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Octahedra and braids

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OCTAHEDRA AND BRAIDS

BY

BIRGER IVERSEN (*)

RÉSUMÉ. — La catégorie homotopique des complexes sur une catégorie additive est une catégorie triangulée au sens de Verdier et satisfait comme telle l'axiome d'*octaèdre*. Beilinson, Bernstein et Deligne ont observé qu'un diagramme d'octaèdre donne naissance à deux suites de Mayer-Vietoris qui, dans les cas connus, sont des triangles distingués.

Dans cet article nous donnons d'une part un exemple d'un octaèdre dont les suites de Mayer-Vietoris ne sont pas des triangles distingués, et d'autre part nous présentons un nombre d'octaèdres utiles dont les suites de Mayer-Vietoris sont des triangles distingués. Ces exemples sont trouvés en essayant d'implémenter la théorie des triangles sur la topologie algébrique. D'après C. T. C. Wall et d'autres on sait qu'une grande partie de l'algèbre utilisée par Eilenberg et Steenrod peut être systématisée en utilisant un type de diagrammes s'appellant des *tresses*. Les octaèdres et les tresses sont au fond la même chose.

ABSTRACT. — The homotopy category of complexes over an additive category is a triangulated category in the sense of Verdier and satisfies as such the *octahedral* axiom. It has been observed by Beilinson, Bernstein and Deligne that an octahedral diagram gives rise to two Mayer-Vietoris sequences and they have pointed out that these sequences in all known cases are distinguished triangles.

In this paper we give on the one hand an example of an octahedron whose Mayer-Vietoris sequences are not distinguished triangles on the other hand present some useful octahedra whose Mayer-Vietoris sequences are distinguished triangles. These examples were found in an attempt to implement the theory of triangles on algebraic topology. It has been observed by C.T.C. Wall and others that large parts of the algebra used by Eilenberg and Steenrod can be systematicised using a type of diagrams called *braids*. Octahedra and braids are really the same thing.

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The homotopy category of complexes over an additive category is a triangulated category in the sense of VERDIER [7] and satisfies as such the *octahedral* axiom. In [1] 1.1.13 BEILINSON, BERNSTEIN and DELIGNE point out that an octahedral diagram gives rise to two "Mayer-Vietoris" sequences, which in all known cases are distinguished triangles: "Nous ignorons ce qu'il en est en général. Si ces triangles devenaient utiles, il y aurait peut-être lieu de renforcer l'axiome TR 4...".

In this paper we shall on the one hand give an example of an octahedron whose Mayer-Vietoris sequences are not distinguished triangles on the other hand present some usefull octahedra whose Mayer-Vietoris sequences are distinguished triangles. These examples were found in an attemp to implement the theory of triangles on algebraic topology along the lines suggested by a paper by C. T. C. WALL [8]. This can be done once the observation is made that an octahedron can be represented as a braid. —Our main result on the positive side, 4.6 provides the *missing link* to lift some long exact sequences in singular homology to the level of triangles, the best example being the homology sequence of an adjunction space [4], p. 117.

Contents. — §1, Triangles; §2, Octahedra; §3, Mayer-Vietoris sequences; §4, Some usefull octahedra; §5, A counterexample.

1. Triangles

In the first four sections of this paper we work in the category of complexes over a fixed additive category or the corresponding homotopy category.

We use the notation $X^{\cdot}=(X, \hat{c})$ for a complex, in the sense that the symbol X, which is $X^{\cdot}$ with the dot removed, denotes the graded object underlying the complex $X^{\cdot}$. For an integer $n\in\mathbb{Z}$, we put $X^{\cdot}[n]=(X[n], (-1)^n\hat{c})$ where $X[n]$ is given by $(X[n])^p=X^{n+p}$. A morphism of complexes $u: X^{\cdot}\to Y^{\cdot}$ will induce a morphism $u[n]: X^{\cdot}[n]\to Y^{\cdot}[n]$ by the convention $(u[n])^p=u^{n+p}$. This will often cause notational inconvenience, and *we write u instead of u[n]*.

