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A scattering theory for the wave equation on Kerr black hole exteriors

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A SCATTERING THEORY FOR THE WAVE EQUATION ON KERR BLACK HOLE EXTERIORS

BY MIHALIS DAFERMOS, IGOR RODNIANSKI
AND YAKOV SHLAPENTOKH-ROTHMAN

ABSTRACT. – We develop a definitive physical-space scattering theory for the scalar wave equation $\square_g \psi = 0$ on Kerr exterior backgrounds in the general subextremal case $|a| < M$. In particular, we prove results corresponding to “existence and uniqueness of scattering states” and “asymptotic completeness” and we show moreover that the resulting “scattering matrix” mapping radiation fields on the past horizon \mathcal{H}^- and past null infinity \mathcal{I}^- to radiation fields on \mathcal{H}^+ and \mathcal{I}^+ is a bounded operator. The latter allows us to give a time-domain theory of superradiant reflection. The boundedness of the scattering matrix shows in particular that the maximal amplification of solutions associated to ingoing finite-energy wave packets on past null infinity \mathcal{I}^- is bounded. On the frequency side, this corresponds to the novel statement that the suitably normalized reflection and transmission coefficients are uniformly bounded independently of the frequency parameters. We further complement this with a demonstration that superradiant reflection indeed amplifies the energy radiated to future null infinity \mathcal{I}^+ of suitable wave-packets as above. The results make essential use of a refinement of our recent proof [30] of boundedness and decay for solutions of the Cauchy problem so as to apply in the class of solutions where only a degenerate energy is assumed finite. We show in contrast that the analogous scattering maps cannot be defined for the class of finite non-degenerate energy solutions. This is due to the fact that the celebrated horizon red-shift effect acts as a blue-shift instability when solving the wave equation backwards.

RÉSUMÉ. – Nous développons une théorie de la diffusion définitive en espace physique pour l'équation scalaire d'onde dans la région extérieure de la métrique de Kerr dans le cas sous-extrémal général $|a| < M$. En particulier, nous prouvons des résultats qui correspondent à « l'existence et l'unicité des états de diffusion » et la « complétude asymptotique » et nous montrons de plus que la matrice de diffusion qui envoie les champs de radiation sur l'horizon passé et l'infini nul passé aux champs sur l'horizon futur et l'infini nul futur est un opérateur borné. Ce dernier point nous permet de donner une théorie de réflexion superradiante dans le domaine temporel. Le fait que la matrice de diffusion est bornée montre en particulier que l'amplification maximale de solutions associées aux paquets d'ondes entrants d'énergie finie sur l'infini nul passé est bornée. En fréquence, cela correspond à l'affirmation nouvelle que les coefficients de réflexion et de transmission, convenablement normalisés, sont bornés uniformément, indépendamment des paramètres de fréquence. Nous complétons ceci de plus avec une démonstration que la réflexion superradiante amplifie effectivement l'énergie rayonnée à l'infini nul futur, pour les paquets d'ondes appropriés comme ci-dessus. Les résultats font usage essentiel d'un raffinement de notre démonstration récente [30] de la bornitude et de la décroissance

des solutions du problème de Cauchy de façon à s'appliquer à la classe de solutions où seulement une énergie dégénérée est supposée finie. Nous montrons en contraste que l'application de diffusion analogue ne peut pas être définie pour la classe de solutions d'énergie finie non dégénérée. C'est dû au fait que le célèbre effet de décalage vers le rouge agit comme une instabilité de décalage vers le bleu quand on résout l'équation d'onde rétrograde.

1. Introduction

Black holes play a central role in our present general relativistic picture of the universe. At the same time, however, they are perhaps the example *par excellence* of a physical object which cannot be observed “directly”. An effective approach to infer both the very presence but also the finer properties of black holes proceeds through the study of the scattering of waves on their exterior. Hence, a theoretical understanding of scattering theory in this context is of paramount importance.

The bulk of the now classical black hole scattering-theory literature concerns only the *fixed-frequency* study of solutions $u_{(\omega,m,\ell)}(r^*)$ to the radial o.d.e.

$$(1) \quad u'' + \omega^2 u = Vu,$$

where $V = V_{(\omega,m,\ell)}(r^*)$, resulting from Carter's remarkable separation [15] of the linear scalar wave equation

$$(2) \quad \square_g \psi = 0$$

on Kerr black hole backgrounds $(\mathcal{M}, g_{a,M})$. One can also consider more complicated systems like the Maxwell Equations or the equations of linearised gravity. See Chandrasekhar's monumental [16] and the monograph [40].

Beyond formal fixed-frequency statements concerning (1), true scattering results in the “time-domain,” describing actual *finite-energy* solutions of (2) and related equations, have only been obtained in various special cases. Let us already mention the pioneering results of Dimock and Kay [33, 35, 34] in the Schwarzschild $a = 0$ case. See also [8, 9]. In the case of rotating Kerr black holes with $a \neq 0$, on the other hand, despite recent progress on the Cauchy problem, first for the $|a| \ll M$ case [28, 4, 69] and then, for the full subextremal range $|a| < M$ in [30], the most basic questions of scattering theory for (2) have remained to this day unanswered. In particular:

- (a) Can one associate a finite-energy solution of (2) to every suitable finite-energy past/future asymptotic state? (*Existence of scattering states.*)
- (b) Is the above association unique, i.e., do two finite-energy solutions having the same asymptotic state necessarily coincide? (*Uniqueness of scattering states.*)
- (c) Do the above solutions parametrised by finite-energy past/future asymptotic states describe the totality of finite-energy solutions ψ to (2)? (*Asymptotic completeness.*)

See the classic [62] for a general introduction to the scattering theory framework in physics.

At the conceptual level, one of the most interesting new phenomena of black hole scattering which arises when passing from the Schwarzschild $a = 0$ to the rotating $a \neq 0$ Kerr case is that of *superradiance*. This already can be seen at the level of the fixed-frequency o.d.e. (1). We review this very quickly for the benefit of the reader familiar with the classical physics literature [16].⁽¹⁾ For each fixed frequency triple (ω, m, ℓ) with $\omega \in \mathbb{R}$, one can define two complex-valued solutions $U_{\text{hor}}(r^*)$ and $U_{\text{inf}}(r^*)$ of (1) so that

$$U_{\text{hor}} \sim e^{-i(\omega - \omega_+ m)r^*} \text{ as } r^* \rightarrow -\infty, \quad U_{\text{inf}} \sim e^{i\omega r^*} \text{ as } r^* \rightarrow \infty,$$

corresponding to the asymptotic behavior of the potential V , which is itself real. Here ω_+ is related to the Kerr parameters a, M by the formula $2M\omega_+(M + \sqrt{M^2 - a^2}) = a$. The linear independence of U_{hor} and U_{inf} is the statement of *mode stability* on the real axis and was proven recently by one of us [67], extending the transformation theory of [73]. By dimensional considerations, this linear independence at one go answers the “fixed frequency” analog of questions (a)–(c) in the affirmative. It follows that since $\overline{U_{\text{inf}}}$ also solves (1), we may write

$$(3) \quad -\frac{\omega \mathfrak{T}}{(\omega - \omega_+ m)} U_{\text{hor}} = \mathfrak{R} U_{\text{inf}} + \overline{U_{\text{inf}}},$$

where $\mathfrak{T} = \mathfrak{T}(\omega, m, \ell)$ and $\mathfrak{R} = \mathfrak{R}(\omega, m, \ell)$ are known as the *transmission* and *reflexion* coefficients. Formally, these coefficients describe the proportion of “energy” at fixed frequency (ω, m, ℓ) transmitted to the horizon and reflected to infinity, respectively, of purely incoming wave from past infinity. With the precise normalization of (3), which will be in fact motivated by the considerations of this paper, the energy identity associated to (1) yields

$$(4) \quad |\mathfrak{R}|^2 + \frac{\omega}{\omega - \omega_+ m} |\mathfrak{T}|^2 = 1.$$

Superradiance, first discussed by Zeldovich [74], corresponds to the fact that, for the frequency range

$$(5) \quad \omega(\omega - \omega_+ m)^{-1} < 0,$$

the transmission coefficient \mathfrak{T} is weighted with a negative sign in (4) allowing thus the reflection coefficient \mathfrak{R} to have norm strictly greater than 1

$$(6) \quad |\mathfrak{R}(\omega, m, \ell)| > 1.$$

That is to say, there is a nontrivial energy amplification factor at fixed frequency. The first estimates for the maximum reflection coefficient in various frequency regimes go back to pioneering work of Starobinskii [68] (see also [70]), but even the statement of the uniform boundedness of $\mathfrak{R}(\omega, m, \ell)$ over all superradiant frequencies (5) has remained an open problem.

In passing from a fixed-frequency scattering theory to a true time-domain scattering theory, the absence of an obvious quantitative frequency-independent control of the coefficient $\mathfrak{R}(\omega, m, \ell)$ presents itself as a fundamental difficulty. Moreover, an additional difficulty is identifying the correct notion of “energy” with respect to which solutions should be defined. In particular, one requires a notion of energy which controls solutions of (2) not

⁽¹⁾ All notations here will be explained in detail in the paper. The reader for which this is unfamiliar can skip directly to the next paragraph!

only in the forward but also in the *backward* direction, i.e., an energy *not* subject to the local red-shift effect associated to the event horizon, which when solving backwards appears as a *blue-shift* instability.

The purpose of this paper is to overcome these difficulties and develop a definitive finite-energy scattering theory for (2) on general subextremal Kerr exteriors $(\mathcal{M}, g_{a,M})$ with $|a| < M$, showing in particular:

The answer to (a), (b) and (c) is yes. Existence and uniqueness of scattering states as well as asymptotic completeness indeed hold for the space of solutions to (2) and scattering states defined by the finiteness of a natural energy flux.

We will understand scattering states in the sense of Friedlander [38] (for the Schwarzschild case in this context, see [59]), and our approach to both constructing and estimating the scattering maps can be thought of as a combination of what in the traditional literature are known as “stationary” and “time-dependent” methods [49, 53]. We will depend heavily on our recent boundedness results [30] for the Cauchy problem for (2), as well as certain decay results of [30], which indeed succeeded in giving a first version of quantitative physical-space control over superradiance, independent of frequency, and also showed that a suitable class of solutions of (2) can be indeed understood as superpositions of solutions of (1) over real frequencies ω . Our argument crucially relied on the fact that *the difficulties of superradiance and trapping are disjoint*. We will in fact, however, here require a certain refinement of the estimates of [30] so as to apply to a degenerate energy not subject to the backwards blue-shift instability. This notion of energy lies behind the particular choice of normalization of the reflection coefficient \mathfrak{R} in (3). Along the way, we shall in particular provide the missing frequency-independent bound on \mathfrak{R} over all superradiant frequencies (5):

$$(7) \quad \sup_{(\omega, m, \ell)} |\mathfrak{R}(\omega, m, \ell)| = S(a, M) < \infty$$

by a finite constant $S(a, M)$ depending only on the Kerr parameters, with $S(a, M) > 1$ if $a \neq 0$.

Our asymptotic completeness results will allow us to define (in the language of Wheeler [72]) an S -matrix \mathcal{S} whose boundedness in the operator norm replaces the usual unitarity property. A suitable restriction of \mathcal{S} will be related to a generalization of the inverse-Fourier transform applied to multiplication by the coefficients \mathfrak{R} and \mathfrak{T} defined by (3). Through this, we will give a definitive physical space (i.e., time-domain) interpretation of superradiant reflection, in particular, showing:

*Superradiant reflection indeed strictly amplifies the energy radiated to infinity of suitably constructed purely ingoing finite-energy wave packets. The maximum amplification factor, however, is **bounded** precisely by the constant $S(a, M)$ of (7).*

Our results leave open the extremal case $g_{a,M}$ for $a = M$ (see [7]). In particular, it is not known whether the limit $\lim_{|a| \rightarrow M} S(a, M)$ is finite.

1.1. Brief overview of the main theorems

We introduce briefly the main theorems of the paper in what follows below. (We will give a more detailed overview together with precise statements of all theorems listed here in boldface type in Section 2.)

1.1.1. *From Schwarzschild to Kerr: the T -energy theory and superradiance.* – The first difficulty in constructing a physical-space scattering theory is identifying what constitutes the “correct” class of finite energy solutions and asymptotic states. In the Schwarzschild case, as admissible solutions to (2) it is natural to consider the class of ψ which have finite *conserved* energy (i.e., finite energy corresponding to the stationary Killing vector field T) on a Cauchy hypersurface. This in turn suggests a corresponding notion of asymptotic states defined in terms of the completion (with respect to the natural T -energy flux) of the set of *Friedlander radiation fields* $r\psi$ on \mathcal{I}^+ (see [38]), complemented by the analogous completion of the set of traces of ψ on the event horizon \mathcal{H}^+ . See Nicolas [59] for a recent formulation of Schwarzschild scattering theory in precisely these terms. This theory can be constructed entirely in the time-domain, i.e., using “time-dependent” methods. (We will in fact give our own self-contained version of the Schwarzschild theory in Section 9.6.)

Turning to the Kerr case, the above conserved energy corresponding to T is clearly unsuitable for a scattering theory, because the inner product it defines is now indefinite, in view of the existence of the well-known *ergoregion* where T is spacelike.⁽²⁾ This is the physical-space origin of the phenomenon of superradiance discussed with respect to (4). Recent progress on understanding the Cauchy problem for (2) on Schwarzschild and Kerr has rested in part on the realization (see [23, 26, 27]) that a more natural energy quantity for understanding forward evolution is that defined by a T -invariant everywhere-timelike vector field N (see (41) in Section 3.4). Even though this N -energy is not conserved, it remains, as proven in our recent work [30] (for the full sub-extremal range of Kerr parameters $|a| < M$), uniformly bounded through a suitable spacelike foliation Σ_s^* of the exterior region and controls in fact a spacetime integral quantity. The good divergence properties of the vector field N are related to the celebrated red-shift effect associated to the horizon \mathcal{H}^+ .

1.1.2. *The N -energy theory and the backwards blue-shift instability.* – Despite its success in the context of the Cauchy problem on Kerr, the above N -energy is again unsuitable for defining a scattering theory, because the helpful red-shift transforms into a lethal blue-shift when trying to associate admissible solutions to their natural asymptotic states, which requires solving the wave equation backwards. See the discussion in [20, 64] and also the more recent comments in [59]. (Note that the appearance of a blue-shift instability in this context is familiar from the phenomenon of Hawking radiation [47, 10, 11, 44, 45]. See also [50].) The first two results of our paper are dedicated to making explicit this obstruction. Our Theorems 1 and 2 together show that while one can naturally associate (using our results

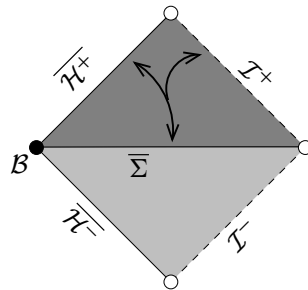
⁽²⁾ Let us note that, in contrast to the wave equation (2), for the Dirac equation, one still has a coercive L^2 -conservation law despite the absence of a globally timelike Killing field. Using this, Häfner and Nicolas [46] have constructed a scattering theory for the Dirac equation on Kerr backgrounds, generalizing [58]. This has been extended to Kerr-Newman-de Sitter backgrounds by Daudé and Nicoleau [32]. In this context, see also Häfner [43] for scattering results concerning a non-superradiant class of solutions of the Klein-Gordon equation for fixed azimuthal mode m .

of [30]) asymptotic states to finite N -energy solutions, this map is not surjective, and thus, one cannot define a one-sided inverse map embodying the existence of scattering states (cf. (a)).

1.1.3. *The V -energy theory.* – The correct setting for a scattering theory on Kerr would then appear to be an energy quantity defined by a vector field V which (like T in Schwarzschild) is null on the horizon and timelike outside. With the help of the additional axisymmetric Killing field Φ , one can in fact construct such a vector field V which can be chosen moreover Killing in a neighborhood of both \mathcal{H}^+ and \mathcal{I}^+ (though not globally Killing!). Even the question of uniform boundedness of solutions assumed to lie only in the energy space defined by V (see (42) in Section 3.4), however, has not previously been answered. (See however, the very related higher-order weighted estimates of Andersson-Blue [4] in the very slowly rotating $|a| \ll M$ case.)

The main results of the present paper (Theorems 3 and 4) succeed in constructing a bounded invertible map \mathcal{F}_+ associating a unique future asymptotic state to each solution with initially bounded V -energy, with two-sided inverse \mathcal{B}_- satisfying

$$(8) \quad \mathcal{B}_- \circ \mathcal{F}_+ = \text{Id}, \quad \mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}.$$



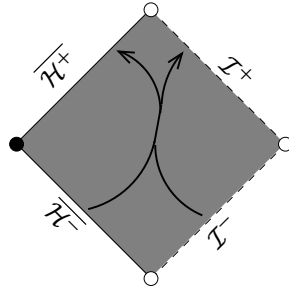
The boundedness of the map \mathcal{F}_+ requires a refinement of our previous boundedness results on the Cauchy problem (see [30]) so as to apply for admissible solutions defined by the finiteness of a suitable V -energy as above. This will require us to revisit the fixed frequency o.d.e. estimates on (1) proven in [30]. What will be the inverse map \mathcal{B}_- is constructed explicitly via the frequency domain by an appropriate superposition of solutions to the fixed frequency o.d.e (1). Again, to infer the boundedness of \mathcal{B}_- one needs to exploit quantitative estimates on (1) adapted from [30], again referring only to the V -energy flux. One may define similar maps \mathcal{F}_- , \mathcal{B}_+ associating solutions to *past* asymptotic states.

In the traditional language of scattering theory, let us note that existence of scattering states (cf. (a)) corresponds to the existence of \mathcal{B}_\mp , uniqueness of scattering states (cf. (b)) to the injectivity of \mathcal{F}_\pm , and asymptotic completeness (cf. (c)) to the surjectivity of \mathcal{B}_\mp . These three statements of course all follow from (8).

1.1.4. *The scattering map \mathcal{S} , superradiant reflection \mathcal{R} and applications.* – The asymptotic completeness results allow us in particular to define a scattering map (S -matrix)

$$\mathcal{S} = \mathcal{F}_+ \circ \mathcal{B}_+$$

taking asymptotic past states to asymptotic future states



which is moreover bounded in the operator norm with respect to the spaces defined by the flux of the V -energy (see Theorem 5).

To connect with the usual discussion of superradiant scattering, we may also define a reflection map \mathcal{R} and a transmission map \mathcal{T} which restricts \mathcal{S} to past asymptotic states with no trace on the past event horizon \mathcal{H}^- and returns only the radiation to future null infinity \mathcal{I}^+ or the future event horizon \mathcal{H}^+ respectively. It follows in particular that \mathcal{R} and \mathcal{T} are also bounded (see Theorem 6). On the other hand, we show that the operator norm of \mathcal{R} satisfies $\|\mathcal{R}\| > 1$ (see Theorem 7), and thus there exist wave packets corresponding to past asymptotic states supported only on \mathcal{I}^- such that the energy radiated to \mathcal{I}^+ is strictly greater than the energy flux on \mathcal{I}^- . As discussed above, this gives a physical space interpretation of superradiance (cf. the numerical [6]). Next, we will show that $\mathcal{T} \oplus \mathcal{R}$ is pseudo-unitary in that it preserves an indefinite inner product associated to the T -energy (Theorem 8). Upon restricting to “non-superradiant” data along \mathcal{H}^- and \mathcal{I}^- the map \mathcal{S} becomes unitary in the standard sense (Theorem 9).

We finally give a “unique continuation” result that finite V -energy solutions are uniquely characterized by their scattering data on any of the “ill-posed” pairs $\mathcal{H}^- \cup \mathcal{H}^+$, $\mathcal{I}^+ \cup \mathcal{I}^-$, $\mathcal{H}^- \cup \mathcal{I}^+$ or $\mathcal{H}^+ \cup \mathcal{I}^-$ (see Theorem 10). This has the interpretation that for this improper notion of asymptotic states, uniqueness of scattering states (b) holds without existence (a).

1.1.5. *Back to the fixed-frequency theory.* – We have already noted that our results will require revisiting the estimates proven in [30] for the radial o.d.e. (1) appearing in Carter’s classical separation of (2). In this sense, our work makes contact back with the formal scattering theory literature [40] concerning (1) at fixed frequency. In particular, our o.d.e. results will yield the uniform boundedness of the reflection and transmission coefficients (Theorem 11), in particular, giving (7). This complements the work of Starobinskii and others (see [68, 70]) aimed at numerically estimating the maximum of these for low fixed values of m, ℓ . Our transmission and reflection maps \mathcal{T} and \mathcal{R} can in fact be represented as a generalized inverse Fourier transform of multiplication by \mathfrak{T} and \mathfrak{R} (Theorem 12). In particular, a posteriori, the boundedness statements of Theorems 6 and 11 are equivalent. This connects the fixed frequency and physical space scattering theories in a very explicit way.

Though our approach to asymptotic completeness will only require us to study the reflection and transmission coefficients for real frequencies, it is also interesting to study the meromorphic continuation of \mathfrak{R} and \mathfrak{T} in appropriate subsets of the complex plane. The work of

Whiting [73] referred to above shows that there are no poles in the upper half plane while the poles in the lower half plane correspond to “quasi-normal modes,” e.g., see [36].

1.2. Related work, future directions of study and further reading

Let us specifically mention here a related recent important advance by Georgescu, Gérard and Häfner [41] which proves scattering results for fixed-azimuthal mode (i.e., fixed m) solutions of the Klein-Gordon equation in the very slowly rotating Kerr-de Sitter case $|a| \ll M, \Lambda$. This is in part based on work on the Cauchy problem due to Dyatlov [36]. For an earlier work concerning superradiant scattering involving a charged Klein-Gordon field in an electrostatic potential, see Bachelot [13, 12]

For additional background on the Cauchy problem on other black hole spacetimes, besides references mentioned previously, we refer the reader to the lecture notes [27].

It would be of significant interest to see if appropriately modified versions of the asymptotic completeness results established in this paper can be extended to other matter models of interest, for example, Maxwell’s equations (see [5] for the state of the art on the corresponding Cauchy problem), and possibly most interestingly, to the equations of linearised gravity (see [19]).

It is worth emphasizing that the understanding of scattering theory for a massive scalar field will require significant modifications due to both the existence of eigenvalues in the upper half-plane [66] and to the inadequacy for massive fields of the Friedlander radiation field notion.

We note finally that the classical scattering theory developed here has applications to quantum field theory on curved backgrounds and the phenomenon of Hawking radiation. See [47, 71, 10, 11, 44, 45].

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2. Detailed overview and statements of the main theorems

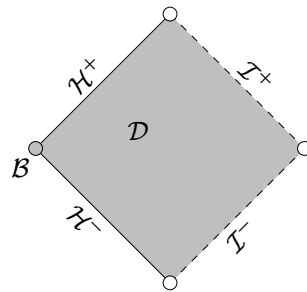
In this section, we will give a more detailed overview of the main results of this paper. We begin in Section 2.1 with the basic setup for our “time-domain” scattering theory. We shall then briefly turn in Section 2.2 to a discussion first of the Schwarzschild $a = 0$ case based on spaces defined by the conserved T -energy, and then of the problem of superradiance in Kerr for $a \neq 0$ which makes this approach impossible. With these preliminaries, we present in Section 2.3 the statements of the main theorems of our scattering theory in the time domain for Kerr in the general subextremal range $|a| < M$. We shall relate this back to the fixed-frequency theory in Section 2.4, stating two additional theorems. In Section 2.5, we make a brief comparison with non-linear scattering problems involving black holes, in particular referring to a recent scattering construction of solutions to the Einstein equations themselves which asymptote in time to the Kerr family [20]. Finally, we shall give in Section 2.6 a section

by section outline of the remainder of the paper, identifying in particular where each of the main theorems is proven.

2.1. The setup for scattering theory in the time-domain

We begin with the basic setup describing our “time-domain” scattering theory in the Kerr black-hole context.

2.1.1. *The exterior region of Kerr.* – We will fix subextremal Kerr parameters $|a| < M$ and consider the Kerr metric $g_{a,M}$ defined on a “domain of outer communication” \mathcal{D} . See Section 3.2 for an explicit representation of this manifold with stratified boundary.



The boundary components \mathcal{H}^\pm correspond to past and future event horizons and meet in the so-called bifurcation sphere \mathcal{B} . (Our convention will be that \mathcal{H}^\pm do *not* contain \mathcal{B} .) Moreover, one can define the two “asymptotic” boundary components future and past null infinity \mathcal{I}^\pm , which, in an auxiliary topology, can indeed be attached to \mathcal{D} as boundary. See Section 4.2.

2.1.2. *Hypersurfaces and forward evolution of smooth data.* – We begin by considering smooth solutions ψ of (2) arising from compactly supported initial data on a suitable hypersurface. We will in fact consider three distinct classes of such data.

When we are only interested in future scattering, it is more natural to focus on solutions parametrised by compactly supported data $(\psi_{\Sigma_0^*}, \psi'_{\Sigma_0^*})$ on a hypersurface

$$\Sigma_0^* = \{t^* = 0\},$$

defined as the level set of a future-horizon penetrating t^* -coordinate. See Section 3.2. Here Σ_0^* is understood as a manifold-with-boundary, so the support of the data can in principle contain the boundary $\Sigma_0^* \cap \mathcal{H}^+$. By general theory, such data give rise to a unique smooth solution ψ of (2) on $\mathcal{R}_{\geq 0} = D^+(\Sigma_0)$. We shall call the map from smooth initial data to solution *forward evolution*:

$$(9) \quad (\psi, \psi') \mapsto \psi.$$

See Proposition 3.6.1.

When we are interested in defining the S -matrix, we need to parameterise solutions ψ by data which determine ψ globally on \mathcal{D} . It is in fact natural to distinguish between two cases. Defining

$$\overset{\circ}{\Sigma} = \{t = 0\}, \quad \overline{\Sigma} = \overset{\circ}{\Sigma} \cup \mathcal{B},$$

where t is the usual Boyer-Lindquist coordinate defined only on the interior of \mathcal{D} , we can consider smooth compactly supported data $(\psi_{\overline{\Sigma}}, \psi'_{\overline{\Sigma}})$ on $\overline{\Sigma}$, or the more restrictive class of

smooth compactly supported data $(\psi_{\overset{\circ}{\Sigma}}, \psi'_{\overset{\circ}{\Sigma}})$ on $\overset{\circ}{\Sigma}$. (The latter, thought of as a special case of the former, must vanish in a neighborhood of \mathcal{B} .) We can now associate in either case a global smooth solution ψ on \mathcal{D} . See Proposition 3.6.2. We will again refer to the map (9) as forward evolution.

The significance of considering the restricted data (i.e., data whose support as a subset of $\overset{\circ}{\Sigma}$ is compact) is that the support of the resulting ψ in \mathcal{D} is disjoint from an open neighborhood of \mathcal{B} . This will be useful technically in defining the backwards map in the frequency domain. It will also facilitate comparison with other results where it has often been this scattering theory that has been implicitly or explicitly considered.

2.1.3. *Radiation fields and horizon traces.* – The most natural formulation of a scattering theory from the point of view of the present problem describes asymptotic states by an appropriate Hilbert space completion (see below) of the future and past radiation fields on \mathcal{I}^\pm augmented by radiation fields on the horizons.

