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FORMALITY OF THE FUNCTION SPACE OF FREE MAPS INTO AN ELLIPTIC SPACE

BY TOSHIHIRO YAMAGUCHI (*)

ABSTRACT. — Let X be an n -connected elliptic space and Y a non rationally contractible, finite-type, q -dimensional CW complex, where $q \leq n$. We show that the function space X^Y of free maps from Y into X is formal if and only if the rational cohomology algebra $H^*(X; \mathbb{Q})$ is free, that is, X has the rational homotopy type of a product of odd dimensional spheres.

RÉSUMÉ. — FORMALITÉ DES ESPACES DE FONCTIONS LIBRES DANS UN ESPACE ELLIPTIQUE. — Soient X un espace elliptique n -connexe et Y un CW complexe non rationnellement contractile, de type fini et de dimension $q \leq n$. Nous montrons que l'espace X^Y des fonctions libres de Y dans X est formel si et seulement si l'algèbre $H^*(X, \mathbb{Q})$ est libre, *i.e.* X a le type d'homotopie rationnelle d'un produit de sphères de dimensions impaires.

1. Introduction

D. Sullivan's minimal model $(\wedge V, d)$ satisfies a nilpotence condition on d , *i.e.*, there is a well ordered basis $\{v_i\}_{i \in I}$ of V such that, $i < j$ if $\deg v_i < \deg v_j$ for each $i, j \in I$ and $d(v_i) \in \wedge V_{< i}$. Here $V_{< i}$ denotes the subspace of V generated by basis elements $\{v_j; j \in I, j < i\}$. According to [9, Def. 1.2], $(\wedge V, d)$ is called *normal* if $\text{Ker}[d|_V] = \text{Ker}(d|_V)$ where

$$\text{Ker}[d|_V] := \{v_i \in V; i \in I, d(v_i) \text{ is cohomologous to zero in } (\wedge V_{< i}, d)\}.$$

Let F , E and B be connected nilpotent spaces and let $\mathcal{M}(B)$ be a normal minimal model. In this paper, we say that a rational fibration [7, p. 200]

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$F \xrightarrow{i} E \xrightarrow{\pi} B$ is *M.N* if there is a KS-extension:

$$(1.1) \quad \begin{array}{ccccc} \mathcal{M}(B) & \xrightarrow{\text{inclusion}} & (\mathcal{M}(B) \otimes \wedge V, D) & \xrightarrow{\text{projection}} & (\wedge V, \bar{D}) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ A^*(B) & \xrightarrow{\pi^*} & A^*(E) & \xrightarrow{i^*} & A^*(F) \end{array}$$

in which $(\mathcal{M}(B) \otimes \wedge V, D)$ is minimal (*i.e.*, D is decomposable) and normal by a suitable change of KS-basis. Here $A^*(X)$ denotes the rational de-Rham complex of a space X , $\mathcal{M}(F) \cong (\wedge V, \bar{D})$ and “ \simeq ” means quasi-isomorphic, *i.e.*, the map induces an isomorphism in cohomology. We remark that “M.N” is a characteristic of the rational fibration but not of the total space.

Many rational fibrations are M.N. For example, the rational fibration given by a KS-extension:

$$(\wedge(x, y), 0) \longrightarrow (\wedge(x, y, z), D) \longrightarrow (\wedge z, 0)$$

with $|x| = 3$ (where $|v|$ means $\deg(v)$ for $v \in V$), $|y| = 3$, $|z| = 5$ and $D(z) = xy$ is M.N. Of course, any rationally trivial fibration is M.N. On the other hand, many rational fibrations are not M.N. For example, in the KS-model of the Hopf fibration $S^3 \rightarrow S^7 \rightarrow S^4$, the model of the total space $(\mathcal{M}(S^4) \otimes \wedge(x_3), D)$ with $|x_3| = 3$ is not even minimal. The rational fibration given by a KS-extension:

$$(\wedge(x, y), d) \longrightarrow (\wedge(x, y, z), D) \longrightarrow (\wedge z, 0)$$

with $|x| = 2$, $|y| = 5$, $|z| = 3$, $D(x) = d(x) = 0$, $D(y) = d(y) = x^3$ and $D(z) = x^2$ is minimal but can not be normal by any change of KS-basis.

In the following, a fibration means a rational fibration. A nilpotent space X or the minimal model $\mathcal{M}(X)$ is called (rationally) *formal* if there is a quasi-isomorphism from $\mathcal{M}(X)$ to $(H^*(X; \mathbb{Q}), 0)$ (see [3]). The reason we consider M.N-type fibrations is that we can then state a necessary (but perhaps not sufficient) condition for the formality of the total space as in [3, Thm 4.1] when the base space is formal (see Lemma 2.3).