Let us be quite specific with respect to the notion of *triangle*, which we use as a synonym to distinguished triangle. —Given a chainwise split

exact sequence of complexes

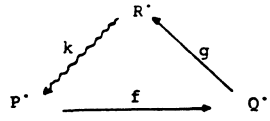
1.1 $$P^{\cdot} \xrightarrow{f} Q^{\cdot} \xrightarrow{g} R^{\cdot}.$$

Choose a section $s: R \to Q$ to $g: Q \to R$ and let $k: R \to P$ [1] denote the morphism given by

1.2 $$fk = \partial s - s\partial.$$

This defines a morphism $k: R^{\cdot} \to P^{\cdot}$ [1] whose homotopy class is independent of s. We call $k: R^{\cdot} \to P^{\cdot}$ [1] the *homotopy* invariant of the sequence 1.1. From these data we can build a diagram in the homotopy category

1.3

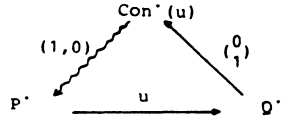


A *triangle* is by definition a diagram in the homotopy category isomorphic to a diagram of the form 1.3.

Given a morphism $u: P^{\cdot} \to Q^{\cdot}$ of complexes. To this we associate the the *mapping cone*

1.4 $$\mathrm{Con}^{\cdot}(u) = \left(P[1] \oplus Q, \begin{pmatrix} -\partial & 0 \\ -u & \partial \end{pmatrix} \right).$$

Which give rise to the *mapping cone* triangle



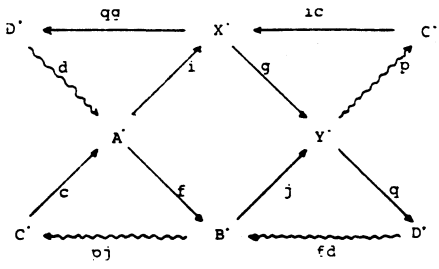
Any triangle is isomorphic to a mapping cone triangle.

Reference: [7], [1], [5], [6]. Notice that we follow the sign convention of Deligne and Verdier as opposed to that of Hartshorne and Illusie. We have changed the mapping cone accordingly, as proposed by BOURBAKI [2].

2. Octahedra

By an *octahedron* we understand a diagram in the homotopy category of the form

2.1

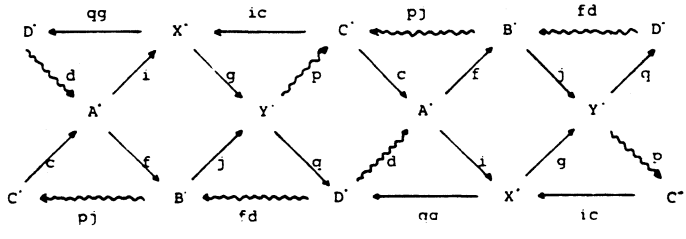


consisting of four triangles and such that

2.2 $$gi=jf, \qquad cp=dq.$$

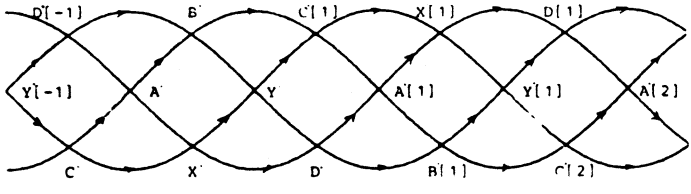
The first of these relations expresses commutativity of the middle diamond in 2.1. The second relation is nicely realized by reflecting the diagram 2.1 in the line AY and attaching it to the old diagram:

2.3



Alternatively the diagram can be represented as a *braid* in the homotopy category

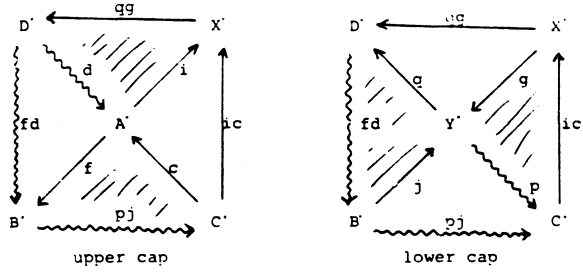
2.4



This representation reveals that an octahedron contains four morphisms of triangles.

To explain the name of the diagram 2.1 we shall represent it as an octahedron

2.5

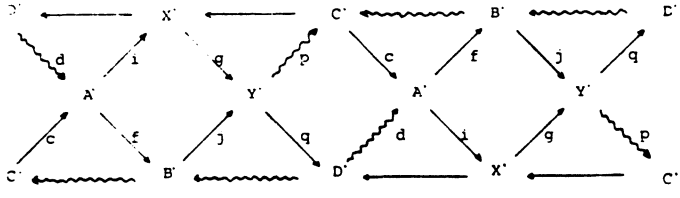


where the shaded areas represent triangles and the unshaded areas represent commutative diagrams. This representation is copied from [1].

