BULLETIN DE LA S. M. F.

PAUL MALLIAVIN
L.A. RUBEL

On small entire functions of exponential type with given zeros

Bulletin de la S. M. F., tome 89 (1961), p. 175-206
<http://www.numdam.org/item?id=BSMF_1961__89__175_0>

Bull. Soc. math. France,
89, 1961, p. 175 à 206.

ON SMALL ENTIRE FUNCTIONS OF EXPONENTIAL TYPE WITH GIVEN ZEROS;

BY

PAUL MALLIAVIN AND L. A. RUBEL (*).

1. Introduction. — In this paper, we give a complete solution of the problem : " If the zeros of an entire function of exponential type are known to include a given sequence of positive real numbers, what can be said about the growth of the function on the imaginary axis? " Our solution of this general problem provides the first solution of some special cases of it that have been studied for some time.

It is well known that an entire function of exponential type that has many real positive zeros cannot be small on the imaginary axis. The initial result along these lines is Carlson's theorem that no such function $f(z)$ can vanish on all the positive integers and also be majorized on the imaginary axis by $|\sin tz|$ if $t < \pi$.

For a sequence Λ of positive real numbers, we denote by $\mathcal{F}(\Lambda)$ the ideal, in the ring of all entire functions of exponential type, of those functions that vanish at least on Λ. (We exclude once and for all the null function $f(z) \equiv 0$ and the ideals containing only the null function.) We introduce an order relation in this system of ideals, $\mathcal{F}(\Lambda) \prec \mathcal{F}(\Lambda')$, meaning that for each $g \in \mathcal{F}(\Lambda')$, we can find an $f \in \mathcal{F}(\Lambda)$ such that $|f(iy)| \leqq |g(iy)|$ for every real y. Crudely stated, $\mathcal{F}(\Lambda) \prec \mathcal{F}(\Lambda')$ if it is easier to construct small entire functions that vanish on Λ than those that vanish on Λ'.

A consequence of our analysis will be that if there exists only one pair f_0, g_0 with $f_0 \in \mathcal{F}(\Lambda)$, $g_0 \in \mathcal{F}(\Lambda')$ such that $|f_0(iy)| \leqq |g_0(iy)|$ for all y, and such that g_0 has no other zeros in the half plane $x > 0$ than those in the sequence Λ', then $\mathcal{F}(\Lambda) \prec \mathcal{F}(\Lambda')$.

(*) The second author was partially supported in this research by the United States Air Force Office of Scientific Research.

The major problem is to decide, by elementary computations on the sequences Λ and Λ', whether $\mathcal{F}(\Lambda) \prec \mathcal{F}(\Lambda')$. First, we define an equivalence relation between ideals, $\mathcal{F}(\Lambda) \sim \mathcal{F}(\Lambda')$, meaning that both $\mathcal{F}(\Lambda) \prec \mathcal{F}(\Lambda')$ and $\mathcal{F}(\Lambda') \prec \mathcal{F}(\Lambda)$.

On the other hand, consider

$$\lambda(x) = \sum_{\substack{\lambda \in \Lambda \\ \lambda \leqq x}} \frac{1}{\lambda}$$

and say that the two sequences Λ and Λ' are equivalent, $\Lambda \sim \Lambda'$, when

$$\lambda(x) - \lambda'(x) = O(1).$$

Then we have the result

$$\mathcal{F}(\Lambda) \sim \mathcal{F}(\Lambda') \quad \text{if and only if } \Lambda \sim \Lambda'.$$

We continue by defining an order relation between the sequences, which is the quotient of the inclusion relation modulo equivalence. This means that $\Lambda \prec \Lambda'$ if there exists Λ'' such that $\Lambda \subset \Lambda''$ and such that $\Lambda'' \sim \Lambda'$.

Then we have the result

$$\mathcal{F}(\Lambda) \prec \mathcal{F}(\Lambda') \quad \text{if and only if } \Lambda \prec \Lambda'.$$

Furthermore, we can decide whether $\Lambda \prec \Lambda'$ by elementary computations : we have $\Lambda \prec \Lambda'$ if and only if

$$\lambda(y) - \lambda(x) \leqq \lambda'(y) - \lambda'(x) + O(1) \quad \text{for all } x, y : x < y.$$

For instance, specializing our results to the case where Λ' is the sequence of all positive integers, and choosing $g_0 \in \mathcal{F}(\Lambda')$ as $g_0(z) = \sin \pi z$, we see that there exists a function $f \in \mathcal{F}(\Lambda)$ satisfying

$$|f(iy)| \leqq \exp(\pi |y|)$$

if and only if

$$\lambda(y) - \lambda(x) \leqq \log y/x + O(1).$$

This statement gives us the first necessary and sufficient condition known for the problem of completeness of $\exp(-\lambda_n z)$ in a horizontal strip, under uniform convergence on compact sets, a problem which has been considered by CARLEMAN [2], KAHANE [5] and LEONTIEV [7], *cf.* [15] p. 135.

Furthermore, using other entire functions than the sine, we can decide whether there exists in $\mathcal{F}(\Lambda)$ functions with assigned growth on the imaginary axis (*see* Theorems 6.2, 6.3, 6.4 and 6.5).

The main technique of construction relies on a method of balayage, or sweeping (see, for instance, [3]). In paragraph 5, a balayage of the measure generated by the zeros is performed using explicit expressions for the sweeping kernels, which are calculated by means of the Mellin transform. This enables us to see exactly what changes we make in the growth when we displace the zeros from Λ to Λ'.

In section 8 we consider the problem of minimizing the overall type of functions in $\mathcal{F}(\Lambda)$. By overall type, we mean

$$h(f) = \limsup_{|z| \to \infty} \frac{\log |f(z)|}{|z|}.$$

This problem is central in the theory of adherent series and in the theory of detection of singularities of functions defined by a Dirichlet series [11]. In the previous work on this subject, the Weierstrass product over the sequence Λ has often been used as a comparison function. We shall show, as a matter of fact, that in a wide class of cases, the Weierstrass product does not minimize the overall type of the functions in $\mathcal{F}(\Lambda)$. An open problem is to find an explicit expression for this minimum type in terms of elementary computations on the sequence Λ.

CONTENTS.

2. Notations, definitions, and classical results. — We study sequences $\Lambda = \{\lambda_n\}$ of positive real numbers λ_n, arranged for convenience in non-decreasing order :

$$\Lambda : \quad 0 < \lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots \tag{2.1}$$

The " characteristic logarithm " $\lambda(t)$ associated with Λ is defined by

$$\lambda(t) = \sum_{\lambda_n \leqq t} \lambda_n^{-1} \tag{2.2}$$

and the counting function $\Lambda(t)$ by

$$(2.3) \qquad \Lambda(t) = \sum_{\lambda_n \leqq t} 1 = \int_0^t s\, d\lambda(s).$$

Some upper densities associated with Λ are the "upper density" $\bar{D}(\Lambda)$, the "upper Poisson density" $\bar{D}_p(\Lambda)$, and the "logarithmic block density" $\bar{D}_L(\Lambda)$, defined respectively by

$$(2.4) \qquad \bar{D}(\Lambda) = \limsup_{t \to \infty} t^{-1} \Lambda(t),$$

$$(2.5) \qquad \bar{D}_p(\Lambda) = \limsup_{y \to \infty} \frac{2}{\pi} \int_0^\infty \frac{y}{t^2 + y^2} \frac{\Lambda(t)}{t}\, dt,$$

$$(2.6) \qquad \bar{D}_L(\Lambda) = \inf_{a > 1} \limsup_{t \to \infty} \frac{\lambda(at) - \lambda(t)}{\log a}.$$

See [13] for some of the properties and interrelationships of these and other densities.)

Throughout the paper, we will assume that

$$(2.7) \qquad \lim_{t \to \infty} \Lambda(t) = \infty,$$

$$(2.8) \qquad \bar{D}(\Lambda) < \infty,$$

i.e., that Λ is an *infinite* sequence and has finite upper density. With suitable interpretations, our theorems are trivially true if Λ is finite, or if the upper density of Λ is infinite. We stress, however, that no extraneous "separation condition" of the sort $\lambda_{n+1} - \lambda_n \geqq \delta > 0$ is assumed, except in section 7, in which we re-prove, by other methods, a result of the preceding section. Also, repetition of the same λ_n is allowed, provided the multiplicity is taken account of.

The notions of equivalence, quasi-inclusion, etc., and the symbols $\sim$ and $\prec$ are defined in paragraph 3, and in the introduction.

We denote by $\mathcal{F}(\Lambda)$ the class of entire functions $f(z)$ of exponential type such that $f(\lambda_n) = 0$ for $n = 0, 1, 2, \ldots$, with the exception of the null function, $f(z) = 0$ for all z, which we exclude. To say that $f(z)$ is of exponential type is to say that

$$(2.9) \qquad |f(z)| \leqq K e^{\tau |z|}$$

for some choice of the constants K and τ, and for all z. We recall [1], Chapter 5, that the growth of an entire function $f(z)$ of exponential type is studied by introducing the "indicator function" $h_f(\theta)$ defined by

$$(2.10) \qquad h_f(\theta) = \limsup_{r \to \infty} r^{-1} \log |f(r e^{i\theta})|,$$

that $h_f(\theta)$ is continuous in θ, and enjoys certain convexity properties. By h_f, the overall type of f, we mean the infimum of those numbers τ for which there is a constant K such that (2.9) holds. The type h_f is also given by

$$h_f = \max_{\theta} h_f(\theta). \tag{2.11}$$

The envelope of the line $x \cos\theta - y \sin\theta - h_f(\theta) = 0$ is a convex curve Γ_f, called the indicator diagram of f. Γ_f is the convex boundary of the set of singularities of the Borel transform $F(z)$ of f

$$F(z) = \int_0^\infty f(t)\, e^{-zt}\, dt. \tag{2.12}$$

If Γ^* is a curve homologous to Γ and lying in the unbounded portion of the complement of Γ, then

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma^*} F(w)\, e^{zw}\, dw. \tag{2.13}$$

We make frequent use of the following theorem of Lindelöf [1], p. 27 :

THEOREM A. — *In order that a sequence z_n of complex numbers be precisely the sequence of zeros of an entire function of exponential type, it is necessary and sufficient that $z_n = 0$ for at most finitely many n and that each of the following two conditions hold*

$$\limsup_{t\to\infty} t^{-1} N(t) < \infty, \tag{2.14}$$

$$\sum_{0<|z_n|<R} z_n^{-1} = O(1), \tag{2.15}$$

where $N(t) = \sum_{0<|z_n|\leq t} 1$. Furthermore, given an entire function of order 1 whose zeros satisfy these conditions, the function must be of exponential type.