The notion of radiation field along \mathcal{I}^+ is due to Friedlander [38] and in our context is given by the following proposition:

PROPOSITION 1. – *If data (ψ, ψ') are smooth of compact support on Σ_0^* , $\overset{\circ}{\Sigma}$ or $\overline{\Sigma}$, then the solution $r\psi$ extends to a smooth function ϕ defined on \mathcal{I}^+ .*

We shall infer the above as an essentially trivial consequence of the r^p estimates of [24]. See Proposition 3.8.1 and Corollary 4.2.1.

The radiation field on the horizon is just the usual restriction of ψ as a smooth function. Let us introduce the notation $\mathcal{H}_{\geq 0}^+ = \mathcal{R}_{\geq 0}$, and $\overline{\mathcal{H}^+} = \mathcal{H}^+ \cup \mathcal{B}$. Since ψ arising from compactly supported data $(\psi_{\Sigma_0^*}, \psi'_{\Sigma_0^*})$ is only defined on $\mathcal{R}_{\geq 0}$, we may define in this case only $\psi|_{\mathcal{H}_{\geq 0}^+} \doteq \psi|_{\mathcal{R}_{\geq 0}}$. In the case of solutions arising from compactly supported data on $\overset{\circ}{\Sigma}$ and $\overline{\Sigma}$, respectively, ψ is of course defined on all of $\overline{\mathcal{H}^+}$; nonetheless, we shall refer to $\psi_{\mathcal{H}^+} \doteq \psi|_{\mathcal{H}^+}$ in the former case and $\psi_{\overline{\mathcal{H}^+}} \doteq \psi|_{\overline{\mathcal{H}^+}}$ in the latter case. This notation reminds us (cf. the remark at the end of Section 2.1.2 above) that in the former case, the support of $\psi_{\mathcal{H}^+}$ is disjoint from a neighborhood of \mathcal{B} in \mathcal{D} , whereas, in the latter case, the support of $\psi_{\overline{\mathcal{H}^+}}$ may contain \mathcal{B} .

To summarize, forward evolution (9) gives rise to a map on smooth compactly supported initial data

$$(10) \quad (\psi|_{\Sigma_0^*, \overset{\circ}{\Sigma}, \text{ or } \overline{\Sigma}}, \psi'|_{\Sigma_0^*, \overset{\circ}{\Sigma}, \text{ or } \overline{\Sigma}}) \mapsto \psi$$

$$\mapsto (\psi|_{\mathcal{H}_{\geq 0}^+, \mathcal{H}^+, \text{ or } \overline{\mathcal{H}^+}} \doteq \psi|_{\mathcal{H}_{\geq 0}^+, \mathcal{H}^+, \text{ or } \overline{\mathcal{H}^+}}, \phi|_{\mathcal{I}^+} \doteq r\psi|_{\mathcal{I}^+})$$

defined by solving the initial value problem for (2) and restricting to the radiation fields. The forward maps of our scattering theory will be constructed by completing the above map with respect to suitably defined energies.

2.1.4. *Vector fields, energies and asymptotic states.* – The states defining scattering theory are associated to energies which are in turn defined by vector fields.

Recall that a general vector field X defines an energy current $\mathbf{J}^X[\psi]$ and an energy flux

$$(11) \quad \int_{\mathcal{S}} \mathbf{J}^X[\psi]$$

through an arbitrary hypersurface \mathcal{S} . (See Section 3.1.)

For appropriate vector fields X for which (11) is nonnegative, the square root of the expression (11) can in turn be used as a norm to define a space

$$(12) \quad \mathcal{E}_{\Sigma_0^*}^X, \quad \mathcal{E}_{\overset{\circ}{\Sigma}}^X, \quad \mathcal{E}_{\bar{\Sigma}}^X$$

by completion of the set of smooth compactly supported data (ψ, ψ') on $\Sigma_0^*, \overset{\circ}{\Sigma}, \bar{\Sigma}$, respectively. (See Section 8.1.) Recall that “compactly supported on $\overset{\circ}{\Sigma}$ ” is a more restrictive assumption than “compactly supported on $\bar{\Sigma}$ ” and thus $\mathcal{E}_{\overset{\circ}{\Sigma}}^X \subset \mathcal{E}_{\bar{\Sigma}}^X$.

Similarly, the flux (11) defines asymptotic spaces

$$(13) \quad \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^X \oplus \mathcal{E}_{\mathcal{I}^+}^X, \quad \mathcal{E}_{\mathcal{H}^+}^X \oplus \mathcal{E}_{\mathcal{I}^+}^X, \quad \mathcal{E}_{\mathcal{H}^+}^X \oplus \mathcal{E}_{\mathcal{I}^+}^X,$$

via completion of the space of radiation fields arising from (10). Here we have that $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^X$ embeds (non-uniquely) into $\mathcal{E}_{\mathcal{H}^+}^X$, and also, $\mathcal{E}_{\mathcal{H}^+}^X \subset \mathcal{E}_{\mathcal{H}^+}^X$.

In this picture, the problems (a)–(c) of scattering theory translate into finding bijective maps between (12) and (13) induced by the completion of forward evolution (10) of smooth data, for a suitable choice of the vector field X . We will not discuss the construction of wave operators in the spirit of [39, 55] as there is no compelling global “reference dynamics” with which to compare; see [59] for a nice discussion of how to construct the latter if desired.

2.2. The T -energy theory and its limitations

Before turning to our main theorems, we briefly review the Schwarzschild $a = 0$ case, as well as the physical space manifestation of the difficulty of superradiance, discussed previously, which arises upon passing to rotating Kerr with $a \neq 0$.

2.2.1. *The Schwarzschild $a = 0$ case.* – In the Schwarzschild case $a = 0$, the stationary Killing field T is timelike in the interior of \mathcal{D} becoming null on $\mathcal{H}^+ \cup \mathcal{H}^-$ and vanishing on \mathcal{B} . Thus the energy defined by T degenerates pointwise. Nonetheless, the completions $\mathcal{E}_{\Sigma}^T, \mathcal{E}_{\mathcal{H}^+}^T$ and $\mathcal{E}_{\mathcal{I}^+}^T$ define Hilbert spaces and one can obtain a unitary isomorphism

$$(14) \quad \mathcal{E}_{\Sigma}^T \cong \mathcal{E}_{\mathcal{H}^+}^T \oplus \mathcal{E}_{\mathcal{I}^+}^T.$$

In our notation, this is the content of the previously known Schwarzschild scattering theory [33, 35, 59].

We will give our own self-contained treatment in Section 9.6. One obtains with no additional difficulty the alternative unitary isomorphisms $\mathcal{E}_{\Sigma_0^*}^T \cong \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^T \oplus \mathcal{E}_{\mathcal{I}^+}^T$ and $\mathcal{E}_{\bar{\Sigma}}^T \cong \mathcal{E}_{\mathcal{H}^+}^T \oplus \mathcal{E}_{\mathcal{I}^+}^T$.

2.2.2. *The case $a \neq 0$ and the ergoregion.* – Turning to the Kerr case $a \neq 0$, there is now a non-empty subset \mathcal{S} of \mathcal{D} known as the *ergoregion* where T is spacelike. In particular, the energy-fluxes $\int_{\Sigma} \mathbf{J}^T[\psi]$, $\int_{\mathcal{H}^+} \mathbf{J}^T[\psi]$ defined by T fail to be positive definite. This is the physical space origin of the phenomenon of superradiance, discussed in the fixed-frequency theory in the context of (4) and (5).

Part of the conceptual difficulty of formulating a scattering theory in the Kerr case is thus to find the correct notion of asymptotic states which replaces those based on \mathcal{E}^T . At the same time, one must understand what property replaces the notion of unitarity in (14) as a means of quantifying the good properties of the scattering map. We turn now to the statements of the main results of this paper that give a definitive resolution of this problem.

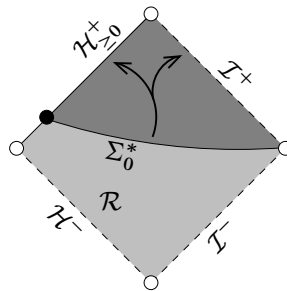
2.3. A scattering theory for Kerr: the main theorems

In this section, we will present in detail the main theorems of our paper concerning physical-space (time-domain) scattering theory for the wave equation (2) on Kerr in the general subextremal case $|a| < M$.

2.3.1. *The N -energy forward map.* – The first candidate replacement for the (degenerate) Schwarzschild T -energy is the so-called N -energy. Here, N is a globally timelike vector field which is T -invariant outside a neighborhood of the bifurcation sphere \mathcal{B} and moreover such that $N = T$ in a neighborhood of \mathcal{I}^+ . The energies (12) associated to this vector field are indeed manifestly positive-definite and pointwise non-degenerate.

The first main theorem defines asymptotic states for all solutions arising from finite N -energy data on the hypersurface Σ_0^* , i.e., in the notation (12), for all solutions parametrised by $\mathcal{E}_{\Sigma_0^*}^N$.

THEOREM 1. – *Forward evolution (10) with data on Σ_0^* extends to a bounded map $\mathcal{F}_+ : \mathcal{E}_{\Sigma_0^*}^N \rightarrow \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N \oplus \mathcal{E}_{\mathcal{I}^+}^T$.*



See Theorem 8.2.1. (Note that $\mathcal{E}_{\mathcal{I}^+}^N = \mathcal{E}_{\mathcal{I}^+}^T$.) For the hard analysis behind the above, the proof relies in particular on a uniform boundedness statement for the energy $\int_{\Sigma_s^*} \mathbf{J}^N$ through a foliation Σ_s^* defined by future-translating Σ_0^* by the flow of T , as well as a weak decay statement, both of which follow from the results of [30] mentioned previously, here quoted as Theorem 3.7.1.

2.3.2. *A blue-shift instability and the non-existence of an N -energy backwards map.* – Satisfactory though the forward theory may be, it turns out that the above N -energy is ill-suited for defining the asymptotic states of a scattering theory. The fundamental origin of this is the red-shift effect on the horizon (so favorable for controlling forward evolution!), which for backwards evolution is now seen as a blue-shift. See [20] and Section 3.1.2 of Sbierski [64]. It turns out that one can show explicitly that the map of Theorem 1 fails to be surjective:

THEOREM 2. – *Already in the Schwarzschild $a = 0$ case, the map \mathcal{F}_+ of Theorem 1 fails to be surjective.*

It follows that there does not exist even a *one-sided* inverse \mathcal{B}_- satisfying $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$; thus, *existence of scattering states* (cf. (a)) does *not* hold in the N -theory. (As we shall see in Section 2.3.4, the above map \mathcal{F}_+ is however injective.)

The above theorem in fact applies in the general Kerr case, ⁽³⁾ but in the present paper, we shall give an elementary purely physical space proof for the case $a = 0$ which exploits monotonicity satisfied by the spherical mean under spatial evolution. Though essentially independent of the rest of the paper, the precise statement proven (Theorem 11.1) is deferred to the end (Section 11), so that it can be interpreted both as a non-surjectivity result with respect to our N -energy scattering theory (Corollary 11.4) and also constructively (Corollary 11.1) using Theorem 4 of our V -energy scattering theory to be discussed below. Let us already remark, however, that the non-surjectivity statement we obtain in Corollary 11.4 is more precise than what we have just stated above. We elaborate briefly below.

First let us note that with the notations of the present paper, the considerations of Section 1.1.6.1 of [20] show that by introducing sufficiently high exponential weights in the spaces defining the scattering data, i.e., considering the spaces $\mathcal{E}_{\mathcal{H}^+}^{e^{\alpha v} N}$ and $\mathcal{E}_{\mathcal{I}^+}^{e^{\alpha u} T}$, then there indeed exists a bounded one-sided inverse

$$(15) \quad \mathcal{B}_- : \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^{e^{\alpha v} N} \oplus \mathcal{E}_{\mathcal{I}^+}^{e^{\alpha u} T} \rightarrow \mathcal{E}_{\Sigma_0^*}^N$$

such that $\mathcal{F}_+ \circ \mathcal{B}_- = \text{id}$. Thus, we do have existence of a *restricted class* of future scattering states.

With this setting, our Theorem 11.1 in fact shows (see Corollary 11.4) that $e^{\alpha v}$ above *cannot* be replaced by $|v|^p$ no matter how large p is taken, i.e., the map \mathcal{F}_+ of Theorem 1 is not surjective as a map $\mathcal{F}_+^{-1}(\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^{|v|^p N} \oplus \{0\}) \rightarrow \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^{|v|^p N} \oplus \{0\}$. The question of precise characterization of the range of \mathcal{F}_+ remains open. We shall return to this issue in Section 2.5.

2.3.3. *The V -energy forward map.* – To define a forward map which one can indeed hope to show is invertible, we must pass to a degenerate energy class which does not see the red-shift at the horizon.

Recall that $g_{M,a}$ admits an additional Killing vector field Φ corresponding to axisymmetry. Although for $a \neq 0$, the vector field T fails to be globally timelike in the interior of \mathcal{D} , the span of T and Φ does form a timelike plane, and the Killing combination $K = T + \omega_+ \Phi$ is timelike in a neighborhood of \mathcal{H}^+ , becoming null on \mathcal{H}^+ itself. (Note that if $a = 0$, then $K = T$, but if $a \neq 0$, then K is spacelike away from the axis of symmetry near

⁽³⁾ See our more recent [31] for the more general case.

\mathcal{I}^+ .) We define a T -invariant vector field V with the property that $V = K$ near \mathcal{H}^+ and $V = T$ near \mathcal{I}^+ and V is timelike in the interior of \mathcal{D} . The energy associated to this vector field is manifestly non-negative definite, though degenerate analogous to the T -energy in the Schwarzschild case. In the case $a \neq 0$, there is necessarily a region where V fails to be Killing.

Our third main theorem is a degenerate V -energy analog of Theorem 1 given by

THEOREM 3. – *Forward evolution (10) extends to bounded maps*

$$\mathcal{F}_+ : \mathcal{E}_{\Sigma_0^*}^V \rightarrow \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T, \quad \mathcal{F}_+ : \mathcal{E}_{\Sigma}^V \rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T, \quad \mathcal{F}_+ : \mathcal{E}_{\Sigma}^V \rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T.$$

See Theorems 8.2.1, 8.2.3 and 8.2.4. The above theorem requires a new version of the boundedness part of Theorem 3.7.1 of [30], depending only on the degenerate energy. This result, which is of independent interest, is stated as Theorem 7.1 and proven in Section 7. The reader can compare with the higher-order weighted boundedness result of Andersson and Blue [4] for the $|a| \ll M$ case, whose degenerate horizon weights are similar to the V -energy. Note that it is an immediate consequence of finite-in-time energy estimates and Theorem 1 that upon replacing both V and K with N , the analogs of all three of the maps in Theorem 3 are bounded. (Recall, however, that N is not T -invariant in a neighborhood of the bifurcation sphere. See Remark 3.4.2 for an expression for $\mathcal{E}_{\Sigma_0^*}^N$.) Finally, we note that it is the second of the maps in Theorem 3 which corresponds to the classical inverse wave operators.

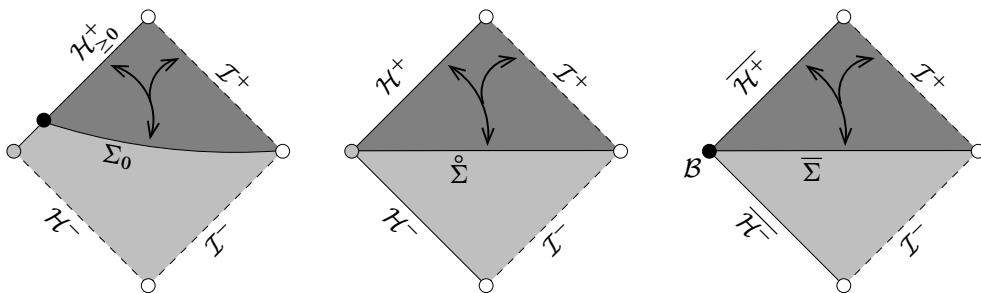
Let us note that the proof of Theorem 7.1 will require us to revisit the quantitative study of the o.d.e. (1) at fixed frequency, on which the original results of [30] were based, in particular in the form of Theorem 6.3.1, and a new result, Theorem 6.2.1, which we will prove here by adapting the proof of [30]. In particular, from these statements, one can already infer novel results on the fixed-frequency scattering; we defer specific discussion of these till Section 2.4.

2.3.4. *The V -energy backwards map.* – Our degenerate-energy class is indeed suitable to construct a bounded inverse of the map of Theorem 3 and thus infer the existence of a satisfactory scattering theory satisfying (a)–(c).

THEOREM 4. – *There exist bounded maps*

$$(16) \quad \mathcal{B}_- : \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\Sigma_0^*}^V, \quad \mathcal{B}_- : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\Sigma}^V, \quad \mathcal{B}_- : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\Sigma}^V,$$

which are two-sided inverses to the maps of Theorem 3, i.e., $\mathcal{B}_- \circ \mathcal{F}_+ = \text{Id}$ and $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$.



See Theorems 9.2.1, 9.1.1 and 9.3.1. As explained in Section 1.1.3, it is the existence of the map \mathcal{B}_- which gives the *existence* of future scattering states (a), the injectivity of \mathcal{F}_+ which gives the *uniqueness* of future scattering spaces (b), and the surjectivity of \mathcal{B}_- corresponds to *asymptotic completeness* of future scattering states (c). We note that the map of Theorem 1 is in fact the restriction of the first map of Theorem 3. Thus a corollary of the above is that the map \mathcal{F} of Theorem 1 is injective. In this sense, for the N -energy theory, one still has uniqueness (b)–but *not* existence (a)!–of scattering states. Cf. the discussion of the ill-posed problems of Section 2.3.8.

Let us note that in our proof, we construct \mathcal{B}_- with the help of the frequency domain, again using our o.d.e. result Theorem 6.2.1, together with a decomposition first given in [30] and which exploits the fact that the span of T and Φ is timelike (See Section 9.1.2), to give us the quantitative statement of boundedness. Due to this use of the frequency domain, it is in fact the map $\mathcal{B}_- : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\Sigma}^V$ which is most natural to construct first.

It is perhaps worth explicitly noting that even to show the existence of \mathcal{B}_- , we require appeal to an o.d.e. result which in essence already embodies the totality of the quantitative decay statement for the Cauchy problem (2). This should emphasize how intricately tied in the Kerr case the problem of boundedness is to the problem of quantitative decay. This is in contrast to many usual problems in scattering theory where “existence of scattering states” (cf. (a)) is a relatively soft result, which can be proven independently of the structure necessary to obtain asymptotic completeness-type statements.

2.3.5. *Existence and boundedness of the S-matrix.* – We will base our discussion here on the scattering theory associated to $\overset{\circ}{\Sigma}$ or $\overline{\Sigma}$. First, note that applying a discrete isometry of \mathcal{D} which interchanges the future and past of $\overset{\circ}{\Sigma}$, we infer analogously to Theorems 3 and 4 the existence of bounded *past* forward maps,

$$\mathcal{F}_- : \mathcal{E}_{\overset{\circ}{\Sigma}}^V \rightarrow \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T, \quad \mathcal{F}_- : \mathcal{E}_{\overline{\Sigma}}^V \rightarrow \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T,$$

and the corresponding bounded two-sided inverses

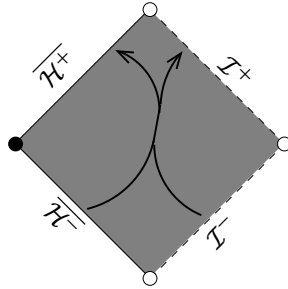
$$\mathcal{B}_+ : \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\overset{\circ}{\Sigma}}^V, \quad \mathcal{B}_+ : \mathcal{E}_{\overline{\mathcal{H}^-}}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\overline{\Sigma}}^V.$$

We thus have both existence and uniqueness for *past* scattering states as well as *past* asymptotic completeness.

The following is then an immediate corollary

THEOREM 5. – *The composition of $\mathcal{S} = \mathcal{F}_+ \circ \mathcal{B}_+$ defines bounded invertible maps*

$$(17) \quad \mathcal{S} : \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T, \quad \mathcal{S} : \mathcal{E}_{\overline{\mathcal{H}^-}}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\overline{\mathcal{H}^+}}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T.$$



The boundedness $\|\mathcal{S}\| \leq C$ of the map \mathcal{S} in the operator norm should be viewed as the quantitative replacement for the usual unitarity property.

2.3.6. *A physical space theory of superradiant reflection.* – Given the scattering map \mathcal{S} , we can now give an account of superradiant reflection in physical space, i.e., in the “time domain”.

Recall the standard physical set-up: One wishes to study the scattering of waves with no ingoing contribution from the past event horizon \mathcal{H}^- and we are interested only in the part of the wave reflected to future null infinity \mathcal{I}^+ . We thus pass from \mathcal{S} to the transmission map \mathcal{T} and reflection map \mathcal{R} defined by

$$(18) \quad \mathcal{T} = \pi_{\mathcal{E}_{\mathcal{H}^+}^K} \circ \mathcal{S}|_{\{0\} \oplus \mathcal{E}_{\mathcal{I}^-}^T}, \quad \mathcal{R} = \pi_{\mathcal{E}_{\mathcal{I}^+}^T} \circ \mathcal{S}|_{\{0\} \oplus \mathcal{E}_{\mathcal{I}^-}^T}$$

where

$$\pi_{\mathcal{E}_{\mathcal{I}^+}^T} : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\mathcal{I}^+}^T, \quad \pi_{\mathcal{E}_{\mathcal{H}^+}^K} : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\mathcal{H}^+}^K$$

are the natural projections. Note that this map does not depend on whether we consider the domain of \mathcal{S} to be either of the choices in (17). The map

$$\mathcal{R} : \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\mathcal{I}^+}^T$$

takes an asymptotic state corresponding to an incoming wave packet supported solely on past infinity \mathcal{I}^- (i.e., with no incoming radiation from \mathcal{H}^-) and maps it to the part of the asymptotic state which is reflected to future null infinity \mathcal{I}^+ (i.e., projecting out the part transmitted to the future horizon \mathcal{H}^+). Similarly, the map

$$\mathcal{T} : \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\mathcal{H}^+}^K$$

takes an asymptotic state corresponding to an incoming wave packet supported solely on past infinity \mathcal{I}^- and maps it to the part of the asymptotic state which is transmitted to the future event horizon \mathcal{H}^+ .

Since $\mathcal{S}|_{\{0\} \oplus \mathcal{E}_{\mathcal{I}^-}^T} = \mathcal{T} \oplus \mathcal{R}$, the boundedness of \mathcal{S} above immediately yields the strictly weaker statement

THEOREM 6. – *The reflection and transmission maps \mathcal{R} and \mathcal{T} are bounded, i.e., $\|\mathcal{R}\|, \|\mathcal{T}\| \leq C$.*

See the first statement of Theorem 10.1.1. In view of the relation with the fixed-frequency theory to be discussed in Section 2.4 below, we have

$$(19) \quad \sup_{(\omega, m, \ell)} |\mathfrak{R}(\omega, m, \ell)| = \|\mathcal{R}\|,$$

and thus, a posteriori, Theorem 6 gives in particular (7). We note however that in the logic of the proof, we will have essentially already used (7) in proving the boundedness of both the maps \mathcal{F}_+ and \mathcal{B}_- .

Let us here already mention a further application of the relationship (19) to our physical-space scattering theory. First, note that general soft o.d.e. theory is sufficient to show that the reflection coefficient satisfies $|\mathfrak{R}(\omega, m, \ell)| > 1$ for any superradiant frequency triple (see Corollary 5.3.1). Thus, one immediately obtains from (19) the statement

THEOREM 7. – *For $a \neq 0$, the reflection map \mathcal{R} has norm strictly greater than 1, i.e., $\|\mathcal{R}\| > 1$.*

See the second statement of Theorem 10.1.1. The above theorem can be viewed as the definitive physical-space interpretation of the phenomenon of superradiant reflection. To connect with the numerical setting often studied (e.g., [6, 52]) in which it is difficult to implement past scattering data on \mathcal{I}^- , we will extract in addition the following somewhat less natural statement concerning Cauchy data on $\mathring{\Sigma}$ via a density argument (see Theorem 10.1.2): *There exists a smooth solution ψ with the property that its T -energy flux through \mathcal{I}^+ is greater than its T -energy flux through $\mathring{\Sigma}$ and moreover, the support of the solution on $\mathring{\Sigma}$ is compact and can be made arbitrarily close to spatial infinity.* Cf. [37]. This addresses in particular some questions raised in [52].

2.3.7. Pseudo-unitarity and non-superradiant unitarity. – As we have already discussed, when $a \neq 0$ one does not have a unitary scattering theory; however, one still expects to recover the conservation of the indefinite inner product associated to the T -energy, provided this inner product is finite.

The T -energy is *not* finite on the full domain of the scattering matrix \mathcal{S} of (17). It is, however, finite if one for instance restricts to past scattering data supported only on \mathcal{I}^- . Recalling the notation (18), one statement of “pseudo-unitarity” is then captured by the following theorem.

THEOREM 8. – *The map $\mathcal{T} \oplus \mathcal{R}$ preserves the T -energy:*

$$\int_{\mathcal{H}^+} \mathbf{J}_\mu^T [\mathcal{T}\phi] n_{\mathcal{H}^+}^\mu + \int_{\mathcal{I}^+} \mathbf{J}_\mu^T [\mathcal{R}\phi] n_{\mathcal{I}^+}^\mu = \int_{\mathcal{I}^-} \mathbf{J}_\mu^T [\phi] n_{\mathcal{I}^-}^\mu.$$

In particular, if the right hand side above is bounded, then the first term on the left hand side, which is unsigned, is integrable. See Theorem 10.2.1.

Note that in the context of quantum field theory on curved backgrounds, one is also interested in the conservation of particle current under the map $\mathcal{T} \oplus \mathcal{R}$; see [71]. A version of Theorem 8 holds also in this setting. See Remark 10.2.3.

If we restrict to past scattering data on $\mathcal{H}^- \cup \mathcal{I}^-$ that are *non-superradiant*, i.e., supported in frequency space outside the superradiant range, then our scattering map \mathcal{S} will indeed be

unitary in the usual sense. For this we define Hilbert spaces $\mathcal{E}_{\mathcal{H}^\pm}^{T, \mathfrak{h}} \oplus \mathcal{E}_{\mathcal{I}^\pm}^{T, \mathfrak{h}}$ by the completion under the inner product

$$(20) \quad \langle (\psi_1, \phi_1), (\psi_2, \phi_2) \rangle = \int_{-\infty}^{\infty} \sum_{m\ell} \left[\omega(\omega - \omega_+ m) \operatorname{Re}(\hat{\psi}_1 \overline{\hat{\psi}_2}) + \omega^2 \operatorname{Re}(\hat{\phi}_1 \overline{\hat{\phi}_2}) \right]$$

of scattering data whose Fourier transforms are supported in the *non-superradiant range*

$$\{(\omega, m, \ell) : \omega(\omega - \omega_+ m) > 0\}.$$

We then have

THEOREM 9. – *The restriction of the first map of (17) extends to a unitarity isomorphism $\mathcal{S} : \mathcal{E}_{\mathcal{H}^-}^{T, \mathfrak{h}} \oplus \mathcal{E}_{\mathcal{I}^-}^{T, \mathfrak{h}} \rightarrow \mathcal{E}_{\mathcal{H}^-}^{T, \mathfrak{h}} \oplus \mathcal{E}_{\mathcal{I}^-}^{T, \mathfrak{h}}$ with respect to the positive definite inner product (20).*

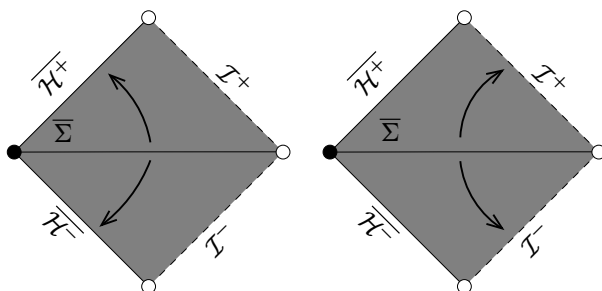
See Theorem 10.2.2. Note that the above theorem retrieves in particular the unitarity of the first map of (17) in the Schwarzschild case $a = 0$ (which we in fact provide an independent treatment of; see Theorem 9.6.2) as well as the unitarity of \mathcal{S} restricted to axisymmetric data in the full $|a| < M$ case.