A fibration $F \rightarrow E \rightarrow B$ is called:

- $\sigma \cdot F$ if it has a rational section;
- *W.H.T* if $\pi_*(E) \otimes Q = (\pi_*(B) \otimes Q) \oplus (\pi_*(F) \otimes Q)$ for the rational number field Q and
- *H.T* if it is rationally trivial (see [11]).

The following lemma expresses the relations among these different types of fibrations.

LEMMA 1.1.

1) “ $M.N$ ” is embedded in the sequence of implications:

$$\sigma \cdot F \implies M.N \implies W.H.T,$$

where the reversed implications are false in general.

2) If a fibration $F \rightarrow E \rightarrow B$ is $\sigma \cdot F$ and E is formal, then B is formal (compare [4, Lemme 2])

Our object of interest is the function space X^Y of free, continuous maps from a connected space Y into a connected space X , endowed with the compact-open topology. Observe that X^Y is infinite dimensional and is connected if X is n -connected and Y is a q -dimensional CW-complex, where $q \leq n$. Furthermore, X^Y is the total space of the fibration:

$$(*) \quad (X, *)^{(Y, *)} \longrightarrow X^Y \xrightarrow{\pi} X,$$

where $(X, *)^{(Y, *)}$ is the function space of pointed maps, and π is the evaluation at the base point. We know that $(*)$ has a section s , where $s(x)$ is the constant map at x . Therefore $(*)$ is $\sigma \cdot F$. When $Y = S^1$, N. Dupont and M. Vigué-Poirrier proved the following formality result.

THEOREM (see [4, Théorème]). — *Let X be a simply connected space where $H^*(X; Q)$ is finitely generated. Then X^{S^1} is formal iff $H^*(X; Q)$ is free, i.e., X has the rational homotopy type of a product of Eilenberg Maclane spaces.*

Our goal in this article is to generalize the theorem of Dupont and Vigué-Poirrier to X^Y , when Y is of finite-type, i.e., $\pi_i(Y) \otimes Q$ is finite-dimensional for all i , provided that X is elliptic, i.e., the total dimensions of $H^*(X; Q)$ and $\pi_*(X) \otimes Q$ are finite. More precisely, we prove the following theorem.

THEOREM 1.2. — *Let X be an n -connected elliptic space, and let Y be a non rationally contractible, finite-type, q -dimensional CW complex, where $q \leq n$. Then X^Y is formal iff $H^*(X; Q)$ is free, i.e., X has the rational homotopy type of a product of odd dimensional spheres.*

In proving Theorem 1.2, we use a model due to Brown and Szczarba [2] for the connected component in X^Y of a map $f: Y \rightarrow X$, which is constructed from minimal models of X , Y and f . We remark that, under the hypotheses of Theorem 1.2, this *non-formalizing tendency* of X^Y does not depend on the rational homotopy type of Y . We cannot easily relax the connectivity hypothesis.

For example, when $X = \mathbb{C}P^2$ and $Y = S^3$, we can see $X_{(0)}^Y \simeq (\mathbb{C}P^2 \times K(Q, 2))_{(0)}$ by the calculation in [2]. In particular, $X_{(0)}^Y$ is formal even though X does not have the rational homotopy type of a product of odd dimensional spheres. Also we must consider each connected component of X^Y in the general case.

In the following sections, our category is CDGA, that is, the objects are commutative differential graded algebras (cdga) over Q , and the morphisms are maps of differential graded algebra. Also, $H^*(\)$ means $H^*(\ ; Q)$ and $I(S)$ denotes the ideal in the algebra A generated by a basis of a subspace S in A . When B is a subalgebra of A and both A and B contain S , then $I(S)$ denotes the ideal in the algebra A and $I_B(S)$ the ideal in the algebra B , unless otherwise noted.

2. Two changes of KS-basis

When a cdga \mathcal{A} is formal, we can choose a minimal model $\mathcal{M} = (\wedge V, d)$ of \mathcal{A} such that $V = \text{Ker}(d|_V) \oplus \text{Ker}(\psi|_V)$ for a quasi-isomorphism $\psi: \mathcal{M} \rightarrow (H^*(\mathcal{A}), 0)$. Therefore, according to [3, Thm 4.1], \mathcal{A} is formal iff there is a complement N to $\text{Ker}(d|_V)$, $V = \text{Ker}(d|_V) \oplus N$, such that any d -cocycle of $I(N)$ is d -exact. We remark this ‘ \mathcal{M} ’ must be a normal minimal model. Conversely, if $\mathcal{M} = (\wedge V, d)$ is a normal minimal model and formal, $H^*(\mathcal{M})$ is generated by $\text{Ker}(d|_V)$ as an algebra (see [9, Lemma 1.8]). Therefore for any quasi-isomorphism $\psi: \mathcal{M} \rightarrow (H^*(\mathcal{A}), 0)$, we have $V = \text{Ker}(d|_V) \oplus \text{Ker}(\psi|_V)$.

