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THE ENDS OF DISCS

BY

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RÉSUMÉ. — On étudie le comportement au bord des applications holomorphes du disque unité de $\mathbb{C}$ dans $\mathbb{C}^n$. On démontre que, avec des hypothèses, si f est injective sur U et si, de plus, f est continue sur $\bar{U}$, alors f est injective sur bU. Aussi, si f, g sont deux applications propres et holomorphes de U dans $\mathbb{B}_2$, la boule unite de $\mathbb{C}^n$, f, g continues sur $\bar{U}$, alors, encore avec des hypothèses, si $f(bU) \cap g(bU)$ est un ensemble de longueur non nulle, les ensembles $f(\bar{U})$ et $g(\bar{U})$ coïncident.

ABSTRACT. — We study the boundary behavior of holomorphic maps f from the unit disc U in $\mathbb{C}$ into $\mathbb{C}^n$. We show that under some hypotheses, if f is injective on U and extends continuously to $\bar{U}$, then f is injective on bU. Also, if f, g are proper holomorphic maps from U into $\mathbb{B}_2$, the unit ball in $\mathbb{C}^2$, that are continuous on $\bar{U}$, then, again with some hypotheses, if $f(bU)$ meets $g(bU)$ in a set positive length, the sets $f(\bar{U})$ and $g(\bar{U})$ coincide.

I. Introduction

In this paper, we study the boundary behavior of maps from the unit disc, U, into $\mathbb{C}^N$. Our results are of three kinds. First, we show that if $f: U \to \mathbb{C}^N$ is a holomorphic map that is injective on U, then f is injective on bU, granted that $f(bU)$ is a smooth simple closed curve and that f is smooth on bU. Some smoothness is necessary, for examples show that

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continuity by itself does not suffice for the conclusion. Our second set of results has to do with the following question: If $f: U \to D$ is a proper holomorphic map, D a bounded, strictly convex domain in $\mathbb{C}^N$, how do the radial images $I_f(\theta) = \{f(re^{i\theta}): 0 < r < 1\}$ approach bD? If $f' \in H^1$, then almost all of the $I_f(\theta)$ approach bD nontangentially. Other results of this general kind are obtained. Finally, we have some boundary uniqueness theorems for discs: In particular if $f, g: \bar{U} \to \bar{\mathbb{B}}_2$ are continuous maps holomorphic in U with $f(bU)$ and $g(bU)$ rectifiable simple closed curves that lie in $b\mathbb{B}_2$, then either $f(U) = g(U)$ or else $f(bU) \cap g(bU)$ has zero length.

Our methods are a mixture of elementary geometric considerations and appeal to some sophisticated results from modern geometric function theory.

II. On injective maps

Our first result if the following.

THEOREM 1. — *If $f: \bar{U} \to \mathbb{C}^N$ is a continuous map that is holomorphic and one-to-one on U, if $f \mid bU$ is of bounded variation, and if $f(bU)$ is contained in a simple closed curve, then $f(bU)$ is a rectifiable simple closed curve, and f is one-to-one on bU.*

Given a continuous map $g: U \to \mathbb{C}^N$, we will denote by $\mathscr{C}_{bU}(g)$ the *global cluster set of g at bU*, which is the set of all points $p \in \mathbb{C}^N$ for which there is a sequence $\{\zeta_j\}_{j=1}^{\infty}$ in U such that $\zeta_j \to bU$ and $g(\zeta_j) \to p$. According to the work of CHIRKA [Ch], if f is holomorphic from U into $\mathbb{C}^N$, and if $\mathscr{C}_{bU}(f)$ is contained in a totally real manifold M of class $\mathscr{C}^k$, $k > 1$, then f extends continuously to $\bar{U}$ and the extension is of class $\mathscr{C}^{k-0}$. It follows that Theorem 1 has the following consequence:

COROLLARY 2. — *If $f: U \to \mathbb{C}^N$ is a holomorphic map that is one-to-one on U, and if $\mathscr{C}_{bU}(f)$ is contained in a simple closed curve of class $\mathscr{C}^k$, $k > 1$, then $f(bU)$ is this simple closed curve, and f is one-to-one on bU.*

Proof of Theorem 1. — Let $f(bU)$ be contained in the simple closed curve Γ. The set $f(bU)$ is connected and is not a point, so it is either an arc contained in Γ or else is all of Γ. Assume $f(bU) = J$ is an arc. By the maximum principle, $f(U)$ is contained in $\hat{J}$, the polynomially convex hull of J. However, as $f \mid bU$ is of bounded variation, J is a rectifiable

arc, whence, by Alexander's theorem [Al], $\hat{J}=J$, and we have a contradiction. Thus, $f(b\,U)=\Gamma$, and, again, Γ must be rectifiable.

We shall need to consider orientation preserving and reversing maps into Γ. To make these notions precise, we proceed as follows. Fix a diffeomorphism ψ from $\mathbb{T}$, the unit circle in $\mathbb{C}$, onto Γ. Let exp denote the map $x \mapsto e^{ix}$ from $\mathbb{R}$ onto $\mathbb{T}$. Given a continuous map g from an interval $I=[a, b] \subset \mathbb{R}$ into Γ, we shall say that g is *orientation preserving* or *orientation reversing* if there is a continuous map $\tilde{g}: I \to \mathbb{R}$ such that $\psi(\exp \tilde{g})=g$ and g is monotonically increasing or monotonically decreasing, respectively. Covering space theory assures the existence of the maps $\tilde{g}$. The map $\tilde{g}$ is not unique, but if one $\tilde{g}$ is monotononically increasing all are, and if one is monotonically decreasing all are, so our notions are well defined.

Consider now the map $f: b\,U \to \Gamma$. We shall show that f *admits no folding*. To formulate this notion carefully, notice that by uniform continuity, there is a $\delta_0 > 0$ so small that if I and J are arcs in $b\,U$ of length no more than δ_0, then $f(I) \cup f(J) \neq \Gamma$.