3. Mayer-Vietoris sequences

Let us consider a quite general octahedron in the homotopy category

3.1



This gives rise to two sequences

3.2 $$A^{\cdot} \xrightarrow{\begin{pmatrix} i \\ -f \end{pmatrix}} X^{\cdot} \oplus B^{\cdot} \xrightarrow{(g,\ j)} Y^{\cdot} \xrightarrow{cp} A^{\cdot}[1],$$

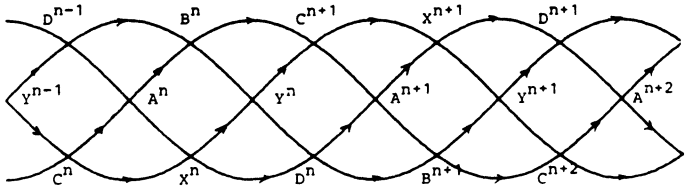
3.3 $$A^{\cdot} \xrightarrow{gi} Y^{\cdot} \xrightarrow{\begin{pmatrix} -q \\ p \end{pmatrix}} C^{\cdot}[1] \oplus D^{\cdot} \xrightarrow{(c,\ d)} A^{\cdot}[1],$$

which we call the *Mayer-Vietoris sequences* of the octahedron. — These are interesting in virtue of

LEMMA 3.4. — *Given an arbitrary object $Z^{\cdot}$ in the homotopy category. Then each of the functors* $\mathrm{Hom}(Z^{\cdot}, -)$, $\mathrm{Hom}(-, Z^{\cdot})$ *transforms the Mayer-Vietoris sequences into long exact sequences of abelian groups.*

This lemma is easily proved by using the braid representation 2.4, *compare Wall* [8]:

LEMMA 3.5. — *Given a braid of abelian groups, i.e. a commutative diagram of the form*



consisting of four long exact sequences. Then the following sequence of abelian groups is exact

$$\to A^n \to X^n \oplus B^n \to Y^n \to A^{n+1} \to$$

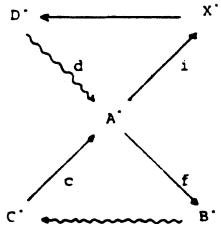
where all morphisms are those of the diagram except $A^n \to X^n$ *which is the one from the diagram with the opposite sign.*

Proof. — Diagram chasing.

Q.E.D.

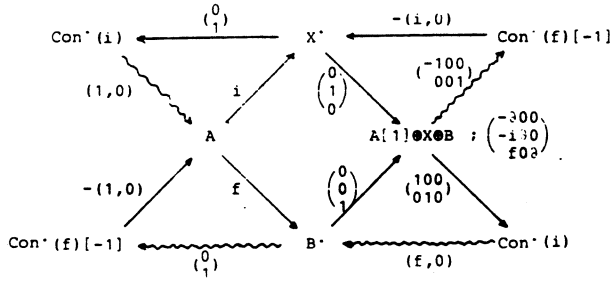
Let us give a proof of the octahedral axiom of Verdier. We offer the excuse, that virtually all details will be used later.

THEOREM 3.6. (Verdier). — *Any diagram consisting of two triangles with a common vertex can be completed to an octahedron whose Mayer-Vietoris sequences are triangles.*



Proof. — Consider the following diagram

3.7



The differential of the complex in the right center is written to the right of the octahedron. —The diamond in the middle is homotopy commutative:

$$\begin{pmatrix}0\\0\\1\end{pmatrix} f - \begin{pmatrix}0\\1\\0\end{pmatrix} i = \begin{pmatrix}\partial & 0 & 0\\ -i\partial & 0 & f\\ 0 & \partial & \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix}\partial$$

To verify that the lower right hand corner is a triangle we can consider the chainwise split exact sequence of complexes

$$B \longrightarrow \begin{matrix}\begin{pmatrix}0\\0\\1\end{pmatrix} & \begin{pmatrix}-\partial & 0 & 0\\ -i\partial & 0 & f\\ 0 & \partial & \end{pmatrix}\end{matrix} A[1]\oplus X\oplus B \xrightarrow{\begin{matrix}\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix} & \begin{pmatrix}-\partial & 0\\ -i\partial & 0\\ 0 & \partial\end{pmatrix}\end{matrix}} A[1]\oplus X$$

