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TWO CARTESIAN PRODUCTS WHICH ARE EUCLIDEAN SPACES

BY

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WHITEHEAD has given an example of a three-dimensional manifold W which is not (homeomorphic to) E^3, Euclidean 3-space [3]. We prove the following theorem about W, the first statement of which is due to A. SHAPIRO.

THEOREM. — *If W is the manifold described below then $W \times E^1$ is homeomorphic to E^4. Also $W \times W$ is homeomorphic to $E^3 \times W$ (which is homeomorphic to E^6).*

That W is not homeomorphic to E^3 was proved in [1], [2]. In [1] it is shown that no cube in W contains W_0 (defined below), which implies W is not E^3. The homeomorphism $W \times E^1 \approx E^4$ can be used to show the existence of a two element (and so compact) group of homeomorphisms of E^4 onto itself whose fixed point set is W. The problem of showing that $W \times W$ is homeomorphic to E^6 was suggested to the author by L. ZIPPIN.

Let W_0, W_1, R_0, R_1 be solid tori with W_0 simply self-linked in the interior of W_1 (*see fig.* 1) and R_0 trivially imbedded in the interior of R_1. Let I_0 and I_1 be closed bounded intervals of E^1 with I_0 contained in the interior of I_1. Let w (resp. r) be a 3-cell in the interior of W_0 (resp. R_0), let e (resp. f, g) be a homeomorphism of E^3 (resp. E^3, E^1) onto itself with $e(W_0) = W_1$ [resp. $f(R_0) = R_1$, $g(I_0) = I_1$] and $e \mid w$ (resp. $f \mid r$) the identity. Let

$$W_n = e^n(W_0), \qquad R_n = f^n(R_0), \qquad I_n = g^n(I_0).$$

Let $W = \bigcup_{n=1}^{\infty} W_n$, we suppose that

$$E^3 = \bigcup_{n=1}^{\infty} R_n, \qquad E^1 = \bigcup_{n=1}^{\infty} I_n.$$

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Let $S = \{ h \mid A : A \subset E^3$, h is a homeomorphism of E^3 onto itself which is the identity outside a compact set $\}$; we further suppose $e \in S$, $f \mid R_i \in S$ and $\lambda'(R_0) = W_0$ for some λ' in S.

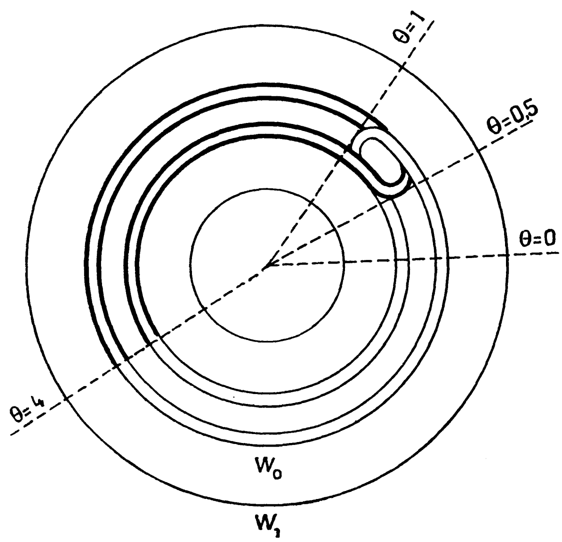

Fig. 1.

PROOF. — We prove both statements simultaneously. Let V_n denote I_n (resp. W_n), V denote E^1 (resp. W). For each positive integer n, we construct a homeomorphism $h_n : W_n \times V_n \to R_n \times V_n$ with the properties

(1) $h_n(W_{n-1} \times V_{n-1}) = R_{n-1} \times V_{n-1}$;

(2) $h_n \mid W_{n-2} \times V_{n-2} = h_{n-1} \mid W_{n-2} \times V_{n-2}$ $(n \geqq 2)$.