The function $W(z) = W(z:\Lambda)$, belonging to $\mathcal{F}(\Lambda)$, is called the Weierstrass product (over Λ) and is defined by

$$W(z:\Lambda) = W(z) = \prod_{n=0}^{\infty} (1 - z^2 \lambda_n^{-2}). \tag{2.16}$$

The logarithm of $|W(z)|$ may be written as a Stieltjes integral,

$$\log|W(z:\Lambda)| = \int_0^\infty \log|1 - r^2 t^{-2} e^{2i\theta}|\, d\Lambda(t), \tag{2.17}$$

where $z = r e^{i\theta}$. For $\theta \neq 0, \pi$, an integration by parts in (2.17) gives

$$(2.18) \qquad \log | W(r e^{i\theta}) | = r \int_0^\infty P(t, \theta) \Lambda(rt) (rt)^{-1} dt,$$

where

$$(2.19) \qquad P(t, \theta) = 2 \frac{1 - t^2 \cos 2\theta}{1 - 2t^2 \cos 2\theta + t^4}.$$

For the upper Poisson density, we have

$$(2.20) \qquad \overline{D}_P(\Lambda) = \pi^{-1} h_W\left(\frac{\pi}{2}\right) = \pi^{-1} \operatorname{type} (W(z:\Lambda)).$$

We define, for $0 < b < \infty$, the arithmetic progression Λ_b by

$$(2.21) \qquad \Lambda_b = \{ 1/b, 2/b, 3/b, \dots \}$$

and observe that

$$\Lambda_b(t) = [bt] = bt + O(1),$$
$$\lambda_b(t) = b \log t + O(1) \quad \text{for } t \geqq 1,$$
$$\overline{D}(\Lambda_b) = \overline{D}_P(\Lambda_b) = \overline{D}_L(\Lambda_b) = b,$$
$$W(z:\Lambda_b) = \frac{\sin \pi b z}{\pi b z},$$
$$h_{W_b}(\theta) = \pi b | \sin \theta |.$$

3. **Quasi-inclusion and the logarithmic block density.** — We write $\Lambda \subset \Lambda'$ to indicate that Λ is a subsequence of Λ', and remark that $\Lambda \subset \Lambda'$ if and only if $\lambda(y) - \lambda(x) \leqq \lambda'(y) - \lambda'(x)$ whenever $x \leqq y$.

DEFINITION 3.1. — *Λ is equivalent to Λ', written $\Lambda \sim \Lambda'$, shall mean that $\lambda'(x) - \lambda(x) = O(1)$.*

DEFINITION 3.2. — *Λ' is a quasi-supersequenee of Λ, written $\Lambda' \succ \Lambda$, shall mean that there exists a sequence Λ'', $\Lambda'' \supset \Lambda$, such that $\Lambda'' \sim \Lambda'$.*

DEFINITION 3.3 — *Λ is a quasi-subsequence of Λ', written $\Lambda \prec \Lambda'$, shall mean that there exists a sequence Λ''', $\Lambda''' \subset \Lambda'$, such that $\Lambda''' \sim \Lambda$.*

Although $\Lambda \prec \Lambda'$ and $\Lambda' \succ \Lambda$ mean two different things, the first corollary of the next lemma resolves this notational difficulty.

LEMMA 3.1. — *$\Lambda' \succ \Lambda$ holds if and only if there exist a constant K such that*

$$(3.1) \qquad \lambda(y) - \lambda(x) \leqq \lambda'(y) - \lambda'(x) + K$$

whenever $0 < x \leqq y < \infty$. Likewise, $\Lambda \prec \Lambda'$ holds if and only if (3.1) is satisfied.

Corollary 1. — $\Lambda \prec \Lambda'$ *if and only if* $\Lambda' \succ \Lambda$.

Corollary 2. — *If* $\Lambda \sim \Lambda_1$ *and* $\Lambda' \sim \Lambda'_1$, *and* $\Lambda \prec \Lambda'$, *then* $\Lambda_1 \prec \Lambda'_1$.

Corollary 3. — *If* $\Lambda_1 \prec \Lambda_2$ *and* $\Lambda_2 \prec \Lambda_3$, *then* $\Lambda_1 \prec \Lambda_3$.

Corollary 4. — *If* $\Lambda \prec \Lambda'$ *and* $\Lambda' \prec \Lambda$, *then* $\Lambda \sim \Lambda'$.

Thus, $\prec$ is a well-defined partial ordering of equivalence classes under $\sim$.

Proof of Lemma 3.1. — That $\Lambda' \succ \Lambda$ and $\Lambda \prec \Lambda'$ each imply (3.1) is trivial. To show that (3.1) implies that $\Lambda' \succ \Lambda$, we define

$$\varphi(x) = \inf_{s \geqq x} \{ \lambda'(s) - \lambda(s) \}. \tag{3.2}$$

It follows from (3.1) that $\varphi(x) \geqq -K$.

Now $\varphi(x)$ is constant except for possible positive jumps at the jumps of $\lambda'(x)$. Let x_0 be a point of discontinuity of φ. Then

$$\varphi(x_0 - 0) = \lambda'(x_0 - 0) - \lambda(x_0 - 0) \text{ and} \tag{3.3}$$

$$\varphi(x_0 + 0) \leqq \lambda'(x_0 + 0) - \lambda(x_0 + 0). \tag{3.4}$$

We denote by $\Delta\varphi(x_0)$ the jump of φ at x_0. Then

$$\Delta\varphi(x_0) \leqq \Delta\lambda'(x_0) - \Delta\lambda(x_0) \leqq \Delta\lambda'(x_0). \tag{3.5}$$

We let

$$\Lambda^*(t) = [\Phi(t)], \tag{3.6}$$

where $[a]$ denotes the integral part of a and

$$\Phi(t) = \int_0^t s \, d\varphi(s), \tag{3.7}$$

and let $\lambda^*(t)$ be the characteristic logarithm of that sequence Λ^* whose counting function [*see* (2.3)] is $\Lambda^*(t)$. The function $\lambda^*(t)$ is constant except possibly at the jumps of $\varphi(t)$, and we have

$$\Delta\lambda^*(x_0) < 1/x_0 + \Delta\varphi(x_0). \tag{3.8}$$

Using (3.5), we get

$$\Delta\lambda^*(x_0) < 1/x_0 + \Delta\lambda'(x_0). \tag{3.9}$$

Furthermore, $x_0 \Delta\lambda^*(x_0)$ and $x_0 \Delta\lambda'(x_0)$ must be integers, so that (3.9) implies

$$\Delta\lambda^*(x_0) \leqq \Delta\lambda'(x_0), \tag{3.10}$$

and this means that Λ^* is a subsequence of Λ'.

We now define

$$(3.11) \qquad \lambda''(x) = \lambda(x) + \lambda^*(x),$$

so that $\lambda''(x)$ is the characteristic logarithm of some sequence $\Lambda'' \supset \Lambda$. To prove that $\Lambda'' \sim \Lambda'$, we must prove that $\delta(x) = O(1)$, where

$$(3.12) \qquad \delta(x) = \lambda(x) + \lambda^*(x) - \lambda'(x).$$

Now

$$(3.13) \qquad \varphi(t) - \varphi(0) = \int_0^t s^{-1}\, d\Phi(s)$$

and

$$(3.14) \qquad \lambda^*(t) = \int_0^t s^{-1}\, d[\Phi(s)].$$

An integration by parts shows that

$$(3.15) \qquad \lambda^*(t) - \varphi(t) = -\varphi(0) + O\left(\frac{1}{t}\right),$$

so that it is enough to prove that $\theta(x) = O(1)$, where

$$(3.16) \qquad \theta(x) = \lambda(x) + \varphi(x) - \lambda'(x) = \lambda(x) - \lambda'(x) + \inf_{s \geqq x} \{\lambda'(s) - \lambda(s)\}.$$

But it is clear that $\theta(x) \leqq 0$, and (3.1) is simply another way of saying that $\theta(x) \geqq O(1)$.

To prove that (3.1) implies $\Lambda \prec \Lambda'$, we put

$$(3.17) \qquad \lambda'''(x) = \lambda'(x) - \lambda^*(x).$$

Since, by (3.10), Λ^* is a subsequence of Λ', there is a sequence Λ''' defined by (3.17), and Λ''' is a subsequence of Λ'. Since we have already shown that $\Lambda''' \sim \Lambda$, i.e., $\delta(x) = O(1)$, our proof is complete.

Logarithmic block density. — Each sequence Λ [*see* (2.1), (2.7) and (2.8)] is a quasi-subsequence of some arithmetic progression Λ_b [*see* (2.21)]. We adopt the conventions $\Lambda_\infty = \{0, 0, 0, \ldots\}$ and $\Lambda_0 = \emptyset$, the empty sequence.

Definition 3.4. — *Given a sequence* Λ, *we shall mean by* $b^*(\Lambda)$ *the infimum of those numbers* b *for which* $\Lambda \prec \Lambda_b$.

Definition 3.5. — *Given a sequence* Λ *and a number* $a > 1$, *we define*

$$\bar{D}_L(\Lambda : a) = (\log a)^{-1} \limsup_{x \to \infty} \{\lambda(ax) - \lambda(x)\},$$

$$\bar{D}_L(\Lambda) = \inf_{a > 1} \bar{D}_L(\Lambda : a).$$

The number $\overline{D}_L(\Lambda)$ is the logarithmic block density of Λ, and was introduced in [13]. A density dual to $\overline{D}_L$ was introduced in [10].

LEMMA 3.2. — $\overline{D}_L(\Lambda) = \lim\limits_{a \to \infty} \overline{D}_L(\Lambda : a)$.