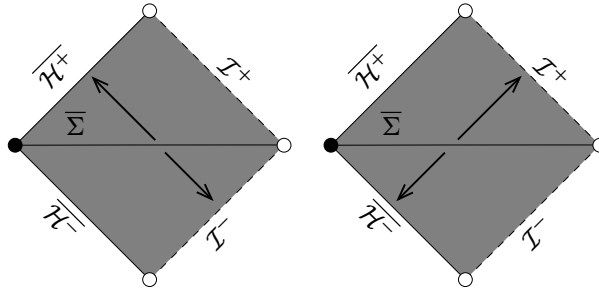
2.3.8. *Uniqueness of scattering states for ill-posed scattering data.* – Finally, we note that our scattering theory allows us to make the following injectivity statements which can be understood as statements just of *uniqueness* of scattering states (cf. (b)) for scattering data determined on any of the four “ill-posed” pairs of asymptotic boundaries $\mathcal{H}^+ \cup \mathcal{H}^-$, $\mathcal{I}^+ \cup \mathcal{I}^-$, $\overline{\mathcal{H}^+} \cup \mathcal{I}^-$ and $\overline{\mathcal{H}^-} \cup \mathcal{I}^+$.

THEOREM 10. – *The maps*

$$\begin{aligned} \mathcal{F} : \mathcal{E}_{\Sigma}^V &\rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{H}^-}^K, & \mathcal{F} : \mathcal{E}_{\Sigma}^V &\rightarrow \mathcal{E}_{\mathcal{I}^+}^T \oplus \mathcal{E}_{\mathcal{I}^-}^T, \\ \mathcal{F} : \mathcal{E}_{\Sigma}^V &\rightarrow \mathcal{E}_{\overline{\mathcal{H}^+}}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T, & \mathcal{F} : \mathcal{E}_{\Sigma}^V &\rightarrow \mathcal{E}_{\overline{\mathcal{H}^-}}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \end{aligned}$$

are all injective.





See Corollary 10.3.1. Together with the previous results, the above implies that finite V -energy solutions are uniquely determined by their fluxes to any pair of the set $\{\mathcal{H}^+, \mathcal{H}^-, \mathcal{I}^+, \mathcal{I}^-\}$. In contrast, however, to the forward maps of Theorem 3, it follows already from general local ill-posedness type results for the wave equation (see e.g., the classic textbook [42]) that *the above maps \mathcal{F} are not surjective*. Thus, one does not have the analog of “existence of scattering states” (cf. (a)) for scattering states parameterized as above. ⁽⁴⁾

2.4. Applications to fixed frequency scattering theory

As we have discussed, the proofs of our theorems of physical space scattering theory required us to revisit our quantitative fixed frequency study of the o.d.e. (1) conducted in [30]. Thus, along the way, we have in fact obtained new results for the fixed-frequency scattering theory initiated by Chandrasekhar [16], as well as a precise connection of the two through the scattering matrix \mathcal{S} . We collect these statements in this section.

2.4.1. *Uniform boundedness of the coefficients \mathfrak{R} and \mathfrak{T} .* – We begin with the statement of the uniform boundedness of the transmission and reflection coefficients.

THEOREM 11. – *The reflection and transmission coefficients as normalized in (3) are uniformly bounded over all frequencies:*

$$(21) \quad \sup_{(\omega, m, \ell)} |\mathfrak{R}(\omega, m, \ell)| \leq C, \quad \sup_{(\omega, m, \ell)} |\mathfrak{T}(\omega, m, \ell)| \leq C.$$

We in fact have a statement for the complete set of coefficients where we also allow for waves normalized to the past horizon. See Theorem 6.2.2.

We will infer the above theorem as an immediate corollary of our o.d.e. estimate Theorem 6.2.1, which itself is an easy adaptation of an estimate of our previous [30]. We emphasize again that this result requires in particular appeal to the real-mode stability theorem of [67].

To connect with the pioneering heuristic work of Starobinski [68], we may define the following constant depending only on the Kerr parameters

$$S(a, M) \doteq \sup_{(\omega, m, \ell)} |\mathfrak{R}(\omega, m, \ell)|,$$

⁽⁴⁾ This is of course in sharp distinction to the fixed-frequency theory, for which “existence of scattering states” associated to $\mathcal{H}^+ \cup \mathcal{H}^-$ and $\mathcal{I}^+ \cup \mathcal{I}^-$, respectively, corresponds precisely to the existence and linear independence of the pairs $U_{\text{hor}}, \bar{U}_{\text{hor}}$ (or alternatively $U_{\text{inf}}, \bar{U}_{\text{inf}}$) described in the beginning of this introduction, on which the whole theory is based.

and by Theorem 11, together with the soft statement Corollary 5.3.1 (mentioned already in the context of Theorem 7), we have

$$1 < S(a, M) < \infty, \quad 0 < |a| < M.$$

(For $a = 0$ we have of course $S(0, M) \leq 1$ and in fact, by an easy high angular frequency ℓ estimate given by Corollary 6.4.1, $S(0, M) = 1$.) It would be very interesting from the point of view of applications, following [68], to find effective upper and lower bounds for $S(a, M)$, and to understand in particular the limit

$$(22) \quad \lim_{|a| \rightarrow M} S(a, M).$$

2.4.2. *Connection with physical-space theory.* – The full scattering map \mathcal{S} defined in Section 2.3.5 can be represented as a generalized Fourier transform involving the transmission and reflection coefficients \mathfrak{T} and \mathfrak{R} defined via (3), together with coefficients $\tilde{\mathfrak{T}}$ and $\tilde{\mathfrak{R}}$ associated to analogously defined solutions U of (1) normalized to the past event horizon \mathcal{H}^- . So as not to define the latter here, for convenience, let us simply state the relations for the physical space transmission and reflection maps \mathcal{T} and \mathcal{R} defined in Section 2.3.6:

THEOREM 12. – *We may represent*

$$\mathcal{R}[\phi] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} a_{\mathcal{I}^-} \mathfrak{R} e^{-it\omega} e^{im\phi} S_{m\ell}(a\omega, \cos\theta) d\omega$$

and

$$\mathcal{T}[\phi] = -\frac{1}{\sqrt{4M\pi r_+}} \int_{-\infty}^{\infty} \sum_{m\ell} \left(\frac{\omega}{\omega - \omega_{+m}} \right) a_{\mathcal{I}^-} \mathfrak{T} e^{-it\omega} e^{im\phi} S_{m\ell}(a\omega, \cos\theta) d\omega.$$

Here

$$-i\omega a_{\mathcal{I}^-} \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \partial_t \phi e^{it\omega} e^{-im\phi} S_{m\ell}(a\omega, \cos\theta) \sin\theta dt d\theta d\phi.$$

In particular, (19) holds.

See Theorem 9.5.3 for the full statement concerning \mathcal{S} .

In fact, a posteriori, in view of Theorem 9.5.3, the statement of Theorem 6.2.2 is equivalent to the boundedness of the map \mathcal{S} of Theorem 6. We note in contrast that the boundedness of the maps \mathcal{F}_+ and \mathcal{B}_+ individually (already asserted) does *not* have an obvious natural interpretation purely in terms of the formal fixed frequency scattering theory. Similarly, the boundedness statement of Theorem 1 (and the boundedness statement of [30] quoted here as Theorem 3.7.1 which concerns boundedness through a spacelike foliation) cannot be directly interpreted purely in terms of the formal fixed frequency scattering theory. These are all distinct manifestations of ways that the phenomenon of “superradiance” allowed by the presence of an ergoregion can be quantified. As with the question of the finiteness of (22), it is a completely open question which if any of these boundedness statements survives in the extremal case $|a| = M$. See [7].

2.5. Nonlinear problems and scattering constructions of dynamical black holes

We make a few comments on scattering theory for non-linear generalizations of (2). Perhaps the ultimate nonlinear such generalization is provided by the Einstein vacuum equations

$$(23) \quad \text{Ric}(g) = 0$$

themselves, where the background geometry is now itself unknown.

The problem of characterizing *all* “admissible” solutions by appropriate asymptotic states may turn out to be too ambitious for equations as nonlinear as (23). The mere constructing of some, however, in the spirit of the map (15), can serve as an important way of obtaining non-trivial examples of solution spacetimes which cannot otherwise easily be inferred to exist. A result in that direction has recently been provided by

THEOREM ([20]). – *Consider asymptotic data on $\mathcal{H}^+ \cup \mathcal{I}^+$ for the Einstein vacuum equations (23), decaying towards Kerr data corresponding to $g_{a,M}$ with $|a| \leq M$ at a sufficiently fast exponential rate. Then there exists a vacuum spacetime (\mathcal{M}, g) attaining the data.*

The spacetimes (\mathcal{M}, g) constructed in the above are in fact the first known examples of dynamical vacuum black holes settling down to Kerr.

The above theorem can be thought of as a non-linear analog of the map (15) (for energies which have additional weights in r however!). In fact, proving the above requires capturing a complicated r^p -hierarchy of decay of various components of curvature which in turn allows one to identify a null condition in the implicit non-linearities in (23). (We note in contrast that without additional special structure, the analog of the above theorem does *not* hold even say for the general scalar semilinear equation of the form $\square_g \psi = Q(\nabla \psi, \nabla \psi)$.) We refer the reader to [20].

In the context of our present paper, let us simply remark that the degenerate V -scattering theory developed here, together with the blow-up result Theorem 11.1 and the upcoming [21], gives further support to the following conjecture of [20]:

CONJECTURE ([20]). – *Consider asymptotic data on $\mathcal{H}^+ \cup \mathcal{I}^+$ as above but which decay to $g_{a,M}$ only at a sufficiently fast inverse polynomial rate. Then there exists a vacuum spacetime (\mathcal{M}, g) attaining the data. For generic such data, \mathcal{H}^+ is a “weak null singularity” across which the metric extends continuously but with Christoffel symbols which fail to be locally square integrable.*

See the discussion of Section 11.3 and [54].

2.6. Outline

The logic of the paper will depart slightly from the order we have presented the main results above. We thus close this introduction with a brief section by section outline of the contents of the remainder of the paper, highlighting in bold where each of the main theorems above are actually proven. (More detailed outlines will be given in the body of the paper at the beginning of each individual section.)

In Section 3, we briefly review the structure of the Kerr spacetime, introduce various conventions, and quote some previous results on forward evolution which will be important,

in particular, we will state general well-posedness results (Propositions 3.6.1–3.6.4), precise versions of our previous boundedness and integrated local decay results of [30] (quoted as Theorem 3.7.1 and the higher order Theorem 3.7.2) as well as a general r^p weighted estimate (Proposition 3.8.1) which we will derive from [24].

In Section 4, we define and establish some basic properties of the radiation fields and energy fluxes along $\mathcal{H}_{\geq 0}^+$ (or $\overline{\mathcal{H}^+}$) and \mathcal{I}^+ for solutions ψ to (2) arising from smooth initial data along Σ_0^* (or $\overline{\Sigma}$) which are compactly supported. The main result is Proposition 4.2.1, which is the precise statement of Proposition 1 above.

In Section 5, we first review Carter’s separation of variables for the wave equation and then define the *radial o.d.e.*, recalling also some results from its basic asymptotic analysis. This will allow us to define the reflection and transmission coefficients (Definition 5.3.2), deduce fixed-frequency superradiant amplification in the form of Corollary 5.3.1, and define the so-called *microlocal radiation fields* (Definition 5.4.1) and *fluxes* (Definition 5.4.2).

In Section 6 we establish various estimates for the radial o.d.e. and give some useful applications. We start by proving Theorem 6.2.1, an estimate for general solutions to the homogeneous radial o.d.e. The proof of Theorem 6.2.1 will heavily rely on our o.d.e. estimates from [30]. Next, in Section 6.3 we establish an important estimate for U_{hor} in the superradiant regime (Proposition 6.3.1). We then use related ideas in Section 6.4 to prove Proposition 6.4.1 which states that for fixed ω and m , \mathfrak{T} vanishes in the large- ℓ limit. In Section 6.5 we show that for each m and ℓ , the reflection coefficient \mathfrak{R} is not identically 0 as a function of ω . In Section 6.6 we prove Proposition 6.6.1, which is the microlocal version of the r^p estimates of [24] (cf. Proposition 3.8.1). The goal of Section 6.7 is to prove Proposition 6.7.1 which establishes uniform estimates, over all frequency parameters, for the rate of convergence of solutions to the radial o.d.e. to their microlocal radiation fields. Finally, in Section 6.8 we prove Propositions 6.8.1 and 6.8.2 which establish that for suitable solutions ψ to the wave equation, the microlocal radiation fields are essentially the Fourier transform of the physical space radiation fields defined in Section 4.

In Section 7, we prove Theorem 7.1, the statement that the total flux to null infinity \mathcal{I}^+ and the degenerate K -flux to the horizon \mathcal{H}^+ of a solution ψ to the wave equation may be controlled by the V -energy of ψ along Σ_0^* . The theorem is stated in Section 7.1, after which the reader impatient to proceed to the construction of our scattering theory may skip to Section 8 below. The proof of Theorem 7.1, which occupies Sections 7.3 is a modification of the proof of Theorem 3.7.1 quoted from [30]. In a brief aside in Section 7.2, we shall state Theorem 7.2, which is the full degenerate-energy analog of our results of [30], quoted as Theorem 3.7.1 above. We emphasize that Theorem 7.2 is not in fact necessary for the rest of the paper, and we defer its proof to Section 9.4, where we can make use of the backwards maps of our scattering theory.

In Section 8, we introduce the \mathcal{E}_{Σ}^V and \mathcal{E}_{Σ}^N spaces, etc., and define the various “forward” maps and establish their boundedness. Theorem 8.2.1, the precise version of Theorem 1, is independent of Section 7, as it relies directly on Theorem 3.7.1 of [30]. Theorem 8.2.2, on the other hand, which together with its corollaries Theorems 8.2.3 and 8.2.4 embodies the precise version of Theorem 3, uses in a fundamental way Theorem 7.1.

In Section 9, we prove first Theorem 9.1.1, then Theorem 9.2.1, then Theorem 9.3.1. This obtains all statements in Theorem 4. As an aside in Section 9.4, we obtain the proof

of Theorem 7.2 referred to above. Next, we construct the “scattering” map \mathcal{S} and show that it is a bounded invertible map from data along $\mathcal{H}^- \cup \mathcal{I}^-$ to data along $\mathcal{H}^+ \cup \mathcal{I}^+$ (Theorem 9.5.2, the precise version of Theorem 5). We then prove Theorem 9.5.3 which establishes a formula for the scattering map \mathcal{S} explicitly exhibiting the roles of the reflexion and transmission coefficients. This formula will in particular establish the relationship between physical space and fixed frequency scattering theories embodied by Theorem 12. Finally, as an additional aside in Section 9.6, we give an alternative self-contained treatment for the Schwarzschild case where it is possible to exploit purely physical space arguments.

In Section 10, we begin by interpreting our scattering results for the reflection operator \mathcal{R} . Theorem 10.1.1 combines the statements of Theorems 6 and 7. We also infer the related Theorem 10.1.2. Following this, we study the pseudo-unitarity properties of \mathcal{S} and prove the corresponding Theorems 10.2.1 and 10.2.2 (cf. Theorems 8 and 9). Finally, our “uniqueness of improper scattering states” results are stated as Theorem 10.3.1, giving Theorem 10.

In Section 11, we prove Theorem 11.1, the statement that solutions ψ of (2) on Schwarzschild whose radiation fields on the horizon \mathcal{H}^+ have a precise polynomial tail and whose radiation fields on \mathcal{I}^+ vanish must necessarily have infinite N -energy on the hypersurface Σ_0^* . This statement can be understood independently of the results concerning our scattering maps, and indeed, Sections 11.1 and 11.2 can be read independently of the rest of the paper. In Sections 11.3 and 11.4 we will then return to the scattering framework of our paper. We first use the backwards map of our V -scattering theory to infer the existence (See Corollary 11.1) of solutions satisfying the assumptions of Theorem 11.1. Finally, we infer Theorem 2 as Corollary 11.4.

For the convenience of the reader, we have provided an index of notation at the end of the paper.

3. Preliminaries

We begin in this section with various preliminaries.

After reviewing our notations for energy currents associated to vector fields in Section 3.1, we will define carefully in Section 3.2 the ambient spacetime \mathcal{D} (and related subsets) on which we will consider the Kerr metric $g_{a,M}$ for subextremal values $|a| < M$. Our conventions for constants depending only on the Kerr parameters will be reviewed in Section 3.3. These follow our conventions from [30]. Some auxiliary useful vector fields will be presented in Section 3.4.

It will be useful to define a hyperboloidal-type foliation S_τ of \mathcal{R} and we shall do this in Section 3.5. The form of the T energy-flux through such a foliation is recorded in Lemma 3.5.1. Section 3.6 states general well posedness results (Propositions 3.6.1–3.6.4) for the wave equation (2) on the Kerr exterior. We shall then quote our boundedness and integrated decay statement from [30] in Section 3.7, as Theorem 3.7.1 and the higher order Theorem 3.7.2. The foliation of Section 3.5 will then allow us in Section 3.8 to easily quote the r^p hierarchy of estimates (introduced in [24]) in the form of Proposition 3.8.1 and the higher order Proposition 3.8.2.

3.1. Currents

Given a general Lorentzian manifold (\mathcal{M}, g) , let Ψ be a sufficiently regular complex function. We define

$$\mathbf{T}_{\mu\nu}[\Psi] \doteq \operatorname{Re} \left(\partial_\mu \Psi \overline{\partial_\nu \Psi} \right) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \operatorname{Re} \left(\partial_\alpha \Psi \overline{\partial_\beta \Psi} \right).$$

Given a sufficiently regular vector field X on \mathcal{M} , we define the currents

$$\begin{aligned} \mathbf{J}_\mu^X[\Psi] &= \mathbf{T}_{\mu\nu}[\Psi] X^\nu, \\ \mathbf{K}^X[\Psi] &= \mathbf{T}_{\mu\nu}[\Psi] \nabla^\mu X^\nu = \frac{1}{2} \mathbf{T}_{\mu\nu}^{(X)} \pi^{\mu\nu}, \\ \mathcal{E}^X[\Psi] &= -\operatorname{Re} \left((\square_g \Psi) \overline{X^\nu \Psi_{,\nu}} \right). \end{aligned}$$

Here ${}^{(X)}\pi^{\mu\nu} \doteq \nabla^\mu X^\nu + \nabla^\nu X^\mu$ is the deformation tensor of X . In particular, $\mathbf{K}^X = 0$ where X is Killing.

Recall the fundamental identity:

$$\nabla^\mu \mathbf{J}_\mu^X[\Psi] = \mathbf{K}^X[\Psi] - \mathcal{E}^X[\Psi].$$

Then the divergence identity between two homologous spacelike hypersurfaces S^- , S^+ , bounding a region \mathcal{C} , with S^+ in the future of S^- , yields

$$(24) \quad \int_{S^+} \mathbf{J}_\mu^X[\Psi] n_{S^+}^\mu + \int_{\mathcal{C}} (\mathbf{K}^X[\Psi] - \mathcal{E}^X[\Psi]) = \int_{S^-} \mathbf{J}_\mu^X[\Psi] n_{S^-}^\mu,$$

where n_{S^\pm} denotes the future directed timelike unit normal, and the induced volume forms are to be understood.

REMARK 3.1.1. – *In general, in integrals we will either write explicitly a volume form or it is to be understood that the integration is with respect to the induced volume form. In the case of a null hypersurface, the volume element depends on the choice of a null generator and is defined so that the divergence theorem holds.*

We direct the reader unfamiliar with the use of energy currents to the concise introductory book [3]. See [17] for a systematic discussion.

3.2. The ambient differentiable structure and the Kerr metric

In this section we will briefly review the background differentiable structure and various convenient coordinate systems for the Kerr spacetime. We direct the reader to [28] and [30] for a more thorough discussion of our conventions and to the books [48] and [61] for a proper introduction to Kerr.

As is well known, the Kerr spacetimes $(\mathcal{M}, g_{a,M})$ are a 2-parameter family of spacetimes which in the parameter range $|a| < M$ may be thought of as the maximal Cauchy development of a Cauchy hypersurface with two asymptotically flat ends. The spacetime \mathcal{M} possesses a bifurcate Killing horizon separating two asymptotically flat exterior regions from a *black hole* and a *white hole* region (in the case $a \neq 0$, then \mathcal{M} is further extendible beyond a smooth Cauchy horizon to a larger spacetime which fails however to be globally hyperbolic and is thus not uniquely determined by initial data). In this paper, we will work on the subregion \mathcal{D} which is the closure of one of the exterior regions \mathcal{M} . The boundary of the region \mathcal{D} consists of the union of two null hypersurfaces \mathcal{H}^+ and \mathcal{H}^- , the *future event horizon* and the *past event*

horizon, along with \mathcal{B} , the bifurcate sphere. Our convention will be that \mathcal{B} is not included in \mathcal{H}^\pm and $\mathcal{H}^+ \cup \mathcal{B} \cup \mathcal{H}^-$ is a bifurcate null hypersurface.

We proceed to describe explicitly the underlying structure and metric. We start with the smooth manifold with boundary

$$(25) \quad \mathcal{R} = \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{S}^2,$$

parameterized by $y^* \in \mathbb{R}_{\geq 0}$, $t^* \in \mathbb{R}$ and a choice of standard spherical coordinates $(\theta^*, \phi^*) \in \mathbb{S}^2$. This coordinate system will be known as ‘‘Kerr-star coordinates’’. Let us denote the coordinate vector field $T = \partial_{t^*}$ and $\Phi = \partial_{\phi^*}$ and let us denote by $\Omega_1, \Omega_2, \Omega_3$ a basis of standard angular momentum operators corresponding to the \mathbb{S}^2 factor of (25).⁽⁵⁾ In particular, the Ω_i span the tangent space of \mathbb{S}^2 .

We define what shall be the future event horizon \mathcal{H}^+ by $\mathcal{H}^+ = \partial\mathcal{R} = \{y^* = 0\}$. It will be useful to adopt the conventions:

$$\begin{aligned} \mathcal{H}_{\geq s}^+ &\doteq \mathcal{H}^+ \cap \{t^* \geq s\}, \\ \mathcal{H}^+(s_1, s_2) &\doteq \mathcal{H}^+ \cap \{t^* \in [s_1, s_2]\}, \\ \mathcal{R}_{\geq s} &\doteq \{t^* \geq s\}, \\ \mathring{\mathcal{R}} &\doteq \text{int}(\mathcal{R}) = \mathcal{R} \setminus \mathcal{H}^+, \\ \mathring{\mathcal{R}}_{\geq s} &\doteq \mathcal{R}_{\geq s} \setminus \mathcal{H}_{\geq s}^+. \end{aligned}$$

Next, given a choice of parameters (a, M) satisfying $|a| < M$, we define a new coordinate function $r = r(y^*)$ on \mathcal{R} (with $\infty > C > \frac{dr}{dy^*} \geq c > 0$) so that $r|_{\mathcal{H}^+} = r_+(a, M)$ where $r_\pm \doteq M \pm \sqrt{M^2 - a^2}$. It is often convenient to replace r with yet another rescaled version, $r^* = r^*(r)$, defined in $\mathring{\mathcal{R}}$, by

$$(26) \quad \frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}, \quad r^*(3M) = 0,$$

where

$$(27) \quad \Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-).$$

Since $r_- < r_+$, it follows that Δ vanishes to first order on \mathcal{H}^+ , and thus the coordinate range $\infty > r > r_+$ covering $\mathring{\mathcal{R}}$ corresponds to the range $\infty > r^* > -\infty$. It will also be useful to sometimes employ what will be an ‘‘approximately null’’ coordinate system $(\tilde{u}, \tilde{v}, \theta, \phi)$ defined by

$$\tilde{u} = \frac{1}{2}(t - r^*), \quad \tilde{v} = \frac{1}{2}(t + r^*).$$

Next, we introduce the new coordinates

$$(28) \quad t(t^*, r) \doteq t^* - \bar{t}(r), \quad \phi(\phi^*, r) \doteq \phi^* - \bar{\phi}(r) \pmod{2\pi}, \quad \theta \doteq \theta^*$$

where $\bar{t}(r)$ and $\bar{\phi}(r)$ are appropriately chosen smooth functions depending on a and M (see [28] and [30] for details) which vanish for sufficiently large r .

⁽⁵⁾ We may take $\Omega_1 = \Phi$ for instance.

In these “Boyer-Lindquist coordinates” (t, r, θ, ϕ) , we finally define the Kerr metric by

$$(29) \quad g_{a,M} = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} (a dt - (r^2 + a^2)d\phi)^2,$$

where $\rho^2 \doteq r^2 + a^2 \cos^2 \theta$. Though a priori (29) is only defined in $\mathring{\mathcal{R}}$, by examining the expression of the metric in Kerr-star coordinates y^* and t^* (see [28] for the computation), one checks easily that $g_{a,M}$ extends smoothly to \mathcal{H}^+ making $(\mathcal{R}, g_{a,M})$ a smooth Lorentzian manifold-with-boundary. Let us note moreover that T and Φ defined previously can be expressed again as coordinate vector fields $T = \partial_t$ and $\Phi = \partial_\phi$, whence it follows from (29) that T and Φ are Killing on \mathcal{R} . These are the so-called stationary and axisymmetric Killing vector fields.

Recall that when $a \neq 0$ the vector field T is *not* everywhere timelike. The region \mathcal{S} where T is spacelike is known as the “ergoregion”. Explicitly, we have

$$(30) \quad \mathcal{S} = \{\Delta - a^2 \sin^2 \theta < 0\}.$$

Note that

$$\mathcal{S} \subset \{r < 2M\}.$$

Let us also recall that in [28] and [30] we chose the function \bar{t} of (28) so that the hypersurfaces $t^* = s$, denoted by Σ_s^* , are spacelike with respect to the Kerr metric as just defined. Furthermore, we will have $\mathcal{R}_{\geq s} = D^+(\Sigma_s^*)$. Let us introduce the notation

$$(31) \quad \mathring{\Sigma} \doteq \{t = 0\}.$$

We have that $\mathring{\Sigma}$ is also spacelike and a Cauchy hypersurface for $\mathring{\mathcal{R}}$.