Suppose we have constructed all the h' s. Then we define

$$\Phi : \quad W \times V \to E^3 \times V$$

as follows. If $(x, y) \in W \times V$, then for some n, $(x, y) \in W_n \times V_n$. Let $\Phi(x, y) = h_{n+1}(x, y)$. By (2) we see that Φ is well-defined, by (1) we see that Φ is onto. Since h_n is a homeomorphism, Φ is also.

Suppose the following lemma is true. Using the lemma, we will construct the h_n.

LEMMA. — *If we are given a homeomorphism $\beta' : w \times V_0 \to R_0 \times V_0$ (into), and if β' has the form $\lambda' \mid w \times I$ where λ' is a homeomorphism in S of W_0 onto R_0, then there is a homeomorphic extension β of β',*

$$\beta : \quad W_1 \times V_1 \to R_1 \times V_1, \qquad \beta(W_0 \times V_0) = R_0 \times V_0,$$

and $\beta \mid \mathrm{Bdry}(W_1 \times V_1) = \lambda \times I$ *for* λ *some homeomorphism in* S *of* W_1 *onto* R_1.

Let λ' be a homeomorphism in S mapping W_0 onto R_0. Let $h_1 = \beta$, the extension of $\beta' = (\lambda' \mid w) \times I$ given by the lemma. We suppose inductively that for n a positive integer greater or equal to 2, h_{n-1} has been constructed, and $h_{n-1} \mid \mathrm{Bdry}(W_{n-1} \times V_{n-1}) = \gamma \times I$, for γ some homeomorphism in S of W_{n-1} onto R_{n-1}. We note that h_1 has this property. Observe that $(\gamma^{-1} \times I)\, h_{n-1}$ is a homeomorphism of $W_{n-1} \times V_{n-1}$ onto itself leaving the boundary pointwise fixed. Let h be the extension of this map to $W_n \times V_n$ which is the identity on $W_n \times V_n$-Interior $(W_{n-1} \times V_{n-1})$. Let r' be a 3-cell with Interior $R_{n-1} \supset r' \supset R_{n-2}$. Let $w' = \gamma^{-1}(r')$. Let $k : W_n \to W_n$ be a homeomorphism in S, $k \mid (W_n\text{-Interior } W_{n-1}) = \text{identity}$, $k(w') \subset w$. Let β be the extension of $\gamma k^{-1} \times I \mid w \times V_{n-1}$ to a homeomorphism of $W_n \times V_n$ onto $R_n \times V_n$ as given by the lemma. Let $h_n = \beta(k \times I)\, h$. We check that h_n satisfies (1) and (2),

$$h_n(W_{n-1} \times V_{n-1}) = \beta(W_{n-1} \times V_{n-1}) = R_{n-1} \times V_{n-1}.$$

If $z \in W_{n-2} \times V_{n-2}$, then $(k \times I)\, h(z) \in w \times V_{n-1}$ and

$$\begin{aligned} h_n(z) &= \beta(k \times I)\, h(z) \\ &= (\gamma k^{-1} \times I)(k \times I)(\gamma^{-1} \times I)\, h_{n-1}(z) = h_{n-1}(z) \end{aligned}$$

as asserted. Also

$$\begin{aligned} h_n \mid \mathrm{Bdry}(W_n \times V_n) &= \beta(k \times I)\, h \mid \mathrm{Bdry}(W_n \times V_n) \\ &= \lambda k \times I \mid \mathrm{Bdry}(W_n \times V_n), \end{aligned}$$

where the last equality arises from the form of β on $\mathrm{Bdry}(W_n \times V_n)$ and the fact that $(k \times I)(\mathrm{Bdry}(W_n \times V_n)) = \mathrm{Bdry}(W_n \times V_n)$. Thus h_n satisfies the induction hypothesis and all the h_n can be defined, if we prove the lemma.