LEMMA 3. — *Let I and I' be disjoint arcs in $b\,U$ each of length less than δ_0. Let the end points of I be p and q, and let those of I' be p' and q' where the notation is chosen so that $p=e^{i\theta_p}$, $q=e^{i\theta_q}$ with $0 \leqslant \theta_p < 2\pi$, $\theta_p < \theta_q < \theta_p + 2\pi$, and $I=\{e^{it}: \theta_p \leqslant t \leqslant \theta_q\}$ and similarly for p', q', and I'. If $f(p') \neq f(q')$, we cannot have*

$$f(p)=f(q') \qquad \text{and} \qquad f(q)=f(p').$$

Proof. — Assume the lemma false. By hypothesis, $f(p') \neq f(q')$. We may, therefore, define a point $b' \in I'$ as the first point we reach when moving along I' from p' toward q' that is carried by f to $f(q')$. It may be that $b'=q'$. Next, define $a' \in I'$ as the first point we reach when moving from b' along I' toward p' that is carried by f to $f(p')$. We may have $a'=p'$. Perform a similar construction in I to find points a and b. We have now that $f(a')=f(p')$, $f(b')=f(q')$, and that the arc $[a', b'] \subset b\,U$ is carried to an arc γ' in Γ whose endpoints are $f(a')$ and $f(b')$. Also, the arc $[a, b] \subset b\,U$ is carried to an arc γ in Γ whose endpoints are $f(a)$ and $f(b)$.

We have $f(a)=f(b')$ and $f(b)=f(a')$.

Also, $\gamma=\gamma'$. This is so, for γ and γ' are arcs in Γ with the same endpoints. If they are not coincident, then their union $\gamma \cup \gamma'$ contains a

simple closed curve, C, but then C must be Γ, and $\gamma \cup \gamma' = \Gamma$, contrary to the choice of I and I' (as *short* arcs). Thus, as asserted, $\gamma = \gamma'$.

If α is a smooth one-form on $\mathbb{C}^N$, we may consider the integral $\int_\gamma \alpha$ where γ is taken to be positively oriented in the sense described above. This means that if we take a map $h\colon [t_0, t_1] \to \gamma$ that is of bounded variation and that satisfies the condition that an $\tilde{h}\colon [t_0, t_1] \to \mathbb{R}$ with $\psi(\exp \tilde{h}) = h$ is necessarily monotonically increasing, then

$$\int_\gamma \alpha = \int_{t_0}^{t_1} h^* \alpha.$$

Thus, if we take coordinates $z_1, \ldots, z_N$ on $\mathbb{C}^N$ with $z_j = x_j + i x_{N+j}$ so that

$$\alpha = \sum_{j=1}^{2N} A_j \, dx_j,$$

then

$$\int_\gamma \alpha = \int_{t_0}^{t_1} \{\sum_{j=1}^{2N} A_j \circ h(t) \, h_j'(t)\} \, dt.$$

If, on the other hand, $g\colon [s_0, s_1] \to \gamma$ is a map of bounded variation and satisfies $g = \psi(\exp \tilde{g})$ with $\tilde{g}\colon [s_0, s_1] \to \mathbb{R}$ monotonically decreasing, then

$$\int_\gamma \alpha = -\int_{s_0}^{s_1} \{A_j \circ g(t) \, g_j'(t)\} \, dt.$$

Let $a = e^{i\theta_a}$, $b = e^{i\theta_b}$ and similarly for a' and b', with

$$\theta_p \leqslant \theta_a < \theta_b \leqslant \theta_q \qquad \text{and} \qquad \theta_{p'} \leqslant \theta_{a'} < \theta_{b'} \leqslant \theta_{q'}.$$

We have two parameterizations of the arc γ, viz.,

$$\varphi\colon \ [\theta_a, \theta_b] \to \gamma \qquad \text{and} \qquad \varphi'\colon [\theta_{a'}, \theta_{b'}] \to \gamma$$

given by

$$\varphi(\theta) = f(e^{i\theta}) \qquad \text{and} \qquad \varphi'(\theta) = f(e^{i\theta}).$$

These parameterizations are of bounded variation, but we do not know them to be injective. However, we have that for a smooth one-form α,

$$\int_\gamma \alpha = \pm \int_{\theta_a}^{\theta_b} \varphi^* \alpha = \pm \int_{\theta_{a'}}^{\theta_{b'}} \varphi'^* \alpha. \tag{1}$$

We shall show that the second and third integrals in (1) differ by sign. To do this, choose liftings $\tilde{\varphi}$ and $\tilde{\varphi}'$ of φ and φ' respectively so that $\varphi = \psi(\exp \tilde{\varphi})$, $\varphi' = \psi(\exp \tilde{\varphi}')$. As $f(a) = f(b')$ and $f(b) = f(a')$, we have $\varphi(\theta_a) = \varphi'(\theta_{b'})$ and $\varphi(\theta_b) = \varphi'(\theta_{a'})$. We may, therefore, choose $\tilde{\varphi}$ and $\tilde{\varphi}'$ so that $\tilde{\varphi}(\theta_a) = \tilde{\varphi}(\theta_{b'})$. Having done so, we must necessarily have $\tilde{\varphi}(\theta_b) = \tilde{\varphi}'(\theta_{a'})$: We have $\varphi(\theta_b) = \varphi'(\theta_{a'})$, so $\tilde{\varphi}(\theta_b) = \tilde{\varphi}(\theta_{a'}) \pmod{2\pi}$. As the intervals I and I' are short, this implies $\tilde{\varphi}(\theta_b) = \tilde{\varphi}'(\theta_{a'})$. We have therefore that $\tilde{\varphi}([\theta_a, \theta_b]) = \tilde{\varphi}'([\theta_{a'}, \theta_{b'}])$.

Let $x_a = \tilde{\varphi}(\theta_a)$, $x_b = \tilde{\varphi}(\theta_b)$ whence $x_a = \tilde{\varphi}'(\theta_{b'})$ and $x_b = \tilde{\varphi}'(\theta_{a'})$. If $\eta = \psi \circ \exp$ so that $\eta : [x_a, x_b] \to \gamma$ is a parameterization, we have

$$\int_{x_a}^{x_b} \eta^* \alpha = \int_{\theta_a}^{\theta_b} \tilde{\varphi}^* \eta^* \alpha = \int_{\theta_a}^{\theta_b} (\eta \circ \tilde{\varphi})^* \alpha = \int_{\theta_a}^{\theta_b} \varphi^* \alpha$$

and

$$\int_{x_a}^{x_b} \eta^* \alpha = \int_{\theta_{b'}}^{\theta_{a'}} \tilde{\varphi}'^* \eta^* \alpha = \int_{\theta_{b'}}^{\theta_{a'}} (\eta \circ \tilde{\varphi}')^* \alpha = -\int_{\theta_{a'}}^{\theta_{b'}} \varphi'^* \alpha.$$

Thus, as claimed,

$$\int_{\theta_a}^{\theta_b} \varphi^* \alpha = -\int_{\theta_{a'}}^{\theta_{b'}} \varphi'^* \alpha. \tag{2}$$

Let now λ be an arc with endpoints a and b and otherwise lying in U, and let λ' be a similar arc with endpoints a' and b'. We suppose $\lambda \cap \lambda' = \emptyset$. Thus, $f(\lambda) \cup f(\lambda')$ is a simple closed curve, Λ, in $\mathbb{C}^N$. Denote by Δ the simply connected domain in U the boundary of which consists of λ together with the arc $[a, b]$ in bU. Let Δ' be the corresponding domain determined by λ' and $[a', b']$.