This sequence has homotopy invariant (f, 0) as it follows from the formula

$$\begin{pmatrix}-\partial & 0 & 0\\ -i\partial & 0 & f\\ 0 & \partial & \end{pmatrix}\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix} - \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}-\partial & 0\\ -i\partial & 0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}(f,0)$$

To verify that the upper right hand corner is a triangle notice that the chainwise split exact sequence

$$X \xrightarrow{\begin{matrix}\begin{pmatrix}0\\1\\0\end{pmatrix} & \begin{pmatrix}-\partial & 0 & 0\\ -i\partial & 0 & f\\ 0 & \partial & \end{pmatrix}\end{matrix}} A[1]\oplus X\oplus B \xrightarrow{\begin{matrix}\begin{pmatrix}-1&0&0\\0&0&1\end{pmatrix} & \begin{pmatrix}-\partial & 0\\ -f & \partial\end{pmatrix}\end{matrix}} A[1]\oplus B$$

has homotopy invariant (i, 0) as it follows from

$$\begin{pmatrix}-\partial & 0 & 0\\ -u\partial & 0 & \\ 0 & \partial & \end{pmatrix}\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix} - \begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix}\begin{pmatrix}-\partial & 0\\ -vu & \partial\end{pmatrix} = \begin{pmatrix}0 & 0\\ -u & 0\\ vu & \end{pmatrix} = \begin{pmatrix}0\\-1\\v\end{pmatrix}(u,0)$$

The commutativity relations 2.2 are clearely satisfied. Let us prove that the Mayer-Vietoris sequence 3.3 is a triangle leaving 3.2 to the reader. Consider the chainwise split exact sequence

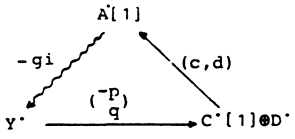
$$\begin{pmatrix} -\partial 0 0 \\ -i\partial 0 \\ \mp 0\partial \end{pmatrix} \quad \begin{pmatrix} 10 & 0 \\ 00 & -1 \\ 10 & 0 \\ 01 & 0 \end{pmatrix} \quad \begin{pmatrix} -\partial 0 & 00 \\ -f\partial & 00 \\ 00 & -\partial 0 \\ 00 & -i\partial \end{pmatrix} \qquad (-1,0,1,0)$$

$$A[1]\oplus X\oplus B \longrightarrow A[1]\oplus B\oplus A[1]\oplus X \longrightarrow A[1] \qquad -\partial$$

The homotopy invariant of this sequence is $\begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix}$ as it follows from the following formula

$$\begin{pmatrix} -\partial 0 & 00 \\ -f\partial & 00 \\ 00 & -\partial 0 \\ 00 & -i\partial \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}(-\partial) = \begin{pmatrix} 10 & 0 \\ 00 & -1 \\ 10 & 0 \\ 01 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix}$$

This proves, with the notation of 3.3, that



is a triangle. Finally turn this triangle once counterclockwise to see that 3.3 is a triangle.

Q.E.D.

4. Some usefull octahedra

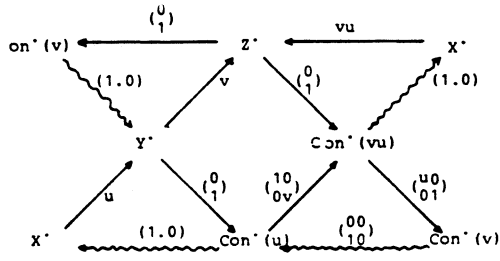
We shall now establish three explicit octahedra whose Mayer-Vietoris sequences are triangles. They will all arize from diagrams in the category of complexes (not the homotopy category). In this context we shall consider the mapping cone as a functor on the category of morphisms of complexes.

These three octahedra have their counterpart in singular homology: The long exact sequence of a triple, Eilenberg-Steenrod [3], I, § 10, triads VII, § 11 and proper (excisive) triads I, § 15.