PROOF. — Our proof uses some familiar ideas [12]. We put

$$\psi(\alpha) = \limsup_{x \to \infty} \{ \lambda(e^{\alpha} x) - \lambda(x) \}.$$

Now

$$\begin{aligned} \psi(\alpha + \beta) &= \limsup_{x \to \infty} \{ \lambda(e^{\alpha+\beta} x) - \lambda(x) \} \\ &= \limsup_{x \to \infty} \{ \lambda(e^{\alpha+\beta} x) - \lambda(e^{\beta} x) + \lambda(e^{\beta} x) - \lambda(x) \} \\ &\leq \limsup_{x \to \infty} \{ \lambda(e^{\alpha+\beta} x) - \lambda(e^{\beta} x) \} \\ &\quad + \limsup_{x \to \infty} \{ \lambda(e^{\beta} x) - \lambda(x) \} = \psi(\alpha) + \psi(\beta). \end{aligned}$$

Fixing s, and writing, for large t, $t = ns + p$, with n a large positive integer and $0 \leq p < s$, we see that

$$\frac{\psi(t)}{t} \leq \frac{n\psi(s) + \psi(p)}{ns + p} = \frac{\psi(s)}{s} \frac{1}{1 + p/n} + \frac{\psi(p)}{ns + p}.$$

Hence, $\limsup\limits_{t \to \infty} t^{-1} \psi(t) \leq s^{-1} \psi(s)$. That is, $\limsup\limits_{a \to \infty} \overline{D}_L(\Lambda : a) \leq \overline{D}_L(\Lambda : s)$, and since s is arbitrary, the lemma is proved.

LEMMA 3.3. $b^*(\Lambda) = \overline{D}_L(\Lambda)$.

PROOF. — First we prove that $\overline{D}_L(\Lambda) \leq b^*(\Lambda)$. If $\Lambda \prec \Lambda_b$, then by Lemma 3.1, there is a constant M_b, depending only on b, such that

$$\lambda(y) - \lambda(x) \leq b \log(y/x) + M_b \tag{3.18}$$

whenever $y \geq x$. We put $y = ax$ and let $x \to \infty$ to get

$$\overline{D}_L(\Lambda : a) \leq b + (\log a)^{-1} M_b.$$

But we now let $a \to \infty$ and apply Lemma 3.2 to obtain the desired conclusion.

To prove $b^*(\Lambda) \leq \overline{D}_L(\Lambda)$, we choose any number $b > \overline{D}_L(\Lambda)$, and shall show that $\Lambda \prec \Lambda_b$ by proving (3.18), which by Lemma 3.1 is sufficient. By Lemma 3.2, we can find a_0 such that

$$\overline{D}_L(\Lambda : a_0) < b$$

and by the definition of $\overline{D}_L(\Lambda : a_0)$ we can find R such that

$$\lambda(a_0 x) - \lambda(x) \leq b \log a_0 \quad \text{for all } x > R. \tag{3.19}$$

Now let $y > x > R$ and denote by N the integer such that

$$a_0^N \leqq y/x < a_0^{N+1}.$$

We have

$$\begin{aligned}\lambda(y) - \lambda(x) &\leqq \lambda(a_0^{N+1}x) - \lambda(x)\\ &= \lambda(a_0^{N+1}x) - \lambda(a_0^N x) + \lambda(a_0^N x) - \lambda(a_0^{N-1}x) + \ldots + \lambda(a_0 x) - \lambda(x)\\ &\leqq (N+1)\, b \log a_0 \leqq b\,\{\log y/x + \log a_0\}.\end{aligned}$$

This implies that

$$(3.20) \qquad \lambda(y) - \lambda(x) \leqq b \log y/x + b \log a_0 + \lambda(R)$$

whenever $y > x > 0$.

4. The main theorem. — In the introduction, we described how the main theorem could be put in the form : $\mathcal{F}(\Lambda) \prec \mathcal{F}(\Lambda')$ if and only if $\Lambda \prec \Lambda'$, where $\mathcal{F}(\Lambda) \prec \mathcal{F}(\Lambda')$ means that each function $g \in \mathcal{F}(\Lambda')$ majorizes at least one $f \in \mathcal{F}(\Lambda)$, in the sense that $|f(iy)| \leqq |g(iy)|$ for all y. We now state the main theorem in a form independent of the special notation of this paper.

THEOREM 4.1. — *Given two sequences Λ and Λ' of positive real numbers, the following three statements are equivalent.*

i. *Given any entire function $g(z)$ of exponential type such that $g(\lambda') = 0$ for each $\lambda' \in \Lambda'$, but such that $g(z)$ does not vanish identically, there exists an entire function $f(z)$ of exponential type such that $f(\lambda) = 0$ for each $\lambda \in \Lambda$, $f(z)$ does not vanish identically, and such that for each real number y,*

$$|f(iy)| \leqq |g(iy)|.$$

ii. *There exists a constant K such that whenever $0 < x \leqq y < \infty$,*

$$\sum_{x<\lambda\leqq y} \frac{1}{\lambda} \leqq \sum_{x<\lambda'\leqq y} \frac{1}{\lambda'} + K.$$

iii. *There exists a single pair f_0, g_0 of entire functions of exponential type, neither of them vanishing identically, such that $f_0(\lambda) = 0$ for each $\lambda \in \Lambda$ and $g_0(\lambda') = 0$ for each $\lambda' \in \Lambda'$, with $g_0(x+iy)$ having no other zeros in the half plane $x > 0$ than those in Λ', and such that $|f_0(iy)| \leqq |g_0(iy)|$ for each real number y.*

REMARK. — The condition in (i) and (iii) above, that $|f(iy)| \leqq |g(iy)|$, has many equivalent forms. For example, $|g(iy)|$ may be replaced there by $O(1+|y|^\alpha)\,|g(iy)|$, or $Q(|y|)\,|g(iy)|$, where $Q(y) = \exp(q(|y|))$, where $q(y)$ increases as $y \uparrow \infty$, and $\int_1^\infty q(y)\, y^{-2}\, dy < \infty$.

PROOF OF THEOREM 4.1. — We leave the proof that (ii) implies (i) for the next section. It is clear that (i) implies (iii); a suitable choice for $g_0(z)$ in (iii) is the Weierstrass product $W(z:\Lambda')$ [*see* (2.16)]. We prove here that (iii) implies (ii). We give the proof for the special choice $g_0(z) = W(z:\Lambda')$ since we shall refer later to some of the estimates in this case, but the same proof works in the general case.

More precisely, if there exists an $f \in \mathcal{F}(\Lambda)$ with

$$|f(iy)| \leq |W(iy:\Lambda')| \quad \text{for all } y, \tag{4.1}$$

then (3.1) holds, and $\Lambda \prec \Lambda'$.

We choose ρ, with $0 < \rho < \lambda_0$ so that all the zeros, $z_n = r_n e^{i\theta_n}$, of $f(z)$ in the right half plane [assuming for convenience that $f(z)$ has no zeros on $z = iy$] satisfy $r_n > \rho$, and write one form of Carleman's theorem, taking $y > x > \rho$, as

$$\Sigma(y) - \Sigma(x) = I(y) - I(x) + J(y) - J(x) + O(1), \tag{4.2}$$

where

$$\Sigma(R) = \Sigma(R:f) = \sum_{r_n \leq R} \left(\frac{1}{r_n} - \frac{r_n}{R^2}\right) \cos\theta_n,$$

$$I(R) = I(R:f) = \frac{1}{2\pi} \int_\rho^R \left(\frac{1}{t^2} - \frac{1}{R^2}\right) \log|f(it) f(-it)|\, dt,$$

$$J(R) = J(R:f) = \frac{1}{\pi R} \int_{-\pi/2}^{\pi/2} \log|f(R e^{i\theta})| \cos\theta\, d\theta.$$

Following [13], we use the estimates

$$\sum \frac{r_n}{R^2} \cos\theta_n = O(1), \tag{4.3}$$

$$J(R) = O(1). \tag{4.4}$$

The estimate (4.3) follows from Theorem A, and (4.4) is a consequence [1], p. 31, of Jensen's theorem. (Of course, these estimates are not valid for functions holomorphic only in $\operatorname{Re} z \geq 0$.)

From (4.3) we obtain

$$\Sigma(y) - \Sigma(x) \geq \lambda(y) - \lambda(x) + O(1), \tag{4.5}$$

and using (4.4) and (4.5) in (4.2) we get

$$\lambda(y) - \lambda(x) \leq \{I(y) - I(x)\} + O(1). \tag{4.6}$$

The next lemma is trivial to prove, but nonetheless useful.

LEMMA 4.1. — *If* $\varphi(t) \geqq 0$ *and* $\rho < x < y$, *then*

$$(4.7)\qquad \int_\rho^y (t^{-2} - y^{-2})\,\varphi(t)\,dt - \int_\rho^x (t^{-2} - x^{-2})\,\varphi(t)\,dt \geqq 0.$$

PROOF. — We rewrite (4.7) as

$$(4.8)\qquad \int_\rho^x (x^{-2} - y^{-2})\,\varphi(t)\,dt + \int_x^y (t^{-2} - y^{-2})\,\varphi(t)\,dt \geqq 0,$$

which is obviously true, since each of the integrands is non-negative.

It follows from (4.1) and Lemma 4.1 that, writing W for $W(z:\Lambda')$,

$$(4.9)\qquad I(y:f) - I(x:f) \leqq I(y:W) - I(x:W).$$

On the other hand, applying Carleman's theorem to W now, whose only zeros in the right half plane are the λ'_n, we see that

$$(4.10)\qquad I(y:W) - I(x:W) = \lambda'(y) - \lambda'(x) + O(1).$$

Combining (4.10) with (4.9) and (4.6), we get

$$\lambda(y) - \lambda(x) \leqq \lambda'(y) - \lambda'(x) + O(1),$$

and the proof is done.

5. **The construction.** — We suppose given two sequences Λ and Λ' with

$$\lambda(y) - \lambda(x) \leqq \lambda'(y) - \lambda'(x) + O(1),$$

and a function $g(z) \in \mathcal{F}(\Lambda')$. We must construct a function $f \in \mathcal{F}(\Lambda)$ with $|f(iy)| \leqq |g(iy)|$ for all y. By Lemma 3.1, we may suppose that $\lambda(t) = \lambda'(t) + O(1)$ since Λ is a subsequence of a sequence Λ'' for which this is true, and $\mathcal{F}(\Lambda'') \subset \mathcal{F}(\Lambda)$.

By the Hadamard factorization theorem, we may write

$$(5.1)\qquad g(z) = g_1(z)\,g_2(z),$$

where

$$(5.2)\qquad g_1(z) = \Pi(1 - z/\lambda'_n)\exp(z/\lambda'_n),$$

$$(5.3)\qquad g_2(z) = cz^k \exp(az)\,\Pi(1 - z/z_n)\exp(z/z_n),$$

where the $z_n \neq 0$ are the zeros of $g(z)$ that are not counted in Λ'.