Define a current T on $\mathbb{C}^N \setminus \Lambda$ by the condition that if α is a smooth, compactly supported 2-form on $\mathbb{C}^N \setminus \Lambda$, then

$$T(\alpha) = \int_{f(\Delta)} \alpha + \int_{f(\Delta')} \alpha. \tag{3}$$

That these integrals exist is seen as follows: α is a sum of terms of form

$A dz_j dz_k$, $B dz_j d\bar{z}_k$ and $C d\bar{z}_j d\bar{z}_k$. Thus, what must be seen is that if G is a smooth function on the plane that vanishes near λ, then the integral

$$\int_\Delta G(z) f_j'(z) \bar{f}_k'(z)\, dz\, d\bar{z}, \tag{4}$$

exists. (Here $f_1, \ldots, f_N$ denote the components of the map f.)

By hypothesis, f is of bounded variation on bU, so each f_j' belongs to the Hardy space H^1 on the unit disc ([Zy], p. 285) and ([Du], p. 42). (This is a theorem of Privalov. *See* [Pr], Chapter I, §4.) It follows that $\int_U |f_j'|^2 < \infty$ (1). Thus, the integrals (4) exist.

The current T is closed because of Stokes's theorem and the result (2). It is of bidegree (1, 1) and positive. The support of T is the set $(f(\bar{\Delta}_1) \cup f(\bar{\Delta}_2)) \setminus \Lambda$, and at each point of this support outside the set γ, which has finite one-dimensional Hausdorff measure, the Lelong number of T is one. Thus, a theorem of KING ([Ki], Th. 5.3.1) applies: There is a one-dimensional variety W in $\mathbb{C}^N \setminus \Lambda$ such that

$$T\alpha = \int_W \alpha$$

for all 2-forms α. We have then that $W \supset f(\Delta) \cup f(\Delta')$ and, consequently, the closure of $f(\Delta) \cup f(\Delta')$ in $\mathbb{C}^N \setminus \Lambda$ necessarily coincides with the variety W.

We shall defer the proof of the following fact for the moment.

LEMMA 4. — *The intersection $f(bU) \cap f(U) = \Gamma \cap f(U)$ has zero one-dimensional Hausdorff measure.*

This is somewhat more than we need just now, but as λ and λ' are contained, except for their endpoints, in U, it implies the existence of an arc L contained in $f([a, b]) \setminus \Lambda = f([a', b']) \setminus \Lambda$. But this leads to a contradiction of the maximum principle.

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(1) If $f_j = \sum a_n z^n$, then $\int_U |f_j'|^2 < \infty$ is equivalent to $\sum n |a_n|^2 < \infty$. The Riemann-Lebesgue lemma yields $na_n \to 0$, and, as f_j is of bounded variation, $\sum |a_n| < \infty$ ([Zy], p. 286). Thus, $\sum n |a_n|^2 < \infty$.

We have that $\bar{W} \setminus W \subset \Lambda$, $\bar{W}$ denoting the closure of W in $\mathbb{C}^N$. As $\Lambda \subset f(U)$, this implies that W is contained in the polynomially convex hull of Γ. By the theory of the Shilov boundary [Ri] we know that the polynomially convex hull of Γ is the same as the polynomially convex hull of the subset Γ_0 of Γ that consists of the closure of the set of peak points for the algebra $\mathscr{P}(\Gamma)$. The arc L is contained in the variety W, so no point of L can be a peak point for the algebra $\mathscr{P}(\Gamma)$. This means that $\Gamma_0 \subset (\Gamma \setminus L)^-$. This is impossible though, for by the theorem of ALEXANDER [A1], the rectifiable arc $(\Gamma \setminus L)^-$ is polynomially convex.

This contradiction completes the proof of Lemma 3.

Proof of Lemma 4. – Put $E = f(bU) \cap f(U)$. If $\Lambda'(E) \neq 0$ – we denote by Λ^r the r-dimensional Hausdorff measure on $\mathbb{C}^N$ – there are points z_0 of metric density of the set E with respect to Λ^1. Let $z_0 = f(0)$. There is a connected open set V_0 in U containing 0 that is mapped by f injectively onto a subvariety $f(V_0)$ of a ball $\mathbb{B}(z_0, r)$ of radius r centered at z_0. (Note, we cannot claim that f is biholomorphic on V_0; $df(0)$ may be 0.) For some choice of $h_1, \ldots, h_s \in O(\overline{\mathbb{B}(z_0, r)})$.

$$f(V_0) = \{z \in \mathbb{B}(z_0, r) : h_1(z) = \ldots = h_s(z) = 0\}.$$

As z_0 is a point of density for E, there is an arc $L \subset bU$ such that $z_0 \in f(L) \subset \mathbb{B}(z_0, r)$ and $\Lambda^1(E \cap f(L)) > 0$. The map $f : bU \to \Gamma$ is absolutely continuous and so sends null sets to null sets. This implies that $\Lambda^1(L \cap f^{-1}(E)) > 0$.

Let L' be an arc in U, except for its endpoints, which are those of L, and which, together with L, bounds a domain D in U that is disjoint from V_0 and that is carried by f into $\mathbb{B}(z_0, r)$. As $h_1, \ldots, h_s$ are holomorphic on $\overline{\mathbb{B}(z_0, r)}$, the functions $h_j \circ f$ are defined on $\bar{D}$ and vanish on the set $f^{-1}(E) \cap L$. As this set has positive length, $h_j \circ f$ vanishes on D whence

$$f(D) \subset \cap_{j=1}^{s} \{h_j = 0\} = f(V_0).$$

By construction, $D \cap V_0 \neq \emptyset$, so we have a contradiction to the assumed injectivity of f on U.

The lemma is proved.

To complete the proof of Theorem 1, we shall need to use part (ii) of the following theorem which is of some independent interest.