PROOF OF LEMMA. — Given $\beta' = \lambda' \mid w \times I : w \times V_0 \to R_0 \times V_0$, we can extend $\lambda' \mid w$ to a homeomorphism in $S\,\lambda$ of W_1 onto R_1. In fact let j be a homeomorphism in S of R_1 onto itself which maps R_0 onto R_0 and $\lambda'(w)$ into r. Let

$$\lambda = j^{-1} f j \lambda' e^{-1}.$$

Then λ is a homeomorphism in S of W_1 onto R_1 aud $\lambda \mid w = j^{-1} f j \lambda' \mid w = \lambda' \mid w$ so λ is the desired extension of $\lambda' \mid w$. It is now sufficient to construct a homeomorphism h of $W_1 \times V_1$ onto itself which leaves $w \times V_0$ pointwise fixed with $h \mid \mathrm{Bdry}(W_1 \times V_1) = \mu \times I$ for some μ in S which maps W_1 onto W_1, and with $h(W_0 \times V_0) = \lambda^{-1}(R_0) \times V_0$. In fact $(\lambda \times I)\, h = \beta$ is a homeomorphism of $W_1 \times V_1$ onto $R_1 \times V_1$, β extends β', and

$$\begin{aligned} \beta(W_0 \times V_0) &= \lambda\lambda^{-1}(R_0) \times V_0 = R_0 \times V_0, \\ \beta \mid \mathrm{Bdry}(W_1 \times V_1) &= \lambda\mu \times I \mid \mathrm{Bdry}(W_1 \times V_1). \end{aligned}$$

The homeomorphism h will be given as the product of four homeomorphism Λ, Σ, Δ and P of $W_1 \times V_1$ onto itself. Λ, Σ and Δ will each leave Bdry $(W_1 \times V_1) \cup (w \times V_0)$ pointwise fixed. Λ will lift the dark portion of W_0, Σ will slide this lifted part away from the link, and Δ will drop the image under $\Sigma\Lambda$ of the dark part of W_0 back into its original plane. We suppose W_1 is $D \times C$ where D is the square $\{(u, v) : 0 \leqq u, v \leqq 20\}$ and C is the circle $\{\theta : 0 \leqq \theta < 2\pi\}$. We suppose that

$$W_0 \subset \{(u, v) : 9 \leqq u, v \leqq 10\} \times C, \qquad w \subset D \times \{\theta : 6 \leqq \theta < 2\pi\},$$

the link in $W_0 \subset D \times \{\theta : .5 \leqq \theta \leqq 1\}$. Let α, β, γ, δ be functions on C, let a, b, c be functions on $[0, 20]$, defined as follows. Let

$$\alpha([0, 2]) = 1, \qquad \alpha([4, 2\pi]) = 0, \qquad \beta(0) = 0,$$
$$\beta([.5, 4]) = 1, \qquad \beta([6, 2\pi]) = 0,$$
$$\gamma([0, 1]) = 0, \qquad \gamma([2, 2\pi]) = 1, \qquad \delta([0, 1]) = 0,$$
$$\delta([1.5, 3]) = 1, \qquad \delta([5, 2\pi]) = 0,$$

and let α, β, γ, δ be linear on intervals for which they are not defined above. Let

$$\alpha(0) = 0, \qquad a([9, 10]) = 1, \qquad a(20) = 0,$$
$$b([0, 10]) = 0, \qquad b([11, 12]) = 1, \qquad b(20) = 0,$$
$$c(0) = 0, \qquad c([9, 12]) = 1, \qquad c(20) = 0,$$

and let a, b, c be linear on intervals for which they are not defined above. Let ε be a continuous map of W_1 into $[0, 1]$ such that $\varepsilon(u, v, \theta) = \alpha(\theta)$ for (u, v, θ) in the dark part of W_0, $\varepsilon = 0$ on the rest of W_0 and on Bdry W_1. If (u, v), $(x, y) \in D$, θ, $\psi \in C$, let

$$\Lambda(u, v, \theta, x, y, \psi) = (u, v, \theta, x, y + 2\varepsilon(u, v, \theta)\, a(x)\, a(y), \psi),$$
$$\Sigma(u, v, \theta, x, y, \psi) = (u, v, \theta + \beta(\theta)\, a(x)$$
$$\times [(1 - \gamma(\theta))\, b(y) + \gamma(\theta)\, c(y)]\, a(u)\, a(v), x, y),$$
$$\Delta(u, v, \theta, x, y, \psi) = (u, v, \theta, x, y - 2\delta(\theta)\, c(y)\, a(x)\, a(u)\, a(v), \psi).$$