Some additional notation from [30]: Note that the definition ∂_r is ambiguous since it depends on the choice of coordinate system. Thus, we define

$$(32) \quad Z^* \doteq \partial_r \quad \text{with respect to coordinates} \quad (t^*, r, \theta^*, \phi^*),$$

$$(33) \quad Z \doteq \partial_r \quad \text{with respect to coordinates} \quad (t, r, \theta, \phi).$$

Note that Z^* is well defined in \mathcal{R} and is transversal to \mathcal{H}^+ while Z is only well defined in $\mathring{\mathcal{R}}$. Finally, we will use ∇ to denote the induced covariant derivative on the \mathbb{S}^2 factor of \mathcal{R} .

Though all explicit computations will take place on the manifold-with-boundary \mathcal{R} defined above, it is of fundamental importance to understand the existence and properties of a further smooth extension to $\mathcal{D} = \mathcal{R} \cup \mathcal{H}^- \cup \mathcal{B}$, which will represent precisely the \mathcal{D} described at the beginning of this section. We will be brief in our presentation; we direct the reader to [61] for a very careful and detailed exposition.

We begin by attaching \mathcal{H}^- . Starting with Boyer-Lindquist coordinates (t, r, θ, ϕ) , one defines a new coordinate system $({}^*t, {}^*\phi, r, {}^*\theta)$ by

$${}^*t(t, r) \doteq t - \bar{t}(r), \quad {}^*\phi(t, r) \doteq \phi - \bar{\phi}(r) \pmod{2\pi}, \quad {}^*\theta \doteq \theta,$$

where \bar{t} and $\bar{\phi}$ are as above. A straightforward computation shows that the metric naturally extends smoothly so as to be defined also at $r = r_+$ in this chart. We may thus use this coordinate chart to extend \mathcal{R} to a larger manifold-with-boundary $\mathcal{R} \cup \mathcal{H}^-$ where \mathcal{H}^- corresponds to the hypersurface $r = r_+$ of this new chart. We shall refer to \mathcal{H}^- as the *past event horizon*.

One may easily check that the Boyer-Lindquist coordinate defined map

$$(34) \quad (t, \phi) \mapsto (-t, -\phi)$$

is an isometry of $\overset{\circ}{\mathcal{R}}$ which smoothly extends to an isometry of $\mathcal{R} \cup \mathcal{H}^-$ and furthermore sends \mathcal{H}^+ to \mathcal{H}^- .

Finally, one may even further extend $\mathcal{R} \cup \mathcal{H}^-$ to a larger Lorentzian manifold $\widetilde{\mathcal{M}}$ so that the boundary of \mathcal{R} (as a subset) in $\widetilde{\mathcal{M}}$ consists of a bifurcate null hypersurface $\mathcal{B} \cup \mathcal{H}^- \cup \mathcal{H}^+$, with $\mathcal{B} \subset \widetilde{\mathcal{M}}$ a sphere. Our region of interest \mathcal{D} described at the beginning of this section is simply then the manifold-with-stratified boundary $\mathcal{D} = \mathcal{R} \cup \mathcal{H}^- \cup \mathcal{B}$. We remark that \mathcal{D} admits a globally regular coordinate system ⁽⁶⁾ $(U^+, V^+, \theta, \phi) \in [0, -\infty) \times [0, \infty) \times \mathbb{S}^2$ so that $\mathcal{H}^+ = \{U^+ = 0, V^+ \in (0, \infty)\}$, $\mathcal{H}^- = \{V^+ = 0, U^+ \in (-\infty, 0)\}$ and $\mathcal{B} = \{(U^+, V^+) = (0, 0)\}$. Moreover, along \mathcal{B} we have

$$(35) \quad g_{U+U^+} = g_{U+\theta} = g_{U+\phi} = g_{V+V^+} = g_{V+\theta} = g_{V+\phi} = 0.$$

We shall not here require the form of the explicit coordinate transformations defining U^+ and V^+ in terms of our previously described charts on $\mathcal{R} \cup \mathcal{H}^-$ but we remark that

$$(36) \quad \overline{\Sigma} \doteq \mathcal{B} \cup \overset{\circ}{\Sigma}$$

is a smooth manifold-with-boundary (with boundary \mathcal{B}) and interior $\overset{\circ}{\Sigma}$. Note that smooth functions (ψ, ψ') “compactly supported on $\overset{\circ}{\Sigma}$ ” extend to smooth compact supported functions on $\overline{\Sigma}$ which moreover vanish in a neighborhood of \mathcal{B} . On the other hand, smooth functions compactly supported on $\overline{\Sigma}$ do not restrict to compactly supported functions on $\overset{\circ}{\Sigma}$.

It will be convenient to introduce the notation

$$(37) \quad \overline{\mathcal{H}^\pm} \doteq \mathcal{H}^\pm \cup \mathcal{B},$$

$$(38) \quad \overline{\mathcal{H}_{\leq \tau}^+} \doteq \overline{\mathcal{H}^+} \cap (\{t^* \leq \tau\}) \cup \mathcal{B}.$$

These are again smooth hypersurfaces-with-boundary, with boundary \mathcal{B} for $\overline{\mathcal{H}^\pm}$ and boundary $\mathcal{B} \cup (\Sigma_0^* \cap \mathcal{H}^+)$ for $\overline{\mathcal{H}_{\leq 0}^+}$. The reader should in particular again contrast the distinct notions of “compactly supported” on \mathcal{H}^+ and $\overline{\mathcal{H}^\pm}$.

We have already noted that the vector fields T and Φ are Killing. The event horizon \mathcal{H}^+ is also a *Killing horizon*: the Killing field given by the linear combination

$$(39) \quad K \doteq T + \omega_+ \Phi,$$

where $\omega_+ \doteq \frac{a}{2Mr_+}$ is the “angular velocity” of the event horizon. The vector field K is null and normal to \mathcal{H}^+ ; thus, \mathcal{H}^+ is in particular a null hypersurface. In integrals associated to energy currents we will denote K by $n_{\mathcal{H}^+}^\mu$. It will be useful to recall that the vector field K restricted to \mathcal{H}^+ coincides with the smooth extension of the coordinate vector field ∂_{r^*} of the (r^*, t, θ, ϕ) coordinate system.

The past event horizon \mathcal{H}^- is also a Killing horizon with a Killing field also given by K . Note however that the restriction of K to \mathcal{H}^- coincides with the smooth extensions of the coordinate vector field $-\partial_{r^*}$ of the (r^*, t, θ, ϕ) coordinate system.

⁽⁶⁾ It is globally regular up to the usual degeneration of spherical coordinates.

Note finally that the vector fields T and Φ and the discrete isometry (34) both extend smoothly to all of \mathcal{D} and

$$(40) \quad K|_{\mathcal{B}} = 0.$$

3.3. Dependence on a and M and conventions on constants

In all propositions to follow, unless otherwise stated, $|a| < M$ are fixed parameters and everything refers to the Kerr metric $g_{M,a}$ on \mathcal{D} as described in the previous section.

Let us briefly review our conventions from [28] and [30] regarding constants depending on the parameters a and M . Large positive constants will be denoted by B , and small positive constants by b . Both constants B and b depend only on M and a lower bound for $M - |a|$, and this dependence is always to be understood even when not mentioned explicitly. Often these constants will blow up $B \rightarrow \infty$, $b^{-1} \rightarrow \infty$ in the extremal limit $|a| \rightarrow M$.

We recall the usual arithmetic properties of b and B :

$$b + b = b, B + B = B, B \cdot B = B, B^{-1} = b, \dots$$

The statement $f \sim g$ will mean

$$bg \leq f \leq Bg.$$

The statement “for R sufficiently large,” etc., without further qualification, will mean “there exists a constant $R_0(a, M)$ such that for $R \geq R_0$ ”.

Lastly, if the constant B or b depends on the value of a yet to be fixed parameter, then that dependence will be explicitly noted. For example, if B depends on a parameter c which has not been fixed, we shall denote it by $B(c)$. Once the constant c is fixed, we then write B .

3.4. Useful vector fields

We recall the following two lemmas proved in [30].

LEMMA 3.4.1. – *The vector field*

$$T + \frac{2Mar}{(r^2 + a^2)^2} \Phi$$

is a smooth vector field in \mathcal{D} , is timelike in $\overset{\circ}{\mathcal{R}}$ and null on \mathcal{H}^\pm .

LEMMA 3.4.2. – *There exists a constant $\epsilon_0 = \epsilon_0(a, M) > 0$ such that the vector field K (39) is timelike for $r \in (r_+, r_+ + \epsilon_0)$.*

These lemmas allow us to make the following definition.

DEFINITION 3.4.1. – *Let $\epsilon_0 > 0$ be from Lemma 3.4.2. Let $\alpha(r)$ be a function such that $V \doteq T + \alpha(r)\Phi$ is a smooth vector field in \mathcal{D} , timelike in $\overset{\circ}{\mathcal{R}}$ and which satisfies*

$$\begin{aligned} V &= K, \text{ when } r \in [r_+, r_+ + \epsilon_0/2], \\ V &= T + \frac{2Mar}{(r^2 + a^2)^2} \Phi, \text{ when } r \in \left[r_+ + \epsilon_0, \frac{M(7 + \sqrt{2})}{4} \right], \\ V &= T, \text{ when } r \geq \frac{M(3 + \sqrt{2})}{2}. \end{aligned}$$

REMARK 3.4.1. – *This vector field will be useful because it is manifestly T -invariant, it is timelike (hence the associated energy fluxes \mathbf{J}^V are positive definite) and because it is Killing for $r \leq r_+ + \epsilon_0/2$ and $r \geq M(3 + \sqrt{2})/2$ (hence the error terms \mathbf{K}^V from the energy identity (24) are supported in $r_+ + \epsilon_0/2 \leq r \leq M(3 + \sqrt{2})/2$).*

It will be useful to observe the following immediate corollary of Lemmas 3.4.1 and 3.4.2.

COROLLARY 3.4.1. – *For every $\epsilon > 0$ sufficiently small and any $r_0 \in (r_+, \infty)$, there exists a vector field $\tilde{V} = \tilde{V}(r_0, \epsilon)$ of the form $T + \tilde{\alpha}(r)\Phi$ for an appropriate function $\tilde{\alpha}(r)$ such that \tilde{V} is a smooth vector field on \mathcal{D} , is timelike in $\tilde{\mathcal{R}}$, is Killing in the region*

$$r \in (\min(r_+, r_0 - \epsilon), r_0 + \epsilon],$$

and is equal to V for r sufficiently close to r_+ and r sufficiently large.

We shall apply the above corollary, for finitely many distinct choices of r_0 , in the context of Section 9.1.2.

In order for our non-degenerate energies to have a fixed meaning, it is useful to fix once and for all a choice of a globally defined smooth timelike vector field on \mathcal{D} .

DEFINITION 3.4.2. – *Let N denote any fixed choice of a smooth timelike vector field on \mathcal{D} which is invariant under the flow of T on the complement of a compact set containing the bifurcate sphere \mathcal{B} , ⁽⁷⁾ and satisfies $N = T$ for sufficiently large r .*

Finally, we note the following easy calculations.

REMARK 3.4.2. – *Fix an open set $\mathcal{U} \subset \mathcal{D}$ containing the bifurcate sphere $\mathcal{B} \subset \mathcal{U}$. Then, for s such that $\Sigma_s^* \cap \mathcal{U} = \emptyset$ we have*

$$(41) \quad \int_{\Sigma_s^*} \mathbf{J}_\mu^N[\psi] n_{\Sigma_s^*}^\mu \sim \|\psi\|_{\dot{H}^1(\Sigma_s^*)}^2 + \|n_{\Sigma_s^*} \psi\|_{L^2(\Sigma_s^*)}^2 \sim \int_{\theta, \phi^*} \int_{r_+}^\infty (|\partial_{t^*} \psi|^2 + |\partial_r \psi|^2 + |\nabla \psi|_{\mathcal{g}}^2) dr dV_{\mathcal{g}}$$

with respect to coordinates $(t^, r, \theta^*, \phi^*)$.*

REMARK 3.4.3. – *We have*

$$(42) \quad \int_{\Sigma_s^*} \mathbf{J}_\mu^V[\psi] n_{\Sigma_s^*}^\mu \sim \int_{\theta, \phi^*} \int_{r_+}^\infty (|\partial_{t^*} \psi|^2 + (1 - \frac{r_+}{r}) |\partial_r \psi|^2 + |\nabla \psi|_{\mathcal{g}}^2) dr dV_{\mathcal{g}}$$

with respect to coordinates $(t^, r, \theta^*, \phi^*)$. We note that the relation (42) would also hold with V replaced by the vector field from Lemma 3.4.1.*

⁽⁷⁾ Note that in view of the vanishing of T on \mathcal{B} , one cannot define such a timelike N which is invariant on *all* of \mathcal{D} .

3.5. A foliation by hyperboloidal hypersurfaces

It will be convenient to have the following explicit foliation of \mathcal{R} by a family of hyperboloidal hypersurfaces.

DEFINITION 3.5.1. – For every $\tau \in \mathbb{R}$ we set

$$S_\tau \doteq \begin{cases} t^* = \tau & r \leq 5M \\ t^* - r^* + \frac{10M}{r} = \tau - (5M)^* + 2 & r > 5M. \end{cases}$$

(This hypersurface could be smoothed out, but this does not in fact make a difference.) Some straightforward, if tedious, calculations yield the following lemma.

LEMMA 3.5.1. – For every $\tau \in \mathbb{R}$, S_τ is a spacelike hypersurface, $\mathcal{R} = \bigcup_{\tau \in \mathbb{R}} S_\tau$, and, for sufficiently large R ,

$$\int_{S_\tau \cap \{r \geq R\}} \mathbf{J}_\mu^T[\psi] n_{S_\tau}^\mu \sim \int_{S_\tau \cap \{r \geq R\}} \left[|\partial_{\bar{v}} \psi|^2 + r^{-2} |\partial_{\bar{u}} \psi|^2 + |\nabla \psi|^2 \right] r^2 \sin \theta \, dv \, d\theta \, d\phi.$$

In comparing the way these two integrals are written, we recall our convention that if no volume form is written explicitly (as on the left hand side of the above), the integration is with respect to the induced volume form.

Later, when r is sufficiently large we will often work in the coordinate system (τ, r, θ, ϕ) associated to the foliation $\{S_\tau\}_{\tau \in \mathbb{R}}$. We will in fact use this coordinate system to define our notion of null infinity \mathcal{I}^+ in Section 4.2.

3.6. Well-posedness

Let us briefly recall some basic well-posedness statements.

First we consider the case of initial data prescribed on Σ_0^* . Recall that $\mathcal{R}_{\geq 0} = \{t^* \geq 0\} = D^+(\Sigma_0^*)$. In the propositions below the H^s and C^k spaces will refer to complex valued functions.

PROPOSITION 3.6.1. – Let $(\psi, \psi') \in H_{\text{loc}}^s(\Sigma_0^*) \times H_{\text{loc}}^{s-1}(\Sigma_0^*)$. Then there exists a unique solution ψ to the wave equation (2) on $\mathcal{R}_{\geq 0}$ such that

$$\psi \in C_{\tau \in [0, \infty)}^0(H_{\text{loc}}^s(\Sigma_\tau^*)) \cap C_{\tau \in [0, \infty)}^1(H_{\text{loc}}^{s-1}(\Sigma_\tau^*)) \cap H_{\text{loc}}^s(\mathcal{H}_{\geq 0}^+),$$

$\psi|_{\Sigma_0^*} = \psi$, and $n_{\Sigma_0^*} \psi|_{\Sigma_0^*} = \psi'$. Furthermore, the solution map depends continuously on the initial data. Finally, we note that if the initial data (ψ, ψ') are smooth, then the solution ψ will be smooth.

Next we consider the case of initial data along $\bar{\Sigma}$. Let us define $\tilde{\Sigma}_\tau$ to be the image of $\bar{\Sigma}$ at time τ of the flow map associated to the vector field N from Definition 3.4.2.

PROPOSITION 3.6.2. – Let $(\psi, \psi') \in H_{\text{loc}}^s(\bar{\Sigma}) \times H_{\text{loc}}^{s-1}(\bar{\Sigma})$. Then there exists a unique solution ψ to the wave equation (2) in \mathcal{D} such that

$$\psi \in C_{\tau \in (-\infty, \infty)}^0(H_{\text{loc}}^s(\tilde{\Sigma}_\tau)) \cap C_{\tau \in (-\infty, \infty)}^1(H_{\text{loc}}^{s-1}(\tilde{\Sigma}_\tau)) \cap H_{\text{loc}}^s(\mathcal{H}^+) \cap H_{\text{loc}}^s(\mathcal{H}^-),$$

$\psi|_{\bar{\Sigma}} = \psi$, and $n_{\bar{\Sigma}} \psi|_{\bar{\Sigma}} = \psi'$. Furthermore, the solution map depends continuously on the initial data. Finally, we note that if the initial data (ψ, ψ') are smooth, then the solution ψ will be smooth.

REMARK 3.6.1. – In the case when the initial data (ψ, ψ') are compactly supported along $\overset{\circ}{\Sigma}$, then a \mathbf{J}^K energy estimate immediately implies that bifurcate sphere \mathcal{B} lies outside the support of the solution ψ produced by Proposition 3.6.2. Also note that if (ψ, ψ') are compactly supported on $\overline{\Sigma}$, then $(\psi|_{\Sigma_0^*}, n_{\Sigma_0^*}\psi|_{\Sigma_0^*})$ are compactly supported on Σ_0^* .

It will also be useful to consider the following two mixed characteristic-spacelike initial value problems. For convenience these will both be stated in the smooth category. The following two propositions can be deduced from the work of Rendall [63]. First we have

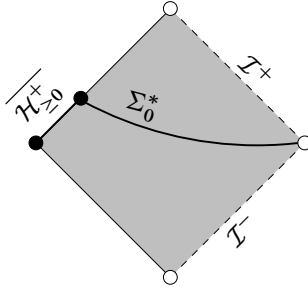
PROPOSITION 3.6.3. – Let $\psi|_{\overline{\mathcal{H}_{\leq 0}^+}}$ be a smooth function on $\overline{\mathcal{H}_{\leq 0}^+}$ and $(\psi_{\Sigma_0^*}, \psi'_{\Sigma_0^*})$ be a pair of smooth functions on Σ_0^* such that there exists a smooth function $\tilde{\Psi}$ on \mathcal{D} satisfying

$$\tilde{\Psi}|_{\overline{\mathcal{H}_{\leq 0}^+}} = \psi|_{\overline{\mathcal{H}_{\leq 0}^+}}, \quad (\tilde{\Psi}|_{\Sigma_0^*}, n_{\Sigma_0^*}\tilde{\Psi}|_{\Sigma_0^*}) = (\psi_{\Sigma_0^*}, \psi'_{\Sigma_0^*}).$$

Then there exists a unique smooth solution ψ to the wave equation (2) in the past of Σ_0^* such that

$$\psi|_{\overline{\mathcal{H}_{\leq 0}^+}} = \psi|_{\overline{\mathcal{H}_{\leq 0}^+}}, \quad (\psi|_{\Sigma_0^*}, n_{\Sigma_0^*}\psi|_{\Sigma_0^*}) = (\psi_{\Sigma_0^*}, \psi'_{\Sigma_0^*}).$$

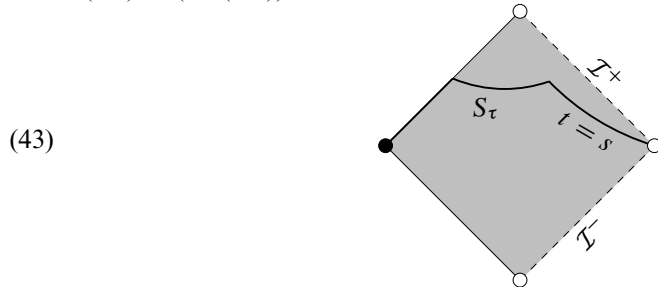
See



Before giving the next proposition, it is useful to define a function $r(\tau, s)$: Let $\tau > -\infty$. Then, for each $s > 0$ sufficiently large we define the value $r(\tau, s)$ to be the largest solution to

$$s - r^*(\tau, s) + \frac{10M}{r(\tau, s)} \doteq \tau - (5M)^* + 2.$$

Observe that the hypersurface $\{t = s\}$ will intersect the hypersurface S_τ along the surface where $(t, r) = (s, r(\tau, s))$. Refer to



We have

PROPOSITION 3.6.4. – *Let $\tau < \infty$, let $\psi|_{\overline{\mathcal{H}_{\leq\tau}^+}}$ be a smooth function on $\overline{\mathcal{H}_{\leq\tau}^+}$ which vanishes in a neighborhood of $S_\tau \cap \mathcal{H}^+$ and let $\Phi_{\{t=s\} \cap \{r \geq r(\tau,s)\}}$ be a smooth compactly supported function on $\{t = s\} \cap \{r \geq r(\tau,s)\}$ which vanishes in a neighborhood of $\{t = s\} \cap \{r = r(\tau,s)\}$. Then there exists a unique smooth solution ψ to the wave equation (2) in the past of $\overline{\mathcal{H}_{\leq\tau}^+} \cup (S_\tau \cap \{r \leq r(\tau,s)\}) \cup (\{t = s\} \cap \{r \geq r(\tau,s)\})$ such that*

$$\begin{aligned} \psi|_{\overline{\mathcal{H}_{\leq\tau}^+}} &= \psi|_{\overline{\mathcal{H}_{\leq\tau}^+}}, \\ (\psi|_{S_\tau \cap \{r \leq r(\tau,s)\}}, n_{S_\tau} \psi|_{S_\tau \cap \{r \leq r(\tau,s)\}}) &= (0, 0), \\ r\psi|_{\{t=s\} \cap \{r \geq r(\tau,s)\}} &= \Phi_{\{t=s\} \cap \{r \geq r(\tau,s)\}}. \end{aligned}$$

In accordance with our conventions (recall Section 3.3), the above propositions refer always to the Kerr metric with fixed parameters $|a| < M$. Let us remark that we have defined the differentiable structure in [30] so that we can assert also the smooth dependence of ψ on our parameters a and M ; this, however, shall play no role in the current paper.

3.7. The non-degenerate boundedness and integrated energy decay statements

In this section, we shall recall the precise boundedness and integrated energy decay statements proved in [30].

First we recall a few additional notations from [30]:

DEFINITION 3.7.1. – *Given s^- satisfying $r_+ < 3M - s^- < \infty$, let us define a cutoff function $\chi(r)$ such that $\chi = 1$ for $r \geq 3M - s^-$ and $\chi = 0$ for $r \leq (r_+ + 3M - s^-)/2$. We then set*

$$\tilde{Z}^* = \chi Z + (1 - \chi)Z^*.$$

DEFINITION 3.7.2. – *Given s^- and s^+ satisfying $r_+ < 3M - s^- < 3M + s^+ < \infty$ we define*

$$(44) \quad \zeta(r) \doteq (1 - 3M/r)^2 (1 - \eta_{[3M-s^-, 3M+s^+]}(r)),$$

where η is the indicator function.

The main result of [30] was

THEOREM 3.7.1 ([30]). – *There exist parameters $s^-(a, M)$ and $s^+(a, M)$ satisfying $r_+ < 3M - s^- < 3M + s^+ < \infty$ such that for all $\delta > 0$, all sufficiently regular solutions ψ*

to (2) on $\mathcal{R}_{\geq 0}$ satisfy the following estimates:

$$(45) \quad \int_{\mathcal{R}_{\geq 0}} \left(r^{-1} \zeta |\nabla \psi|^2 + r^{-1-\delta} \zeta |T\psi|^2 + r^{-1-\delta} |\tilde{Z}^* \psi|^2 + r^{-3-\delta} |\psi - \psi_\infty|^2 \right) \leq B(\delta) \int_{\Sigma_0^*} \mathbf{J}_\mu^N[\psi] n_{\Sigma_0^*}^\mu,$$

$$(46) \quad \int_{\mathcal{H}_{\geq 0}^+} \left(\mathbf{J}_\mu^N[\psi] n_{\mathcal{H}^+}^\mu + |\psi - \psi_\infty|^2 \right) \leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^N[\psi] n_{\Sigma_0^*}^\mu,$$

$$(47) \quad \int_{\Sigma_s^*} \mathbf{J}_\mu^N[\psi] n_{\Sigma_s^*}^\mu \leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^N[\psi] n_{\Sigma_0^*}^\mu, \quad \forall s \geq 0,$$

where $4\pi\psi_\infty^2 = \lim_{r' \rightarrow \infty} \int_{\Sigma_0^* \cap \{r=r'\}} r^{-2} |\psi|^2$.

REMARK 3.7.1. – We remind the reader that due to the existence of trapped null geodesics, the estimate (45) would be false if the cut-off ζ was removed. Furthermore, the fact that one can indeed prove an estimate with this degeneration relies on the fact that the trapping is sufficiently unstable. See the discussion in [30].

We also proved the following higher order version of Theorem 3.7.1:

THEOREM 3.7.2 ([30]). – With $s^\pm(a, M)$ as above, then for all $\delta > 0$, $j \geq 1$, all sufficiently regular solutions ψ to (2) on $\mathcal{R}_{\geq 0}$ satisfy the following estimates:

$$(48) \quad \int_{\mathcal{R}_{\geq 0}} r^{-1-\delta} \zeta \sum_{1 \leq i_1 + i_2 + i_3 \leq j} |\nabla^{i_1} T^{i_2} (\tilde{Z}^*)^{i_3} \psi|^2 + r^{-1-\delta} \sum_{1 \leq i_1 + i_2 + i_3 \leq j-1} \left(|\nabla^{i_1} T^{i_2} (\tilde{Z}^*)^{i_3+1} \psi|^2 + |\nabla^{i_1} T^{i_2} (Z^*)^{i_3} \psi|^2 \right) \leq B(\delta, j) \int_{\Sigma_0^*} \sum_{0 \leq i \leq j-1} \mathbf{J}_\mu^N[N^i \psi] n_{\Sigma_0^*}^\mu,$$

$$(49) \quad \int_{\mathcal{H}_{\geq 0}^+} \sum_{0 \leq i \leq j-1} \mathbf{J}_\mu^N[N^i \psi] n_{\mathcal{H}^+}^\mu \leq B(j) \int_{\Sigma_0^*} \sum_{0 \leq i \leq j-1} \mathbf{J}_\mu^N[N^i \psi] n_{\Sigma_0^*}^\mu,$$

$$(50) \quad \int_{\Sigma_s^*} \sum_{0 \leq i \leq j-1} \mathbf{J}_\mu^N[N^i \psi] n_{\Sigma_s^*}^\mu \leq B(j) \int_{\Sigma_0^*} \sum_{0 \leq i \leq j-1} \mathbf{J}_\mu^N[N^i \psi] n_{\Sigma_0^*}^\mu, \quad \forall s \geq 0.$$

REMARK 3.7.2. – Sufficiently regular may be taken to mean that the initial data lies in $H_{\text{loc}}^s(\Sigma_0^*)$ for s suitably large and that the right hand sides of each inequality are finite.

REMARK 3.7.3. – Recall that a straightforward elliptic estimate would yield

$$(51) \quad \int_{\Sigma_s^*} \sum_{0 \leq i \leq j-1} \mathbf{J}_\mu^N[N^i \psi] n_{\Sigma_s^*}^\mu \sim \sum_{1 \leq i \leq j} \|\psi\|_{\dot{H}^i(\Sigma_s^*)}^2 + \|n_{\Sigma_s^*} \psi\|_{\dot{H}^{i-1}(\Sigma_s^*)}^2.$$

REMARK 3.7.4. – In view of the discrete isometry (34), one immediately obtains versions of Theorem 3.7.1 and Theorem 3.7.2 for solutions defined in the past of the hypersurface $*t = 0$.