THEOREM 5. — *Let f, $g\colon \bar{U} \to \mathbb{C}^N$ be continuous maps holomorphic on U. Assume that both are of bounded variation on bU and that both are injective on U and also on bU. Let λ_f, λ_g be open arcs in bU taken by f and g, respectively, onto a rectifiable open arc Λ. Put*

$$\psi = (g \mid \lambda_g)^{-1} \circ (f \mid \lambda_f).$$

(i) If ψ reverses orientation, then the set

$$E = [f(U) \cup g(U) \cup \Lambda) \setminus [f(bU \setminus \lambda_f) \cup g(bU \setminus \lambda_g)],$$

is a variety in the domain

$$\Omega = \mathbb{C}^N \setminus [f(bU \setminus \lambda_f) \cup g(bU \setminus \lambda_g)].$$

(ii) *If ψ preserves orientation, then there exist neighborhoods W_f of λ_f in $\lambda_f \cup U$ and W_g of λ_g in $\lambda_g \cup U$ such that $f(W_g) = g(W_g)$ and $f = g \circ \varphi$ for some biholomorphic map $\varphi\colon W_f \to W_g$.*

The point is that either the discs $f(U)$ and $g(U)$ abut along Λ so as to form a variety or else they coincide near Λ. Of course, either case can occur.

Proof. — As f and g are of bounded variation on bU, $f(U)$ and $g(U)$ both have finite area.

Assume that ψ reverses orientation. Define a current T on Ω by

$$T\alpha = \int_{f(U)} \alpha + \int_{g(U)} \alpha,$$

for all smooth, compactly supported two-forms α on Ω. The *structure theorem* ([Ha], p. 337) implies that T is a holomorphic chain: For some locally finite family V_j, $j = 1, 2, \ldots$, of varieties V_j in Ω and for some integers m_j, which we may take to be nonzero,

$$T = \sum_j m_j [V_j].$$

We see that the only possible values for m_j are one and two.

The arc Λ is contained in a single one of the V_j, say V_1. It follows that every global branch of $f(U) \setminus (f(bU) \cup g(bU))$ whose closure (in $\mathbb{C}^N$) meets Λ is necessarily contained in V_1, and similarly, every global branch of $g(U) \setminus (f(bU) \cup g(bU)$ whose closure meets Λ is contained in V_1. In

each case, there is only one of these global branches: In the case of f, it is the image, A_f under f of the component of $U \setminus f^{-1}(f(bU) \cup g(bU))$ that abuts λ_f, and in the case of g, it is the image, A_g, under g of the component of $U \setminus g^{-1}(f(bU) \cup g(bU))$ that abuts λ_g, because f and g are both injective on bU.

We now have that $V_1 \supset (A_f \cup A_g)$. The orientation hypothesis implies that A_f and A_g lie on opposite sides of Λ in V_1. It follows then that the variety V effects the continuation of $f(U) \cup g(U)$ through Λ.

It remains to deal with the case that ψ preserves orientation. In this case, define the current S on Ω by

$$S\alpha = \int_{f(U)} \alpha - \int_{g(U)} \alpha.$$

Again by the structure theorem, S is a holomorphic chain: For some locally finite family V_j, $j=1, \ldots,$ of distinct one-dimensional varieties in Ω and some choice of integers $m_1, m_2, \ldots$

$$S\alpha = \sum_j m_j [V_j].$$

We may suppose that each of the V_j's meets $f(U) \cup g(U)$ in an open subset; with this assumption, we cannot suppose that all the m_j's are nonzero.

By virtue of the definition of S, the only possible values of m_j are -1, 0, and 1.

With A_f and B_f as above, we again have $A_f \cup B_f$ contained in a single one of the V_j's, say V_1. The orientation hypothesis implies that A_f and B_f lie on the same side of Λ in V_1. Thus, A_f and B_f meet in an open set, whence the integer m_1 is 0. This implies that A_f and B_f coincide.

The theorem is proved.

We now complete the proof of Theorem 1. We know that the map f is locally injective from bU to Γ, and thus, it is a covering map, say of μ sheets. We must prove μ to be one. If not, we can find disjoint arcs λ_1, and λ_2 in bU that are carried by f homeomorphically onto an arc $\gamma \subset \Gamma$. As $f^{-1}(\Gamma) \cap U$ has length zero, we can find rectifiable arcs λ_1' and λ_2' which, except for their endpoints, are contained in U and such that $\lambda_1' \cup \lambda_1$ bounds a domain D_1 and $\lambda_2' \cup \lambda_2$ a domain D_2 with f one-to-one on D_1 and on bD_1 and also one-to-one on D_2 and on bD_2. If

we denote by φ_j, $j=1, 2$, a conformal map from U to D_j, and then apply Theorem 5 to the maps $f\circ\varphi_1$ and $f\circ\varphi_2$, we find that $\varphi_1(U)$ and $\varphi_2(U)$ meet, i. e., $f(D_1)\cap f(D_2)\neq\emptyset$. This contradicts the assumption that f is injective on U.

Theorem 1 is proved.

In connection with Theorem 1, it may be useful to cite the map $g\colon \bar{U}\to\mathbb{C}^2$ given by $g(\zeta)=(\zeta^3, 1-|\zeta|^2\zeta)$ which is one-to-one on U but which realizes bU as a three-sheeted covering of the simple closed curve $\{(e^{i\theta}, 0)\colon \theta\in\mathbb{R}\}$. Theorem 1 depends in an essential way on the holomorphicity of the map.

We cannot say, in general, when the hypotheses of Theorem 1 are satisfied, but there is the following situation in which they are.

THEOREM 6. — *Let D be a bounded convex domain in $\mathbb{C}^N$, and let $f\colon U\to D$ be a proper, holomorphic map that extends continuously to $\bar{U}$. If $f(bU)$, a subset of bD, is contained in a rectifiable simple closed curve, then the map $f|bU$ is of bounded variation and is locally injective. If f is injective on U, then f is injective on bU and hence on $\bar{U}$.*

If we invoke the embedding theorem given by FORNAESS [Fo] and HENKIN [HCh], we see that this theorem is true also when D is a strongly pseudoconvex domain.

The proof of this result depends on a preliminary fact about polynomial convexity.

LEMMA 7. — *Let $K\subset\bar{D}$ be a polynomially convex, compact set that meets bD at a single point. If λ is a rectifiable arc in bD, then $K\cup\lambda$ is polynomially convex.*

We will defer the proof of this for the moment and proceed directly to the proof of the Theorem. Denote by Γ a rectifiable simple closed curve in bD that contains $f(bU)$. Again, as rectifiable arcs are necessarily polynomially convex, we must have $\Gamma=f(bU)$.