If $V_i = I_i$, we identify I_0 with $\{10\} \times [9, 10] \times \{0\} \subset W_1$ and I_1 with $\{10\} \times [0, 20] \times \{0\} \subset W_1$. Then Λ, Σ, and Δ map $W_1 \times I_1$ onto itself and $h' = \Delta\Sigma\Lambda \mid W_1 \times I_1$ (resp. $h' = \Delta\Sigma\Lambda$) is a homeomorphism of $W_1 \times V_1$ onto itself which leaves $(\text{Bdry}(W_1 \times V_1)) \cup (w \times V_0)$ pointwise fixed. For $(x, y, \psi) \in V_0$, $\Delta\Sigma\Lambda(W_0 \times (x, y, \psi)$ is trivially imbedded in $W_1 \times (x, y, \psi)$ and the projection W'_0 on W_1 of $\Delta\Sigma\Lambda(W_0 \times (x, y, \psi))$ is independent of x, y, ψ in V_0. To see this it is sufficient to compute $\Delta\Sigma\Lambda(u, v, \theta, x, y, \psi)$ for (u, v, θ) in W_0, x, y in $[9, 10]$ and θ a point of non-linearity of α, β, γ or δ. Suppose we have a homeomorphism ρ' of W_1 onto W_1 which leaves Bdry $W_1 \cup w$ pointwise fixed, and with $\rho'(W'_0) = \lambda^{-1}(R_0)$. Define $P = \rho' \times I$: $W_1 \times V_1 \to W_1 \times V_1$, define $h = Ph'$. Then h has the necessary properties.

Since $\lambda^{-1}(R_0)$ is trivially imbedded in W_1, it is in a 3-cell in the interior of W_1. There is a homeomorphism g' of E^3 onto itself leaving E^3-W_1 pointwise fixed and such that $g'(W'_0)$ and $\lambda^{-1}(R_0)$ both lie in a 3-cell u in the interior of W_1. It is evident that there is a homeomorphism in S mapping W_0 onto W'_0 and so there is a homeomorphism g'' in S of E^3 onto itself mapping $g'(W'_0)$ onto $\lambda^{-1}(R_0)$. We can find a 3-cell U outside of which g'' is the identity and a homeomorphism φ mapping U onto u which is the identity on $\lambda^{-1}(R_0)\cup g'(W'_0)$. Define $g=$ identity outside u, $g=\varphi g''\varphi^{-1}$ on u. Then $h=gg'$ is a homeomorphism leaving boundary W_1 fixed and mapping W'_0 onto $\lambda^{-1}(R_0)$. Since $w\subset$ Interior W'_0, $h(w)\subset$ Interior $\lambda^{-1}(R_0)$ and since $w\subset$ Interior $\lambda^{-1}(R_0)$ there is a homeomorphism i of E^3 onto itself leaving $E^3-\lambda^{-1}(R_0)$ fixed and mapping $h(w)$ into w. Let U_0, u_0 be 3-cells, with $U_0\supset W_1$, $\lambda^{-1}(R_0)\supset u_0$, Interior $u_0\supset w$ and let φ_0 be a homeomorphism of U_0 onto u_0 leaving w pointwise fixed. Let $j=\varphi_0(ih)^{-1}\varphi_0^{-1}$ on u_0, $j=$ identity on W_1-u_0. Then $\rho'=jih$ is a homeomorphism of W_1 onto W_1,

$$\rho'(W'_0)=ji\lambda^{-1}(R_0)=\lambda^{-1}(R_0),$$

$\rho'\mid$ Bdry $W_1=$ identity and $\rho'\mid w=\varphi_0(ih)^{-1}\varphi_0^{-1}ih\mid w=\varphi_0\mid w=$ identity. This completes the proof.

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