Following [8, p. 5], we use the term “change of KS-basis” in this paper as follows. Suppose that

$$(B^*, d_B) \longrightarrow (B^* \otimes \wedge V, \delta) \longrightarrow (\wedge V, \bar{\delta})$$

is a KS-extension with KS-basis $\{v_i\}_{i \in I}$, *i.e.*, a well-ordered basis of V such that $i < j$ if $|v_i| < |v_j|$ for each $i, j \in I$ and $\delta(v_i) \in B^* \otimes \wedge V_{<i}$. Define a map of algebras $\phi: B^* \otimes \wedge V \rightarrow B^* \otimes \wedge V$ by setting

$$\phi|_B = \text{id}_B \quad \text{and} \quad \phi(v_i) = v_i + \chi_i$$

on basis elements of V , where $\chi_i \in B^* \otimes \wedge V_{<i}$ (To be exact, this is different from the definition of “KS-change of basis” of [8, p. 5] since χ_i may not be contained in $B^+ \otimes \wedge V$.) Finally, define a new differential D on $B^* \otimes \wedge V$ by

$$D = \phi^{-1} \circ \delta \circ \phi.$$

Then we have an isomorphism of KS-extensions

$$(2.1) \quad \begin{array}{ccccc} (B^*, d_B) & \xrightarrow{\text{incl.}} & (B^* \otimes \wedge V, D) & \xrightarrow{\text{proj.}} & (\wedge V, \bar{D}) \\ \downarrow = & & \phi \downarrow \cong & & \bar{\phi} \downarrow \cong \\ (B^*, d_B) & \xrightarrow{\text{incl.}} & (B^* \otimes \wedge V, \delta) & \xrightarrow{\text{proj.}} & (\wedge V, \bar{\delta}), \end{array}$$

where $D|_{B^*} = \delta|_{B^*} = d_B$.

In this section we introduce two changes of KS-basis. If the fibration (1.1) is $M.N$, the normal minimal model $\mathcal{M}(E) \cong (\mathcal{M}(B) \otimes \wedge V, D)$ is given by a change of KS-basis that we denote ϕ_1 , one of two basis changes studied in this section.

Proof of Lemma 1.1.

1) The implication $(\sigma \cdot F \Rightarrow M.N)$ is given in terminology of (1.1) with KS-basis $\{v_i\}_{i \in I}$ as follows. We know that (1.1) is $\sigma \cdot F$ iff $Dv - \bar{D}v \in \mathcal{M}^+(B) \otimes \wedge^+ V$ for $v \in V$ (see [10, VI.6.(1)]). Therefore the minimality follows. Suppose there are $\{v_i\}_{i \in J}$ with $J \subset I$ such that Dv_i is cohomologous to 0. For $i \in J$, we can change KS-basis inductively, as $\phi_1(v_i) = v_i - \chi_i$ if $Dv_i = D(\chi_i)$ where $\chi_i \in B^* \otimes \wedge V_{<i}$ and $\phi_1(v_i) = v_i$ for $i \in I - J$. Put $\tilde{D} = \phi_1^{-1} \circ D \circ \phi_1$ and then we have $\tilde{D}(v_i) = 0$ for $i \in J$. Thus we have $\text{Ker}[\tilde{D}|_V] = \text{Ker}(\tilde{D}|_V)$. We put again $D = \tilde{D}$. Since again $Dv - \bar{D}v \in \mathcal{M}^+(B) \otimes \wedge^+ V$ for $v \in V$, we have for $\mathcal{M}(B) = (\wedge V_B, d_B)$

$$\text{Ker}[D|_{V_B}] = \text{Ker}[d_B|_{V_B}] = \text{Ker}(d_B|_{V_B}) = \text{Ker}(D|_{V_B}).$$

The implication $(M.N \Rightarrow W.H.T)$ is clear from the decomposability of D . On the other hand, the first and last examples in Section 1 provide counter-examples to the first and second converses, respectively.

2) From 1), we can assume $(\wedge(V_B \oplus V), D)$ of (1.1) is minimal and normal. Since E is formal, there is a complement N to $\text{Ker}(D|_{V_B \oplus V})$ in $V_B \oplus V$ such that any D -cocycle of $I(N)$ is D -exact since $H^*(E)$ is generated by $\text{Ker}(D|_{V_B \oplus V})$ (see [3], [9]). Then $N \cap V_B$ is a complement to $\text{Ker}(D|_{V_B}) = \text{Ker}(d_B|_{V_B})$ in V_B . From $(D - \bar{D})(V) \subset \wedge^+ V_B \otimes \wedge^+ V$ and $D|_{V_B} = d_B$, we see any d_B -cocycle of $I_B(N \cap V_B)$ is d_B -exact, where $I_B(S)$ is the ideal of $\wedge V_B$ generated by a subset S of V_B . The formality of B follows again from [3, Thm 4.1]. \square

As in the proof of Lemma 1.1, part 1), given any KS-extension (1.1), we can change KS-basis

$$\phi_1: (\wedge(V_B \oplus V), \tilde{D}) \cong (\wedge(V_B \oplus V), D)$$

so that $\text{Ker}[\tilde{D}|_V] = \text{Ker}(D|_V)$. We put again $D = \tilde{D}$.