By uniform continuity, there is $\delta>0$ small enough that if $p, q\in bU$ are distinct points that satisfy dist $(p, q)<\delta$, then if L_{pq} denotes the shorter of the two arcs in bU determined by p and q, then $f(L_{pq})$ is a proper subset of Γ and hence is an arc.

Suppose $p, q\in bU$ to satisfy dist $(p, q)<\delta$ and $f(p)=f(q)$. Let L_{pq} be as in the preceding paragraph, and let $\lambda\subset U\cup\{p, q\}$ be an arc joining p to q. If $J=f(\lambda)$, then J is a closed curve, not necessarily simple, contained

in $D \cup \{f(p)\}$, and its polynomially convex hull, $\hat{J}$, is contained in $D \cup \{f(p)\}$. Denote by L the arc $f(L_{pq})$.

The lemma implies that $L \cup \hat{J}$ is polynomially convex whence $L \setminus \hat{J}$ is open in $(L \cup \hat{J})\hat{} = L \cup \hat{J}$. However, if $z_0 \in L \setminus \hat{J}$, say $z_0 = f(p_0)$ with $p_0 \in L_{pq}$, and if λ_0 is short radial arc in U that terminates at p_0, then z_0 lies in the closure of the subset $f(\lambda_0 \setminus \{p_0\})$ of U.

This is impossible, however, for if Δ denotes the simply connected open subset of U bounded by $\lambda \cup L_{pq}$, then $\lambda_0 \setminus \{p_0\} \subset \Delta$, and the maximum principle shows that $f(\lambda_0) \subset (L \cup J)\hat{} = L \cup \hat{J}$. This contradiction implies that f is locally injective on bU.

That f is of bounded variation is now essentially clear. As Γ is rectifiable, there is a map $\psi\colon \mathbb{R} \to \Gamma$ that is periodic with period one, that is of bounded variation and that is injective on [0, 1). Let $E\colon \mathbb{R} \to bU$ be the exponential map given by $E(t) = e^{2\pi i t}$. By covering space theory, there is a map $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ with $\psi \circ \tilde{f} = f \circ E$. As f is locally injective, $\tilde{f}$ is injective, i. e., monotonic. Locally we may write $f = \psi \circ \tilde{f} \circ E^{-1}$. As ψ is of bounded variation, $\tilde{f}$ monotonic and E^{-1} smooth, we recognize that f is of bounded variation.

If now we assume that f is one-to-one on U, we may apply Theorem 1 to conclude that f is one-to-one on bU and hence one-to-one on $\bar{U}$.

The Theorem is proved, except for the verification of the lemma.

Proof of the Lemma. — STOLZENBERG [St] proved that if the compact set X is polynomially convex and if F is a finite union of $\mathscr{C}^1$ curves such that the map $H^1(X \cup F, \mathbb{Z}) \to H^1(X, \mathbb{Z})$ induced by the inclusion $X \to X \cup F$ is an isomorphism, then $X \cup F$ is polynomially convex. ALEXANDER [Al] proved that rectifiable arcs are polynomially convex, and as he remarked, only minor alterations of his proof are required to obtain the more general version of Stolzenberg's theorem in which F is replaced by a finite union of rectifiable arcs.

Having proved Theorem 6, we should remark that in this context, further regularity theorems follow from the theory of boundary regularity for minimal surfaces. See the papers of NITSCHE [Ni 1, 2].

The proofs of Theorems 1 and 6 have made use of the assumed regularity of Γ at several points. On the one hand, it is a moral certainty that our regularity hypotheses are not optimal, but, on the other hand, some regularity hypothesis is required as the following example shows.

EXAMPLE 8. — *There is a map* $\varphi: \bar{U} \to \mathbb{C}^N$, *for some* N, *that is continuous on* $\bar{U}$, *holomorphic on* U, *that is one-to-one and regular on* U *and that carries* bU *onto a simple closed curve* Γ *in* $\mathbb{C}^N$ *in such a way that every point of* Γ *is the image of exactly two points in* bU.

To construct the example, let Λ be an arc in $\mathbb{C}$ of locally positive area. Let $\mathscr{A}_\Lambda$ be the algebra of continuous functions on the Riemann sphere, $\mathbb{C}^*$, that are holomorphic off Λ. According to WERMER [We] and RUDIN [Ru] finitely many functions, say $f_1, \ldots, f_r$ in $\mathscr{A}_\Lambda$ separate points on $\mathbb{C}^*$. Let the endpoints of Λ be ζ_0 and $\zeta_1 \in \mathbb{C}$. For $i, j, k = 1, \ldots, r$, let

$$F_{ijk}(\zeta) = (f_i(\zeta) - f_i(\zeta_0))(f_j(\zeta) - f_j(\zeta_1))(f_k(\zeta) - f_k(\infty)).$$

For each choice of $\zeta \in \mathbb{C}^* \setminus \{\zeta_0, \zeta_1, \infty\}$ there are i, j, k so that $F_{ijk}(\zeta) \neq 0$. Set $F_{ij}(\zeta) = (f_i(\zeta) - f_i(\zeta_0))\ (f_j(\zeta) - f_j(\zeta_1))$, and let $\tilde{F}_{ijk}(\zeta) = \zeta F_{ijk}(\zeta)$. The $2r^3 + r^2$ functions F_{ijk}, $\tilde{F}_{ijk}$ and F_{ij}, $1 \leqslant i, j, k \leqslant r$ separate points on $\mathbb{C}^*$ except that they identity the points ζ_0 and ζ_1. To the F's we have constructed, we can adjoin a finite number of functions $G_1, \ldots, G_s \in \mathscr{A}_\Lambda$ with the properties that for every i, $G_i(\zeta_0) = G_i(\zeta_1)$ and for every $\zeta \in \mathbb{C}^*$, one of G's is holomorphic with nonvanishing derivative at ζ. Let $t = 2r^3 + r^2 + s$, and let $g_1, \ldots, g_t$ be a relabelling of the F_{ijk}, the $\tilde{F}_{ijk}$, the F_{ij} and the G_i.

The Riemann map ψ from U to $\mathbb{C}^* \setminus \Lambda$ extends continuously to $\bar{U}$; without loss of generality we may suppose that $\psi(1) = \zeta_0$, $\psi(-1) = \zeta_1$. Then ψ carries $b^+U = bU \cap \{z \in \mathbb{C}: \operatorname{Im} z \geqslant 0\}$ homeomorphically onto Λ and also carries $b^-U = bU \cap \{z \in \mathbb{C}: \operatorname{Im} z \leqslant 0\}$ homeomorphically onto Λ.

The map $\varphi = (g_1 \circ \psi, \ldots, g_t \circ \psi): U \to \mathbb{C}^t$ has the desired properties.