Writing $\log|g_1(iy)|$ as a sum of logarithms, and that sum as a Stieltjes integral, we get

$$(5.4)\qquad \log|g_1(iy)| = \frac{1}{2}\int_0^\infty \log(1 + y^2 t^{-2})\,t\,d\lambda'(t).$$

The following lemma will give us an explicit expression for the swept measure on the imaginary axis of a measure carried by the real axis, the potential kernel being $\log|1-z^2|$. This technique of sweeping, analogous to the technique of [10] but different, will be the main tool of our construction; it will allow us to " move the zeros " from the real axis to the imaginary axis.

LEMMA 5.1. — *Let $d\Delta$ be a measure with compact support contained in an interval $[\varepsilon, \varepsilon^{-1}]$ for some $\varepsilon > 0$. Then there exists a function $\varphi(t)$ defined on $(0, \infty)$ such that*

$$\int \log(1+y^2t^{-2})\, d\Delta(t) = \int \log|1-y^2t^{-2}|\, \varphi(t)\, dt \tag{5.5}$$

and

$$|\varphi(t)| < 2 \sup_x \left| \int_0^x s^{-1}\, d\Delta(s) \right|. \tag{5.6}$$

PROOF. — Since we are solving a Mellin-type convolution equation, we shall compute some Mellin transforms, although our goal is (5.11), which can be verified directly by an easy contour integration. We define $T_\theta(z)$, for $0 \leqq \theta \leqq \pi/2$ by

$$T_\theta(z) = \int_0^\infty \log|1+u^2e^{-2i\theta}|\, u^{z-1}\, du. \tag{5.7}$$

If z is real, then

$$T_\theta(z) = \operatorname{Re} \int_0^\infty \log(1+u^2e^{-2i\theta})\, u^{z-1}\, du, \tag{5.8}$$

and by a routine contour integration we get

$$T_\theta(z) = \frac{\pi}{z \sin \pi z/2} \cos\theta z, \qquad -2 < \operatorname{Re} z < 0. \tag{5.9}$$

Similarly, we have the identity

$$\frac{T_0(z)}{T_{\pi/2}(z)} = \frac{1}{\cos \pi z/2} = \frac{2}{\pi} \int_0^\infty \frac{u}{u^2+1} u^{z-1}\, du. \tag{5.10}$$

Hence

$$\log|1+x^2| = \frac{2}{\pi} \int_0^\infty \left\{ \log\left|1-\frac{x^2}{t^2}\right| \right\} \frac{t}{t^2+1} \frac{dt}{t}. \tag{5.11}$$

By Fubini's theorem

$$\begin{aligned} \int \log\left(1+\frac{x^2}{r^2}\right) d\Delta(r) \\ = \frac{2}{\pi} \int \log\left|1-\frac{x^2}{w^2}\right| \left\{ \int \frac{w/t}{w^2/t^2+1}\, d\Delta(t) \right\} \frac{dw}{w}. \end{aligned} \tag{5.12}$$

We therefore define

$$\varphi(w) = \frac{2}{\pi}\int \frac{t^2}{w^2+t^2}\,\frac{d\Delta(t)}{t} \tag{5.13}$$

and (5.12) asserts (5.5) in another form. The bound (5.6) on $\varphi(w)$ follows from integrating (5.13) by parts

$$\varphi(w) = \frac{2}{\pi}\int_0^\infty \frac{d\Delta(t)}{t} - \frac{2}{\pi}\int_0^\infty \left\{\int_0^x \frac{d\Delta(t)}{t}\right\} d_x\left(\frac{x^2}{x^2+w^2}\right). \tag{5.14}$$

Thus

$$|\varphi(w)| < \left\{1 + \int_0^\infty d_x\left(\frac{x^2}{x^2+w^2}\right)\right\} \sup_x \left|\int_0^x \frac{d\Delta(t)}{t}\right|, \tag{5.15}$$

since $x^2(x^2+w^2)^{-1}$ is increasing, and we have proved the lemma.

We now choose

$$\delta(t) = \frac{1}{2}\{\lambda'(t) - \lambda(t)\}, \qquad d\Delta(t) = t\,d\delta(t), \tag{5.16}$$

but cannot apply Lemma 5.1 to $d\Delta$ since its support may not be compact We truncate the support by defining

$$\delta_k(t) = \begin{cases} \delta(t) & \text{if } t \leq k \\ \delta(k) & \text{if } t > k \end{cases} \qquad d\Delta_k(t) = t\,d\delta_k(t), \tag{5.17}$$

with the same convention for $\lambda(t)$ and $\lambda'(t)$.

We now apply Lemma 5.1 to $d\Delta_k$, and conclude that there exists functions $\varphi_k(t)$ such that

$$\int \log(1+y^2t^{-2})\,d\Delta_k(t) = \int \log|1-y^2t^{-2}|\,\varphi_k(t)\,dt. \tag{5.18}$$

Now

$$|\varphi_k(t)| \leq B, \tag{5.19}$$

where B is a constant that is independent of k, namely, from (5.6) and the equivalence of Λ and Λ',

$$B = 2\sup|\lambda(t) - \lambda'(t)|.$$

On putting

$$L_k(y) = \int \log(1+y^2t^{-2})\,\frac{1}{2}\,t\,d\lambda_k(t) + \int \log|1-y^2t^{-2}|\,d\Phi_k(t), \tag{5.20}$$

where

$$d\Phi_k(t) = \varphi_k(t)\,dt, \tag{5.21}$$

we have

$$(5.22) \qquad L_k(y) = \frac{1}{2}\int \log(1+y^2t^{-2})\, t\, d\lambda'_k(t).$$

Hence, by (5.4),

$$(5.23) \qquad \lim_{k\to\infty} L_k(y) = \log|g_1(iy)|.$$

At this point, the idea is to find an entire function $F(z)$ for which the hypothetical formula

$$\log|f(iy)| = \lim_{k\to\infty}\int \log|1-y^2t^{-2}|\, d\Phi_k(t)$$

holds in some appropriate sense. First, however, the limit need not exist, but a simple selection argument with normal families will handle this difficulty. Also, the measures $d\Phi_k(t) = \varphi_k(t)\, dt$ are unsuitable since they need not be positive, and cannot be discrete. (It is easy to see that all the $d\Phi_k(t)$ are positive only in case $\Lambda \subset \Lambda'$, a trivial case.) But first we show that adding a constant to φ_k, in order, to make $d\Phi_k(t)$ positive, does not change L_k. Then we show that the resulting measure may be made discrete with little loss of precision.

Resuming the construction, we define

$$(5.24) \qquad \psi_k(t) = B + \varphi_k(t),$$

and by (5.19) conclude that

$$(5.25) \qquad \psi_k(t) \geq 0 \quad \text{for all } t.$$

A contour integration, or (5.9), establishes that

$$(5.26) \qquad \int \log|1-y^2t^{-2}|\, dt = 0,$$

so that

$$(5.27) \quad L_k(y) = \int \log(1+y^2t^{-2})\,\frac{1}{2}\, t\, d\lambda_k(t) + \int \log|1-y^2t^{-2}|\, d\Psi_k(t),$$

where $d\Psi_k(t) = \psi_k(t)\, dt$. We now let $\Psi_k^*(t) = [\Psi_k(t)]$, the integral part of $\Psi_k(t)$, and we define $L_k^*(y)$ by (5.27), with Ψ_k replaced by Ψ_k^* there.

Lemma 5.2. — *There is a constant β, independent of k, such that for all $y > 1$,*

$$(5.28) \quad \int \log|1-y^2t^{-2}|\, d\Psi_k^*(t) \leq \int \log|1-y^2t^{-2}|\, d\Psi_k(t) + \beta\log|y|.$$

Proof. — It is, clearly, enough to apply the next lemma with $\Psi_k(t) = \nu(t)$ and $\Psi_k^*(t) = n(t)$. That β is independent of k rests on the facts that $|(d/dt)\Psi_k(t)|$ and $|\Psi_k(t) - \Psi_k^*(t)|$ are bounded independently of k.

LEMMA A. — *Suppose that* $\nu(r)$ *is a continuously differentiable function for* $0 < r < \infty$, *that* $0 \leq \nu'(r) < B < \infty$, *that* $n(r)$ *is non-decreasing and that*

$$\nu(r) \geq n(r) > \nu(r) - K$$

for some constant K. *Then*

$$\int \log|1 - y^2 t^{-2}|\, dn(t) \leq \int \log|1 - y^2 t^{-2}|\, d\nu(t) + O(\log y)$$

as $y \to \infty$.

REMARK. — There is a proof of this result in [6]. We give here a somewhat different proof.

PROOF. — Putting $\rho(r) = \nu(r) - n(r)$, we must show that

$$\int \log|1 - y^2 t^{-2}|\, d\rho(t) \leq O(\log y).$$

Now, denoting Cauchy principal values by $P.\ V.$, we have

$$\int \log|1 - y^2 t^{-2}|\, d\rho(t) = P.\ V. \int \frac{2y^2}{y^2 - t^2} \rho(t) \frac{dt}{t}.$$
$$= \int_0^{1/4B} + \int_{1/4B}^{y-1} + P.\ V. \int_{y-1}^{y+1} + \int_{y+1}^{\infty}$$

On the interval $[0, 1/4B]$, $\rho(t) < 4Bt$ so that $\left| \int_1^{1/4B} \right| < 2$ for $|y| > y_0$, say. Hence

$$\left| \int_{1/4B}^{y-1} \right| < 4 \int_{\frac{1}{4By}}^{1-\frac{1}{y}} \left| \frac{1}{1-u^2} \right| \frac{du}{u} = O(\log|y|).$$

In the same way,

$$\int_{y+1}^{\infty} = O(\log|y|).$$

Finally,

$$P.\ V. \int_{y-1}^{y+1} = P.\ V. \int_{y-1}^{y+1} \frac{y^2}{y^2 - t^2} \nu(t) \frac{dt}{t} - P.\ V. \int_{y-1}^{y+1} \frac{y^2}{y^2 - t^2} n(t) \frac{dt}{t}.$$

Making the substitution $t = y - h$ at the left of y and $t = y + h$ at the right, we may write these last integrals in the form

$$\int_0^1 \{ H_y(-h) - H_y(h) \} \frac{dh}{h},$$

where

$$H_y(u) = \frac{y^2}{(y+u)(2y+u)} n(u)$$

with a corresponding formula for the integral involving $\nu(t)$.

Since $\nu(t)$ has a bounded derivative, then so has the corresponding H, and thus the integral involving $\nu(t)$ is $O(1)$. For the term with $n(t)$, we write

$$H_y(u) = \frac{1}{2} n(y+u) + \left(\frac{y^2}{(y+u)(2y+u)} - \frac{1}{2}\right) n(y+u).$$

The contribution of the first term is non-positive because $n(t)$ is non-decreasing, and the contribution of the second term is $O(1)$, by an easy estimate. Hence the lemma is proved.