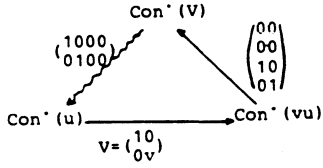
The octahedron of two composable morphisms

Two composable morphisms $u: X^{\cdot} \to Y^{\cdot}$ and $v: Y^{\cdot} \to Z^{\cdot}$ give rise to an *octahedron whose Mayer-Vietoris sequences are exact.*

4.1



Verification. — The octahedron 4.1 is in fact isomorphic to the octahedron 3.7 but we shall proceed directly. The main point is to establish the triangle in the lower right hand corner. We shall derive this from the standard mapping cone based on $V=\begin{pmatrix}1 & 0\\ 0 & v\end{pmatrix}$.



and the following identification of $\mathrm{Con}^{\cdot}(v)$ and $\mathrm{Con}^{\cdot}(V)$

4.2

$$\mathrm{Con}^{\cdot}(v) \underset{(0\,1u\,0\,/\,0\,0\,0\,1)}{\overset{\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix}}{\rightleftarrows}} \mathrm{Con}^{\cdot}(V)$$

using the explicit homotopy

$$\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix}\begin{pmatrix}0&1&u&0\\0&0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}-1&0&0&0\\0&0&u&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}=$$

$$\begin{pmatrix}\partial&0&0&0\\u-\partial&0&0&0\\-1&0-\partial&0\\0-v-vu&\partial\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}\partial&0&0&0\\u-\partial&0&0&0\\-1&0-\partial&0\\0-v-vu&\partial\end{pmatrix}$$

To establish the Mayer-Vietoris sequence 3.2 we notice that the chainwise split exact sequence

$$Y^{\cdot} \xrightarrow{\begin{pmatrix} 0 \\ -1 \\ v \end{pmatrix}} \mathrm{Con}^{\cdot}(u) \oplus Z^{\cdot} \xrightarrow{\begin{pmatrix} 100 \\ 0v1 \end{pmatrix}} \mathrm{Con}^{\cdot}(vu)$$

has homotopy invariant $(u, 0)$ as it follows from

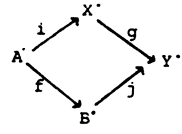
$$\begin{pmatrix} -\partial 00 \\ -i\partial 0 \\ f0\partial \end{pmatrix}\begin{pmatrix} -10 \\ 00 \\ 01 \end{pmatrix} - \begin{pmatrix} -10 \\ 00 \\ 01 \end{pmatrix}\begin{pmatrix} -\partial 0 \\ -f\partial \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(i, 0)$$

We leave the sequence 3.3 to the reader.

THE OCTAHEDRON OF A SQUARE

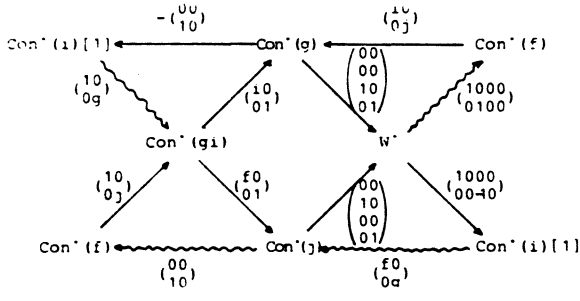
A commutative diagram in the category of complexes

4.3



give rise to an *octahedron whose Mayer-Vietoris sequences are triangles:*

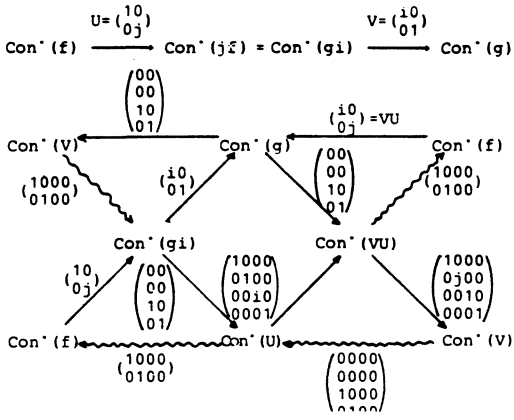
4.4



where the upper right hand triangle is a straight mapping cone triangle, i. e.

$$W^{\cdot}: A[2] \oplus B[1] \oplus X[1] \oplus Y, \begin{pmatrix} \partial & 0 & 0 & 0 \\ f & -\partial & 0 & 0 \\ -i & 0 & -\partial & 0 \\ 0 & -j & -g & \partial \end{pmatrix}$$

Verification. – Let us write down the octahedron 4.1 for the two composable morphisms