Notice that the degree of the map $\varphi \mid bU \to \Gamma$ is zero, as it must be since φ factors through the contractible space Λ.

III. On the approach to the boundary

In this section we consider a holomorphic map f from the unit disc U in $\mathbb{C}$ into a convex, smoothly bounded domain D in $\mathbb{C}^N$ and examine how the radial images

$$I_f(\theta)=\{f(re^{i\theta}): 0<r<1\},$$

approach the boundary under various hypotheses on f. The principal result is the following:

THEOREM 9. — *Let D be a bounded strictly convex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, and let $f: U \to D$ be a holomorphic map such that $f(1)=\lim_{r\to 1^-} f(r)$ exists and lies in bD:*

(a) If $\lim_{r\to 1^-} f'(r)$ exists, it is nonzero and transverse to bD at $f(1)$.

(b) If $f'(r)$ is bounded on $(0, 1)$, then $f(r)$ approaches $f(1)$ nontangentially as $r \to 1^-$, i. e., there exists a constant K such that

$$|f(r)-f(1)| \leqslant K \operatorname{dist}(f(r), bD).$$

(c) If $\int_0^1 |f'(r)|^2\, dr < \infty$, then for $r \to 1^-$,

$$|f(r)-f(1)| = o\{\operatorname{dist}(f(r), bD)\}^{1/2}.$$

Remarks. — 1. The condition of strict convexity is understood in the analytic sense that there be a defining function for the domain D with positive definite Hessian along bD. This implies that for some fixed $R > 0$, given a point $p \in bD$, the ball of radius R in $\mathbb{C}^N$ that meets D and whose boundary passes through p and is tangent at p to bD in fact contains the whole domain D. Thus, to prove the theorem, it suffices to prove it in case that D is the ball $\mathbb{B}_N$. On the other hand, having the result for general strictly convex domains shows, by way of the embedding theorem of Fornaess and Henkin that the corresponding result for strongly pseudoconvex domains is true. (In fact, as the question is entirely local, it would suffice to use the fact that a strongly pseudoconvex domain is strictly convex, in suitable local coordinates, near each of its boundary points.)

2. Geometrically, the conclusion of (*b*) is that $f(r)$ approaches $f(1)$ through a cone with vertex at the point $f(1)$. The conclusion of (*c*) is that $f(r)$ approaches $f(1)$ through *every* ball Ω contained in D and tangent to bD at $f(1)$.

Proof of Theorem 9. — As we have noted in Remark 1, it suffices to consider the case that D is the ball $\mathbb{B}_N$. Also, as the automorphisms of

$\mathbb{B}_N$ extend to biholomorphic maps of a neighborhood of $\bar{\mathbb{B}}_N$, we may, without loss of generality, suppose that $f(0)=0$.

Notice to begin with, that if $f'(r)$ is bounded in $(0, 1)$, then for some $c>0$,

$$1-|f(r)| \geq c|f(r)-f(1)|, \qquad 0<r<1. \tag{1}$$

If (1) is false, then for some se sequence $\{r_n\}_{n=1}^{\infty}$, $r_n \to 1^-$, we have

$$1-|f(r_n)| = o(f(r_n)-f(1)). \tag{2}$$

As $f(0)=0$, the Schwarz lemma yields

$$1-|f(z)| \geq 1-|z|, \tag{3}$$

whence (2) implies that as $n \to \infty$,

$$\left|\frac{1-r_n}{f(1)-f(r_n)}\right| \to 0. \tag{4}$$

However, (4) is impossible: As f' is bounded, we have

$$|f(r_n)-f(1)| = \left|\int_{r_n}^{1} f'(r)\,dr\right| \leq (1-r_n)\sup_{0<r<1}|f'(r)|,$$

which precludes (4). Thus, (1) holds, and we have proved (b). Also, as

$$\sup_{r<t<1}|f'(t)| \geq \frac{|f(1)-f(r)|}{1-r} \geq \frac{1-r}{1-r} = 1,$$

it follows that $\lim_{r\to 1^-} f'(r)$, if it exists, cannot be zero.

That $f'(1)=\lim_{r\to 1} f'(r)$, when it exists, is not transverse to $b\mathbb{B}_N$ is seen as follows. We have that $f'(1)$ is not zero; assume it tangent to $b\mathbb{B}_N$ at $f(1)$ so that if $\langle\ ,\ \rangle$ denotes the Hermitian inner product on $\mathbb{C}^N$, then $\operatorname{Re}\langle f'(1), f(1)\rangle = 0$. We know that $f(r)\to f(1)$ nontangentially, so that for some K,

$$|f(1)-f(r)| \leq K(1-|f(r)|).$$

This yields

$$\frac{|f(1)-f(r)|}{1-r} \leqslant \frac{K}{1-r}(\langle f(1), f(1)\rangle - \langle f(r), f(r)\rangle)$$

$$= K\left(\langle f(1), \frac{f(1)-f(r)}{1-r}\rangle + \langle \frac{f(1)-f(r)}{1-r}, f(r)\rangle\right)$$

$$\to K 2\,\mathrm{Re}\langle f(1), f'(1)\rangle = 0,$$

which is inconsistent with (3). Thus, $f'(1)$ is not tangent to $b\mathbb{B}_N$ at $f(1)$, and we have (2).

It remains to prove (*c*). To this end, let $0 < r < r' < 1$. Then

$$|f(r')-f(r)| = \left|\int_r^{r'} f'(\xi)\,d\xi\right| \leqslant \sqrt{r'-r}\left(\int_r^{r'} |f'(\xi)|^2\,d\xi\right)^{1/2}.$$

If we let $r' \to 1^-$, we find that

$$f(1)-f(r) = o(\sqrt{1-r}).$$

Then, by (3), we get

$$\frac{f(1)-f(r)}{\{\mathrm{dist}(f(r), b\mathbb{B}_N)\}^{1/2}} = \frac{f(1)-f(r)}{\sqrt{1-r}}\,\frac{\sqrt{1-r}}{\sqrt{1-|f(r)|}} = o(1),$$

as we wished to prove.

This completes the proof of the theorem.

Notice that in step (*c*), if we assume instead that if $\int_0^1 |f'(r)|^p\,dr < \infty$, $1 < p < \infty$, we find that

$$|f(r)-f(1)| = o\{\mathrm{dist}(f(r), bD)\}^{1/q},$$

where q is the index conjugate to p.

Another transversality result, in the general direction of (*a*), was given in [GS1].

As a corollary of (*a*), we have the following fact.