We consider now the polynomials $P_k(z)$ defined by

$$\log|P_k(z)| = \int \log|1 + z^2 t^{-2}|\, d\Psi_k^*(t). \tag{5.29}$$

Lemma 5.3. — *There is a constant* β', *independent of* k, *such that for all* z,

$$\log|P_k(z)| \leq \beta' |z|. \tag{5.30}$$

Proof. — Putting $z = x + iy$, we see that

$$\int \log|1 + z^2 t^{-2}|\, d\Psi_k^*(t) \leq \int \log|1 + x^2 t^{-2}|\, d\Psi_k^*(t).$$

But since $\Psi_k^*(t) = \left[\int_0^t \{B + \varphi_k(s)\}\, ds\right]$, and since $|\varphi_k(t)| \leq B$ by (5.19), the proof is immediate by an integration by parts.

Since the family $\{P_k(z)\}$ is therefore uniformly bounded in each disc $|z| < R$, it is consequently a normal family and we may extract a sequence $\{P_{k_j}(z)\}$, that converges to an entire function as $k_j \to \infty$. We call this entire function $F(z)$

$$F(z) = \lim_{j \to \infty} P_{k_j}(z). \tag{5.31}$$

Because $P_k(0) = 1$ for each k, it follows that $F(0) = 1$. From Lemma 5.3 we conclude that $F(z)$ is of exponential type. Since each $P_k(z)$ has only imaginary zeros, so has $F(z)$. Furthermore, $F(z)$ is an even function.

Let $i\Gamma = \{i\gamma_n\}$ be the zeros of $F(z)$. Since $F(z)$ is of exponential type, Γ has finite upper density. Thus,

$$F(z) = \Pi(1 + z^2 \gamma_n^{-2}). \tag{5.32}$$

We can, at last, define $f(z)$ by

$$f(z) = f_1(z)\,F(z)\,g_2(z), \tag{5.33}$$

where

$$f_1(z) = \Pi(1 - z/\lambda_n)\exp(z/\lambda_n). \tag{5.34}$$

As a consequence of the estimates (5.23) and (5.28) we see that

$$\log|f(iy)| \leqq \log|g(iy)| + O(\log|y|). \tag{5.35}$$

As indicated in the remarks following the statement of Theorem 4.1, (5.35) is as good, for our purposes, as $|f(iy)| \leqq |g(iy)|$. The additional term $O(\log|y|)$ is easily removed from (5.35). One way is to multiply the function $f(z)$ by $a\{(iz)^{-1}\sin iz\}^b$, with a suitable choice of a and b.

It is not immediately obvious, though, that $f(z)$ is of exponential type, since $f_1(z)$ and $g_2(z)$ will not in general be of exponential type, although they are certainly of order 1. To prove that $f(z)$ is of exponential type, we appeal to Theorem A, § 2.

Let us denote by a_n and b_n the zeros, other than the origin, of $f(z)$ and $g(z)$ respectively. Then we see that

$$\begin{aligned} O(1) &= \sum_{|b_n|\leqq R} b_n^{-1} = \sum_{\substack{|b_n|\leqq R \\ b_n \notin \Lambda'}} b_n^{-1} + \lambda'(R) = \sum_{\substack{|b_n|\leqq R \\ b_n \notin \Lambda'}} b_n^{-1} + \lambda(R) + \{\lambda'(R) - \lambda(R)\} \\ &= \sum_{\substack{|b_n|\leqq R \\ b' \notin \Lambda'}} b_n^{-1} + \lambda(R) + O(1) = \sum_{|a_n|\leqq R} a_n^{-1} + O(1), \end{aligned}$$

by observing first that $\lambda'(R) - \lambda(R) = O(1)$ by hypothesis, and then that the zeros of $f(z)$ other than the origin fall into three categories : i. those b_n not counted in Λ'; ii. the elements of Λ, and iii. the zeros of $F(z)$. The zeros of $F(z)$ contribute nothing to Σa_n^{-1} since $F(z)$ is even. Thus, we have verified (2.15) for $f(z)$. The condition (2.14) is even easier to verify. Now a second appeal to Theorem A tells us that $f(z)$ is of exponential type, and our proof is done.

6. Special majorants. — In this section, we derive, as consequences of the main theorem, the conditions on Λ that correspond to the usual kind of majorant for $|f(iy)|$. The same remarks that follow the statement of the main theorem apply here.

THEOREM 6.1. — *Given a sequence Λ and a number $b \geqq 0$, in order that there exists an $f \in \mathcal{F}(\Lambda)$ with*

$$|f(iy)| \leqq \exp(\pi b|y|), \tag{6.1}$$

it is necessary and sufficient that Λ *be a quasi-subsequence of* Λ_b, *or equivalently* (*by lemma* 3.1) *that there exist a constant K such that*

$$\lambda(y) - \lambda(x) \leqq b \log(y/x) + K \tag{6.2}$$

whenever $0 < x \leqq y < \infty$.

Proof. — In theorem 4.1, choose $\Lambda' = \Lambda_b$ and $g(z) = \sin \pi b z$.

Remark. — The analogous result [1]. p. 157 of Fuchs, for functions holomorphic in the right half plane only, is that

$$\lambda(x) \leqq b \log x + K,$$

instead of (6.2), is necessary and sufficient.

Theorem 6.2. — *Given a sequence* Λ *and a number* $b \geqq 0$, *in order that there exist, for each* $\varepsilon > 0$, *an* $f \in \mathcal{F}(\Lambda)$ *with*

$$h_f(\pm \pi/2) \leqq \pi b + \varepsilon \tag{6.3}$$

it is necessary and sufficient that

$$\bar{D}_{\mathrm{L}}(\Lambda) \leqq b. \tag{6.4}$$

Proof. — Theorem 4.1 and Lemma 3.3.

Remark. — This theorem was conjectured in [13], where necessity was proved in general, and sufficiency in the case that Λ is a sequence of distinct positive integers and $b = 1$.

Theorem 6.3. — *Given a sequence* Λ *and a number* $b \geqq 0$, *in order that there exist an* $f \in \mathcal{F}(\Lambda)$ *with*

$$h_f\left(\pm \frac{\pi}{2}\right) \leqq \pi b \tag{6.5}$$

it is necessary and sufficient that there is a function $\delta(y)$ *such that* $\delta(y) \to 0$ *as* $y \to \infty$, *and*

$$\lambda(y) - \lambda(x) \leqq \{ b + \delta(y) \} \log y/x + O(1) \tag{6.6}$$

whenever $0 < x \leqq y < \infty$.

The proof of this result uses Theorem 6.5 and we defer it to the end o this section.

Theorem 6.4. — *Given a bounded mesurable function* $q(t)$, *suppose that there exists an* $f \in \mathcal{F}(\Lambda)$ *with*

$$|f(iy)| \leqq \exp\{ \pi |y| q(|y|) \} \tag{6.7}$$

for all y. Then

$$\lambda(y) - \lambda(x) \leqq \int_x^y q(t)\, t^{-1}\, dt + K \tag{6.8}$$

whenever $0 < x \leqq y < \infty$, *where K is a constant.*

PROOF. — By (4.9), with $Q(y) \equiv \exp\{\pi |y| q(|y|)\}$ replacing $W(iy)$, and using the fact that

$$I(y:Q) - I(x:Q) = \int_x^y q(t)\, t^{-1}\, dt + O(1), \tag{6.9}$$

we immediately obtain (6.8).

THEOREM 6.5. — *Let* $q(y)$ *be a positive, bounded, and measurable function that is slowly oscillating in the sense that*

$$q(ty) - q(y) \to 0 \qquad \textit{as} \quad y \to \infty \tag{6.10}$$

uniformly for $\varepsilon \leqq t \leqq \varepsilon^{-1}$ *for each* $\varepsilon > 0$. *Then given a sequence* Λ *for which* (6.8) *holds, there exists an* $f \in \mathcal{F}(\Lambda)$ *such that*

$$|f(iy)| \leqq \exp\{\pi |y| q(|y|) + o(|y|)\} \tag{6.11}$$

REMARKS. — A simple condition that implies (6.10) is

$$\frac{d}{dy} q(y) = o\left(\frac{1}{y}\right) \qquad \textit{as} \quad y \to \infty. \tag{6.12}$$

Some restriction like (6.10) is to be expected. The gap between Theorem 6.4 and Theorem 6.5 is the term $o(|y|)$ in (6.11).

PROOF OF THEOREM 6.5. — Let us consider the comparison sequence Λ_q defined by

$$\lambda_q(t) = \int_0^t s^{-1}\, d[Q(s)], \qquad \text{where} \quad Q(t) = \int_0^t q(s)\, ds.$$

Then

$$\lambda_q(t) = \int_1^t q(s)\, s^{-1}\, ds + o(1) + a, \tag{6.13}$$

where a is a constant. Now let $W_q(z) = W(z:\Lambda_q)$ [*see* (2.16)]. By Theorem 4.1 there exists an $f \in \mathcal{F}(\Lambda_q)$ with

$$|f(iy)| \leqq |W_q(iy)|. \tag{6.14}$$

But, as in [6], we have the estimate

$$\log |W_q(iy)| = \pi |y| q(|y|) + o(|y|), \tag{6.15}$$

and the theorem is proved. To obtain (6.15), write (taking $y > 0$ for convenience)

$$(6.16)\qquad \log|W_q(iy)| - \pi y\, q(y) = \int_0^\infty \log(1 + y^2 t^{-2})\, d_t\{t\lambda_q(t) - \pi y\, q(y)\, t\}.$$

Using (6.13) and an integration by parts, we obtain

$$(6.17)\qquad \log|W_q(iy)| - \pi y\, q(y) = y\int_0^\infty \log(1 + t^{-2})\{q(ty) - q(y)\}\, dt + o(y).$$

Choosing $\varepsilon > 0$, and taking evident estimates on $\int_0^\varepsilon$ and $\int_{\varepsilon^{-1}}^\infty$, and using the hypothèses (6.10) on $\int_\varepsilon^{\varepsilon^{-1}}$, we get (6.15),

Proof of theorem 6.3. — We remark first that (6.6) is easily shown to be implied by

$$(6.18)\qquad \lambda(y) - \lambda(x) \leqq b \log y/x + \int_x^y \varepsilon(t)/t\, dt + O(1),$$

where $\varepsilon(t)$ is a continuous function that approaches 0 as $t \to \infty$.