Next we introduce the second type change of KS-basis, which we denote ϕ_2 .

LEMMA 2.1. — *Let E be a formal space and $F \rightarrow E \rightarrow B$ a W.H.T fibration, with KS-model $(\wedge V_B, d_B) \rightarrow (\wedge(V_B \oplus V), D) \rightarrow (\wedge V, \bar{D})$. Then, for any complement N to $\text{Ker}(D|_V)$ in V , there is a change KS-basis*

$$\phi_2: (\wedge(V_B \oplus V), \tilde{D}) \cong (\wedge(V_B \oplus V), D)$$

such that any \tilde{D} -cocycle of $I(N)$ is \tilde{D} -exact.

Proof. — Since the fibration is W.H.T, $(\Lambda(V_B \oplus V), D)$ is a minimal model of E with a KS-basis $\{v_i\}_{i \in I}$ of V . Let $\psi: (\Lambda(V_B \oplus V), D) \rightarrow (H^*(E), 0)$ be a quasi-isomorphism and $K = \text{Ker}(\psi|_V)$ with the sub-KS-basis $\{v_i\}_{i \in I_1}$. Then K is a complement of $\text{Ker}(D|_V)$ in V such that any D -cocycle of $I(K)$ is D -exact (see [3], [9]). Let $\{v_j\}_{j \in I_2}$ be the sub-KS-basis of $\text{Ker}(D|_V)$ where we assume that I is indexed by $i > j$ if $|v_i| = |v_j|$ for $i \in I_1$ and $j \in I_2$. Here $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. Then we can choose a basis of the given complement N to $\text{Ker}(D|_V)$ as $\{v_i + \sum_{j \in J_i} a_{ij} v_j\}_{i \in I_1}$ with some $a_{ij} \in Q$. Here $J_i = \{j \in I_2; |v_j| = |v_i|\}$. There is a regular linear transformation $\phi_2: V \rightarrow V$ given by

$$\phi_2(v_i) = v_i - \sum_{j \in J_i} a_{ij} v_j \text{ for } i \in I_1 \quad \text{and} \quad \phi_2(v_j) = v_j \text{ for } j \in I_2.$$

Extend it to an algebra map

$$\phi_2: \Lambda(V_B \oplus V) \longrightarrow \Lambda(V_B \oplus V)$$

by $\phi_2|_{V_B} = \text{id}_{V_B}$ and define $\tilde{D} = \phi_2^{-1} D \phi_2$. Then N is also a complement to $\text{Ker}(\tilde{D}|_V)$ in V and equals $\text{Ker}(\psi \phi_2|_V)$. If an element w of $I(N)$ is a \tilde{D} -cocycle, then $[w] = \psi \phi_2(w) = 0$ and w is \tilde{D} -exact since $\psi^* \phi_2^*$ is an isomorphism on cohomology and $\text{Ker}[\tilde{D}|_V] = \text{Ker}(D|_V)$. \square

COROLLARY 2.2. — *Let $\mathcal{M} = (\Lambda V, d)$ be a normal minimal model. If \mathcal{M} is formal, for any complement N to $\text{Ker}(d|_V)$ there is a change of basis $(\Lambda V, \tilde{d}) \cong (\Lambda V, d)$ of V so that any \tilde{d} -cocycle of $I(N)$ is \tilde{d} -exact.*

Proof. — It follows by applying Lemma 2.1 when the base space is the one-point space. \square

Let a fibration $F \rightarrow E \rightarrow B$ be M.N. Let $\mathcal{M}(B) = (\Lambda V_B, d_B)$ be normal and $(\Lambda V_B, d_B) \xrightarrow{i} (\Lambda(V_B \oplus V), D)$ a KS-extension, which is normal by a suitable KS-basis change ϕ_1 . Let B be formal. Then there is a quasi-isomorphism $\rho_B: (\Lambda V_B, d_B) \rightarrow (H^*(B), 0)$ embedded in the commutative diagram:

$$(2.2) \quad \begin{array}{ccc} (\Lambda V_B, d_B) & \xrightarrow{i} & (\Lambda(V_B \oplus V), D) \\ \rho_B \downarrow & & \downarrow \rho \\ (H^*(B), 0) & \xrightarrow{i'} & (H^*(B) \otimes \Lambda V, D'), \end{array}$$