From 4.2 we get the homotopy equivalence

$$\mathrm{Con}^{\cdot}(j) \underset{(01f0\ 0001)}{\overset{\begin{pmatrix}00\\10\\00\\01\end{pmatrix}}{\rightleftarrows}} \mathrm{Con}^{\cdot}(U)$$

in a similar way we get the homotopy equivalence

$$\begin{pmatrix}\partial & 0\\ i-\partial\end{pmatrix};\ \mathrm{Con}^{\cdot}(i)[1] \underset{(10\ 00\\ 00-10)}{\overset{\begin{pmatrix}1&0\\0&g\\0&-1\\0&0\end{pmatrix}}{\rightleftarrows}} \mathrm{Con}^{\cdot}(V) \quad ;\begin{pmatrix}\partial & 0 & 0 & 0\\ gi-\partial & 0 & 0\\ -i & 0-\partial & 0\\ 0-1-g & \partial\end{pmatrix}$$

as is seen from the homotopy

$$\begin{pmatrix}1&0\\0&g\\0&-1\\0&0\end{pmatrix}(10\ 00\\ 00-10) - \begin{pmatrix}1000\\0100\\0010\\0001\end{pmatrix} = \begin{pmatrix}0&0&0&0\\0&-1-g&0\\0&0&0&0\\0&0&0-1\end{pmatrix} =$$

$$\begin{pmatrix}\partial & 0 & 0 & 0\\ gi-\partial & 0 & 0\\ -i & 0-\partial & 0\\ 0-1-g & \partial\end{pmatrix}\begin{pmatrix}0000\\0001\\0000\\0000\end{pmatrix} + \begin{pmatrix}0000\\0001\\0000\\0000\end{pmatrix}\begin{pmatrix}\partial & 0 & 0 & 0\\ gi-\partial & 0 & 0\\ -i & 0-\partial & 0\\ 0-1-g & \partial\end{pmatrix}$$

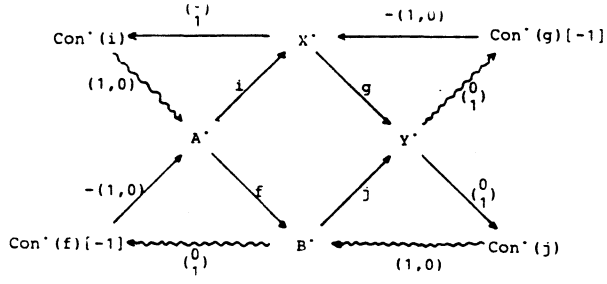
The rest of the details are left to the reader. Let us remark that the detour over 4.1 was chosen in order to establish the Mayer-Vietoris sequences as triangles.

Remark. — In case the diagram 4.3 is merely commutative up to homotopy an octahedron similar to 4.4 can be established by a triple application of Verdier's "octahedral axiom" 3.6. The cost is lost control over the morphisms. The precise form 4.4 is use in the proof of 4.6.

THE OCTAHEDRON OF AN EXCISIVE SQUARE

The commutative diagram 4.3 in the category of complexes give rise to the following diagram containing 4 triangles

4.5



THEOREM 4.6. — *If one of the following* 4 *induced morphisms is a homotopy equivalence*

$$\mathrm{Con}^{\bullet}(i) \to \mathrm{Con}^{\bullet}(j),$$

$$\mathrm{Con}^{\bullet}(f) \to \mathrm{Con}^{\bullet}(g),$$

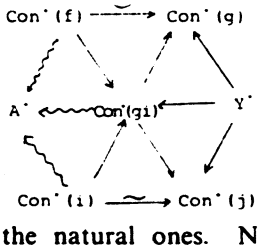
$$\mathrm{Con}^{\bullet}(i) \oplus \mathrm{Con}^{\bullet}(f) \to \mathrm{Con}^{\bullet}(gi),$$

$$\mathrm{Con}^{\bullet}(gi) \to \mathrm{Con}^{\bullet}(j) \oplus \mathrm{Con}^{\bullet}(g),$$

then so are the three others. In the affirmative case the diagram 4.5 *is an* octahedron *once we identify* $\mathrm{Con}^{\bullet}(i)$ *with* $\mathrm{Con}^{\bullet}(j)$ *and* $\mathrm{Con}^{\bullet}(f)$ *with* $\mathrm{Con}^{\bullet}(g)$. *Moreover, the Mayer-Vietoris sequences of this octahedron are triangles.*

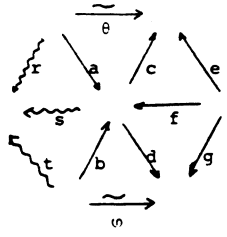
Proof. — Consider the octahedron 4.4. It follows from Lemma 3.4 that each of the four conditions are equivalent to $W^{\bullet} \sim 0$, which proves the first part.