To get (6.18) from (6.5), we see that (6.5) implies that

$$\log|f(iy)| \leqq \pi b|y| + o(y).$$

Writing $o(t)$ as $t\varepsilon(t)$ for a suitable choice of the function $\varepsilon(t)$, and applying Theorem 6.4, we obtain (6.18)

Conversely let us suppose that (6.6) holds. We may, without loss of generality, further suppose that $\delta(t)$ is positive, $\delta(t)$ derivable, decreasing, and $\delta'(t) = o(t^{-1})$.

Then $n(t) = bt + t\,\delta(t)$ is a growing function, let Λ_1 the sequence defined by $\Lambda_1(t) = [n(t)]$.

Then we have by Lemma 3.1 and (6.6) that $\Lambda \prec \Lambda_1$. Since $t^{-1}\Lambda_1(t) \to b$, we have

$$\log|W(iy : \Lambda_1)| = b|y| + o(y),$$

then Theorem 4.1 completes the proof.

7. An alternative construction. — Another method of construction, combining a method of Fuchs [1], p. 157 with a method of Macintyre [9], is possible. It requires the additional hypothesis (7.1) and yields less general results. Also, the method of section 5 is more explicit about the location of the zeros of the function constructed than this section is. Nevertheless, it sheds additional light on the problem, and from a new angle.

In this section *only*, we assume the separation condition

$$\lambda_{n+1} - \lambda_n \geqq \gamma > 0 \tag{7.1}$$

and will construct the function required by Theorem 6.1 under the hypothesis (6.2).

FUCHS' LEMMA. — *Under* (7.1), *the function*

$$H(z) = \prod \frac{\lambda_n - z}{\lambda_n + z} \exp(2z/\lambda_n) \tag{7.2}$$

is holomorphic for $x = \operatorname{Re} z \geqq 0$ *and satisfies there*

$$\log|H(z)| \leqq 2x\lambda(|z|) + Ax, \tag{7.3}$$

$$\log|H(z)| \geqq 2x\lambda(|z|) + Bx, \qquad z \in \mathfrak{Z}, \tag{7.4}$$

where

$$\mathfrak{Z} = \bigcap_{n=0}^{\infty} \{z : |z - \lambda_n| \geqq \gamma/3\}, \tag{7.5}$$

where A and B are finite constants.

We consider now the function $G(z)$ defined by

$$G(z) = H(z)/\Gamma(2 + 2bz), \tag{7.6}$$

where Γ is the Euler gamma function. By Lemma 3.1, we may as well assume that $\lambda(r) = b\log r + O(1)$. We then have

$$|G(iy)| \leqq K'(1 + y^2)^{-1} \exp \pi b|y|, \tag{7.7}$$

$$\log|G(z)| \leqq O(|z|), \tag{7.8}$$

$$\log|G(z)| \geqq \eta|z|, \qquad z \in \mathfrak{Z}, \tag{7.9}$$

where K' and η are constants.

By the Pólya-Macintyre representation theory [9] for holomorphic functions of exponential type in a half plane, there exists a function $\gamma(s) = \gamma(\sigma + it)$, defined on the semi-infinite rectangle C_R

$$C_R : \left\{ \begin{array}{ll} t = \pm \pi b, & \sigma < R \\ \sigma = R, & |t| \leqq \pi b \end{array} \right\}, \tag{7.10}$$

where R is the constant implied by $O(|z|)$ in (7.8), such that

$$G(z) = \int_{C_R} \gamma(s)\, e^{sz}\, ds. \tag{7.11}$$

Following a construction of Macintyre, we define $G_T(z)$ by

$$G_T(z) = \int_{C^*} \gamma(s)\, e^{sz}\, ds, \tag{7.12}$$

where $C^* = C^*(T)$ is the part of C_R lying in the half plane $\sigma \geqq T$. The function $G_T(z)$ is an entire function of exponential type, since C^* is bounded, and it satisfies

$$| G_T(z) - G(z) | = O(e^{Tx}). \tag{7.13}$$

By (7.9) and (7.13), we have

$$\left| \frac{G_T(z) - G(z)}{G(z)} \right| < \frac{1}{2} e^{-Mx}, \qquad z \in \mathfrak{Z}, \tag{7.14}$$

for any preassigned positive M, provided that T is chosen near enough to $-\infty$.

By Rouche's theorem, $G_T(z)$ and $G(z)$ have the same number of zeros in each of the circles

$$| z - \lambda_n | \leqq e^{-\delta\lambda_n}. \tag{7.15}$$

Writing the Hadamard product for $G_T(z)$,

$$G_T(z) = cz^m e^{az} \Pi(1 - z/z_n) \exp(z/z_n) \Pi(1 - z/\lambda_n^*) \exp(z/\lambda_n^*), \tag{7.16}$$

where the λ_n^* are the zeros of $G_T(z)$ in the circles (7.15), we define

$$f(z) = cz^m e^{az} \Pi(1 - z/z_n) \exp(z/z_n) \Pi(1 - z/\lambda_n) \exp(z/\lambda_n). \tag{7.17}$$

By the argument at the end of paragraph 5, we see that $f \in \mathcal{F}(\Lambda)$. Now because

$$| \lambda_n - \lambda_n^* | \leqq e^{-\delta\lambda_n} \leqq e^{-\gamma\delta n}, \tag{7.18}$$

an easy estimate shows that

$$\log | f(iy) | \leqq \log | G_T(iy) | + O(\log | y |). \tag{7.19}$$

From (7.7) and (7.13), we see that f satisfies (6.1), with an error factor of polynomial growth, which can be removed just as the same error was removed from (5.35).

8. Overall type.

THEOREM 8.1. — *Given a sequence Λ of positive real numbers, the Weierstrass product $W(z)$ has the smallest overall type of any function in $\mathcal{F}(\Lambda)$ if and only if*

$$\overline{D}_L(\Lambda) = \overline{D}_P(\Lambda). \tag{8.1}$$

A sequence Λ [*see* (10.1)], of distinct positive integers, for which $\bar{D}_L(\Lambda) \neq \bar{D}_P(\Lambda)$, was given in [13], p. 424. We remark that Theorem 8.1 is, after Theorem 6.2, equivalent to the statement that $W(z)$ has the smallest overall type if and only if is has the smallest type on the imaginary axis.

It is, perhaps, surprising that $W(z)$ need not minimize the overall type, since Jensen's theorem shows that $W(z)$ minimizes the mean type $\bar{h}$ of all $f \in \mathcal{F}(\Lambda)$, where

$$\bar{h} = \limsup_{r \to \infty} \frac{1}{2\pi r} \int_{-\pi}^{\pi} \log |f(r e^{i\theta})| \, d\theta.$$

A pertinent minimal property that W does have is given in Theorem 4.1, namely: given Λ and Λ_1, if there exists an $f_0 \in \mathcal{F}(\Lambda_1)$ such that

$$|f_0(iy)| \leq |W(iy)|$$

for all y, then for each $g \in \mathcal{F}(\Lambda)$, there exists an $f \in \mathcal{F}(\Lambda_1)$ such that $|f(iy)| \leq |g(iy)|$ for all y. Stated crudely, this says that W is as hard to minorize on the imaginary axis as any other function in $\mathcal{F}(\Lambda)$.

Since it has become the pratice in the literature, in dealing with overall type, to use $W(z)$ as a comparison function, the effect of Theorem 8.1 is to show that results obtained in this way, despite their quantitative formulation, are not sharp. How the minimum overall type depends on Λ is an unsolved problem.

A peculiar consequence of Theorem 8.1 and Theorem 6.3 together is that if $\bar{D}_P(\Lambda) = \bar{D}_L(\Lambda)$, then (6.6) holds, namely

$$\lambda(y) - \lambda(x) \leq \{ \bar{D}_L(\Lambda) + \varepsilon(y) \} \log y/x.$$

Our proof of theorem 8.1 requires precise information on the indicator diagram of W. It is well known that $h(\theta) = h_W(\theta)$ is a non-decreasing function of θ in $0 < \theta < \pi/2$. We prove in the next theorem that $h(\theta)$ is *strictly increasing* there. The estimates used to prove it can be made precise, to find a function $\rho(\theta) = \rho(\theta : h(\pi/2))$ which is strictly increasing in $0 < \theta < \pi/2$, such that

$$\rho(\pi/2) = 1, \qquad \text{and} \qquad h(\theta) \leq \rho(\theta)\, h(\pi/2).$$

This should be compared with a result of Levin [8], p. 329, exhibiting a function $\rho^*(\theta)$ such that

$$h(\theta) \leq \rho^*(\theta)\, \bar{D}(\Lambda).$$

Theorem 8.2. — *Given a sequence Λ of positive real numbers, $h_w(\theta)$ is a strictly increasing function of θ for $0 < \theta < \pi/2$, with the exception, of course, of the case $h_w(\theta) = 0$ for all θ.*

PROOF. — We choose θ_1 in $(0, \pi/2)$ and let $r_m = r_m(\theta_1)$ be a sequence tending to $+\infty$ such that

$$(8.2) \qquad \lim_{m\to\infty} r_m^{-1} \log|W(r_m e^{i\theta_1})| = h(\theta_1).$$

We observe that if $h(\theta_1) = 0$ then, as is well known, $h(\theta) = 0$ for all θ, the case that we have excluded. To see this, we notice first of all that $h(\theta)$ is certainly non-decreasing in $(0, \pi/2)$ because $|1 - r^2 e^{2i\theta}|$ is an increasing function of θ there. Also, $h(\theta) \geqq 0$ everywhere. Thus, $h(\theta_1) = 0$ implies that $h(0) = 0$, which in turn implies that $h(\theta) = h(\pi/2)|\sin\theta|$ for all θ, since h is a supporting function (*see* section 2). Thus, if $h(\theta_1) = 0$ then, in particular, $h(\theta_1) = h(\pi/2)|\sin\theta_1|$, which implies that $h(\pi/2) = 0$ so that $h(\theta) = 0$ for all θ, as asserted.