COROLLARY 10. — *Let D be a bounded strictly convex domain in $\mathbb{C}^N$ with bD of class $\mathscr{C}^2$. If $f: U \to D$ is a proper holomorphic map that extends continuously to $\bar{U}$ and that satisfies the condition that $f(bU)$ is a rectifiable*

simple closed curve, then for almost all θ, *the radial image* $I_f(\theta)$ *approaches* $f(e^{i\theta})$ *nontangentially.*

Again, the result extends immediately to strongly pseudoconvex domain.

Proof. — By Theorem 6, $f \mid bU$ is of bounded variation, whence by invoking Privalov's theorem again, we find that $\lim_{r \to 1} f'(re^{i\theta})$ exists for almost all θ. This implies the result, by Theorem 9.

COROLLARY 11. — *Let* D *be a bounded strictly convex domain in* $\mathbb{C}^N$ *with* bD *of class* $\mathscr{C}^2$. *If* $f: U \to D$ *is a proper holomorphic map such that* $f(U)$ *has finite area, then for almost all* θ, *the radial image* $I_f(\theta)$ *approaches* $f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ *through every ball contained in* D *and tangent to* bD *at* $f(e^{i\theta})$.

Proof. — The condition that $f(U)$ have finite area is the condition that

$$\int_U \int |f'(re^{i\theta})|^2 \, r \, dr \, d\theta < \infty \quad (^2).$$

Thus, for almost all θ, $\int_0^1 |f'(re^{i\theta})|^2 \, dr < \infty$, and the result follows from (*c*) of Theorem 9.

IV. Supplementary remarks

The matters we take up here fall under the rubric of boundary uniqueness theory. As in Theorem 5, the general idea is that if f and g are continuous maps from $\bar{U}$ to $\mathbb{C}^N$, holomorphic in U, and if f and g satisfy some additional condition at bU, then from the hypothesis that $f(bU) \cap g(bU)$ be large follows the conclusion that $f(U) = g(U)$ or else that $f(U) \cup g(U)$ continues across $f(bU) \cap g(bU)$ to give a variety. Of course, hypotheses are necessary, but there are some nontrivial conclusions of this sort to be drawn on the basis of what we have done above. We have the following simply stated result.

(^{2}) The integral gives the area, *counted with multiplicity*, of $f(U)$. As f is proper, the multiplicity is uniformly bounded off a countable set.

THEOREM 12. — *Let f and g be proper holomorphic maps from U to $\mathbb{B}_2$ that extend continuously to $\bar{U}$. If $f(bU)=\Gamma_1$ and $g(bU)=\Gamma_2$ with Γ_1, Γ_2 rectifiable simple closed curves, and if $\Gamma_1 \cap \Gamma_2$ has positive length, then $f(U)=g(U)$.*

Here, of course, *length* is in the sense of the one-dimensional Hausdorff measure with respect to the standard metric on $\mathbb{C}^2$.

Proof. — By Theorem 6, $f \mid bU$ and $g \mid bU$ are both of bounded variation, and so f', $g' \in H^1(U)$. It follows — recall footnote (1) — that f and g both have finite Dirichlet integral, so the varieties $f(U)$ and $g(U)$ have finite area.

By the result of BERNDTSSON [Be], there is a bounded holomorphic function F on $\mathbb{B}^2$ with $f(U)=V_F=\{z\in\mathbb{B}_2 : F(z)=0\}$.

Let $\Sigma=f(bU)\cap g(bU)$ so that Σ is a set of positive length. The map $g \mid bU$ is of bounded variation and so is absolutely continuous. This implies that the set $g^{-1}(\Sigma)=\Sigma_g$ is a subset of bU of positive length. As $g'\in H^1(U)$, $g'(e^{i\theta})=\lim_{r\to 1^-} g'(re^{i\theta})$ exists and is finite for almost every $e^{i\theta}\in\Sigma_g$. Theorem 9 *b* implies that the radial image $I_g(\theta)$ approaches $g(e^{i\theta})$ nontangentially.

Also, if $\Sigma_f=f^{-1}(\Sigma)$, then Σ_f has positive length and for almost every $e^{i\theta}\in\Sigma_f$, $I_f(\theta)$ approaches $f(e^{i\theta})$ nontangentially. It follows from Čirka's generalization of the Lindelöf theorem ([Ru2], p. 171) that the function $F\circ g$ has radial limits zero a.e. on the set Σ_g. Thus, $F\circ g$ vanishes identically whence $g(U)\subset V_F=f(U)$. By symmetry, $f(U)\subset g(U)$, and the theorem is proved.

The preceding argument is not long, though it is based on several earlier results. The assumed boundary regularity was used explicitly in the proof, as it must be: Recall that in [GS2] proper holomorphic maps $\varphi: U\to\mathbb{B}_2$ are constructed with the property that $\mathcal{C}_{bU}(\varphi)=b\mathbb{B}_2$. Of course, it is not clear what the minimal smoothness hypothesis is that will ensure the validity of the conclusion.

We know that to obtain a result like Theorem 5, some element of smoothness is required. By the examples of RUDIN [Ru1] and WERMER [We], disjoint analytic discs can abut along arcs (of large Hausdorff dimension) and yet neither coincide nor be analytic continuations of

each other ([3]). It is not clear how much smoothness is required. We have the following result that relaxes the smoothness requirement, but imposes a supplementary geometric condition.

THEOREM 13. — *Let f and g be continuous mappings from $\bar{U}$ to $\bar{D}$, D a domain in $\mathbb{C}^N$, f and g holomorphic in U. Suppose there to be an open ball Ω in $\mathbb{C}^N$ centered at a point of bD such that $f(bU)\cap\bar{\Omega}=g(bU)\cap\bar{\Omega}=\Lambda$ is an arc that projects homeomorphically into the z_1-plane onto an arc of class $\mathscr{C}^{1+\varepsilon}$ for some $\varepsilon>0$. If Λ consists entirely of peak points for the algebra $A(D)$ ([4]), then $f(U)=g(U)$.*

Remark. — An arc in bD that projects homeomorphically onto a smooth arc certainly need not itself be smooth.