3.8. The r^p estimates

It will be useful to exploit the hierarchy of “ r^p estimates” from [24]. For our purposes, it is convenient to apply these estimates in the following form.

PROPOSITION 3.8.1. – *Let R be sufficiently large. Then for all $\tau_1 < \tau_2$, $p \in [0, 2]$, and all ψ sufficiently regular solutions to (2) on $\mathcal{D}(\tau_1, \tau_2) \doteq J^+(S_{\tau_1}) \cap J^-(S_{\tau_2})$, then setting $\varphi \doteq (r^2 + a^2)^{1/2}\psi$ and keeping Remark 3.1.1 in mind, we have*

$$\begin{aligned} & \int_{S_{\tau_2} \cap \{r \geq R\}} \left[r^p |\partial_{\bar{v}}\varphi|^2 + r^{p-2} |\nabla\varphi|^2 + r^{-2} |\partial_{\bar{u}}\varphi|^2 \right] \sin\theta \, dv \, d\theta \, d\phi \\ & + \int_{\mathcal{D}(\tau_1, \tau_2) \cap \{r \geq R\}} \left[pr^{p-1} |\partial_{\bar{v}}\varphi|^2 + ((2-p)r^{p-1} + r^{-1}) |\nabla\varphi|^2 + r^{p-4} |\varphi|^2 \right. \\ & \quad \left. + r^{-2} |\partial_{\bar{u}}\varphi|^2 \right] \sin\theta \, du \, dv \, d\theta \, d\phi \\ & \leq B \int_{\mathcal{D}(\tau_1, \tau_2) \cap \{R \leq r \leq R+1\}} r^p \left[|T\varphi|^2 + |Z\varphi|^2 + |\nabla\varphi|^2 \right] \sin\theta \, du \, dv \, d\theta \, d\phi \\ & + B \int_{S_{\tau_1} \cap \{r \geq R\}} \left[r^p |\partial_{\bar{v}}\varphi|^2 + r^{p-2} |\nabla\varphi|^2 + r^{-2} |\partial_{\bar{u}}\varphi|^2 \right] \sin\theta \, dv \, d\theta \, d\phi. \end{aligned}$$

Proof. – One combines the estimates of [24] with an energy estimate, Hardy inequalities, and a Morawetz estimate. This is a special case of a more general computation done in detail in [56] for the general setting of asymptotically flat spacetimes. \square

REMARK 3.8.1. – *Note that one may easily check that the boundary terms of the $p = 1$ estimate relate to the spacetime terms of the $p = 2$ estimate in such a way as to allow one to combine Theorem 3.7.2 with the iterated pigeon hole argument of [24] in order to conclude for instance that*

$$\int_{S_\tau} \mathbf{J}_\mu^N[\psi] n_{S_\tau}^\mu \leq B\tau^{-2} E_0[\psi] \quad \forall \tau > 0,$$

where $E_0[\psi]$ denotes a weighted second order energy of ψ along Σ_0^* .

Note that we will not require such quantitative decays results in this paper.

We will also need to commute with angular momentum operators $\Omega^{(\alpha)}$. We obtain

PROPOSITION 3.8.2. – *For every multi-index α , let $\Omega^{(\alpha)}$ denote an arbitrary product of angular momentum operators as defined in Section 3.2, and set $\varphi^{(\alpha)} \doteq \Omega^{(\alpha)}(r^2 + a^2)^{1/2}\psi$. For all sufficiently large R , multi-indices α , $\tau_1 < \tau_2$, and $p \in [0, 2]$, we obtain the estimate of Proposition 3.8.1 with φ replaced by $\varphi^{(\alpha)}$.*

Proof. – If $a = 0$, this is of course immediate since then $[\Omega^{(\alpha)}, \square_g] = 0$. Otherwise, one proceeds inductively in $|\alpha|$ and observes that the error terms arising from $[\Omega^{(\alpha)}, \square_g]$ have sufficiently strong r decay so as to be either absorbed by good bulk terms on the left hand side of the estimate or controlled by the previous step. \square

4. Radiation fields and energy fluxes

In this section, we will define the radiation fields along $\mathcal{H}_{\geq 0}^+$ or $\overline{\mathcal{H}^+}$ and \mathcal{I}^+ for solutions ψ to the wave equation (2) arising from smooth initial data along Σ_0^* or $\overline{\Sigma}$ which are compactly supported. Since $\overset{\circ}{\Sigma} \subset \overline{\Sigma}$, this *a fortiori* defines the radiation field for solutions with compactly supported data along $\overset{\circ}{\Sigma}$.

The considerations at the horizon are straightforward and will be given in Section 4.1. The finiteness of both the non-degenerate and degenerate radiation fluxes follows as a soft application of Theorem 3.7.1 quoted in the previous section.

Null infinity will be handled in Section 4.2. We will first have to explicitly define \mathcal{I}^+ as an additional boundary which can be attached to \mathcal{D} (Definition 4.2.1). The main result is Proposition 4.2.1, which gives the statement of Proposition 1 of Section 2.1.3. We shall then relate the radiation field as defined to the limiting energy flux of ψ along \mathcal{I}^+ . Theorem 3.7.1 immediately implies the latter is finite (see Theorem 4.2.1), and according to Proposition 4.2.2 it can be computed from the radiation field.

4.1. The horizon

4.1.1. *The radiation field along $\mathcal{H}_{\geq 0}^+$ and $\overline{\mathcal{H}^+}$.* – We begin with the radiation field along the horizon.

DEFINITION 4.1.1. – *Given a solution ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported, the **radiation field of ψ along $\mathcal{H}_{\geq 0}^+$** is simply defined to be the restriction of ψ to the horizon $\mathcal{H}_{\geq 0}^+$.*

Similarly, we have

DEFINITION 4.1.2. – *Given a solution ψ to (2) on \mathcal{R} arising from smooth initial data along $\overline{\Sigma}$ which are compactly supported, the **radiation field of ψ along $\overline{\mathcal{H}^+}$** is simply defined to be the restriction of ψ to the horizon $\overline{\mathcal{H}^+}$.*

REMARK 4.1.1. – *Note that it follows immediately from Proposition 3.6.2 that the radiation field is smooth along the horizon.*

REMARK 4.1.2. – *If the initial data for ψ is compactly supported on $\overset{\circ}{\Sigma}$, then Remark 3.6.1 implies that the radiation field for ψ is supported in \mathcal{H}^+ .*

REMARK 4.1.3. – *Of course, given a solution ψ to (2) defined in the past of $\{^*t = 0\}$, one may make an analogous definition for the radiation field along $\mathcal{H}_{\geq 0}^-$. Similarly, one may define the radiation field along $\overline{\mathcal{H}^-}$ for a solution ψ to (2) arising from smooth initial data along $\overline{\Sigma}$.*

4.1.2. *The energy flux through $\mathcal{H}_{\geq 0}^+$ and $\overline{\mathcal{H}^+}$.* – We next define the non-degenerate energy flux along the horizon.

DEFINITION 4.1.3. – *Given a solution ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported, the **non-degenerate N -energy flux of ψ through $\mathcal{H}_{\geq 0}^+$** is defined by*

$$\int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_{\mu}^N[\psi] n_{\mathcal{H}^+}^{\mu}.$$

REMARK 4.1.4. – *Note that Theorem 3.7.1 implies that this energy flux is finite.*

Observe that a straightforward computation shows that

$$\int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_{\mu}^N[\psi] n_{\mathcal{H}^+}^{\mu} \sim \int_{\mathcal{H}_{\geq 0}^+} [|K\psi|^2 + |\nabla\psi|^2].$$

In particular, all of the derivatives are tangent to the horizon; thus one may think of the non-degenerate flux as depending only on the radiation field.

Finally, we define the degenerate flux along the horizon.

DEFINITION 4.1.4. – *Given a solution ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported, the **degenerate K -energy flux of ψ through $\mathcal{H}_{\geq 0}^+$** is defined by*

$$\int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_{\mu}^K[\psi] n_{\mathcal{H}^+}^{\mu}.$$

A straightforward computation shows that

$$\int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_{\mu}^K[\psi] n_{\mathcal{H}^+}^{\mu} = \int_{\mathcal{H}_{\geq 0}^+} |K\psi|^2.$$

Similarly,

DEFINITION 4.1.5. – *Given a solution ψ to (2) on \mathcal{R} arising from smooth initial data along $\overline{\Sigma}$ which are compactly supported, the **degenerate K -energy flux of ψ through \mathcal{H}^+** is defined by*

$$\int_{\mathcal{H}^+} \mathbf{J}_{\mu}^K[\psi] n_{\mathcal{H}^+}^{\mu}.$$

A straightforward computation shows that

$$\int_{\mathcal{H}^+} \mathbf{J}_{\mu}^K[\psi] n_{\mathcal{H}^+}^{\mu} = \int_{\mathcal{H}^+} |K\psi|^2.$$

4.2. Null infinity

We first define \mathcal{I}^+ as a suitable additional boundary which can be attached to our space-time.

DEFINITION 4.2.1. – *As a differentiable manifold we define*

$$\mathcal{I}^+ \doteq \mathbb{R} \times \mathbb{S}^2$$

and parameterize \mathcal{I}^+ in the standard fashion by coordinates (τ, θ, ϕ) . Next, we extend our background differentiable structure \mathcal{R} to a manifold with boundary

$$\tilde{\mathcal{R}} \doteq \mathcal{R} \cup \mathcal{I}^+$$

by declaring that for every sufficiently large R and open set $\mathcal{U} \subset \mathcal{I}^+$, the set

$$\mathcal{U}_R \doteq \{(\tau, r, \theta, \phi) : r > R \text{ and } (\tau, \theta, \phi) \in \mathcal{U}\}$$

is open (where (τ, r, θ, ϕ) are the coordinates associated to the foliation $\{S_\tau\}_{\tau \in \mathbb{R}}$ which we defined in Section 3.5), identifying \mathcal{I}^+ with the points $(\tau, \infty, \theta, \phi)$, and then covering the sets \mathcal{U}_R by a coordinate chart $(\tau, s, \theta, \phi) \in \mathbb{R} \times [0, 1) \times \mathbb{S}^2$ via the map

$$(\tau, s, \theta, \phi) \mapsto (\tau, Rs^{-1}, \theta, \phi).$$

REMARK 4.2.1. – *Note that for every fixed (τ, θ, ϕ) there exists a unique limit $\lim_{r \rightarrow \infty} (\tau, r, \theta, \phi) \in \mathcal{I}^+$, and, if we denote these limits by $(\tau, \infty, \theta, \phi)$, then the map $(\tau, \theta, \phi) \mapsto (\tau, \infty, \theta, \phi)$ is a diffeomorphism from $\mathbb{R} \times \mathbb{S}^2$ to \mathcal{I}^+ .*

REMARK 4.2.2. – *The above “pedestrian” definition of \mathcal{I}^+ is completely equivalent to the usual one involving a conformal compactification (see [48]).*

DEFINITION 4.2.2. – *Apply the discrete isometry $(t, \phi) \mapsto (-t, -\phi)$ to the foliation $\{S_\tau\}$ to define a new foliation $\{\tilde{S}_\tau\}$:*

$$\tilde{S}_\tau \doteq \begin{cases} -^*t = \tau & r \leq 5M \\ -^*t - r^* + \frac{10M}{r} = \tau + ^*(5M) + 2 & r > 5M. \end{cases}$$

Repeating the construction above with respect to this new foliation then defines past null infinity \mathcal{I}^- . Proceeding in an analogous fashion to Definition 4.2.1, \mathcal{I}^- may be glued to $\tilde{\mathcal{R}}$ as a suitable boundary.

Lastly, it will be useful to introduce the notations

$$\mathcal{I}_{\geq s}^+ \doteq \{(\tau, \theta, \phi) : \tau \geq s\}, \quad \mathcal{I}_{\leq s}^+ \doteq \{(\tau, \theta, \phi) : \tau \leq s\}.$$

4.2.1. *The radiation field along \mathcal{I}^+ .* – Recall that given a function ψ , in Section 3.8 we introduced the notation

$$\begin{aligned}\varphi &\doteq (r^2 + a^2)^{1/2}\psi, \\ \varphi^{(\alpha)} &\doteq \Omega^{(\alpha)}\varphi.\end{aligned}$$

We now have the following straightforward corollary of Propositions 3.8.1 and 3.8.2.

PROPOSITION 4.2.1. – *For all solutions ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported, and each $(\tau, \theta, \phi) \in \mathbb{R} \times \mathbb{S}^2$, the function*

$$\varphi(\tau, \infty, r, \theta) \doteq \lim_{r \rightarrow \infty} \varphi(\tau, r, \theta, \phi) = \lim_{r \rightarrow \infty} (r^2 + a^2)^{1/2}\psi(\tau, r, \theta, \phi)$$

is well defined, and is in fact a smooth function on \mathcal{I}^+ .

Proof. – Let $r_2 > r_1$. The fundamental theorem of calculus, Cauchy-Schwarz, and a Sobolev inequality on \mathbb{S}^2 imply

$$\begin{aligned}&|\varphi(\tau, r_2, \theta, \phi) - \varphi(\tau, r_1, \theta, \phi)|^2 \\ &\leq B \left(\sum_{|\alpha| \leq 2} \int_{\mathbb{S}^2} \left| \varphi^{(\alpha)}(\tau, r_2, \theta, \phi) - \varphi^{(\alpha)}(\tau, r_1, \theta, \phi) \right| \sin \theta \, d\theta \, d\phi \right)^2 \\ &\leq B \sum_{|\alpha| \leq 2} \left(\int_{S_\tau \cap \{r \geq r_1\}} \left[\left| \partial_{\bar{v}} \varphi^{(\alpha)} \right| + r^{-2} \left| \partial_{\bar{u}} \varphi^{(\alpha)} \right| \right] \sin \theta \, dr \, d\theta \, d\phi \right)^2 \\ &\leq B r_1^{-2} \sum_{|\alpha| \leq 2} \int_{S_\tau \cap \{r \geq r_1\}} \left[r^2 \left| \partial_{\bar{v}} \varphi^{(\alpha)} \right|^2 + r^{-2} \left| \partial_{\bar{u}} \varphi^{(\alpha)} \right|^2 \right] \sin \theta \, dr \, d\theta \, d\phi.\end{aligned}$$

In the second inequality we have used the fundamental theorem of calculus along S_τ and expressed the resulting derivative in terms of $\partial_{\bar{v}}$ and $\partial_{\bar{u}}$.

Now we conclude the proof of existence of the function $\varphi(\tau, \infty, \phi, \theta)$ by observing that Proposition 3.8.2 implies that this last quantity is bounded by $B(\tau)r_1^{-2}$.

Smoothness of φ as a function on \mathcal{I}^+ follows in a straightforward manner by applying the above argument to $\partial_\tau^i \Omega^{(\alpha)}\varphi$, for $i \in \mathbb{Z}_{\geq 0}$ and $|\alpha| \in \mathbb{Z}_{\geq 0}$. \square

REMARK 4.2.3. – *If we combine the proof of Proposition 4.2.1 with Theorem 3.7.1 we may easily conclude that for any $\tau_0 \in \mathbb{R}$, $(r^2 + a^2)^{1/2}\psi$ converges to its limit $\varphi|_{r=\infty}$ in $L^\infty_{\mathbb{R}_{\geq \tau_0} \times \mathbb{S}^2}$.*

Following [38] and using the previous proposition, we may now define the radiation field along \mathcal{I}^+ .

DEFINITION 4.2.3. – *Given a solution ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported, the **radiation field of ψ along \mathcal{I}^+** is defined to be the function $\varphi(\tau, \infty, \theta, \phi)$.*

REMARK 4.2.4. – *Note that any solution ψ to (2) on \mathcal{D} arising from smooth initial data along $\bar{\Sigma}$ which are compactly supported is, a fortiori, a solution to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported (cf. Remark 3.6.1). Thus, this definition of the radiation field may be applied to such solutions.*

REMARK 4.2.5. – Of course, given a solution ψ to (2) defined in the past of $\{t^* = 0\}$, one may analogously define the radiation field along \mathcal{I}^- . In particular, smooth compactly supported data on $\bar{\Sigma}$ give rise to radiation fields along both \mathcal{I}^+ and \mathcal{I}^- .

REMARK 4.2.6. – In passing, we observe that the weighted estimates of Proposition 3.8.2 would allow us to easily show that the radiation field decays along null infinity:

$$|\varphi(\tau, \infty, \theta, \phi)| \leq B\tau^{-1/2}\sqrt{E_0[\psi]} \quad \forall \tau > 0,$$

where E_0 is a weighted higher order energy along Σ_0^* . Again, we emphasize that we shall not need to use such quantitative decay rates in this paper.

REMARK 4.2.7. – Finally, it is worth remarking that in the case of a massive scalar field, the radiation field for any solution arising from regular localized initial data will always vanish [51]. Thus one clearly needs a different approach in that setting.

4.2.2. The energy flux through \mathcal{I}^+ . – In this section we will define the energy flux to future null infinity \mathcal{I}^+ for solutions to the wave equation (2) arising from smooth initial data along Σ_0^* which are compactly supported. Recall that Σ_s^* denotes the hypersurface $\{t^* = s\}$. We begin with the following lemma:

LEMMA 4.2.1. – Given a solution ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported, then for every $\tau > 0$, the following limit exists:

$$(52) \quad \lim_{s \rightarrow \infty} \int_{\Sigma_s^* \cap J^-(S_\tau)} \mathbf{J}_\mu^T[\psi] n_{\Sigma_s^*}^\mu.$$

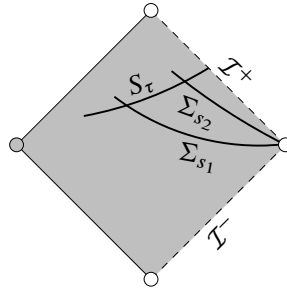
Proof. – First of all, observe that for sufficiently large s , depending on τ , the integration in (52) occurs far outside the ergoregion (30), so that in particular, T is a timelike Killing vector field in the region under consideration. With this in mind, a \mathbf{J}^T energy estimate implies that

$$\int_{S_\tau} \mathbf{J}_\mu^T[\psi] n_{S_\tau}^\mu < \infty.$$

Consequently,

$$\lim_{s \rightarrow \infty} \int_{S_\tau \cap J^+(\Sigma_s^*)} \mathbf{J}_\mu^T[\psi] n_{S_\tau}^\mu = 0.$$

Let $s_1 < s_2$ both be sufficiently large. Refer to the figure below:



It now suffices to observe the following immediate consequence of a \mathbf{J}^T energy estimate:

$$\left| \int_{\Sigma_{s_2} \cap J^-(S_\tau)} \mathbf{J}_\mu^T[\psi] n_{\Sigma_{s_2}}^\mu - \int_{\Sigma_{s_1} \cap J^-(S_\tau)} \mathbf{J}_\mu^T[\psi] n_{\Sigma_{s_1}}^\mu \right| \leq \int_{S_\tau \cap J^+(\Sigma_{s_1})} \mathbf{J}_\mu^T[\psi] n_{S_\tau}^\mu. \quad \square$$

REMARK 4.2.8. – Observe that this lemma holds for essentially any asymptotically flat spacetime possessing a suitable notion of future null infinity; in particular, we do not appeal to Theorem 3.7.1.

REMARK 4.2.9. – We observe that one may easily check that if one considers smooth solutions which satisfy $\int_{\bar{\Sigma} \cap \{r \geq R\}} \mathbf{J}_\mu^T[\psi] n_\Sigma^\mu < \infty$ for all sufficiently large R , but are not necessarily compactly supported, then an easy modification of the proof of Lemma 4.2.1 shows that for all $\tau_0 < \tau_1$, the limit

$$\lim_{s \rightarrow \infty} \int_{\Sigma_s^* \cap J^-(S_{\tau_1}) \cap J^+(S_{\tau_0})} \mathbf{J}_\mu^T[\psi] n_{\Sigma_s^*}^\mu$$

exists.

Lemma 4.2.1 allows us to make the following definitions.

DEFINITION 4.2.4. – Given a solution ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported, and $\tau > -\infty$, **the energy flux of ψ through $\mathcal{I}_{\leq \tau}^+$** is defined by

$$(53) \quad \int_{\mathcal{I}_{\leq \tau}^+} \mathbf{J}_\mu^T[\psi] n_{\mathcal{I}^+}^\mu \doteq \lim_{s \rightarrow \infty} \int_{\Sigma_s^* \cap J^-(S_\tau)} \mathbf{J}_\mu^T[\psi] n_{\Sigma_s^*}^\mu.$$

REMARK 4.2.10. – There is, of course, great flexibility in the choice of the hypersurfaces S_0 and Σ_0^* , but we will forgo a systematic treatment of which choice of hypersurfaces leaves the limit (53) unchanged.

Since $\tau_1 < \tau_2$ implies that $J^-(S_{\tau_1}) \subset J^-(S_{\tau_2})$, it immediately follows that $\int_{\mathcal{I}_{\leq \tau}^+} \mathbf{J}_\mu^T[\psi] n_{\mathcal{I}^+}^\mu$ is an increasing function of τ . Thus, we can make the following definition.

DEFINITION 4.2.5. – Given a solution ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported, **the (total) flux of ψ through null infinity \mathcal{I}^+** is defined by

$$(54) \quad \int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi] n_{\mathcal{I}^+}^\mu \doteq \lim_{\tau \rightarrow \infty} \lim_{s \rightarrow \infty} \int_{\Sigma_s^* \cap J^-(S_\tau)} \mathbf{J}_\mu^T[\psi] n_{\Sigma_s^*}^\mu \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

REMARK 4.2.11. – As with Definition 4.2.4, we note that this definition also makes sense for essentially any spacetime possessing a suitable notion of future null infinity.

Now we observe the following immediate consequence of Definition 4.2.5 and Theorem 3.7.1.

THEOREM 4.2.1. – All sufficiently regular solutions ψ to (2) on $\mathcal{R}_{\geq 0}$ satisfy

$$\int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi] n_{\mathcal{I}^+}^\mu \leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^N[\psi] n_{\Sigma_0^*}^\mu.$$

In particular, in the case of smooth compactly supported initial data of Σ_0^* or $\bar{\Sigma}$, the total flux to null infinity (54) is finite.

Finally, the next proposition establishes the expected connection between the radiation field along null infinity with the energy flux to null infinity

PROPOSITION 4.2.2. – Given a solution ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data along Σ_0^* which are compactly supported, we have

$$\int_{(-\infty, \tau) \times \mathbb{S}^2} |\partial_\tau \varphi(\infty, \tau, \theta, \phi)|^2 \sin \theta \, d\tau \, d\theta \, d\phi = \int_{\mathcal{I}_{\leq \tau}^\pm} \mathbf{J}_\mu^T[\psi] n_{\mathcal{I}^\pm}^\mu \quad \forall \tau \in (-\infty, \infty].$$

Proof. – First of all, a straightforward computation gives

$$\begin{aligned} \int_{\Sigma_s^* \cap J^-(S_\tau)} \mathbf{J}_\mu^T[\psi] n_{\Sigma_s^*}^\mu &= \int_{\Sigma_s^* \cap J^-(S_\tau)} |\partial_\tau \varphi|^2 \sin \theta \, dv \, d\theta \, d\phi \\ &+ O\left(\int_{\Sigma_s^* \cap J^-(S_\tau)} [|\partial_{\bar{v}} \varphi|^2 + |\nabla \varphi|^2] \sin \theta \, dv \, d\theta \, d\phi\right) \text{ as } s \rightarrow \infty. \end{aligned}$$

Now we simply observe that Proposition 3.8.1 (with any choice of $p \in (0, 2]$) implies that we can find a (dyadic) sequence $\{s_i\}_{i=1}^\infty$ such that $\lim_{i \rightarrow \infty} s_i = \infty$ and

$$\lim_{i \rightarrow \infty} \int_{\Sigma_{s_i} \cap J^-(S_\tau)} [|\partial_{\bar{v}} \varphi|^2 + |\nabla \varphi|^2] \sin \theta \, dv \, d\theta \, d\phi = 0. \quad \square$$

5. Carter’s separation and the microlocal radiation fields

As in our previous work [30], estimates obtained by exploiting Carter’s separation of the wave equation (2) will play a fundamental role in our analysis. In this section, we quote a number of results from [67] and [30] concerning the theory of the radial o.d.e (1) for real frequencies ω and its relation to (2). (In Section 6 to follow, we will then obtain various refinements of the quantitative o.d.e. estimates of [30] which will be fundamental for our arguments.)

We begin in Section 5.1 by reviewing our relevant formalism based on the Fourier transform of “sufficiently integrable” solutions (Definition 5.1.1); the reader should consult [30] for more details.

We shall then quote in Section 5.2 some results from [67] concerning the asymptotics of solutions of (1), which in particular allow us to define the special solutions U_{hor} , U_{inf} referred to in the introduction. We state Proposition 5.2.2, the microlocal version of the energy identity (we will consider more general currents in Section 6.1 below).

The Wronskian \mathfrak{W} , as well as the reflection \mathfrak{R} and transmission coefficients \mathfrak{T} referred to already (together with their dual coefficients $\tilde{\mathfrak{R}}$ and $\tilde{\mathfrak{T}}$), are all defined in Section 5.3, appealing to the real-mode stability theorem of [67]. We then obtain Corollary 5.3.1 which gives that the strict inequality (6) indeed holds for any superradiant frequency and establish a fundamental solution formula for the radial o.d.e. in Proposition 5.3.1.

Finally, our separation will allow us to define the “microlocal” radiation fields and fluxes in Section 5.4. (Later, in Section 6.8, these will be related to the radiation fields and degenerate-energy fluxes defined in physical space.)

⁽⁸⁾ The point being that $\int_1^\infty \frac{|f(x)|}{x} dx < \infty$ implies that there exists a sequence $\{x_i\}_{i=1}^\infty$ with $x_i \in [2^i, 2^{i+1}]$ such that $\lim_{i \rightarrow \infty} f(x_i) = 0$.

5.1. Separating the wave equation

We begin by recalling the following definition.

DEFINITION 5.1.1. – We say that a smooth function $\Psi : \overset{\circ}{\mathcal{R}} \rightarrow \mathbb{C}$ is “sufficiently integrable” if for every $j \geq 1$, we have

$$\sum_{0 \leq j_1 + j_2 + |\alpha| \leq j} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \left| \nabla^{\alpha} \partial_r^{j_1} T^{j_2} \Psi \right|^2 \sin \theta \, dt \, d\theta \, d\phi \in L_{\text{loc}}^{\infty}(r^*).$$

REMARK 5.1.1. – We note that this definition is in fact weaker than that given in [30].

REMARK 5.1.2. – Observe that it follows immediately from Proposition 3.6.2, Theorem 3.7.1 and Remark 3.7.4 that any solution to the wave equation arising from smooth compactly supported initial data along $\bar{\Sigma}$ is sufficiently integrable in the sense of Definition 5.1.1 (cf. Remark 3.6.1).