Hence, for all sufficiently large m, we have

$$(8.3) \qquad r_m^{-1}\log|W(r_m e^{i\theta_1})| = r_m^{-1}\int_0^{\infty} \log|1 - r^2 t^{-2} e^{2i\theta_1}|\, d\Lambda(t) \geqq \frac{1}{2} h(\theta_1).$$

On estimating $\int_0^{r_m\varepsilon}$ and $\int_{r_m/\varepsilon}^{\infty}$ by integrating by parts [*see* (2.18)], and using (2.8), we see that there exists an $\varepsilon > 0$ for which

$$(8.4) \qquad r_m^{-1}\int_{r_m\varepsilon}^{r_m/\varepsilon} \log|1 - r_m^2 t^{-2} e^{2i\theta}|\, d\Lambda(t) > \frac{1}{4} h(\theta_1)$$

for all sufficiently large m, since $h(\theta_1) > 0$. But

$$(8.5) \qquad r_m^{-1}\int_{r_m\varepsilon}^{r_m/\varepsilon} \log|1 - r_m^2 t^{-2} e^{2i\theta}|\, d\Lambda(t) \leqq M(\varepsilon)\, r_m^{-1}\int_{r_m\varepsilon}^{r_m/\varepsilon} d\Lambda(t),$$

where

$$(8.6) \qquad M(\varepsilon) = M(\varepsilon, \theta_1) = \max_{\varepsilon \leqq \xi \leqq \varepsilon^{-1}} \log|1 - \xi^2 e^{2i\theta_1}|.$$

We thus have the useful fact that

$$(8.7) \qquad r_m^{-1}\int_{r_m\varepsilon}^{r_m/\varepsilon} d\Lambda(t) \geqq \delta > 0$$

for all sufficiently large m, where δ is independent of m.

Now, by the mean value theorem of the differential calculus, if $0 < \theta_1 < \theta_2 < \pi/2$, there is a value of θ, $\theta_1 \leqq \theta \leqq \theta_2$, for which

$$(8.8) \qquad r^{-1}\log|W(r e^{i\theta_2})| = r^{-1}\log|W(r e^{i\theta_1})| + (\theta_2 - \theta_1)\, \varphi(\theta, r),$$

where

$$(8.9) \qquad \varphi(\theta, r) = \frac{\partial}{\partial\theta}\{r^{-1}\log|W(re^{i\theta})|\}.$$

A simple calculation shows us that

$$(8.10) \qquad \varphi(\theta, r) = r^{-1}\int_0^\infty Q(\theta, t/r)\, d\Lambda(t),$$

where

$$(8.11) \qquad Q(\theta, s) = \frac{2s^2 \sin 2\theta}{s^4 - 2s^2\cos 2\theta + 1}$$

But $Q(\theta, t)$ is non-negative for all t, and is uniformly bounded away from o, say $Q(\theta, t) \geqq \eta > 0$ for $\theta_1 \leqq \theta \leqq \theta_2$ and $\varepsilon \leqq t \leqq \varepsilon^{-1}$, so that

$$(8.12) \qquad \varphi(\theta, r_m) \geqq r_m^{-1}\int_{r_m\varepsilon}^{r_m/\varepsilon} Q(\theta, t/r_m)\, d\Lambda(t) \geqq \eta\, r_m^{-1}\int_{r_m\varepsilon}^{r_m/\varepsilon} d\Lambda(t).$$

Hence, by (8.7), when m is sufficiently large, we have

$$(8.13) \qquad \varphi(\theta, r_m) \geqq \eta\delta > 0.$$

Putting now $r = r_m(\theta_1)$ in (8.8), and using (8.2) and (8.13), we get

$$(8.14) \qquad h(\theta_2) \geqq h(\theta_1) + (\theta_2 - \theta_1)\,\eta\delta,$$

and our proof is done.

Proof of Theorem 8.1. — We remark first of all that by Theorem A, Λ must have finite upper density [*see* (2.14)] if $\mathcal{F}(\Lambda)$ is to be non-empty. Now if Λ does have finite upper density then $W \in \mathcal{F}(\Lambda)$, and $h_W = h_W(\pi/2) = \pi \bar{D}_P(\Lambda)$. It follows, then, from Theorem 6.2 that $\bar{D}_L(\Lambda) \leqq \bar{D}_P(\Lambda)$. Thus, if $\bar{D}_L(\Lambda) = \bar{D}_P(\Lambda)$, then for every $f \in \mathcal{F}(\Lambda)$ we have

$$h_f \geqq h_f(\pm\pi/2) \geqq h_W(\pm\pi/2) = h_W,$$

where h_f denotes the overall type of f [*see* (2.11)]. But this means that W has the smallest overall type of functions in $\mathcal{F}(\Lambda)$ and we have proved the sufficiency.

In the other direction, supposing that $\bar{D}_L(\Lambda) < \bar{D}_P(\Lambda)$, we must construct a function $\Phi(z) \in \mathcal{F}(\Lambda)$ with smaller overall type than that of W.

To this end, we let b be a number with $\bar{D}_L(\Lambda) < b < \bar{D}_P(\Lambda)$. By Theorem 6.2 there is then a function $f \in \mathcal{F}(\Lambda)$ with $h_f(\pm\pi/2) < \pi b$, and hence

$$(8.15) \qquad h_f\left(\pm\frac{\pi}{2}\right) < h_W\left(\frac{\pi}{2}\right).$$

At this point we require a modification of the construction of paragraph 5 which permits us to choose an *even* function $f(z)$, whose zeros lie only on the coordinate axes. More specifically, we can choose $f(z)$ of the form

$$(8.16)\qquad f(z) = W(z)\,F(z),$$

where $F(z)$ is the function (5.32), the sequence $\Gamma = \{\gamma_n\}$ of positive real numbers being chosen to satisfy the hypothetical formula

$$\int \log(1+y^2 t^{-2})\,t\,d\lambda(t) + \int \log|1-y^2t^{-2}|\,t\,d\gamma(t) = \int \log(1+y^2t^{-2})\,t\,d\lambda_b(t).$$

As in paragraph 5, this formula cannot actually be satisfied, but the same techniques used there are just as effective here. Indeed, the details of the construction are easier.

LEMMA 8.1. — *Let* $u(\theta)$ *and* $v(\theta)$ *be continuous real-valued functions on* $0 \leqq \theta \leqq \pi/2$ *such that*

$$(8.17)\qquad v(\pi/2) > v(\theta) \qquad \textit{for}\quad 0 \leqq \theta < \pi/2,$$

$$(8.18)\qquad u(\pi/2) < v(\pi/2).$$

Then there is a number α_0, $0 < \alpha_0 < 1$, *such that*

$$(8.19)\qquad \max_{0\leqq\theta\leqq\pi/2} \{\alpha u(\theta) + (1-\alpha)\,v(\theta)\} < v(\pi/2),$$

whenever $0 < \alpha \leqq \alpha_0$.

We stress the *strict* inequalities in (8.17), (8.18) and (8.19). We omit the easy proof of this lemma. To apply the lemma, we take $u(\theta) = h_f(\theta)$ and $v(\theta) = h_W(\theta)$. The point of Theorem **8.2** is that the hypothesis (8.17) is satisfied, and the hypothesis (8.18) is a restatement of (8.15). In particular, we choose a *rational* number $\alpha = p/q$ such that

$$(8.20)\qquad \max_{0\leqq\theta\leqq\pi/2} \left\{\frac{p}{q} h_f(\theta) + \frac{q-p}{q} h_W(\theta)\right\} < h_W(\pi/2).$$

We consider now the entire function

$$(8.21)\qquad g(z) = \{f(z/q)\}^p \{W(z/q)\}^{q-p}.$$

On calculating $h_g(\theta)$, and applying (8.20), we get

$$(8.22)\qquad h_g(\theta) < h_W(\pi/2) \quad \text{for all } \theta.$$

Unfortunately, the function $g(z)$ vanishes not on Λ but on the (closely related) sequence Λ^* instead, where

$$(8.23)\quad \Lambda^* = q\lambda_0,\ q\lambda_0,\ \ldots,\ q\lambda_0,\ q\lambda_1,\ q\lambda_1,\ \ldots,\ q\lambda_1,\ q\lambda_2,\ q\lambda_2,\ \ldots,\ q\lambda_2,\ \ldots,$$

where $q\lambda_n$ occurs q times for each n. In other words, $g(z)$ has a q-fold zero at each of the points $q\lambda_0, q\lambda_1, \ldots$. But it is not hard to replace $g(z)$ by a function $\Phi(z) \in \mathcal{F}(\Lambda)$ with the same indicator,

$$(8.24) \qquad h_\Phi(\theta) = h_g(\theta) \quad \text{for all } \theta.$$

We may write

$$(8.25) \qquad g(z) = \Pi\{(1 - z^2(q\lambda_n)^{-2})\}^q \{\Pi(1 + z^2(q\gamma_n)^{-2})\}^p,$$

where $\pm i\gamma'_n$ are the imaginary zeros of $f(z)$. For simplicity, we rewrite (8.25) as

$$(8.26) \qquad g(z) = \Pi\{1 - z^2(q\lambda_n)^{-2}\}^q \Pi(1 + z^2\alpha_n^{-2}),$$

with the appropriate choice of α_n. Our function $\Phi(z) \in \mathcal{F}(\Lambda)$ will have the form

$$(8.27) \qquad \Phi(z) = \Pi(1 - z^2\lambda_n^{-2})\,\Pi(1 + z^2\beta_n^{-2}),$$

where we shall now see how to choose the sequence $B = \{\beta_n\}$.

From (8.26) and (8.27) we get

$$(8.28) \quad r^{-1}\log|\Phi(re^{i\theta})| = r^{-1}\log|W(re^{i\theta})| + r^{-1}\int \log|1 + r^2t^{-2}e^{2i\theta}|\,dB(t),$$

$$(8.29) \qquad r^{-1}\log|g(re^{i\theta})| = (r/q)^{-1}\log|W(re^{2i\theta}/q)| + r^{-1}\int \log|1 + r^2t^{-2}e^{2i\theta}|\,dA(t),$$

where

$$A(t) = \sum_{\alpha_n \leq t} 1 \qquad \text{and} \qquad B(t) = \sum_{\beta_n \leq t} 1.$$

We choose B so that

$$(8.30) \qquad B(t) = q^{-1}A(qt) + o(t).$$

For example, (8.30) could achieved by choosing for B the sequence consisting of every q^{th} term from the sequence $\alpha_1/q, \alpha_2/q, \alpha_3/q, \ldots$, with an error $O(1)$ in (8.30). Integrating by parts as in (2.18), and taking the obvious estimate based on (8.30), we get

$$(8.31) \qquad r^{-1}\log|g(re^{i\theta})| = (r/q)^{-1}\log|\Phi((r/q)e^{i\theta})| + o(r),$$

for those θ that are not integral multiples of $\pi/2$. The desired result, (8.24) therefore holds with these possible exceptions, and by the continuity of $h(\theta)$ must then hold for all θ.