Proof of the Theorem. — Let $f=(f_1, \ldots, f_N)$, $g=(g_1, \ldots, g_N)$, and let

$$\lambda_f=f^{-1}(\Omega\cap bD), \qquad \lambda_g=g^{-1}(\Omega\cap bD),$$

so that λ_f and λ_g are unions of arcs in bU. According to Čirka's work [Ch] (*cf.* [BG]), the functions f_1 and g_1 are of class $\mathscr{C}^1$ on λ_f and λ_g, respectively. If we denote by Λ_1 the projection of Λ into the z_1-plane, then it follows that at for some $\zeta_0\in\Lambda_1$, we can find points $z_f\in\lambda_f$ and $z_g\in\lambda_g$ such that

$$f(z_f)=\zeta_0=g(z_g) \qquad \text{and} \qquad f_1'(z_f)\neq 0, \qquad g_1'(z_g)\neq 0.$$

Consequently, there are closed Jordan domains Δ_f and Δ_g in $U\cup\lambda_f$ and $U\cup\lambda_g$, respectively, Δ_f a neighborhood of z_f in $U\cup\lambda_f$, Δ_g a neighborhood of z_g in $U\cup\lambda_g$, such that $\Delta_f\cap bU$ and $\Delta_g\cap bU$ are arcs and such that f_1 is one-to-one on Δ_f, g_1 is one-to-one on Δ_g. If we choose Δ_f and Δ_g properly, we shall have that $f_1(\Delta_f\cap bU)=g_1(\Delta_g\cap bU)$, a certain subarc of Λ_1, say λ_1.

Two cases are possible.

First, it may be that $f_1(\Delta_f)$ and $g_1(\Delta_g)$ abut λ_1 from opposite sides. Define $\Phi_f\colon\Delta_f\to\mathbb{C}^N$ and $\Phi_g\colon\Delta_g\to\mathbb{C}^N$ by

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([3]) It seems to be an open question whether it is possible to realize such an arc in the boundary of the ball (or a strictly convex domain).

([4]) As usual, $A(D)$ denotes the algebra of functions continuous on $\bar{D}$, holomorphic in D.

$$\Phi_f(\zeta)=f\circ f_1^{-1}(\zeta) \qquad \text{and} \qquad \Phi_g(\zeta)=g\circ g_1^{-1}(\zeta).$$

Then Φ_f and Φ_g are analytic continuations of each other across λ_1. If $\Phi=\Phi_f$ in Δ_f, Φ_g in Δ_g, then Φ is holomorphic from a neighborhood of ζ_0 into $\bar{D}$ with $\Phi(\zeta_0)\in bD$. As each point of bD is a peak point for $A(D)$, this entails a contradiction to the maximum principle.

Thus, $f(\Delta_f)$ and $g(\Delta_g)$ abut λ_1 from the same side. This implies that the maps Φ_f and Φ_g, which agree along the arc λ_1 agree near λ_1, and so the varieties $f(U)$ and $g(U)$ share a common open subset. As both are irreducible subvarieties of D, we must have equality, whence the result we seek.

One might wonder, on the basis of Theorem 12, whether there is a version of Theorem 5 in which the arc Λ is replaced by a set of positive length. The following example suggests that there is no such conclusion to be drawn under any reasonably general hypotheses.

EXAMPLE 14. — Let J' and J'' be infinitely differentiable simple closed curves in the plane that bound mutually disjoint domains Ω' and Ω'' but that meet in the set $E=J'\cap J''$, which is a perfect, totally disconnected set of positive length.

The construction of the example depends on the following fact:

LEMMA 15. — *Let J be a simple closed infinitely differentiable curve in the plane. Let $\{\zeta_k\}_{k=1}^{\infty}$ be a sequence in the unbounded component of $\mathbb{C}\setminus J$ that has a proper closed subset S of J as its cluster set. If the sequence $\{c_k\}_{k=1}^{\infty}$ is chosen properly, the function*

$$G(\zeta)=\sum_{k=1}^{\infty}\frac{c_k}{\zeta-\zeta_k},$$

will be of class A^{∞} on D, the bounded component of $\mathbb{C}\setminus J$ and will continue analytically across no point of S.

Proof. — If we merely assume $\sum_{k=1}^{\infty}|c_k|<\infty$, then the series defining G will converge uniformly on compacta in $\mathbb{C}\setminus(S\cup\{\zeta_1,\zeta_2,\ldots\})$, so the sum function, G, will be holomorphic there. Moreover, under the assumption that none of the c_k's be zero, each of the ζ_k's is a pole for G, so every point of S is a limit point of poles for G. Thus, G will continue analytically across no point of S. To be certain that $G\in A^{\infty}(D)$, we need only remark that each of the fractions $1/(\zeta-\zeta_k)$, qua function of ζ, lies in

$A^\infty(D)$, and then note that given an arbitrary Fréchet space $\mathscr{E}$ and an arbitrary sequence $\{e_j\}_{j=1}^\infty$ in $\mathscr{E}$, if the sequence $\{\alpha_j\}_{j=1}^\infty$ of nonzero numbers converges rapidly enough to zero, then the series $\sum_j \alpha_j e_j$ converges in $\mathscr{E}$.

The lemma is proved.

Denote by ψ' and ψ'' conformal maps from U onto Ω' and Ω'', respectively. Since the curves J' and J'' are infinitely differentiable, the maps ψ' and ψ'' are of class $A^\infty(U)$.

Let $\Omega_1, \Omega_2, \ldots$ be an enumeration of the bounded components of $\mathbb{C}\setminus(J' \cup J'')$ other than Ω' and Ω'', and for each j, let ζ_j be a point of Ω_j. The set E is perfect and totally disconnected, so each point of E is a limit point of the sequence $\{\zeta_j\}_{i=1}^\infty$, and these are its only limits points.

By the lemma, if $\{c_j\}_{j=1}^\infty$ is chosen correctly, then the function F given by

$$F(\zeta)=\sum_{j=1}^\infty \frac{c_j}{\zeta-\zeta_j},$$

will be of class A^∞ on both Ω' and Ω'' but will continue analytically across no point of E.

Define $\Psi', \Psi'': \bar{U} \to \mathbb{C}^2$ by

$$\Psi'(z)=(\psi'(z), F(\psi'(z))), \qquad \Psi''(z)=(\psi''(z), F(\psi''(z))).$$

Thus,

$$\Psi'(\bar{U}) \cup \Psi''(\bar{U})=\{(\zeta, F(\zeta)): \zeta \in J' \cup J'' \cup \Omega' \cup \Omega''\}.$$

These smoothly bounded analytic discs abut along the set $\tilde{E}=\{(\zeta, F(\zeta): \zeta \in E\}$ which has positive length. However, no one-dimensional analytic variety contains $\Psi'(\bar{U}) \cup \Psi''(\bar{U})$, for such a variety would provide an analytic continuation of F through the set E. As we have constructed F so as to admit no such continuation, the assertion follows.

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