Next, we recall the oblate spheroidal harmonics

$$\{S_{m\ell}(v, \cos \theta) e^{im\phi}\}_{m\ell}, \quad v \in \mathbb{R},$$

which are the eigenfunctions of the self-adjoint operator

$$P(v) f = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} f \right) - \frac{\partial^2 f}{\partial \phi^2} \frac{1}{\sin^2 \theta} - v^2 \cos^2 \theta f$$

on $L^2(\sin \theta \, d\theta \, d\phi)$. We denote the corresponding eigenvalues by $\lambda_{m\ell}^{(v)} \in \mathbb{R}$ where $m \in \mathbb{Z}$ and $l \geq |m|$. The labeling is uniquely determined by requiring that $\lambda_{m\ell}^{(v)}$ depends smoothly on v and setting $\lambda_{m\ell}^{(0)} = \ell(\ell + 1)$.⁽⁹⁾ These satisfy

$$(55) \quad \lambda_{m\ell}^{(v)} + v^2 \geq |m|(|m| + 1),$$

$$(56) \quad \lambda_{m\ell}^{(v)} + v^2 \geq 2|m v|.$$

Because of the above relations, it is often convenient to work with

$$\Lambda_{m\ell}(v) \doteq \lambda_{m\ell}(v) + v^2.$$

Let Ψ be sufficiently integrable in the sense of Definition 5.1.1. Then, setting $v = a\omega$, where a is the Kerr parameter, for each $\omega \in \mathbb{R}$, we decompose

$$\Psi(t, r, \theta, \phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} e^{-i\omega t} \Psi_{m\ell}^{(a\omega)}(r) S_{m\ell}(a\omega, \cos \theta) e^{im\phi} \, d\omega.$$

The sufficiently integrable assumption implies that for each fixed r , this equality may be interpreted in $L_t^2 L_{\mathbb{S}^2}^2$. Now define

$$(57) \quad F = \square_g \Psi.$$

The sufficiently integrable assumption implies that we may define the coefficients $(\rho^2 F)_{m\ell}^{(a\omega)}(r)$ as above (recall that $\rho^2 = r^2 + a^2 \cos^2 \theta$).

Carter’s formal separation [15] of the wave operator yields:

⁽⁹⁾ See Proposition B.1 of [66] for a proof that this does indeed uniquely determine $\{\lambda_{m\ell}^{(v)}\}$.

PROPOSITION 5.1.1. – *Let Ψ be sufficiently integrable in the sense of Definition 5.1.1, and let F be defined by (57). Then*

$$(58) \quad \Delta \frac{d}{dr} \left(\Delta \frac{d\Psi_{m\ell}^{(a\omega)}}{dr} \right) + (a^2 m^2 + (r^2 + a^2)^2 \omega^2 - 4Mr a \omega m - \Delta \Lambda_{m\ell}) \Psi_{m\ell}^{(a\omega)} = \Delta (\rho^2 F)_{m\ell}^{(a\omega)}.$$

Note that the sufficiently integrable assumption allows us to interpret this equality for each r in $L^2_{\omega} l^2_{m\ell}$.

REMARK 5.1.3. – *It will turn out to suffice that we study smooth solutions to the o.d.e. (58). See the discussion in Definition 5.4.1.*

Using the Definition (26) of r^* and setting

$$(59) \quad u_{m\ell}^{(a\omega)}(r) = (r^2 + a^2)^{1/2} \Psi_{m\ell}^{(a\omega)}(r),$$

$$(60) \quad H_{m\ell}^{(a\omega)}(r) = \frac{\Delta (\rho^2 F)_{m\ell}^{(a\omega)}(r)}{(r^2 + a^2)^{3/2}},$$

we obtain

$$(61) \quad \frac{d^2}{(dr^*)^2} u_{m\ell}^{(a\omega)} + (\omega^2 - V_{m\ell}^{(a\omega)}(r)) u = H_{m\ell}^{(a\omega)},$$

where

$$(62) \quad V_{m\ell}^{(a\omega)}(r) = \frac{4Mr a m \omega - a^2 m^2 + \Delta \Lambda_{m\ell}}{(r^2 + a^2)^2} + \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2 + a^2)^3} - \frac{3\Delta^2 r^2}{(r^2 + a^2)^4}.$$

We will often refer to (61) as the “radial o.d.e.”

As in [30], we shall often suppress the dependence of u , H and V on $a\omega$, m , ℓ in our notation. We will also use the notation

$$(63) \quad ' = \frac{d}{dr^*}.$$

Note that

$$r' = \frac{\Delta}{r^2 + a^2}.$$

5.2. Asymptotic analysis of the radial o.d.e.

In this section we will collect various facts concerning the asymptotic analysis of the radial o.d.e. (61). In view of our applications and Remark 5.1.3, all results stated will concern smooth solutions. We will omit proofs as the material is standard (see, e.g., [60]).

PROPOSITION 5.2.1. – *Fix parameters $(\omega, m, \ell) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$ with $\omega \neq 0$ and $\omega \neq \omega_+ m$, and let u be a smooth solution of the radial o.d.e. (61)*

$$u'' + (\omega^2 - V)u = H,$$

where $H(r)$ smoothly extends to $r = r_+$ and vanishes for large r (of course, by using the relation $\frac{d}{dr} = \frac{r^2 + a^2}{\Delta} \frac{d}{dr^}$, the smoothness condition at $r = r_+$ can be translated to a condition on the*

limits of $\frac{d^k H}{d(r^*)^k}$ as $r^* \rightarrow -\infty$). Then there exist unique complex numbers $a_{\mathcal{H}^+}$, $a_{\mathcal{H}^-}$, $a_{\mathcal{I}^+}$, and $a_{\mathcal{I}^-}$, depending on u , such that

$$(64) \quad u = a_{\mathcal{I}^+} e^{i\omega r^*} + a_{\mathcal{I}^-} e^{-i\omega r^*} + O(r^{-1}) \text{ as } r \rightarrow \infty,$$

$$(65) \quad u = a_{\mathcal{H}^+} e^{-i(\omega - \omega_+ m)r^*} + a_{\mathcal{H}^-} e^{i(\omega - \omega_+ m)r^*} + O(r - r_+) \text{ as } r \rightarrow r_+.$$

Here the $O(r^{-1})$ and $O(r - r_+)$ are both preserved upon differentiation in r^* .

Next, we turn to the “microlocal energy identity”.

PROPOSITION 5.2.2. – Fix parameters $(\omega, m, \ell) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$ with $\omega \neq 0$ and $\omega \neq \omega_+ m$, and let u be a smooth solution of the radial o.d.e. (61) with $H(r^*)$ compactly supported in r^* . Then, we have

$$\omega^2 |a_{\mathcal{I}^+}|^2 - \omega^2 |a_{\mathcal{I}^-}|^2 + \omega(\omega - \omega_+ m) |a_{\mathcal{H}^+}|^2 - \omega(\omega - \omega_+ m) |a_{\mathcal{H}^-}|^2 = \omega \int_{-\infty}^{\infty} \text{Im}(H\bar{u}) dr^*.$$

Proof. – We recall the microlocal energy current from [30]:

$$Q^T \doteq \omega \text{Im}(u' \bar{u}),$$

which satisfies

$$(Q^T)' = \omega \text{Im}(H\bar{u}).$$

(The above is of course the most basic energy current associated to (61). We will discuss this and several other currents in Section 6.1). The proposition then follows immediately from the fundamental theorem of calculus and the expansions (64) and (65). \square

It will be useful to introduce the following definitions.

DEFINITION 5.2.1. – Let $(\omega, m, \ell) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$. Then we define $U_{\text{hor}}(r^*, \omega, m, l)$ to be the unique function satisfying

1. $U_{\text{hor}}'' + (\omega^2 - V) U_{\text{hor}} = 0$.
2. $U_{\text{hor}} \sim e^{-i(\omega - \omega_+ m)r^*}$ near $r^* = -\infty$.⁽¹⁰⁾
3. $|U_{\text{hor}}(-\infty)|^2 = 1$.

REMARK 5.2.1. – Note that this definition makes sense even when $\omega - \omega_+ m = 0$ or $\omega = 0$; see, e.g., the discussion in Appendix C.1 of [66].

REMARK 5.2.2. – The physical space interpretation of U_{hor} is that the expression $e^{-it\omega} e^{im\phi} S_{m\ell}(\theta) U_{\text{hor}}(r^*)$ corresponds to an amplitude normalized solution of the wave equation “frequency localized” to (ω, m, ℓ) , with a vanishing energy flux along \mathcal{H}^- and a finite energy flux on any compact subset of \mathcal{H}^+ .

DEFINITION 5.2.2. – For $\omega \neq 0$, define $U_{\text{inf}}(r^*, \omega, m, l)$ to be the unique function satisfying

1. $U_{\text{inf}}'' + (\omega^2 - V) U_{\text{inf}} = 0$.
2. $U_{\text{inf}} \sim e^{i\omega r^*}$ near $r^* = \infty$.⁽¹¹⁾

⁽¹⁰⁾ More precisely, the requirement is that $U_{\text{hor}} e^{i(\omega - \omega_+ m)r^*}$ extends to $r = r_+$ as a smooth function of r .

⁽¹¹⁾ More precisely, this means that U_{inf} exhibits a (generally divergent) asymptotic expansion $U_{\text{inf}} = e^{i\omega r^*} \sum_{i=0}^{\infty} \frac{A_i}{r^i}$ as $r \rightarrow \infty$.

3. $|U_{\text{inf}}(\infty)|^2 = 1.$

REMARK 5.2.3. – *The physical space interpretation of U_{inf} is that $e^{-it\omega} e^{im\phi} S_{m\ell}(\theta) U_{\text{inf}}(r^*)$ corresponds to an amplitude normalized solution of the wave equation “frequency localized” to (ω, m, ℓ) , with a vanishing energy flux along \mathcal{I}^- and a finite energy flux on any compact subset of \mathcal{I}^+ .*

REMARK 5.2.4. – *When $H = 0$, by exploiting the linear independence of the pairs $\{U_{\text{hor}}, \overline{U_{\text{hor}}}\}$ and $\{U_{\text{inf}}, \overline{U_{\text{inf}}}\}$, one may easily check that expansions (64) and (65) may be written as the identities*

$$\begin{aligned} u &= a_{\mathcal{I}^+} U_{\text{inf}} + a_{\mathcal{I}^-} \overline{U_{\text{inf}}}, \\ u &= a_{\mathcal{H}^+} U_{\text{hor}} + a_{\mathcal{H}^-} \overline{U_{\text{hor}}}. \end{aligned}$$

Cf. footnote (4).

REMARK 5.2.5. – *In the usual language of scattering theory, one would refer to $\{U_{\text{hor}}, \overline{U_{\text{hor}}}, U_{\text{inf}}, \overline{U_{\text{inf}}}\}$ as Jost functions [62].*

PROPOSITION 5.2.3. – *The constructions of U_{hor} and U_{inf} imply that for each $k \geq 0$,*

$$\left\| \frac{d^k}{d(r^*)^k} U_{\text{hor}} \right\|_{L_{r^*}^\infty} \leq B(\omega, m, \ell, k),$$

and $\frac{d^k}{d(r^)^k} U_{\text{hor}}$ depends analytically on ω . Similarly, if we additionally assume that $\omega \neq 0$, we also have*

$$\left\| \frac{d^k}{d(r^*)^k} U_{\text{inf}} \right\|_{L_{r^*}^\infty} \leq B(\omega, m, \ell, k),$$

and $\frac{d^k}{d(r^)^k} U_{\text{inf}}$ depends analytically on $\omega \in \mathbb{R} \setminus \{0\}$.*

5.3. The Wronskian and the reflection and transmission coefficients

DEFINITION 5.3.1. – *For $\omega \neq 0$, we define $\mathfrak{W}(\omega, m, \ell)$ to be the Wronskian of U_{hor} and U_{inf} :*

$$\mathfrak{W} \doteq U'_{\text{inf}} U_{\text{hor}} - U_{\text{inf}} U'_{\text{hor}}.$$

REMARK 5.3.1. – *Note that one may easily check that \mathfrak{W} does not depend on r^* and vanishes if and only if U_{hor} and U_{inf} are linearly dependent.*

Using the transformation theory of Whiting from [73], in [67], the following was shown:

THEOREM 5.3.1 ([67]). – *For all $(\omega, m, \ell) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$ with $\omega \neq 0$ we have*

$$\mathfrak{W}(\omega, m, \ell) \neq 0,$$

and thus the functions U_{hor} and U_{inf} are linearly independent.

REMARK 5.3.2. – *In the language of spectral theory the non-vanishing of the Wronskian rules out embedded resonances on the real axis; it can then be used to establish a suitable “limiting absorption principle,” see [67]. Previously, Whiting showed that there do not exist eigenvalues in the upper half-plane [73]. His result can be recovered from Theorem 5.3.1 and a continuity argument in the parameter a . Similarly, our proof of Theorem 3.7.1 in [30], which exploits a continuity argument in the parameter a , also appeals directly only to Theorem 5.3.1. It is this continuity principle that allows our spectral analysis to only concern real frequencies.*

The non-vanishing of the Wronskian will allow us to define the *reflection* and *transmission* coefficients. First we need the following lemma which follows immediately from Remark 5.3.1 and the non-vanishing of the Wronskian.

LEMMA 5.3.1. – *For $\omega \neq 0$ and $\omega \neq \omega_+m$, there exists a unique set of complex numbers $\mathfrak{R}(\omega, m, \ell)$, $\tilde{\mathfrak{R}}(\omega, m, \ell)$, $\mathfrak{T}(\omega, m, \ell)$ and $\tilde{\mathfrak{T}}(\omega, m, \ell)$ which satisfy*

$$(66) \quad \frac{\mathfrak{T}}{-i(\omega - \omega_+m)} U_{\text{hor}} = \frac{\mathfrak{R}}{i\omega} U_{\text{inf}} + \frac{\overline{U_{\text{inf}}}}{i\omega},$$

$$(67) \quad \frac{\tilde{\mathfrak{T}}}{i\omega} U_{\text{inf}} = \frac{\tilde{\mathfrak{R}}}{-i(\omega - \omega_+m)} U_{\text{hor}} + \frac{\overline{U_{\text{hor}}}}{-i(\omega - \omega_+m)},$$

Now we can define the reflection and transmission coefficients.

DEFINITION 5.3.2. – *The complex numbers \mathfrak{R} and $\tilde{\mathfrak{R}}$ are called the **reflection** coefficients, and \mathfrak{T} and $\tilde{\mathfrak{T}}$ are called the **transmission** coefficients.*

REMARK 5.3.3. – *If one considers a solution to the wave equation which is “sourced” with a flux along \mathcal{I}^- equal to 1 and no energy along \mathcal{H}^- and which is furthermore approximately localized to the frequency (ω, m, ℓ) , then \mathfrak{R} measures the amount of energy “reflected” back to future null infinity \mathcal{I}^+ , and \mathfrak{T} measures the energy “transmitted” to the future event horizon \mathcal{H}^+ . There is a similar interpretation for $\tilde{\mathfrak{R}}$ and $\tilde{\mathfrak{T}}$. Our Theorem 9.5.3 will make these interpretations rigorous.*

REMARK 5.3.4. – *One often sees the reflection and transmission coefficients \mathfrak{R} and \mathfrak{T} defined so that they measure the amplitude transmitted to the future event horizon and reflected to future null infinity of a wave of amplitude 1 along \mathcal{I}^- , see e.g., Section 28 of [16]. However, in the context of scattering theory for finite energy solutions, one does not expect to control the radiation fields ψ and ϕ in L^2 along \mathcal{H}^\pm and \mathcal{I}^\pm , hence an energy normalization is most natural.*

Applying Proposition 5.2.2 immediately yields

COROLLARY 5.3.1. – *Fix a frequency triple (ω, m, ℓ) which satisfy $\omega \neq 0$ and $\omega \neq \omega_+m$. Then*

$$|\mathfrak{R}|^2 + \frac{\omega}{\omega - \omega_+m} |\mathfrak{T}|^2 = 1.$$

In particular, if

$$(68) \quad \omega(\omega - \omega_+m) < 0,$$

i.e., the parameters are superradiant, then

$$|\mathfrak{R}|^2 > 1.$$

Proof. – For the second statement, it suffices to note that the basic local existence theory for the radial o.d.e. implies that $\mathfrak{F} \neq 0$ (see [60]). \square

Though the reflection and transmission coefficient have a nice interpretation in terms of the scattering of waves coming from \mathcal{H}^- and \mathcal{I}^- , for technical reasons they are not always the most convenient way to parameterize solutions to the radial o.d.e. Instead we shall often use the following quantities.

DEFINITION 5.3.3. – For $\omega \neq 0$ and $\omega \neq \omega_+ m$, we define the complex numbers $\mathfrak{A}_{\mathcal{I}^+}(\omega, m, \ell)$, $\mathfrak{A}_{\mathcal{I}^-}(\omega, m, \ell)$, $\mathfrak{A}_{\mathcal{H}^+}(\omega, m, \ell)$, and $\mathfrak{A}_{\mathcal{H}^-}(\omega, m, \ell)$ by

$$\begin{aligned} U_{\text{hor}} &= \mathfrak{A}_{\mathcal{I}^+} e^{i\omega r^*} + \mathfrak{A}_{\mathcal{I}^-} e^{-i\omega r^*} + O(r^{-1}) \text{ as } r^* \rightarrow \infty, \\ U_{\text{inf}} &= \mathfrak{A}_{\mathcal{H}^+} e^{-i(\omega - \omega_+ m)r^*} + \mathfrak{A}_{\mathcal{H}^-} e^{i(\omega - \omega_+ m)r^*} + O(r - r_+) \text{ as } r \rightarrow r_+. \end{aligned}$$

Observe that $\mathfrak{A}_{\mathcal{I}^+}(\omega, m, \ell)$, $\mathfrak{A}_{\mathcal{I}^-}(\omega, m, \ell)$, $\mathfrak{A}_{\mathcal{H}^+}(\omega, m, \ell)$, and $\mathfrak{A}_{\mathcal{H}^-}(\omega, m, \ell)$ must obey the following constraints.

LEMMA 5.3.2. – We have

$$\begin{aligned} \mathfrak{A}_{\mathcal{I}^+} \mathfrak{A}_{\mathcal{H}^+} + \mathfrak{A}_{\mathcal{I}^-} \overline{\mathfrak{A}_{\mathcal{H}^-}} &= 1, \\ \mathfrak{A}_{\mathcal{I}^+} \mathfrak{A}_{\mathcal{H}^+} + \overline{\mathfrak{A}_{\mathcal{I}^-}} \mathfrak{A}_{\mathcal{H}^-} &= 1, \\ \mathfrak{A}_{\mathcal{I}^+} \mathfrak{A}_{\mathcal{H}^-} + \mathfrak{A}_{\mathcal{I}^-} \overline{\mathfrak{A}_{\mathcal{H}^+}} &= 0, \\ \mathfrak{A}_{\mathcal{I}^-} \mathfrak{A}_{\mathcal{H}^+} + \mathfrak{A}_{\mathcal{H}^-} \overline{\mathfrak{A}_{\mathcal{I}^+}} &= 0. \end{aligned}$$

Proof. – We may write

$$\begin{aligned} (69) \quad U_{\text{hor}} &= \mathfrak{A}_{\mathcal{I}^+} U_{\text{inf}} + \mathfrak{A}_{\mathcal{I}^-} \overline{U_{\text{inf}}} \\ &= \mathfrak{A}_{\mathcal{I}^+} (\mathfrak{A}_{\mathcal{H}^+} U_{\text{hor}} + \mathfrak{A}_{\mathcal{H}^-} \overline{U_{\text{hor}}}) + \mathfrak{A}_{\mathcal{I}^-} (\overline{\mathfrak{A}_{\mathcal{H}^+}} \overline{U_{\text{hor}}} + \overline{\mathfrak{A}_{\mathcal{H}^-}} U_{\text{hor}}) \\ &= (\mathfrak{A}_{\mathcal{I}^+} \mathfrak{A}_{\mathcal{H}^+} + \mathfrak{A}_{\mathcal{I}^-} \overline{\mathfrak{A}_{\mathcal{H}^-}}) U_{\text{hor}} + (\mathfrak{A}_{\mathcal{I}^+} \mathfrak{A}_{\mathcal{H}^-} + \mathfrak{A}_{\mathcal{I}^-} \overline{\mathfrak{A}_{\mathcal{H}^+}}) \overline{U_{\text{hor}}}. \end{aligned}$$

Similarly,

$$U_{\text{inf}} = (\mathfrak{A}_{\mathcal{H}^+} \mathfrak{A}_{\mathcal{I}^+} + \overline{\mathfrak{A}_{\mathcal{I}^-}} \mathfrak{A}_{\mathcal{H}^-}) U_{\text{inf}} + (\mathfrak{A}_{\mathcal{H}^+} \mathfrak{A}_{\mathcal{I}^-} + \overline{\mathfrak{A}_{\mathcal{I}^+}} \mathfrak{A}_{\mathcal{H}^-}) \overline{U_{\text{inf}}}.$$

The lemma follows immediately. \square

The following relationships may be easily verified in a similar fashion to Lemma 5.3.2.

LEMMA 5.3.3. – We have

$$\begin{aligned} \mathfrak{W} &= 2i\omega \mathfrak{A}_{\mathcal{I}^-}, & \mathfrak{W} &= 2i(\omega - \omega_+ m) \mathfrak{A}_{\mathcal{H}^-}, \\ \mathfrak{R} &= -\overline{\mathfrak{A}_{\mathcal{H}^+}} (\mathfrak{A}_{\mathcal{H}^-})^{-1}, & \mathfrak{F} &= -\frac{(\omega - \omega_+ m)}{\omega} (\mathfrak{A}_{\mathcal{I}^-})^{-1}, \\ \tilde{\mathfrak{R}} &= -\overline{\mathfrak{A}_{\mathcal{I}^+}} (\mathfrak{A}_{\mathcal{I}^-})^{-1}, & \tilde{\mathfrak{F}} &= -\frac{\omega}{(\omega - \omega_+ m)} (\mathfrak{A}_{\mathcal{H}^-})^{-1}. \end{aligned}$$

We close the section with a final remark:

REMARK 5.3.5. – *By exploiting the underlying analyticity (cf. Corollary 6.5.1), one can in fact define the reflection and transmission coefficients almost everywhere without the real mode stability result of [67], quoted here as Theorem 5.3.1. Given this, we see that Theorem 5.3.1 is equivalent to the statement that the reflection and transmission coefficients are bounded on any compact set of frequencies, with a bound depending however on the set. The fact the reflection and transmission coefficients are uniformly bounded over all frequencies is the content of Theorem 6.2.2, to be proven in Section 6.2.*

We end this section with a final corollary of Theorem 5.3.1 which concerns a fundamental-solution representation of solutions u of (61) with vanishing $a_{\mathcal{H}^-} = a_{\mathcal{T}^-} = 0$.

PROPOSITION 5.3.1. – *Let u be a smooth solution to the radial o.d.e. (61) with a right hand side H such that $H(r)$ smoothly extends to $r = r_+$ and vanishes for large r , and such that u satisfies $a_{\mathcal{H}^-} = a_{\mathcal{T}^-} = 0$. Then u is given by the following explicit formula:*

(70)

$$u(r^*) = \mathfrak{W}^{-1} \left(U_{\text{inf}}(r^*) \int_{-\infty}^{r^*} U_{\text{hor}}(x^*) H(x^*) dx^* + U_{\text{hor}}(r^*) \int_{r^*}^{\infty} U_{\text{inf}}(x^*) H(x^*) dx^* \right).$$

Proof. – Given the non-vanishing of the Wronskian (Theorem 5.3.1), this is a trivial computation. \square

REMARK 5.3.6. – *In the language of spectral theory, (70) is simply a formula for the continuous extension of the resolvent to the real axis.*

5.4. The microlocal radiation fields and fluxes

We are now ready to define the microlocal radiation fields. As the name suggests, the definition of the microlocal radiation fields relies on the Fourier transform; hence, we will only be able to define the microlocal radiation fields for a solution ψ if it is defined on all of $\mathring{\mathcal{R}}$, not just $\mathring{\mathcal{R}}_{\geq 0}$.

DEFINITION 5.4.1. – *For all solutions ψ to (2) on $\mathring{\mathcal{R}}$, which are sufficiently integrable in the sense of Definition 5.1.1, we may apply Carter's separation to ψ and define the corresponding function u . An easy argument (one can slightly modify the proof of Lemma 5.4.1 of [30]) implies that for almost every ω and every (m, ℓ) , u will be a smooth solution to the radial o.d.e. (61) with $H = 0$. In particular, we may apply Proposition 5.2.1 and easily show that the corresponding $a_{\mathcal{T}^\pm}(\omega, m, \ell)$ and $a_{\mathcal{H}^\pm}(\omega, m, \ell)$ are measurable functions of (ω, m, ℓ) .*

The microlocal radiation field along \mathcal{I}^\pm associated to ψ is then defined almost everywhere by the measurable function

$$a_{\mathcal{T}^\pm}(\omega, m, \ell) : \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|} \rightarrow \mathbb{C},$$

and the microlocal radiation field along \mathcal{H}^\pm associated to ψ is defined almost everywhere by the measurable function

$$a_{\mathcal{H}^\pm}(\omega, m, \ell) : \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|} \rightarrow \mathbb{C}.$$

We also have the corresponding total fluxes.

DEFINITION 5.4.2. – For all solutions ψ to (2) on $\mathring{\mathcal{R}}$, which are sufficiently integrable in the sense of Definition 5.1.1, the total microlocal energy flux through \mathcal{I}^\pm associated to ψ is given by

$$\int_{-\infty}^{\infty} \sum_{m\ell} \omega^2 |a_{\mathcal{I}^\pm}|^2 d\omega \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

and the total microlocal (degenerate) energy flux through \mathcal{H}^\pm associated to ψ is given by

$$\int_{-\infty}^{\infty} \sum_{m\ell} (\omega - \omega_+ m)^2 |a_{\mathcal{H}^\pm}|^2 d\omega \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

These latter fluxes will be related in Section 6.8 below to the flux to \mathcal{I}^+ and the degenerate K -energy flux to \mathcal{H}^+ defined previously in Sections 4.2.2 and 4.1.2, respectively.

6. Estimates for the radial o.d.e. and applications

In this section we will produce estimates for the radial o.d.e. (61) and give some useful applications. These estimates are refinements of estimates originally proven in [30].

In Section 6.1 we review the separated current template from [25] and [30]. These currents form the essential ingredients for all of the o.d.e. estimates of this section.

In Section 6.2 we start by proving Theorem 6.2.1 which is a general estimate for solutions to the radial o.d.e. with a vanishing right hand side; the proof of Theorem 6.2.1 will heavily rely on Theorem 8.1 from [30]. As a corollary we will obtain the uniform boundedness of all reflexion and transmission coefficients (Theorem 6.2.2). This gives in particular Theorem 11 of Section 2.4.1. We will also obtain a Wronskian bound (Proposition 6.2.1) which will be used in Section 7.

In Section 6.3 we will prove Proposition 6.3.1 which gives asymptotic control of U_{hor} in the superradiant regime as $r \rightarrow r_+$ independent of the frequency parameters. Proposition 6.3.1 plays an important role in Section 7. The proof of Proposition 6.3.1 will require us to quote a special case of Theorem 8.1 from [30] (here given as Theorem 6.3.1).

Next, using closely related ideas, in Section 6.4 we will prove Proposition 6.4.1 which states that for fixed ω and m , the large- ℓ limit of \mathfrak{T} must vanish. As a corollary, we deduce that $\lim_{\ell \rightarrow \infty} |\mathfrak{R}| = 1$.

