $G = \mathcal{O}_L(1-n)$ since $\mathcal{O}_L(1-n)$ is a pure sheaf.

If the dimension of the support of $\mathcal{H}^0(G)$ is equal to 0, we have

$$\operatorname{rk}(F) = \operatorname{rk}(\mathcal{H}^{-1}(G)), \quad c_1(F) = c_1(\mathcal{H}^{-1}(G)) + 1.$$

If $\mathrm{rk}(F) \neq 0$, then the inequalities

$$\mu_H(\mathscr{H}^{-1}(G)) \leq 0 < \mu_H(F)$$

imply $c_1(F) = 1$ and $c_1(\mathscr{H}^{-1}(G)) = 0$. Hence we have $\operatorname{Im} Z_{(0,H)}(G) = 0$. If $\mathrm{rk}(F) = 0$, then $\mathscr{H}^{-1}(G) = 0$ and we also have $\operatorname{Im} Z_{(0,H)}(G) = 0$. Hence we have

$$\phi(G) = 1 > \phi(\mathscr{O}_L(1-n)).$$

Thus G does not break the $\sigma_{(0,H)}$-stability of $\mathscr{O}_L(1-n)$. □

By this lemma and the isomorphism (23) we have an isomorphism

$$M_B(\alpha_0, \tilde{\theta}^0) = \{\Phi\left(\mathscr{O}_L(1-n)[1]\right) \mid L \subset \mathbb{P}^2\colon \text{ line } \} \cong \mathbb{P}^2.$$

On the other hand, by [17, Theorem 5.8 and (5.29)], we have an isomorphism

$$\begin{aligned} \lambda\colon M_{\mathbb{P}^2}(0,1,n) = \{\mathscr{O}_L(1-n) \mid L \subset \mathbb{P}^2\colon \text{ line } \} &\cong M_{\mathbb{P}^2}^{n+1}(n+1,1,n), \\ \mathscr{O}_L(1-n) &\mapsto \lambda\left(\mathscr{O}_L(1-n)\right). \end{aligned}$$

Here a semistable sheaf $\lambda\left(\mathscr{O}_L(1-n)\right)$ is defined by the exact sequence

$$0 \to \mathbb{C}^{n+1} \otimes \mathscr{O}_{\mathbb{P}^2} \to \lambda\left(\mathscr{O}_L(1-n)\right) \to \mathscr{O}_L(1-n) \to 0,$$

where we have $\mathbb{C}^{n+1} \cong \mathrm{Hom}_{\mathbb{P}^2}(\mathscr{O}_{\mathbb{P}^2}, \lambda\left(\mathscr{O}_L(1-n)\right))$. By Proposition 3.4 (i), this implies that for any line L we have

$$\Phi(\mathscr{O}_L(1-n)[1]) \in M_-^0(\alpha_0) = M_+^0(\alpha_0).$$

Hence we get the following lemma.

Lemma A.8. — *Every object in $M_B(\alpha_0, \tilde{\theta}^0)$ is both θ_--stable and θ_+-stable.*

If we put $s_0 := -\frac{1}{2n-1}$, then we have $\tilde{\theta}^{s_0} \in W_0$ and $\tilde{\theta}^{s_0+\varepsilon} \in \mathscr{C}_-$, $\tilde{\theta}^{s_0-\varepsilon} \in \mathscr{C}_+$ for $\varepsilon > 0$ small enough. By Lemma A.8, we see that there exists no wall on the line $\{\tilde{\theta}^s \mid s_0 - \varepsilon \leq s \leq 0\}$. Thus we see that the ray W_0 is not a wall, and that $\tilde{\theta}^0$, $\mathscr{C}_-$, $\mathscr{C}_+$ and W_0 are contained in a single chamber. As a consequence we have an isomorphism

$$M_-(\alpha_0) = M_+(\alpha_0) = M_B(\alpha_0, \tilde{\theta}^0) \cong \mathbb{P}^2.$$

This completes the proof of Proposition 3.11.

BIBLIOGRAPHY

[1] E. Arbarello, M. Cornalba, P. Griffiths & J. Harris – *Geometry of algebraic curves*, vol. I, Springer, 1984.

[2] D. Arcara, A. Bertram & M. Lieblich – "Bridgeland-stable moduli spaces for K-trivial surfaces", *J. Eur. Math. Soc. (JEMS)* **15** (2013), p. 1–38.

[3] C. Birkar, P. Cascini, C. Hacon & J. McKernan – "Existence of minimal models for varieties of log general type", *J. Amer. Math. Soc.* **23** (2010), p. 405–468.

[4] A. I. Bondal – "Representation of associative algebras and coherent sheaves", *Izv. Akad. Nauk SSSR Ser. Mat.* **53** (1989), p. 25–44, English transl. in Math. USSR-Izv. 34(1):23-44, 1990.

[5] T. Bridgeland – "Stability conditions on triangulated categories", *Ann. of Math.* **166** (2007), p. 317–345.

[6] V. I. Danilov & A. G. Khovanskii – "Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers", *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), p. 925–945, English transl. in Math. USSR-Izv. 29(2):279-298, 1987.

[7] J. M. Drézet – "Group de picard des variétés de modules de faisceaux semi-stables sur $\mathbb{P}_2(\mathbb{C})$", *Ann. de l'Institut Fourier.* **38** (1988), p. 105–168.

[8] G. Ellingsrud & S. A. Strømme – "Towards the Chow ring of the Hilbert scheme on $\mathbb{P}^2$", *J. reine angew. Math.* **441** (1993), p. 33–44.

[9] D. Huybrechts & M. Lehn – *Geometry of moduli spaces of sheaves*, Aspects in Mathematics, vol. E31, Vieweg, 1997.

[10] J. Le Potier – "À propos de la construction de l'espace de modules des faisceaux semi-stables sur le plan projectif", *Bull. Soc. Math. France* **122** (1994), p. 363–369.

[11] ———, *Lectures on vector bundles*, Cambridge Stud. Adv. Math, vol. 54, Cambridge Univ. Press, 1997.

[12] J. Li – *Compactification of moduli of vector bundles over algebraic surfaces*, World. Sci. Publishing, 1997.

[13] M. Maruyama – *On a compactification of a moduli space of stable vector bundles on a rational surface*, Kinokuniya, 1988.

[14] H. Nakajima & K. Yoshioka – "Perverse coherent sheaves on blow-up. II. Wall-crossing and Betti numbers formula", *J. Algebraic Geom.* **20** (2011), p. 47–100.

[15] R. Ohkawa – "Moduli of Bridgeland semistable objects on $\mathbb{P}^2$", *Kodai Math. J* **33** (2010), p. 329–366.

[16] K. Yoshioka – "The Betti numbers of the moduli space of stable sheaves of rank 2 on $\mathbb{P}^2$", *J. reine angew, Math.* **453** (1994), p. 193–220.

[17] ———, "A note on moduli of vector bundles on rational surfaces", *J. Math. Kyoto Univ.* **43** (2003), p. 139–163.