9. Applications. — We give here two applications of the earlier results of this paper. The first is a theorem of Szasz-Müntz type for functions holomorphic in a horizontal strip. The second, a variant of the first, is a best-possible gap theorem for power series. There are numerous possible other applications of the same kind.

THEOREM **9.1**. — *Let $\mathcal{H}_b$ be the space of functions holomorphic in the horizontal strip $|\mathrm{Im}\, z| < \pi b$ and continuous in $|\mathrm{Im}\, z| \leqq \pi b$, under the compact open topology (uniform convergence on each compact subset of $|\mathrm{Im}\, z| \leqq \pi b$). Let Λ be an arbitrary sequence of distinct positive real numbers. In order that the collection of functions $\{ e^{-\lambda_n z} \}$ be incomplete in $\mathcal{H}_b$, it is necessary and sufficient that (6.2) hold, i.e., that $\Lambda \prec \Lambda_b$.*

PROOF. — We reduce this problem to the uniqueness problem for entire functions that is solved in Theorem **6.1**. If $\{ \exp(-\lambda_n z) \}$ is not complete, then applying the Hahn-Banach theorem to the locally convex vector space C_b of continuous functions in the closed strip under the compact open topology, we select a measure $d\mu$, whose support is contained in a compact subset of the strip, and a function $g \in \mathcal{H}_b$ such that

$$\int \exp(-\lambda_n w)\, d\mu(w) = 0 \qquad (n = 0, 1, 2, \ldots), \tag{9.0}$$

but such that

$$\int g(w)\, d\mu(w) \neq 0. \tag{9.1}$$

We put

$$f(z) = \int \exp(-zw)\, d\mu(w), \tag{9.2}$$

to obtain a function $f(z)$ that satisfies the hypotheses of Theorem **6.1**, unless $f(z) \equiv 0$. We show now that $f(z) \equiv 0$ contradicts (9.1), so that by Theorem **6.1** we may conclude that $\Lambda \prec \Lambda_b$.

If $f(z) \equiv 0$, then $f^{(p)}(0) = 0$ for all $p = 0, 1, 2, \ldots$, that is,

$$\int z^p\, d\mu(z) = 0 \qquad (p = 0, 1, 2, \ldots).$$

This contradicts (9.1) since by an easy extension of Runge's theorem, the function $g(z)$ can be uniformly approximated by polynomials on a closed rectangle containing the support of $d\mu$.

Conversely, if $\Lambda \prec \Lambda_b$, then by a trivial modification of Theorem **6.1**, we construct an $f \in \mathcal{F}(-\Lambda)$ such that

$$|f(iy)| < \frac{1}{y^2+1} \exp|\pi b y|. \tag{9.3}$$

The Borel transform [*see* (**2**.12)] $F(w)$ of f is continuous on the lines $\mathrm{Im}(w) = \pm \pi b$ since

$$\int_0^\infty |f(\pm iy)| \exp(-\pi b y)\, dy < \infty.$$

Let R be the perimeter of a rectangle, two of whose sides lie along $|\operatorname{Im} w| = \pi b$, and which encloses the indicator diagram of f. Then, defining $d\mu(w)$ as the restriction to R of the measure $(2\pi i)^{-1} F(w)\, dw$, we see by (2.13) that

$$f(z) = \int \exp(zw)\, d\mu(w).$$

Since $f(-\lambda_n) = 0$, we see that $d\mu$ satisfies (9.0). But since, say, $f(z_0) \neq 0$, we may choose $g(w) = \exp(z_0 w)$ so that (9.1) holds. It follows that $g(w)$ cannot be approximated on R by linear combinations of $\{\exp(-\lambda_n w)\}$, that is $\{\exp(-\lambda_n w)\}$ is not complete.

Theorem 9.1 holds, with essentially the same proof, if we put $b = 0$. Thus, a necessary and sufficient condition for the completeness of $\{\exp(-\lambda_n x)\}$ in the space of continuous functions on the real line, with the compact open topology, is that $\lambda(x) \to \infty$. We remark that this is the same condition as the one given by MUNTZ [15] for the completeness of $\{\exp(-\lambda_n x)\}$ in the space $C_0(R^+)$ of continuous functions on $(0, \infty)$ that tend to zero at infinity, with the uniform topology. These two theorems show that the problems are equivalent on the line.

But for the space of functions holomorphic in $|\operatorname{Im} z| < \pi b$, continuous on $|\operatorname{Im} z| = \pi b$, and vanishing at infinity, with the topology of uniform convergence in every closed right half strip, the problem of closure is different. There, the associated uniqueness problem is for functions holomorphic only in a half plane, and FUCHS [4] showed that the condition for completeness is that

$$\limsup_{x \to \infty} \{\lambda(x) - b \log x\} = +\infty.$$

The next result is a logarithmic variant of Theorem 9.1 and can be proved by applying Theorem 6.1 in the manner described in [13], p. 422.

THEOREM 9.2. — *Let Λ be a sequence of distinct positive integers, and let $b \geq 0$ be given. In order that there exist a function $G(z)$, not identically zero, that is holomorphic in a "keyhole" region consisting of a neighborhood of $z = 0$, a neighborhood of $z = \infty$, and the angle $|\arg z| < \pi b$, and continuous on the closure of this region, whose Taylor series expansion about the origin has the form*

$$G(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}, \tag{9.4}$$

it is necessary and sufficient that Λ^*, the set of positive integers complementary to Λ, satisfy $\Lambda^* \prec \Lambda_b$.

10. An example. — The example of this section illustrates Theorem 6.1. The method used to construct it is different from the methods of sections 5 and 7. It relies on the introduction of additional real zeros instead of the imaginary ones used in section 5.

The following sequence Λ was suggested in [13], p. 428 as a test sequence :

$$\Lambda = \bigcup_{k=0}^{\infty} \{ n : 4^k \leq n < 2.4^k \}. \tag{10.1}$$

It is simple to verify that

$$\Lambda \sim \Lambda^* \sim \Lambda_{1/2} \tag{10.2}$$

where Λ^* is the sequence of positive integers complementary to Λ, and $\Lambda_{1/2} = \{ 2n \}$. Theorem 6.1 asserts that there is an $f \in \mathcal{F}(\Lambda)$ with

$$|f(iy)| \leq e^{\pi |y|/2}. \tag{10.3}$$

In this special case we can choose $f(z)$ as

$$f(z) = \{ \Pi(1 - z/\lambda_n) \exp(z/\lambda_n) \} \{ \Pi(1 + z/\lambda_n^*) \exp(-z/\lambda_n^*) \}. \tag{10.4}$$

By Theorem A and (10.2), $f(z)$ is of exponential type, and it is easy to see that

$$|f(iy)| = \left| \frac{\sin \pi iy}{\pi iy} \right|^{1/2}, \tag{10.5}$$

whence (10.3).

Incidentally, the function

$$g(z) = 2\pi z \, e^{\pi i z} \{ f(z) \}^2$$

is an example of a function of exponential type for which $|g(x + iy)|$ approaches a limit as $y \to \infty$ for $x = 0$ but for no other value of x. The first such function $g(z)$ was constructed by SCHAEFFER [14] in response to a research problem posed by BOAS.

BIBLIOGRAPHY.

[1] BOAS, Jr (Ralph Philip). — *Entire functions.* — New York, Academic Press, 1954 (*Pure and applied Mathematics.* A Series of Monographs and Textbooks, 5).

[2] CARLEMAN (Torsten). — Ueber die Approximation analytischer Funktionen durch lineare Aggregate von vergegeben Potenzen, *Arkiv for Mat. Astron. och Fys.*, t. 17, 1922-1923, n° 9, 30 pages.

[3] CARTAN (Henri). — Théorie générale du balayage en potentiel newtonien, *Ann. Univ. Grenoble*, N. S., t. 22, 1946, p. 221-280.

[4] FUCHS (W. H. J.). — A generalization of Carleman's theorem, *J. London Math. Soc.*, t. 21, 1946, p. 106-110.

[5] KAHANE (Jean-Pierre). — Sur quelques problèmes d'unicité et de prolongement, relatifs aux fonctions approchables par des sommes d'exponentielles, *Ann. Inst. Fourier*, Grenoble, t. 5, 1953-1954, p. 39-130 (*Thèse Sc. Math.*, Paris, 1954).

[6] KAHANE (J.-P.) and RUBEL (L. A.). — On Weierstrass products of zero type on the real axis, *Illinois J. Math.*, t. 4, 1960, p. 584-592.

[7] LEONTIEV (A. F.). — Series of Dirichlet polynomials and their applications [in Russian], *Trudy mat. Inst. Steklova*, t. 39, 1951, 214 pages.

[8] LEVIN (B. J.). — *The distribution of the zeros of entire functions* [in Russian]. — Moskva, Gosudarstvennoe Izdatel'stvo techniko-teoreticeskoj Literatury, 1956.

[9] MACINTYRE (A. J.). — Laplace's transformation and integral functions, *Proc. London math. Soc.*, Series 2, t. 45, 1938, p. 1-20.

[10] MALLIAVIN (Paul). — Sur la croissance radiale d'une fonction méromorphe, *Illinois J. Math.*, t. 1, 1957, p. 259-296.

[11] MANDELBROJT (Szolem). — *Séries adhérentes, régularisation des suites, applications*. — Paris, Gauthier-Villars, 1952 (Collection de Monographies sur la Théorie des Fonctions).

[12] PÓLYA (Georg). — Untersuchungen über Lücken und Singularitäten von Potenzreihen, *Math. Z.*, t. 29, 1929, p. 549-640.

[13] RUBEL (L. A.). — Necessary and sufficient conditions for Carlson's theorem on entire functions, *Trans. Amer. math. Soc.*, t. 83, 1956, p. 417-429.

[14] SCHAEFFER (A. C.) — Entire functions, *Pacific J. Math.*, t. 6, 1956, p. 351-362.

[15] SCHWARTZ (Laurent). — Étude des sommes d'exponentielles, 2e édition, Paris, 1959.

(Manuscrit reçu le 5 janvier 1961.)

Paul MALLIAVIN,
Professeur à la Faculté des Sciences de Caen,
Institute for Advanced Study [Princeton],
252, rue de Rivoli, Paris (1er);

L. A. RUBEL,
Associate Professor at the University of Illinois
and Visiting Associate Professor at Columbia University,
Columbia University, New York 27, N. Y., U. S. A.