In Section 6.5 we will approve Proposition 6.5.1 which states that for each fixed m and ℓ , the reflection coefficient \mathfrak{R} is not identically 0 as a function of ω . Using analyticity of \mathfrak{R} , one corollary will be that \mathfrak{R} can only vanish at isolated points.

In Section 6.6 we will interpret the weighted r^p hierarchy of estimates of [24] (given previously as Proposition 3.8.1 of Section 3.8) directly at the level of the o.d.e. (1). The main result is Proposition 6.6.1. We will then use this in Section 6.7 to give a quantitative estimate on the rate of convergence of the microlocal radiation field (Proposition 6.7.1). Using these results, in Section 6.8, we will succeed in relating the microlocal radiation fields of Section 5.4 with the physical-space definitions given previously in Section 4.

6.1. The separated current templates

In this section we will recall the separated current template from [25] and [30]. All of our o.d.e. estimates will be based on suitable combinations of these currents.

PROPOSITION 6.1.1. – *Fix parameters $(\omega, m, \ell) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$ with $\omega \neq 0$, and let u be a smooth solution of the radial o.d.e. (61)*

$$u'' + (\omega^2 - V)u = H.$$

Let $h(r^*)$ be a C^2 function, $y(r^*)$ be a C^1 function and $z(r)$ a C^1 function of r . Set $\tilde{V} = V - V|_{r=r^+}$. Then we define the \mathcal{Q}^h current

$$\mathcal{Q}^h[u] \doteq h \operatorname{Re}(u' \bar{u}) - \frac{1}{2} h' |u|^2,$$

the \mathcal{L}^y current

$$\mathcal{L}^y[u] \doteq y \left(|u'|^2 + (\omega^2 - V) |u|^2 \right),$$

the microlocal redshift current

$$(71) \quad \mathcal{Q}_{\text{red}}^z[u] \doteq z |u' + i(\omega - \omega_+ m)u|^2 - z \tilde{V} |u|^2,$$

the microlocal r^p current

$$(72) \quad \mathcal{Q}_{r^p}^z[u] \doteq z |u' - i\omega u|^2 - zV |u|^2,$$

the microlocal T -energy current

$$(73) \quad \mathcal{Q}^T[u] \doteq \omega \operatorname{Im}(u' \bar{u})$$

and the microlocal K -energy current

$$(74) \quad \mathcal{Q}^K[u] \doteq (\omega - \omega_+ m) \operatorname{Im}(u' \bar{u}).$$

We have

$$(75) \quad (\mathcal{Q}^h[u])' = h \left(|u'|^2 + (V - \omega^2) |u|^2 \right) - \frac{1}{2} h'' |u|^2 + h \operatorname{Re}(u \bar{H}),$$

$$(76) \quad (\mathcal{L}^y[u])' = y' \left(|u'|^2 + (\omega^2 - V) |u|^2 \right) - yV' |u|^2 + 2y \operatorname{Re}(u' \bar{H}),$$

$$(77) \quad (\mathcal{Q}_{\text{red}}^z[u])' = z' |u' + i(\omega - \omega_+ m)u|^2 - (z\tilde{V})' |u|^2 + 2z \operatorname{Re} \left(H \overline{u' + i(\omega - \omega_+ m)u} \right),$$

$$(78) \quad (\mathcal{Q}_{r^p}^z[u])' = z' |u' - i\omega u|^2 - (zV)' |u|^2 + 2z \operatorname{Re} \left(H \overline{u' - i\omega u} \right),$$

$$(\mathcal{Q}^T[u])' = \omega \operatorname{Im}(H \bar{u}),$$

$$(79) \quad (\mathcal{Q}^K[u])' = (\omega - \omega_+ m) \operatorname{Im}(H \bar{u}).$$

The identities above follow by direct computation. Note that we have already used the \mathcal{Q}^T current (73) in Proposition 5.2.2.

REMARK 6.1.1. – *Note that the microlocal r^p current appears for the first time in this paper. The reader may find it illuminating to compare (71) with (72).*

6.2. The microlocal ILED estimate and applications

Recall that the microlocal radiation fields $a_{\mathcal{I}^\pm}$ and $a_{\mathcal{H}^\pm}$ were defined in Definition 5.4.1. With this notation, in our previous work [30], estimates for the radial o.d.e. (61), in all frequency ranges, with a non-zero right hand side H and u satisfying $a_{\mathcal{H}^-} = a_{\mathcal{I}^-} = 0$ played a fundamental role in the proof of Theorem 3.7.1. Besides depending on Theorem 3.7.1 as stated, in the present paper we will also require Theorem 6.2.1, which is a variant of the o.d.e. estimates of [30], concerning now solutions to the *homogeneous* radial o.d.e. (61) where we do *not* however assume that $a_{\mathcal{H}^-} = a_{\mathcal{I}^-} = 0$. We will thus prove this latter theorem in the present section, referring to constructions in our [30]. We will close the section with two corollaries of Theorem 6.2.1: Theorem 6.2.2 which gives the boundedness of the reflection and transmission coefficients and Proposition 6.2.1 which gives a uniform bound on the Wronskian.

6.2.1. *The microlocal ILED estimate for the homogeneous radial o.d.e.*– We will prove here the following variant of Theorem 8.1 of [30] which applies to solutions of the *homogeneous* o.d.e. (61) (with $H = 0$) but allows general asymptotics $a_{\mathcal{H}^\pm} \neq 0, a_{\mathcal{I}^\pm} \neq 0$.

THEOREM 6.2.1. – *There exist parameters s_- and s_+ satisfying $r_+ < 3M - s_- < 3M + s_+ < \infty$ such that for all $-\infty < R_-^* < R_+^* < \infty$, the following is true. Given (ω, m, ℓ) satisfying $\omega \neq 0$ and $\omega \neq \omega_+ m$, there exists a parameter $r_{\text{trap}}(\omega, m, \ell)$ with*

$$r_{\text{trap}} = 0 \quad \text{or} \quad r_{\text{trap}} \in [3M - s_-, 3M + s_+],$$

such that for all smooth solutions u to the radial o.d.e. (61) with vanishing right hand side $H = 0$,

(80)

$$\begin{aligned} (\omega - \omega_+ m)^2 |a_{\mathcal{H}^-}|^2 + \omega^2 |a_{\mathcal{I}^-}|^2 + \int_{R_-^*}^{R_+^*} \left[|u'|^2 + \left((1 - r_{\text{trap}} r^{-1})^2 (\omega^2 + \Lambda) + 1 \right) |u|^2 \right] dr^* \\ \leq B(R_-^*, R_+^*) \left[(\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2 + \omega^2 |a_{\mathcal{I}^+}|^2 \right]. \end{aligned}$$

REMARK 6.2.1. – *Recall that the degeneration due to the $(1 - r_{\text{trap}} r^{-1})^2$ term arises because of trapping. See the discussion in [30].*

REMARK 6.2.2. – *Note that applying the theorem to \bar{u} yields the same statement with the roles of $a_{\mathcal{H}^-}$ and $a_{\mathcal{I}^-}$ interchanged with $a_{\mathcal{H}^+}$ and $a_{\mathcal{I}^+}$.*

REMARK 6.2.3. – *Let us emphasize that even though we require $\omega \neq 0$ and $\omega \neq \omega_+ m$ in order to define $a_{\mathcal{H}^\pm}$ and $a_{\mathcal{I}^\pm}$, the constant $B(R_-^*, R_+^*)$ is according to our conventions in Section 3.3 independent of the frequency parameters and in particular does not blow up in either of the limits $\omega \rightarrow 0$ or $\omega \rightarrow \omega_+ m$. The fact that we have a uniform estimate as $\omega \rightarrow 0$ does not follow from Theorem 5.3.1; a separate low-frequency argument is necessary, see [30].*

Proof. – We recall that in [30] we studied solutions to radial o.d.e. (61) with a non-zero right hand side H and u satisfying $a_{\mathcal{H}^-} = a_{\mathcal{I}^-} = 0$, whereas here $H = 0$ but all $a_{\mathcal{H}^\pm}, a_{\mathcal{I}^\pm}$ are in general nontrivial.

We begin with the important observation that in (version 2! of) [30] in the proof of Theorem 8.1 we used microlocal currents (see Section 6.1) where the functions f, h , etc. were all *bounded* as $r^* \rightarrow \pm\infty$. The currents which led to the positive bulk of the microlocal

ILED statement produced (1) a term associated to the inhomogeneity H and (2) boundary terms which were proportional to $(\omega - \omega_+ m)^2 |u(-\infty)|^2$ and $\omega^2 |u(\infty)|^2$. These boundary terms were eventually controlled with suitable applications of (cut-off versions of) the Q^T and Q^K currents. In order for this to work, one key point was that under the assumptions $a_{\mathcal{H}^-} = a_{\mathcal{I}^-} = 0$ we have

$$(81) \quad \left| Q^T|_{r=r_+} \right| = |\omega(\omega - \omega_+ m) |u(-\infty)||^2, \quad Q^T_{r=\infty} = \omega^2 |u(\infty)|^2,$$

$$(82) \quad Q^K_{r=r_+} = (\omega - \omega_+ m)^2 |u(-\infty)|^2, \quad \left| Q^K_{r=\infty} \right| = |\omega(\omega - \omega_+ m) |u(\infty)||^2.$$

Now we consider the case of a solution u to the radial o.d.e. (61) with a vanishing right hand side H but where we make no assumption about the vanishing or non-vanishing of $a_{\mathcal{H}^\pm}$ and $a_{\mathcal{I}^\pm}$. Since all of the multipliers discussed above are bounded, we immediately observe that we may apply all the currents from (version 2 of) [30] to u .

The term associated to the inhomogeneity in the resulting identity, of course, now vanishes since $H = 0$.

However, every application of the microlocal energy currents Q^K and Q^T will yield now various boundary terms each of which will be proportional to one of $(\omega - \omega_+ m)^2 |a_{\mathcal{H}^-}|^2$, $(\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2$, $\omega^2 |a_{\mathcal{I}^+}|^2$, or $\omega^2 |a_{\mathcal{I}^-}|^2$. Furthermore, the term proportional to $(\omega - \omega_+ m)^2 |a_{\mathcal{H}^-}|^2$ will always enter with the opposite sign of the term proportional to $(\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2$. An analogous relation holds for the terms proportional to $\omega^2 |a_{\mathcal{I}^+}|^2$ and $\omega^2 |a_{\mathcal{I}^-}|^2$. In particular, we do not have (81) and (82) and we cannot hope to prove an estimate with all of the microlocal radiation fields on the left hand side⁽¹²⁾. We are thus forced to always put the boundary terms associated to one of the pairs $(a_{\mathcal{H}^-}, a_{\mathcal{I}^-})$ and $(a_{\mathcal{H}^+}, a_{\mathcal{I}^+})$ on the right hand side.

Given these observations, the following estimate immediately follows from the proof of Theorem 8.1 of [30]:

$$(83) \quad (\omega - \omega_+ m)^2 |a_{\mathcal{H}^-}|^2 + \omega^2 |a_{\mathcal{I}^-}|^2 + \int_{R_-^*}^{R_+^*} \left[|u'|^2 + \left((1 - r_{\text{trap}} r^{-1})^2 (\omega^2 + \Lambda) + 1 \right) |u|^2 \right] dr^* \\ \leq B(R_-^*, R_+^*) \left[(\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2 + \omega^2 |a_{\mathcal{I}^+}|^2 \right. \\ \left. + 1_{\{\omega_{\text{low}} \leq |\omega| \leq \omega_{\text{high}}\} \cap \{\Lambda \leq \epsilon_{\text{width}}^{-1} \omega_{\text{high}}^2\}} |a_{\mathcal{H}^-}|^2 \right],$$

for a parameter $r_{\text{trap}}(\omega, m, \ell)$ satisfying

$$r_{\text{trap}} = 0 \quad \text{or} \quad r_{\text{trap}} \in [3M - s_-, 3M + s_+],$$

where $1_{\{\omega_{\text{low}} \leq |\omega| \leq \omega_{\text{high}}\} \cap \{\Lambda \leq \epsilon_{\text{width}}^{-1} \omega_{\text{high}}^2\}}$ denotes the indicator function for the set

$$\mathcal{F}_b \doteq \{(\omega, m, \ell) : \omega_{\text{low}} \leq |\omega| \leq \omega_{\text{high}} \text{ and } \{\Lambda \leq \epsilon_{\text{width}}^{-1} \omega_{\text{high}}^2\}\}.$$

The ω_{low} , ω_{high} and ϵ_{width} are fixed constants which arise during the proof of Theorem 8.1. Thus we have established (80) for frequencies $(\omega, m, \ell) \notin \mathcal{F}_b$.

⁽¹²⁾ This is not so surprising of course, because if we could prove such an estimate we would deduce that u had to vanish!

In order to finish the proof we need to show that $(\omega, m, \ell) \in \mathcal{F}_b$ implies

$$|a_{\mathcal{H}^-}|^2 \leq B \left[(\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2 + \omega^2 |a_{\mathcal{I}^+}|^2 \right].$$

Consider the solution u^\dagger to the radial o.d.e. defined by

$$u^\dagger = \left(a_{\mathcal{I}^+} \overline{\mathfrak{W}^{-1}(2i\omega)} \right) \overline{U_{\text{hor}}} + \left(a_{\mathcal{H}^+} \overline{\mathfrak{W}^{-1}(2i(\omega - \omega_+ m))} \right) \overline{U_{\text{inf}}},$$

and let $a_{\mathcal{H}^+}^\dagger$ and $a_{\mathcal{I}^+}^\dagger$ denote the microlocal radiation fields of u^\dagger .

Observe that Lemma 5.3.3 implies that

$$a_{\mathcal{H}^+}^\dagger = a_{\mathcal{H}^+}, \quad a_{\mathcal{I}^+}^\dagger = a_{\mathcal{I}^+}.$$

Thus, applying Theorem 5.3.1 and Remark 5.3.1 to $u - u^\dagger$ implies that $u = u^\dagger$. Using the explicit definition of u^\dagger , appealing to Theorem 5.3.1 again and using the compactness of \mathcal{F}_b , we immediately conclude that

$$\begin{aligned} |a_{\mathcal{H}^-}|^2 &= \left| a_{\mathcal{H}^-}^\dagger \right|^2 \leq B \mathfrak{W}^{-2} \left[\omega^2 |a_{\mathcal{I}^+}|^2 + (\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2 \right] \\ &\leq B \left[\omega^2 |a_{\mathcal{I}^+}|^2 + (\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2 \right]. \end{aligned} \quad \square$$

6.2.2. *Uniform boundedness of \mathfrak{R} and \mathfrak{T} .* – Applying Theorem 6.2.1 to the solutions U_{hor} and U_{inf} immediately implies that the reflection and transmission coefficients are bounded uniformly in (ω, m, ℓ) .

THEOREM 6.2.2. – *The reflection and transmission coefficients are uniformly bounded:*

$$|\mathfrak{R}|^2 + \left| \tilde{\mathfrak{R}} \right|^2 + |\mathfrak{T}|^2 + \left| \tilde{\mathfrak{T}} \right|^2 \leq B.$$

Proof. – We simply note that by the definition of \mathfrak{R} and \mathfrak{T} , there exists a solution u to the radial o.d.e. such that

$$a_{\mathcal{H}^+} = \frac{\mathfrak{T}}{-i(\omega - \omega_+ m)}, \quad a_{\mathcal{H}^-} = 0, \quad a_{\mathcal{I}^+} = \frac{\mathfrak{R}}{i\omega}, \quad a_{\mathcal{I}^-} = \frac{1}{i\omega}.$$

Theorem 6.2.1 immediately yields

$$|\mathfrak{T}|^2 + |\mathfrak{R}|^2 = (\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2 + \omega^2 |a_{\mathcal{I}^+}|^2 \leq B \left[(\omega - \omega_+ m)^2 |a_{\mathcal{H}^-}|^2 + \omega^2 |a_{\mathcal{I}^-}|^2 \right] \leq B.$$

An analogous argument applies for $\tilde{\mathfrak{R}}$ and $\tilde{\mathfrak{T}}$. □

The above in particular already yields Theorem 11 of Section 2.4.1.

6.2.3. *A bound for the Wronskian \mathfrak{W} .* – We close the section with a uniform bound on the Wronskian which will be useful in Section 7.

PROPOSITION 6.2.1. – *For all $(\omega, m, \ell) \in \mathbb{R} \setminus \{0\} \times \mathbb{Z} \times \mathbb{Z}_{\geq|m|}$ we have*

$$\frac{\omega^2 + \omega_+^2 m^2}{|\mathfrak{W}|^2} \leq B.$$

Proof. – We first apply Theorem 6.2.1 with $u = \overline{U}_{\text{hor}}$. In this case we have

$$a_{\mathcal{H}^-} = 1, \quad a_{\mathcal{H}^+} = 0, \quad a_{\mathcal{I}^-} = \overline{\mathfrak{A}_{\mathcal{I}^+}}, \quad a_{\mathcal{I}^+} = \overline{\mathfrak{A}_{\mathcal{I}^-}} = -\frac{\overline{\mathfrak{W}}}{2i\omega}.$$

In the last equality we have appealed to Lemma 5.3.3. Theorem 6.2.1 then implies

$$(\omega - \omega_+ m)^2 + \omega^2 |\mathfrak{A}_{\mathcal{I}^+}|^2 \leq B |\mathfrak{W}|^2.$$

Dividing through by \mathfrak{W}^2 implies

$$(84) \quad \frac{(\omega - \omega_+ m)^2}{|\mathfrak{W}|^2} \leq B.$$

Next we apply Theorem 6.2.1 with $u = \overline{U}_{\text{inf}}$. In this case we have

$$\begin{aligned} a_{\mathcal{H}^-} &= \overline{\mathfrak{A}_{\mathcal{H}^+}}, & a_{\mathcal{H}^+} &= \overline{\mathfrak{A}_{\mathcal{H}^-}} = -\frac{\overline{\mathfrak{W}}}{2i(\omega - \omega_+ m)}, \\ a_{\mathcal{I}^-} &= \overline{\mathfrak{A}_{\mathcal{I}^+}} = 0, & a_{\mathcal{I}^+} &= \overline{\mathfrak{A}_{\mathcal{I}^-}} = 1. \end{aligned}$$

Again we have appealed to Lemma 5.3.3. Theorem 6.2.1 then implies

$$(\omega - \omega_+ m)^2 |\mathfrak{A}_{\mathcal{H}^+}|^2 + \omega^2 \leq B |\mathfrak{W}|^2.$$

Dividing through by \mathfrak{W}^2 yields

$$(85) \quad \frac{\omega^2}{|\mathfrak{W}|^2} \leq B.$$

Since

$$\omega_+^2 m^2 = (\omega - \omega_+ m - \omega)^2 \leq B \left[(\omega - \omega_+ m)^2 + \omega^2 \right],$$

it is clear that (84) and (85) conclude the proof. \square

6.3. Superradiant estimates for U_{hor}

The frequency range defined below will play an important role in our arguments.

DEFINITION 6.3.1. – *For every $\epsilon > 0$ we define the set $\mathcal{F}_{\#}^{(\epsilon)}$ by*

$$\mathcal{F}_{\#}^{(\epsilon)} \doteq \{(\omega, m, \ell) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq|m|} : am\omega > 0 \text{ and } |\omega| - |\omega_+ m| < \epsilon |m| \}.$$

REMARK 6.3.1. – *Observe that if we set $\epsilon = 0$, then $\mathcal{F}_{\#}^{(\epsilon)}$ would exactly correspond to the superradiant frequencies (68). When $\epsilon > 0$ is small, then $\mathcal{F}_{\#}^{(\epsilon)}$ contains all frequencies which are “close” to being superradiant. These frequencies will later pose the most serious difficulties in the analysis of Section 7.*

REMARK 6.3.2. – *Note that for frequencies in $\mathcal{F}_{\#}^{(\epsilon)}$ we have $\Lambda \geq b(\epsilon) (1 + \omega^2)$.*

In this section, we shall prove

PROPOSITION 6.3.1. – *Let E_{hor} be defined by*

$$U_{\text{hor}}(r^*) = e^{-i(\omega - \omega_+ m)r^*} + E_{\text{hor}}(r^*).$$

Then $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$ for a sufficiently small $\epsilon > 0$ and Λ sufficiently large imply

$$(86) \quad |E_{\text{hor}}| \leq \frac{B(\epsilon) |\mathfrak{W}|}{\sqrt{\Lambda}} \sqrt{r - r_+},$$

for sufficiently small $r - r_+$.

REMARK 6.3.3. – *Note that the $\sqrt{\Lambda}$ factor above represents a “gain of a derivative” over what one would expect to prove if we were not restricting to $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$.*

This proposition will be of fundamental importance in Section 7. To prove it, we will need again to return to our o.d.e theory for (61). We begin with some preliminaries reviewing some additional results and notation from [30]. The proof proper will be contained in Section 6.3.4.

6.3.1. *An inhomogeneous ILED in the superradiant regime.* – The following estimate is a special case of Theorem 8.1 from [30].⁽¹³⁾

THEOREM 6.3.1 ([30]). – *Let $\epsilon > 0$ be sufficiently small and $-\infty < R_-^* < R_+^* < \infty$, then there exists a constant $B(R_-^*, R_+^*, \epsilon)$ such that for all smooth solutions u to the radial o.d.e. (61) with a smooth compactly supported right hand side H , u satisfying $a_{\mathcal{H}^-} = a_{\mathcal{I}^-} = 0$, and frequencies $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$ with Λ sufficiently large, we have*

$$(87) \quad (\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2 + \omega^2 |a_{\mathcal{I}^+}|^2 + \int_{R_-^*}^{R_+^*} \left[|u'|^2 + \Lambda |u|^2 \right] dr^* \leq B(R_-^*, R_+^*, \epsilon) \int_{-\infty}^{\infty} \left(|Hu'| + \sqrt{\Lambda} |Hu| \right) dr^*.$$

REMARK 6.3.4. – *Note that the integrand on the right hand side of (87) does not degenerate (cf. (80) below). This is because the $(\epsilon$ -enlarged) superradiant frequency range $\mathcal{F}_{\#}^{(\epsilon)}$ is not trapped. See the discussion in [30] regarding the fortuitous disjointness of the difficulties of superradiance and trapping.*

⁽¹³⁾ Note that we have strengthened the statement of Theorem 8.1 in version 2 of [30] with Theorems 6.3.1 and 6.2.1 in mind.

6.3.2. *Properties of the potential V in the superradiant regime.* – We recall the following two propositions proved in [30].

PROPOSITION 6.3.2. – *Let $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$ for a sufficiently small $\epsilon > 0$. Then there exists a unique r value r_{\max} where the potential V of (62) achieves its maximum. Furthermore, there exists $\delta > 0$, independent of the frequency parameters, such that*

$$(V - \omega^2)|_{r \in [r_{\max} - \delta, r_{\max} + \delta]} \geq b\Lambda.$$

Furthermore, r_{\max} is uniformly bounded away from r_+ and ∞ .

PROPOSITION 6.3.3. – *Let $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$ for a sufficiently small $\epsilon > 0$, then there exists $\delta_1 > 0$, independent of the frequency parameters, such that*

$$r \in [r_+, r_+ + \delta_1] \Rightarrow \frac{dV}{dr} \geq b(\epsilon)\Lambda.$$

6.3.3. *An improved estimate in the superradiant regime.* – We begin by applying Theorem 6.2.1 to U_{hor} and refer to Lemma 5.3.3 concerning the Wronskian. We obtain

COROLLARY 6.3.1. – *For all frequencies $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$ for a sufficiently small $\epsilon > 0$ and sufficiently large Λ , and for any constants $-\infty < R_-^* < R_+^* < \infty$, we have*

$$(88) \quad (\omega - \omega_+ m)^2 \cdot 1 + \omega^2 |\mathfrak{A}_{\mathcal{I}^+}|^2 + \int_{R_-^*}^{R_+^*} [|U'_{\text{hor}}|^2 + \Lambda |U_{\text{hor}}|^2] dr^* \leq B(R_-^*, R_+^*, \epsilon) |\mathfrak{W}|^2.$$

Proposition 6.3.2 allows us to “gain a derivative” in comparison with Corollary 6.3.1 in the following lemma.

LEMMA 6.3.1. – *There exists $r_1^* > -\infty$ such that for all frequencies $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$ with a sufficiently small $\epsilon > 0$ and Λ sufficiently large, and $r_0^* < r_1^*$, we have*

$$(89) \quad \int_{r_0^*}^{r_1^*} [|U'_{\text{hor}}|^2 + \Lambda |U_{\text{hor}}|^2] dr^* \leq B(r_0^*, \epsilon) \frac{|\mathfrak{W}|^2}{\Lambda}.$$

Proof. – Let u be an arbitrary smooth solution to the homogeneous radial o.d.e. (61), with $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$ for a sufficiently small $\epsilon > 0$ and Λ sufficiently large. Let \tilde{h} be a smooth positive function supported in $[r_{\max} - \delta, r_{\max} + \delta]$ which is identically 1 within $[r_{\max} - \delta/2, r_{\max} + \delta/2]$. Then set $h \doteq \Lambda \tilde{h}$. Using (75), we obtain

$$(90) \quad \begin{aligned} \Lambda \int_{r_{\max} - \delta/2}^{r_{\max} + \delta/2} [|u'|^2 + \Lambda |u|^2] dr^* &\leq \int_{-\infty}^{\infty} (\mathcal{Q}^h[u])' + B(\epsilon)\Lambda \int_{r_{\max} - \delta}^{r_{\max} + \delta} |u|^2 \\ &= B(\epsilon)\Lambda \int_{r_{\max} - \delta}^{r_{\max} + \delta} |u|^2 \\ &\leq B(\epsilon) [(\omega - \omega_+ m)^2 |a_{\mathcal{H}^+}|^2 + \omega^2 |a_{\mathcal{I}^+}|^2]. \end{aligned}$$

In the last line we used Theorem 6.2.1.

In particular, applying the estimate (90) to U_{hor} and then appealing to Corollary 6.3.1 implies

$$(91) \quad \int_{r_{\max}-\delta/2}^{r_{\max}+\delta/2} \left[|U'_{\text{hor}}|^2 + \Lambda |U_{\text{hor}}|^2 \right] dr^* \leq \frac{B(\epsilon) |\mathfrak{W}|^2}{\Lambda}.$$

Now let χ be a function which is identically 1 on $[r_+, r_{\max} - \delta/2]$ and identically 0 on $[r_{\max} + \delta/2, \infty)$, and then set $\tilde{u} \doteq \chi U_{\text{hor}}$. We have

$$\begin{aligned} \tilde{u}'' + (\omega^2 - V) \tilde{u} &= \chi'' U_{\text{hor}} + 2\chi' U'_{\text{hor}} \doteq \tilde{H}, \\ \tilde{a}_{\mathcal{H}^-} &= \tilde{a}_{\mathcal{I}^-} = 0. \end{aligned}$$

Thus, taking r_1^* sufficiently negative, applying Theorem 6.3.1 to \tilde{u} yields

$$\begin{aligned} \int_{r_0^*}^{r_1^*} \left[|U'_{\text{hor}}| + \Lambda |U_{\text{hor}}|^2 \right] dr^* &\leq B(r_0^*, \epsilon) \int_{r_{\max}-\delta/2}^{r_{\max}+\delta/2} \left[|U'_{\text{hor}}|^2 + \Lambda |U_{\text{hor}}|^2 \right] dr^* \\ &\leq \frac{B(r_0^*, \epsilon) |\mathfrak{W}|^2}{\Lambda}. \end{aligned}$$

In the last line we used the estimate (91). □

Now we are ready for the following lemma.