which is push out in CDGA, *i.e.*, $D'|_V := (\rho_B \otimes 1) \circ D|_V$. Here i and i' are inclusions and ρ is a quasi-isomorphism since ρ_B is [1]. There is a complement N_B to $\text{Ker}(d_B|_{V_B})$ in V_B such that any d_B -cocycle of $I_B(N_B)$ is d_B -exact (see [3], [9]). We remark that

$$N_B = \text{Ker}(\rho_B|_{V_B}) = \text{Ker}(\rho|_{V_B \oplus V}).$$

LEMMA 2.3. — *Let E and B be formal spaces and $F \rightarrow E \rightarrow B$ be an $M.N$ fibration, with KS -model $(\Lambda V_B, d_B) \rightarrow (\Lambda(V_B \oplus V), D) \rightarrow (\Lambda V, \bar{D})$. Then, for any complement N to $\text{Ker}(D|_V)$ in V , there is a change of KS -basis*

$$\phi_2: (\Lambda(V_B \oplus V), \tilde{D}) \cong (\Lambda(V_B \oplus V), D)$$

such that any \tilde{D} -cocycle of $I(N_B \oplus N)$ is \tilde{D} -exact.

Proof. — Let $\psi: (\Lambda(V_B \oplus V), D) \rightarrow (H^*(E), 0)$ be a quasi-isomorphism and $\rho: (\Lambda(V_B \oplus V), D) \rightarrow (H^*(B) \otimes \Lambda V, D')$ a quasi-isomorphism as in (2.2). If $b \in N_B$, $\rho(b) = 0$ since $\text{Ker}(\rho_B|_{V_B}) = \text{Ker}(\rho|_{V_B})$. Then

$$0 = \psi^* \rho^{*-1}[\rho(b)] = [\psi(b)] = \psi(b)$$

in $H^*(E)$. Hence $N_B \subset \text{Ker}(\psi|_{V_B})$. We can change KS -basis by some ϕ_2 for a given complement N to $\text{Ker}(D|_V)$ in V as in Lemma 2.1, so that $N = \text{Ker}(\psi\phi_2|_V)$. Then $\phi_2|_{V_B} = \text{id}_{V_B}$ and therefore we have

$$\text{Ker}(\psi\phi_2|_{V_B \oplus V}) = \text{Ker}(\psi|_{V_B}) \oplus \text{Ker}(\psi\phi_2|_V) \supset N_B \oplus \text{Ker}(\psi\phi_2|_V) = N_B \oplus N.$$

Thus we have that any \tilde{D} -cocycle of $I(N_B \oplus N)$ is \tilde{D} -exact since $\psi\phi_2$ is a quasi-isomorphism and

$$\begin{aligned} \text{Ker}[\tilde{D}|_{V_B \oplus V}] &= \text{Ker}[D\phi_2|_{V_B \oplus V}] = \text{Ker}[D|_{V_B \oplus V}] \\ &= \text{Ker}(D|_{V_B \oplus V}) = \text{Ker}(D\phi_2|_{V_B \oplus V}) = \text{Ker}(\tilde{D}|_{V_B \oplus V}). \quad \square \end{aligned}$$

3. Proof of Theorem 1.2

We begin this section by recalling the construction of the model of X^Y due to Brown and Szczarba [2]. Let $(\Lambda V, d)$ a free cdga and (B, d_B) a finite-type cdga. Let (B_*, d_*) be the differential graded coalgebra with $B_q = \text{Hom}(B^{-q}, Q)$. The differential d_* on B_* is the dual of d_B and the coproduct $\partial: B_* \rightarrow B_* \otimes B_*$ is the dual of multiplication. Let $\Lambda(\Lambda V \otimes B_*)$ be the free cdga generated by the vector space $\Lambda V \otimes B_*$ with the differential induced by the tensor product differential \tilde{d} on $\Lambda V \otimes B_*$, and let I be the ideal in $\Lambda(\Lambda V \otimes B_*)$ generated by $1 \otimes 1 - 1$ and by the all elements of the form

$$v_1 v_2 \otimes \beta - \sum_i (-1)^{|v_2| \cdot |\beta_i|} (v_1 \otimes \beta_i)(v_2 \otimes \beta'_i)$$

with $v_1, v_2 \in \Lambda V$, $\beta_i, \beta'_i \in B_*$ and $\partial\beta = \sum_i \beta_i \otimes \beta'_i$. Then there is a natural isomorphism

$$\kappa: \Lambda(V \otimes B_*) \cong \Lambda(\Lambda V \otimes B_*)/I$$

as graded algebras, induced by the inclusion $V \otimes B_* \rightarrow \Lambda V \otimes B_*$ (see [2, Thm 3.5]). Note that $\tilde{d}(I) \subset I$. Define δ on $\Lambda(V \otimes B_*)$ by $\delta = \kappa^{-1}d\kappa$. For example, if $dv = v_1v_2$ where $v_1, v_2 \in V$ and $\partial\beta = \sum_i \beta_i \otimes \beta'_i$ (see [2, p. 6]),

$$\delta(v \otimes \beta) = \sum_i (-1)^{|v_2|+|\beta_i|} (v_1 \otimes \beta_i) \cdot (v_2 \otimes \beta'_i) + (-1)^{|v|} v \otimes d_*(\beta).$$