Let us now assume $W^{\bullet} \sim 0$. The main point is to prove the second of the commutativity relations 2.2. Consider the following commutative diagram in the *homotopy category*.



where all morphisms are the natural ones. Notice that all diagonal morphisms are zero as it follows from 4.12, and that the central part of the diagram contains a direct sum. It follows that the two morphisms

from A' to Y' obtained by skirting the sides of the hexagon differ in sign only: Let us put names to the morphism in the diagram



Notice the identity of the centre can be written

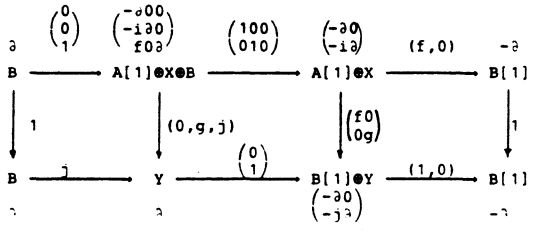
$$1 = a\ \overset{-1}{\theta}\ c + b\ \overset{-1}{\varphi}\ d,$$

by the direct sum assumption. This gives

$$sf = sa\ \overset{-1}{\theta}\ cf + sb\ \overset{-1}{\varphi}\ df = r\ \overset{-1}{\theta}\ e + t\ \overset{-1}{\varphi}\ g$$

and the relation follows since $sf = 0$.

In order to prove that the Mayer-Vietoris sequences of this octahedron are triangles we shall establish an isomorphism from the octahedron 3.7 to the octahedron 4.5. Let us start with the lower right hand triangles



The middle square is homotopy commutative:

$$\begin{pmatrix} f0 \\ 0g \end{pmatrix}\begin{pmatrix} 100 \\ 010 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0,g,j) = \begin{pmatrix} f0 & 0 \\ 00 & -j \end{pmatrix} =$$

$$\begin{pmatrix} -\partial 0 \\ -j\partial \end{pmatrix}\begin{pmatrix} 001 \\ 000 \end{pmatrix} + \begin{pmatrix} 001 \\ 000 \end{pmatrix}\begin{pmatrix} -\partial 00 \\ -i\partial 0 \\ f0\partial \end{pmatrix}$$

The remaining squares are obviously commutative. Thus we have established a morphism of triangles. By assumption two of the sides of this morphism are homotopy equivalences. This implies that we are facing

an equivalence of triangles. The upper right hand triangles are handled in a similar way.

Q.E.D.

Remark 4.7. — In case our basic category is abelian, Theorem 4.6 remains valid if we replace "homotopy equivalence" by "quasi-isomorphism". The output is an octahedron in the derived category whose Mayer-Vietoris sequences are triangles. This follows by a slight modification of the proof of 4.6.

5. A counterexample

In this section we shall construct an octahedron in which one of the Mayer-Vietoris sequences fails to be a triangle. — Let R denote the commutative ring

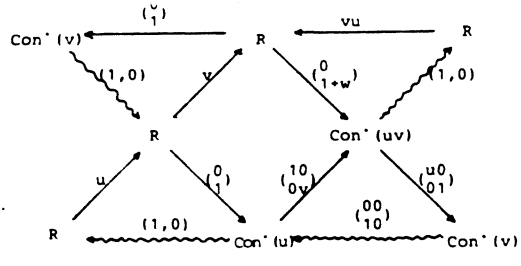
$$5.1 \qquad \mathbb{C}[[S, T, U, V]]/\{V(SU-TV), (T-U)UV^2\}$$

and let u, v, w denote the residue classes of U, V and TV respectively. Let us notice that the first generator gives

$$5.2 \qquad w \equiv 0 \bmod v, \qquad vw \equiv 0 \bmod uv.$$

Consider the following *octahedron* build over the category of free R-modules

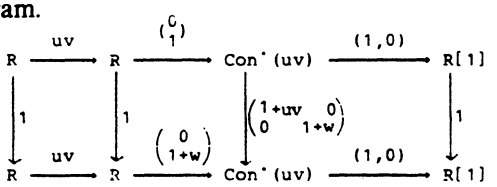
5.3



VERIFICATION OF THE OCTAHEDRON

The congruences 5.2 ensure validity of the needed commutativity relations. To prove that the upper right hand corner of 5.3 is a triangle

we must first remark that the second generator of our ideal provides the relation $(uv-w)\,uv=0$. This allows us to write the following homotopy commutative diagram.