In the following, we suppose that X is n -connected and finite-type, with $\mathcal{M}(X) = (\Lambda V, d)$ (so that $V = \bigoplus_{i>n} V^i$), and Y is a non rationally contractible, finite-type, q -dimensional CW complex, where $q \leq n$. Let $\mathcal{M}(Y) = (\Lambda V_Y, d_Y)$ and $\mathcal{M}_*(Y) = \text{Hom}(\mathcal{M}(Y), Q)$. Then

$$(\Lambda(\Lambda V \otimes \mathcal{M}_*(Y)), \tilde{d})/I \cong (\Lambda(V \otimes \mathcal{M}_*(Y)), \delta) \cong (\Lambda W, \delta) \otimes \mathcal{C} \simeq (\Lambda W, \delta),$$

where $W \subset V \otimes \mathcal{M}_*(Y)$ with

$$W \equiv V \otimes \{\text{the cohomology classes of } d_Y\text{-cocycles}\}_* \cong V \otimes H_*(Y)$$

as vector spaces and \mathcal{C} a contractible cdga. Here a basis of W is inductively constructed so that $V \otimes y_* \subset W$ for $y \in V_Y$ with $d_Y(y) = 0$ and $\delta(W) \subset \Lambda W$ (see [2]). According to [2, Thm 1.5], the minimal model of X^Y is given by $\mathcal{M}(X^Y) \cong (\Lambda W, \delta)$.

Write $W = V \oplus W_+$, where $W_+ := W \cap (V \otimes \mathcal{M}_+(Y))$. Then a KS-model of the fibration (*) (see Section 1) for a normal minimal model $\mathcal{M}(X) = (\Lambda V, d)$ is given as

$$(**) \quad (\Lambda V, d) \longrightarrow (\Lambda V \otimes \Lambda W_+, \delta) \longrightarrow (\Lambda W_+, \bar{\delta}),$$

where $(\Lambda W, \delta) = (\Lambda V \otimes \Lambda W_+, \delta)$ is minimal but may not be normal. Since (*) has a section, it is M.N by Lemma 1.1, part 1). Then there is a KS-basis change of (**) that can be given as follows:

$$(3.1) \quad \begin{array}{ccccc} (\Lambda V, d) & \xrightarrow{i} & (\Lambda V \otimes \Lambda W_+, D) & \longrightarrow & (\Lambda W_+, \bar{D}) \\ & \downarrow = & \phi_1 \downarrow \cong & & \bar{\phi}_1 \downarrow \cong \\ (\Lambda V, d) & \longrightarrow & (\Lambda V \otimes \Lambda W_+, \delta) & \longrightarrow & (\Lambda W_+, \bar{\delta}), \end{array}$$

where $(\Lambda W, D) = (\Lambda V \otimes \Lambda W_+, D)$ is normal with $D = \phi_1^{-1}\delta\phi_1$.

In the following, we suppose that X is elliptic (*i.e.*, $\dim_Q V < \infty$) and X^Y is formal, which implies that X is formal by Lemma 1.1, part 2). If X is elliptic, it is known that $H^*(X)$ is a Poincaré algebra (see [6]). Furthermore, if X is elliptic

and formal, it is known that $(\wedge V, d)$ is two stage, *i.e.*, $V = V_0 \oplus V_1$ with $dV_0 = 0$ and $dV_1 \subset \wedge V_0$ (see [5]). Therefore we can put

$$W_{(0)} = \{V_0 \otimes \mathcal{M}_*(Y)\} \cap W \quad \text{and} \quad W_{(1)} = \{V_1 \otimes \mathcal{M}_*(Y)\} \cap W.$$

Then $W = W_{(0)} \oplus W_{(1)}$, $V_0 \subset W_{(0)}$, $V_1 \subset W_{(1)}$, $\delta W_{(0)} = 0$, and $\delta W_{(1)} \subset \wedge W_{(0)}$. Then $\phi_1|_{\text{Ker}(\delta|_W)} = \text{id}_{\text{Ker}(\delta|_W)}$ and especially $\phi_1|_{W_{(0)}} = \text{id}_{W_{(0)}}$.

For the quasi-isomorphism $\rho_X: (\wedge V, d) \rightarrow (H^*(X), 0)$ with $\text{Ker}(\rho_X|_V) = V_1$, there are the push outs:

$$(3.2) \quad \begin{array}{ccc} (\wedge V, d) & \longrightarrow & (\wedge V \otimes \wedge W_+, \delta) \\ \downarrow \rho_X & & \downarrow \eta \\ (H^*(X), 0) & \longrightarrow & (H^*(X) \otimes \wedge W_+, \delta') \end{array}$$

and for the KS-basis change $\phi_2: (\wedge V \otimes \wedge W_+, \tilde{D}) \cong (\wedge V \otimes \wedge W_+, D)$ corresponding to a certain complement N to $\text{Ker}(D|_V)$ in V as in Lemma 2.1,

$$(3.3) \quad \begin{array}{ccc} (\wedge V, d) & \xrightarrow{i} & (\wedge V \otimes \wedge W_+, \tilde{D}) \\ \downarrow \rho_X & & \downarrow \tilde{\rho} \\ (H^*(X), 0) & \xrightarrow{i'} & (H^*(X) \otimes \wedge W_+, \tilde{D}'), \end{array}$$

where ρ_X , η and $\tilde{\rho}$ are quasi-isomorphisms.

CLAIM. — $(H^*(X) \otimes \wedge W_+, \tilde{D}') \cong (H^*(X) \otimes \wedge W_+, \delta')$ as *cdgas*.

Proof of Claim. — Since (3.3) is a push out, there is a map $(\phi_1\phi_2)'$ such that the following commutes:

$$(3.4) \quad \begin{array}{ccc} (\wedge V \otimes \wedge W_+, \tilde{D}) & \xrightarrow{\phi_1\phi_2} & (\wedge V \otimes \wedge W_+, \delta) \\ \downarrow \tilde{\rho} & & \downarrow \eta \\ (H^*(X) \otimes \wedge W_+, \tilde{D}') & \xrightarrow{(\phi_1\phi_2)'} & (H^*(X) \otimes \wedge W_+, \delta'). \end{array}$$

On the other hand, since (3.2) is a push out, there is a map $(\phi_2^{-1}\phi_1^{-1})'$ such that the following commutes:

$$(3.5) \quad \begin{array}{ccc} (\wedge V \otimes \wedge W_+, \delta) & \xrightarrow{\phi_2^{-1}\phi_1^{-1}} & (\wedge V \otimes \wedge W_+, \tilde{D}) \\ \downarrow \eta & & \downarrow \tilde{\rho} \\ (H^*(X) \otimes \wedge W_+, \delta') & \xrightarrow{(\phi_2^{-1}\phi_1^{-1})'} & (H^*(X) \otimes \wedge W_+, \tilde{D}'). \end{array}$$

Then $(\phi_2^{-1}\phi_1^{-1})' \circ (\phi_1\phi_2)' = \text{id}$ and $(\phi_1\phi_2)' \circ (\phi_2^{-1}\phi_1^{-1})' = \text{id}$ by universality. Hence $(\phi_1\phi_2)'$ is an isomorphism in (3.4). \square

Proof of Theorem 1.2. — The *if* part is obvious since $\delta = 0$ if $d = 0$. The *only if* part is shown as follows. Suppose $V_1 \neq 0$. Let $v = v_i$ be a non-zero basis element in a basis $\{v_j\}_{j \in I}$ of V_1 , where $i = \max\{j \in I; |v_j| = |v|\}$, and let y be a non-zero basis element of V_Y with $d_Y(y) = 0$ for $\mathcal{M}(Y) = (\wedge V_Y, d_Y)$. Such an element y surely exists since Y is not rationally contractible and since V_Y has a well-ordered basis $\{y_i\}_i$ such that $d_Y(y_i) \in \wedge(V_{Y < i})$. Then we can regard $v \otimes y_*$ as a basis element of $W_{(1)}$ with the index k of the basis for some k and $\{v_j \otimes y_*\}_{j < i} \subset W_{< k}$ from the construction of W .

Suppose $\delta(v \otimes y_*) = \delta(\chi)$ for some $\chi \in \wedge W_{< k}$. We can uniquely write

$$\chi = \sum_{j < i} \theta_j(v_j \otimes y_*) + \mu$$

for $\theta_j \in \wedge V_{< i}$ and $\mu \notin \wedge V \otimes (V \otimes y_*)$. Then

$$0 = \delta(v \otimes y_*) - \delta\chi = \kappa^{-1} \tilde{d}\kappa((v - \theta) \otimes y_*) = \kappa^{-1}(d(v - \theta) \otimes y_*)$$

for $\theta = \sum_{j < i} \theta_j v_j$ and $\delta(\mu) = 0$ since

(a) $\tilde{d}(V \otimes z_*) \subset \wedge V \cdot (V \otimes z_*)$ for any $z \in V_Y$, since $d_*(z_*) = z_* \circ d_Y = 0$ due to the decomposability of d_Y , and

(b) $\tilde{d}(V \otimes z_*) \subset \wedge V \cdot (V \otimes (\wedge^{>1} V_Y)_*) \oplus \wedge V \cdot \wedge^{>1}(V \otimes \mathcal{M}_+(Y))$ for any $z \in \wedge^{>1} V_Y$.