Notice that the first row is a triangle. Thus it suffices to prove that the vertical arrows are homotopy equivalences. This means that

5.4 $\qquad 1+w$ is a unit mod uv,

To see this write $w=av$ and $vw=cuv$, as we may by 5.2. Then

$$(1+w)(1-av)=1-avw=1-acuv\equiv 1 \bmod uv.$$

The Mayer-Vietoris sequence

5.5 $\quad R \xrightarrow{\begin{pmatrix}0\\-1\\v\end{pmatrix}} \mathrm{Con}^{\bullet}(u)\oplus R \xrightarrow{\begin{pmatrix}1&0&0\\0&v&1+w\end{pmatrix}} \mathrm{Con}^{\bullet}(uv) \xrightarrow{(u,0)} R[1]$

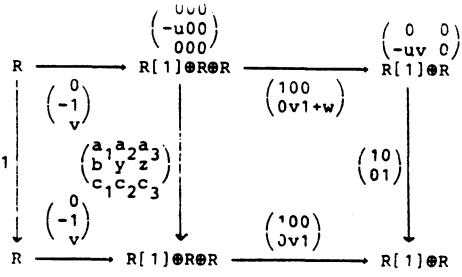
is not a triangle:

Assume the Mayer-Vietoris sequence is a triangle. Then we can compare this with the triangle derived from 4.1

$$R \xrightarrow{\begin{pmatrix}0\\-1\\v\end{pmatrix}} \mathrm{Con}^{\bullet}(u)\oplus R \xrightarrow{\begin{pmatrix}100\\0v1\end{pmatrix}} \mathrm{Con}^{\bullet}(uv) \xrightarrow{(u,0)} R[1]$$

and deduce the existence of a commutative diagram in the homotopy category

5.6



Notice, that the horisontal arrow in the right hand corner is a chainwise split epimorphism, which implies that we may assume the right hand square to be commutative, that is

$$\begin{pmatrix}1&0&0\\0&v&1+w\end{pmatrix}=\begin{pmatrix}1&0&0\\0&v&1+w\end{pmatrix}\begin{pmatrix}a_1&a_2&a_3\\b&y&z\\c_1&c_2&c_3\end{pmatrix}=\begin{pmatrix}a_1&a_2&a_3\\vb+c_1&vy+c_2&vz+c_3\end{pmatrix}$$

Thus our transformation has the form

$$\begin{pmatrix}1&0&0\\b&y&z\\-vb&v-vy&1+w-vz\end{pmatrix}$$

and must commute with the differential

$$\begin{pmatrix}0&0&0\\-u&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\b&y&z\\-vb&v-vy&1+w-vz\end{pmatrix}=\begin{pmatrix}1&0&0\\b&y&z\\-vb&v-vy&1+w-vz\end{pmatrix}\begin{pmatrix}0&0&0\\-u&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\-yu&0&0\\vu(y-1)&0&0\end{pmatrix}$$

which gives the condition $\underline{u-yu=0}$. The first square in the diagram 5.6 must be homotopy commutative, i.e.

$$\begin{pmatrix}0\\-1\\v\end{pmatrix}\sim\begin{pmatrix}1&0&0\\b&y&z\\-vb&v-vy&1+w-vz\end{pmatrix}\begin{pmatrix}0\\-1\\v\end{pmatrix}=\begin{pmatrix}0\\-y+zv\\-v+vy+v+vw-v^2z\end{pmatrix}$$

This gives us $\underline{-v+vy+vw-v^2z=0}$ and a homotopy between the following two morphisms from R to $\mathrm{Con}^{\cdot}(u)$

$$\begin{pmatrix}0\\-1\end{pmatrix}\sim\begin{pmatrix}0\\-y+zv\end{pmatrix}$$

which shows that $\underline{y-zv-1\equiv 0 \bmod u}$.

If we collect the three underlined equations together we see that the following inhomogeneous linear system

5.7
$$\begin{bmatrix} u & 1 & -v \\ 0 & u & 0 \\ 0 & v & -v \end{bmatrix}^2 \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 1 \\ u \\ v-wv \end{Bmatrix}$$

must admit a solution in R. However *the system* 5.7 *have no solutions in R:*

A look at the second equation shows that

$$y = 1 + \text{terms of degree} \geqslant 3.$$

Substitute this into the third equation to see that if we let t denote the residue class of T, then

$$z = t + \text{terms of degree} \geqslant 2.$$

Substitute these two relations into the first equation to get

$$xu = vt + \text{terms of degree} \geqslant 3.$$

This implies that VT is divisible by U which is absurd.

An octahedron where both Mayer-Vietoris sequences fails to be triangles can be obtained by taking the direct sum of the octahedron above and its dual.

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