Since the derivation $() \otimes y_* : \wedge V \rightarrow \wedge W$ is injective, $d(v) = d(\theta)$ for $\theta \in \wedge V_{< i}$, which contradicts the normality of $(\wedge V, d)$. Thus $\delta(v \otimes y_*)$ is not cohomologous to zero.

We see therefore

$$\phi_1(v \otimes y_*) = v \otimes y_*$$

in (3.1) from the definition of change of KS-basis in the proof of Lemma 1.1 (1). Also

$$D(v \otimes y_*) = \phi_1^{-1} \delta \phi_1(v \otimes y_*) = \phi_1^{-1} \delta(v \otimes y_*) = \delta(v \otimes y_*) \neq 0$$

since $\delta(W_{(1)}) \subset \wedge W_{(0)}$ and $\phi_1|_{W_{(0)}} = \text{id}_{W_{(0)}}$. Hence $v \otimes y_* \notin \text{Ker}(D|_W)$. Then, from Lemma 2.1, we can change KS-basis $\phi_2 : (\wedge W, \tilde{D}) \cong (\wedge W, D)$, so that any \tilde{D} -cocycle of $I(N)$ is \tilde{D} -exact, for some subspace N of $W_{(1)} \cap W_+$ with $v \otimes y_* \in N$. We fix a particular N .

Let $[w]$ be the fundamental class of $H^*(X)$. Then $[w] \cdot (v \otimes y_*)$ is a δ' -cocycle of $H^*(X) \otimes \wedge W_+$. In fact, if $dv = \sum_i a_i v_{i_1} \cdots v_{i_{n_i}}$ for $v_i \in V_0$ and $a_i \in Q$,

$$\begin{aligned} \delta'([w] \cdot (v \otimes y_*)) &= [w] \cdot (\rho_X \otimes 1) \delta(v \otimes y_*) \\ &= \sum_i \sum_{1 \leq j \leq n_i} \pm a_i [wv_{i_1} \cdots \hat{v}_{i_j} \cdots v_{i_{n_i}}] \cdot (v_{i_j} \otimes y_*) \end{aligned}$$

must be zero since the degree of $wv_{i_1} \cdots \hat{v}_{i_j} \cdots v_{i_{n_i}}$ is always greater than the formal dimension of X .

Let $\phi_2(v \otimes y_*) = v \otimes y_* + c$ with c a D -cocycle. Since $\phi_1|_{W_{(0)}} = \text{id}_{W_{(0)}}$, we obtain

$$0 = Dc = \phi_1^{-1} \delta \phi_1(c) = \delta \phi_1(c),$$

i.e., $\phi_1(c)$ is a δ -cocycle. Then $[w](v \otimes y_* + \phi_1(c))$ is a δ' -cocycle but cannot be δ' -exact since $v \otimes y_* + \phi_1(c)$ contains a non-zero element of $W_{(1)} \cap W_+$ due to the definition of change of KS-basis and since

- (a) $\delta'(W_+) \subset H^*(X) \otimes (W_{(0)} \cap W_+)$, and
- (b) $\delta'(\Lambda^{>1}W_+) \subset H^*(X) \otimes \Lambda^{>1}W_+$.

Then, according to the Claim above, $[w](v \otimes y_*)$ is a non-exact \tilde{D}' -cocycle, since

$$(\phi_1 \phi_2)'([w](v \otimes y_*)) = [w](\phi_1(v \otimes y_*) + \phi_1(c)) = [w]((v \otimes y_*) + \phi_1(c))$$

in (3.4). Since $\tilde{\rho}$ is a quasi-isomorphism in (3.3), there exists a non-exact \tilde{D} -cocycle $w \cdot (v \otimes y_*) + \xi$ in ΛW , such that $\tilde{\rho}(\xi) = 0$. Since

$$\text{Ker}(\tilde{\rho}|_{V \oplus W_+}) = \text{Ker}(\rho_X|_{V \oplus W_+}) = V_1,$$

we obtain that $\xi \in I(V_1)$ and thus $w \cdot (v \otimes y_*) + \xi \in I(V_1 \oplus N)$. This contradicts Lemma 2.3. Hence $V_1 = 0$.

Since $\dim_Q H^*(X) < \infty$, this means $V_0 = V_0^{\text{odd}}$ and $H^*(X) = \Lambda(V_0^{\text{odd}})$, i.e., X has the rational homotopy type of a product of odd dimensional spheres if $V_0 \neq 0$ and is rationally contractible if $V_0 = 0$. \square

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