LEMMA 6.3.2. – *There exists a constant $\tilde{\epsilon} > 0$ such that for all frequencies $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$ with a sufficiently small $\epsilon > 0$ and Λ sufficiently large, we have*

$$\int_{-\infty}^{(r_+ + \tilde{\epsilon})^*} |U'_{\text{hor}} + i(\omega - \omega_+ m) U_{\text{hor}}|^2 (r - r_+)^{-1} dr^* \leq \frac{B(\epsilon) |\mathfrak{W}|^2}{\Lambda}.$$

Proof. – We consider the microlocal redshift current (71) with

$$z \doteq -\frac{\Lambda}{V} \chi(r),$$

where χ is a bump function which is identically 1 for $r \in [r_+, r_+ + \tilde{\epsilon}]$ and 0 for $r \in [2\tilde{\epsilon}, \infty)$, for a small positive constant $\tilde{\epsilon}$ to be determined. We obtain from (77) the estimate

$$(92) \quad \begin{aligned} \int_{-\infty}^{(r_+ + \tilde{\epsilon})^*} \left[z' |U'_{\text{hor}} + i(\omega - \omega_+ m) U_{\text{hor}}|^2 \right] dr^* \\ \leq B(\epsilon, \tilde{\epsilon}) \int_{(r_+ + \tilde{\epsilon})^*}^{(r_+ + 2\tilde{\epsilon})^*} \left[|U'_{\text{hor}}|^2 + \Lambda |U_{\text{hor}}|^2 \right] dr^* - Q_{\text{red}}^z|_{r=r_+}. \end{aligned}$$

If $\tilde{\epsilon} > 0$ is small enough, then via Proposition 6.3.3 we see that $r \in (r_+, r_+ + \tilde{\epsilon}]$ implies that $z' \geq b(r - r_+)^{-1}$. In particular, after fixing a small choice of $\tilde{\epsilon}$, we may combine (92) and Lemma 6.3.1 to conclude

$$(93) \quad \int_{-\infty}^{(r_+ + \tilde{\epsilon})^*} |U'_{\text{hor}} + i(\omega - \omega_+ m) U_{\text{hor}}|^2 (r - r_+)^{-1} dr^* \leq \frac{B(\epsilon) |\mathfrak{W}|^2}{\Lambda} - B Q_{\text{red}}^z|_{r=r_+}.$$

We conclude the proof by noting that

$$-Q_{\text{red}}^z|_{r=r_+} = -\Lambda \leq 0. \quad \square$$

6.3.4. *Proof of Proposition 6.3.1.* – Finally, Lemma 6.3.2 easily allows us to prove Proposition 6.3.1.

Proof of Proposition 6.3.1. – Let Λ be sufficiently large. Recall the definition (128) of E_{hor} . It follows that

$$E'_{\text{hor}} = U'_{\text{hor}} + i(\omega - \omega_+ m)U_{\text{hor}}.$$

Assuming $r - r_+$ sufficiently small, we then have

$$\begin{aligned} |E_{\text{hor}}(r^*)| &\leq \int_{-\infty}^{r^*} |E'_{\text{hor}}| ds^* = \int_{-\infty}^{r^*} |U'_{\text{hor}} + i(\omega - \omega_+ m)U_{\text{hor}}| ds^* \\ &\leq B \int_{r_+}^r |U'_{\text{hor}} + i(\omega - \omega_+ m)U_{\text{hor}}| (s - r_+)^{-1} ds \\ &\leq B \sqrt{r - r_+} \sqrt{\int_{r_+}^r |U'_{\text{hor}} + i(\omega - \omega_+ m)U_{\text{hor}}|^2 (s - r_+)^{-2} ds} \\ &\leq B \sqrt{r - r_+} \sqrt{\int_{-\infty}^{r^*} |U'_{\text{hor}} + i(\omega - \omega_+ m)U_{\text{hor}}|^2 (s - r_+)^{-1} ds^*} \\ &\leq \frac{B(\epsilon) |\mathfrak{W}|}{\sqrt{\Lambda}} \sqrt{r - r_+}. \quad \square \end{aligned}$$

6.4. The large- ℓ limit of \mathfrak{T}

It is useful to observe that \mathfrak{T} must vanish in the large- ℓ limit.

PROPOSITION 6.4.1. – For each fixed value of ω and m satisfying $\omega - \omega_+ m \neq 0$, we have

$$\lim_{\ell \rightarrow \infty} \mathfrak{T}(\omega, m, \ell) = 0, \quad \lim_{\ell \rightarrow \infty} \tilde{\mathfrak{T}}(\omega, m, \ell) = 0.$$

Proof. – We will only consider the case of \mathfrak{T} as the proof for $\tilde{\mathfrak{T}}$ is exactly the same.

Fix a pair ω and m such that $\omega - \omega_+ m \neq 0$. Next, pick and fix some value of $r_0 \in (r_+, \infty)$. Then, for all sufficiently large ℓ , there will exist a $\delta > 0$ such that

$$(94) \quad (V - \omega^2)|_{r \in [r_0 - \delta, r_0 + \delta]} \geq b\Lambda.$$

The basic intuition is that for Λ sufficiently large, this large potential barrier will prevent the transmissions of waves to \mathcal{H}^+ . To make this rigorous, we observe that an examination of the beginning of proof of Lemma 6.3.1 shows that (94) implies that if ℓ is sufficiently large

$$(95) \quad \int_{r_0 - \delta/2}^{r_0 + \delta/2} [|U'_{\text{hor}}|^2 + \Lambda |U_{\text{hor}}|^2] dr^* \leq \frac{B}{\Lambda} \int_{r_0 - \delta}^{r_0 + \delta} [|U'_{\text{hor}}|^2 + \Lambda |U_{\text{hor}}|^2] dr^*.$$

Keeping in mind that Lemma 5.3.3 implies

$$U_{\text{hor}} = -\frac{\Re(\omega - \omega_+ m)}{\omega \mathfrak{T}} e^{i\omega r^*} - \frac{(\omega - \omega_+ m)}{\omega \mathfrak{T}} e^{-i\omega r^*} + O(r^{-1}) \text{ as } r \rightarrow \infty,$$

an application of Theorem 6.2.1 implies that

$$(96) \quad \int_{r_0 - \delta}^{r_0 + \delta} [|U'_{\text{hor}}|^2 + \Lambda |U_{\text{hor}}|^2] dr^* \leq B \left[1 + \frac{(\omega - \omega_+ m)^2 |\Re|^2}{|\mathfrak{T}|^2} \right].$$

Combining this with (95) implies

$$(97) \quad \int_{r_0-\delta/2}^{r_0+\delta/2} \left[|U'_{\text{hor}}|^2 + \Lambda |U_{\text{hor}}|^2 \right] dr^* \leq \frac{B(\omega, m, r_0)}{\Lambda} \left[1 + \frac{(\omega - \omega_+ m)^2 |\mathfrak{R}|^2}{|\mathfrak{T}|^2} \right].$$

Intuitively, the estimate (97) shows that U_{hor} must be small near the large potential barrier.

We now want to use an energy estimate to show that if U_{hor} is small near the potential barrier, then \mathfrak{T} must be small. We thus consider the microlocal K -energy current (74) from Proposition 6.1.1. Now let $\chi(r)$ denote a cut-off function which is identically 1 for $r \in [r_+, r_0 - \delta/2]$ and identically 0 for $r \in [r_0 + \delta/2, \infty)$. Then, keeping (97) and (79) in mind,

$$(98) \quad (\omega - \omega_+ m)^2 = \int_{-\infty}^{\infty} (\chi Q^K)' dr^* \leq B \int_{r_0-\delta/2}^{r_0+\delta/2} \left[|U'_{\text{hor}}|^2 + |U_{\text{hor}}|^2 \right] dr^* \leq \frac{B}{\Lambda} \left[1 + \frac{(\omega - \omega_+ m)^2 |\mathfrak{R}|^2}{|\mathfrak{T}|^2} \right].$$

Now we may multiply (98) through by \mathfrak{T} , divide through by $(\omega - \omega_+ m)^2$, take $\ell \rightarrow \infty$ and apply Theorem 6.2.2 to conclude that

$$\lim_{\ell \rightarrow \infty} \mathfrak{T}(\omega, m, \ell) = 0. \quad \square$$

The following corollary follows easily from Proposition 6.4.1.

COROLLARY 6.4.1. – *For each fixed value of ω and m satisfying $\omega - \omega_+ m \neq 0$, we have*

$$\lim_{\ell \rightarrow \infty} \mathfrak{R}(\omega, m, \ell) = 1, \quad \lim_{\ell \rightarrow \infty} \tilde{\mathfrak{R}}(\omega, m, \ell) = 1.$$

Proof. – This follows immediately from Corollary 5.3.1 and Proposition 6.4.1. □

6.5. Nonvanishing of \mathfrak{R}

The next proposition shows that for any fixed m and ℓ , the reflection coefficient \mathfrak{R} cannot be identically 0.

PROPOSITION 6.5.1. – *For each m and ℓ , there exists ω such that $\mathfrak{R}(\omega, m, \ell) \neq 0$ and $\tilde{\mathfrak{R}}(\omega, m, \ell) \neq 0$.*

Proof. – We will only consider the case of \mathfrak{R} since $\tilde{\mathfrak{R}}$ is treated in a similar fashion.

Fix a choice of m and ℓ . Then, for the sake of contradiction, assume that $\mathfrak{R}(\omega, m, \ell)$ is identically 0 in ω . We first consider the case when $\omega_+ m \neq 0$. Then Corollary 5.3.1 implies

$$\frac{\omega}{\omega - \omega_+ m} |\mathfrak{T}|^2 = 1.$$

Then we get a contradiction by considering any ω such that $\omega(\omega - \omega_+ m) < 0$.

The case when $\omega_+ m = 0$ is a bit more subtle. First of all, observe that the vanishing of $\mathfrak{R}(\omega, m, \ell)$ implies that for each ω , we can construct a (non-zero!) solution $u = u(r^*, \omega, m, \ell)$ to the radial o.d.e. such that $u \sim e^{-i\omega r^*}$ as $r^* \rightarrow -\infty$ and $u \sim e^{-i\omega r^*}$ as $r^* \rightarrow \infty$. By direct inspection, one finds that the estimates of Section 8.7.1 of (version 2! of) [30] go through for such a solution (see Remark 8.7.1 at the end of Section 8.7.1), and in particular prove that for ω sufficiently small, u must vanish. This contradiction finishes the proof. □

COROLLARY 6.5.1. – *The reflection coefficients \mathfrak{R} and $\tilde{\mathfrak{R}}$ cannot vanish on an open set of ω .*

Proof. – Standard o.d.e. theory implies that for each fixed m and ℓ , \mathfrak{R} and $\tilde{\mathfrak{R}}$ are analytic in $\omega \in \mathbb{R} \setminus \{0\}$, and, because Proposition 6.5.1 implies that they are not identically 0, we conclude that they can only vanish at isolated points in ω . \square

6.6. The microlocal r^p estimate

In this section we will establish an analog of Proposition 3.8.1 for the function u , using the microlocal r^p current (72).

The following proposition is the microlocal analog of Proposition 3.8.1.

PROPOSITION 6.6.1. – *Fix parameters $(\omega, m, \ell) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$ with $\omega \neq 0$, and let u be a smooth solution of the radial o.d.e. (61)*

$$u'' + (\omega^2 - V)u = H,$$

such that $H(r^*)$ is compactly supported and the constant $a_{\mathcal{T}^-}$ from Proposition 5.2.1 vanishes. Then, for all $p \in [0, 2]$ and sufficiently large R (independent of (ω, m, ℓ) !),

$$\begin{aligned} & \int_{R+1}^{\infty} \left[r^{p-1} |u' - i\omega u|^2 + [(2-p)r^{p-3}\Lambda + r^{p-4}] |u|^2 \right] dr^* \\ & \leq B \int_R^{R+1} r^p (1 + \omega^2 + \Lambda) |u|^2 dr^* + B \int_R^{\infty} |H| [r^p |u' - i\omega u| + |u'|] dr^*. \end{aligned}$$

In the case $p = 2$, then we may moreover add the term $\Lambda |a_{\mathcal{T}^+}|^2$ to the left hand side.

Proof. – We begin by observing that a further asymptotic analysis (see Appendix A of [67]) of u yields

$$u = a_{\mathcal{T}^+} e^{i\omega r^*} \left(1 + \frac{C}{r} + O(r^{-2}) \right) \text{ as } r \rightarrow \infty,$$

where $C \in \mathbb{C}$ is a constant independent of u but depending on (ω, m, ℓ) . In particular, we find that

$$u' - i\omega u = O(r^{-2}) \text{ as } r \rightarrow \infty.$$

Next, let $R < \infty$ be sufficiently large and let $z = \chi r^p$ where $p \in [0, 2]$ and χ is a cut-off function which is monotonically increasing, identically 0 for $r \leq R$, and identically 1 for $r \geq R + 1$. Keeping in mind that

$$V = \frac{\Lambda}{r^2} + \frac{2M [1 - (\Lambda - 2am\omega)]}{r^3} + O(r^{-4}) \text{ as } r \rightarrow \infty,$$

we find that

$$\begin{aligned} Q_{r^p}^z[u]|_{r=\infty} &= 0 & \text{if } p \in [0, 2), \\ Q_{r^p}^z[u]|_{r=\infty} &= -\Lambda |a_{\mathcal{T}^+}|^2 & \text{if } p = 2. \end{aligned}$$

Furthermore, recalling that by (56) we have

$$\Lambda \geq 2|am\omega|,$$

one may easily check that r sufficiently large and $p \in [0, 2]$ imply

$$-(r^p V)' \geq b [(2-p)r^{p-3}\Lambda + r^{p-4}] - B \frac{\Lambda}{r^{p-4}}.$$

Thus, applying the fundamental theorem of calculus to the identity (78) yields

$$(99) \quad \int_{R+1}^{\infty} [pr^{p-1} |u' - i\omega u|^2 + [(2-p)r^{p-3}\Lambda + r^{p-4}] |u|^2] \\ \leq B \int_R^{R+1} r^p (1 + \omega^2 + \Lambda) |u|^2 dr^* + B \int_R^{\infty} r^p |H| |u' - i\omega u| dr^* \\ + B \int_{R+1}^{\infty} \frac{\Lambda}{r^{p-4}} |u|^2 dr^*,$$

where in the case $p = 2$ we may add $\Lambda |a_{\mathcal{I}^+}|^2$ to the left hand side.

It remains to estimate the last term on the right hand side of (99). (Note that for any $p \in [0, 2)$ we could take R sufficiently large depending on p and absorb the troublesome term onto the left hand side. However, this cannot work in the case $p = 2$.) Let $\tilde{\chi}$ be a cut-off which is identically 0 for $r \in [r_+, R]$ and identically 1 on $[R + 1, \infty)$. Then, taking R sufficiently large and applying the fundamental theorem of calculus to the identity (76) with $y = \tilde{\chi}$ easily yields

$$\int_{R+1}^{\infty} \frac{\Lambda}{r^3} |u|^2 dr^* \leq B \int_R^{R+1} (1 + \omega^2 + \Lambda) |u|^2 dr^* + B \int_R^{\infty} |H| |u'| dr^*,$$

and thus concludes the proof. □

6.7. A quantitative estimate on the rate of convergence of the microlocal radiation field

The following proposition will be used in Section 6.8 below and also in Section 9.1.2.

PROPOSITION 6.7.1. – *Fix parameters $(\omega, m, \ell) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$ with $\omega \neq 0$, and let u be a smooth solution of the radial o.d.e. (61) with a right hand side H vanishing for sufficiently large r^* , such that the constant $a_{\mathcal{I}^-}$ from Proposition 5.2.1 vanishes. Then there exists a sufficiently large constant R , independent of the frequency parameters, such that for every $\epsilon > 0$*

$$\left| \omega \left(u - e^{i\omega r^*} a_{\mathcal{I}^+} \right) \Big|_{r=r_0} \right|^2 \leq B(\epsilon) r_0^{-2+\epsilon} \int_R^{R+1} (1 + \Lambda^3) |u|^2 dr^*, \quad \forall r_0 \geq R.$$

REMARK 6.7.1. – *Note that if we allowed the constants B and R to depend on the frequency parameters, standard o.d.e. theory (e.g., see [60]) would allow one to replace $-2 + \epsilon$ with the sharp exponent -2 .*

REMARK 6.7.2. – *As far as the applications of Proposition 6.7.1 are concerned the only thing important about the Λ dependence is that it is polynomial.*

Proof. – Set

$$E \doteq u - e^{i\omega r^*} a_{\mathcal{I}^+}.$$

Recall that standard o.d.e. theory implies that $E = O(r^{-1})$ as $r \rightarrow \infty$ (where the implied constant may depend on (ω, m, ℓ)).

Next, we observe that one may find a sufficiently large $R < \infty$ not depending on the frequency parameters so that $r \geq R$ implies $|V| \leq B \left(\frac{\Lambda}{r^2} + \frac{1}{r^3} \right)$.

A simple computation gives

$$E'' + \omega^2 E = VE + e^{i\omega r^*} a_{\mathcal{I}^+} V.$$

Variation of parameters⁽¹⁴⁾ then implies

$$E(r) = - \int_{r^*}^{\infty} \left(\frac{e^{i\omega(r^*-s^*)} - e^{-i\omega(r^*-s^*)}}{2i\omega} \right) \left(V(s)E(s) + e^{i\omega s^*} a_{\mathcal{I}^+} V(s) \right) ds^*.$$

In particular,

$$(100) \quad |\omega E(r)|^2 \leq B \left[\left(\int_{r^*}^{\infty} \frac{(1+\Lambda)|E(s)|}{s^2} ds^* \right)^2 + |a_{\mathcal{I}^+}|^2 \left(\frac{\Lambda^2}{r^2} + \frac{1}{r^4} \right) \right].$$

Now we consider the two terms on the right hand side of (100) separately. For the first term, we begin by observing that $e^{i\omega r^*} (e^{-i\omega r^*} E)' = u' - i\omega u$. Keeping this in mind, we have

$$(101) \quad \begin{aligned} \left(\int_{r^*}^{\infty} \frac{|E(s)|}{s^2} ds^* \right)^2 &\leq B(\epsilon) \int_{r^*}^{\infty} \frac{|E(s)|^2}{s^{3-\epsilon}} ds^* \\ &= B(\epsilon) \int_{r^*}^{\infty} \frac{|e^{-i\omega s^*} E(s)|^2}{s^{3-\epsilon}} ds^* \\ &\leq B(\epsilon) \int_{r^*}^{\infty} \frac{|(e^{-i\omega s^*} E(s))'|^2}{s^{1-\epsilon}} ds^* \\ &\leq B(\epsilon) r^{-2+\epsilon} \int_{r^*}^{\infty} s |u' - i\omega u|^2 ds \\ &\leq (1+\Lambda) B(\epsilon) r^{-2+\epsilon} \int_R^{R+1} |u|^2 dr^*. \end{aligned}$$

In the third inequality we used a standard Hardy inequality, and in the final inequality we appealed to Proposition 6.6.1.

For the second term in (100), we first note that Proposition 6.6.1 with $p = 2$ gives

$$(102) \quad \Lambda \frac{|a_{\mathcal{I}^+}|^2}{r^2} \leq B\Lambda r^{-2} \int_R^{R+1} |u|^2 dr^*.$$

For the lower order term we use

$$(103) \quad \frac{|a_{\mathcal{I}^+}|^2}{r^4} \leq B \int_{r^*}^{\infty} \frac{|a_{\mathcal{I}^+}|^2}{s^5} ds^* \leq B \int_{r^*}^{\infty} \left[\frac{|u|^2 + |E|^2}{s^5} \right] ds^* \leq B(1+\Lambda) r^{-3} \int_R^{R+1} |u|^2 dr^*.$$

In the last inequality we used the estimates done in (101) and Proposition 6.6.1.

Combining (100), (101), (102), and (103) concludes the proof. \square

⁽¹⁴⁾ More concretely, we define a function \tilde{E} by the formula given, note that $(E - \tilde{E})'' + \omega^2(E - \tilde{E}) = 0$, observe the trivial fact that any solution to $g'' + \omega^2 g = 0$ which satisfies $g = O(r^{-1})$ must be identically 0, and deduce that $E = \tilde{E}$.

6.8. Relation to the physical space radiation fields

Definition 5.4.1 is motivated by the following propositions.

PROPOSITION 6.8.1. – *For all smooth solutions ψ to (2) on \mathcal{D} arising from smooth compactly supported data along $\bar{\Sigma}$, let $a_{\mathcal{I}^+}(\omega, m, \ell)$ be the microlocal radiation field along \mathcal{I}^+ . Then $\omega a_{\mathcal{I}^+} \in L^2_{\omega} l^2_{m\ell}$ and*

$$\partial_{\tau}\varphi(\tau, \infty, \theta, \phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} \omega e^{-i\omega\tau} a_{\mathcal{I}^+}(\omega, m, \ell) S_{m\ell}(a\omega, \cos\theta) e^{im\phi} d\omega.$$

Recall that $\varphi(\tau, \infty, \theta, \phi)$ denotes the radiation field of ψ along future null infinity \mathcal{I}^+ .

Proof. – First of all, as noted in Remark 5.1.2, ψ is sufficiently integrable in the sense of Definition 5.1.1 and thus the microlocal radiation field $a_{\mathcal{I}^+}$ is a well defined measurable function.

Now, define

$$\begin{aligned} \psi_+ &\doteq \chi(t^*)\psi, \\ \psi_- &\doteq (1 - \chi(t^*))\psi, \end{aligned}$$

where $\chi(x)$ is a cutoff function which is identically 0 for $x \leq 0$ and is identically 1 for $x \geq 1$ (the apparent asymmetry in the use of a cutoff depending on t^* will not be a problem).

We shall denote $(r^2 + a^2)^{1/2} \psi$, $(r^2 + a^2)^{1/2} \psi_+$, and $(r^2 + a^2)^{1/2} \psi_-$ by φ , φ_+ , and φ_- respectively.

The following facts are immediate consequences of ψ 's compact support along $\bar{\Sigma}$ and the finite speed of propagation.

1. $\varphi_+|_{\mathcal{I}^+} = \varphi|_{\mathcal{I}^+}$.
2. $\varphi_-|_{\mathcal{I}^-} = \varphi|_{\mathcal{I}^-}$.
3. $\psi = \psi_+ + \psi_-$.
4. $\square_g \psi_+$ vanishes for large r .
5. $\square_g \psi_-$ vanishes for large r .

Next, we observe the following immediate consequence of Proposition 3.8.1 with $p = 1$ and Theorem 3.7.1 (note that the compact support of ψ 's initial data implies that the norms on the right sides of the estimates of Theorem 3.7.1 are finite and thus the right hand side of the estimate of Proposition 3.8.1 is uniformly bounded as $\tau_2 \rightarrow \infty$):

$$(104) \quad \int_{-\infty}^{\infty} \int_{r \geq R} \int_0^{2\pi} \int_0^{\pi} \left| (\partial_t + \partial_{r^*}) \left((r^2 + a^2)^{1/2} \psi_+ \right) \right|^2 \sin\theta dt dr d\theta d\phi < \infty,$$

where R is sufficiently large. Applying the discrete isometry $(t, \phi) \mapsto (-t, -\phi)$ and repeating the above argument implies

$$(105) \quad \int_{-\infty}^{\infty} \int_{r \geq R} \int_0^{2\pi} \int_0^{\pi} \left| (\partial_t - \partial_{r^*}) \left((r^2 + a^2)^{1/2} \psi_- \right) \right|^2 \sin\theta dt dr d\theta d\phi < \infty,$$

where R is sufficiently large.

Noting that ψ_{\pm} are easily seen to be sufficiently integrable in the sense of Definition 5.1.1, we may apply Carter’s separation to ψ_+ and ψ_- and define u_+ and u_- . Now we observe that Plancherel and (104) are easily seen to imply the existence of a dyadic sequence $\{r_n\}$ such that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \sum_{m\ell} |u'_+ - i\omega u_+|^2 |_{r=r_n} = 0.$$

In turn, upon passing to a subsequence, this implies that for almost every ω and every (m, ℓ) we have

$$\lim_{n \rightarrow \infty} |u'_+ - i\omega u_+| |_{r=r_n} = 0.$$

Finally, combining this with Proposition 5.2.1 clearly implies that $u_+ \sim e^{i\omega r^*}$ as $r \rightarrow \infty$. Similarly, we observe that $u_- \sim e^{-i\omega r^*}$ as $r \rightarrow \infty$. Since we clearly have $u = u_+ + u_-$, we finally conclude that for almost every ω and each (m, ℓ) we have

(106)
$$u_+ = a_{\mathcal{I}^+} e^{i\omega r^*} + O(r^{-1}) \text{ as } r \rightarrow \infty,$$

(107)
$$u_- = a_{\mathcal{I}^-} e^{-i\omega r^*} + O(r^{-1}) \text{ as } r \rightarrow \infty.$$

Observe that the Fourier transform in τ of $\partial_{\tau} \varphi_+$ is given by $(e^{-i\omega r^*} + O(\frac{\omega}{r})) \omega u_+$ as $r \rightarrow \infty$. Furthermore, observe that Theorem 3.7.2 and Plancherel are easily seen to imply that

$$\int_{-\infty}^{\infty} \sum_{m\ell} \int_R^{R+1} (1 + \Lambda^3 + \omega^2) |u_+|^2 dr^* d\omega < \infty.$$

Thus we may apply Proposition 6.7.1 to conclude that $\omega e^{-i\omega r^*} u_+$ and $\omega^2 e^{i\omega r^*} u_+$ converge in $L^2_{\omega} l^2_{m\ell}$ as $r \rightarrow \infty$ to $\omega a_{\mathcal{I}^+}$ and $\omega^2 a_{\mathcal{I}^+}$ respectively. In particular, the Fourier transform in τ of $\partial_{\tau} \varphi_+$ converges to $\omega a_{\mathcal{I}^+}$ in $L^2_{\omega} l^2_{m\ell}$ as $r \rightarrow \infty$. Plancherel then implies that any subsequence $\{\partial_{\tau} \varphi_+\}_{r_n}$ is Cauchy in $L^2_{\mathbb{R} \times \mathbb{S}^2}$. Now, recalling that $\partial_{\tau} \varphi_+(\tau, r, \theta, \phi)$ converges to $\partial_{\tau} \varphi_+(\tau, \infty, \theta, \phi)$ in $L^{\infty}_{\mathbb{R} \times \mathbb{S}^2}$ (see Remark 4.2.3 and keep in mind that the finite speed of propagation implying φ_+ is only supported along $\tau \geq \tau_0$ for some $\tau_0 \in \mathbb{R}$), we conclude, using the uniqueness of L^p limits, that $\partial_{\tau} \varphi_+(\tau, r, \theta, \phi)$ converges to $\partial_{\tau} \varphi_+(\tau, \infty, \theta, \phi)$ in $L^2_{\mathbb{R} \times \mathbb{S}^2}$. Finally, continuity of the Fourier transform on L^2 implies that

$$\partial_{\tau} \varphi_+(\tau, \infty, \theta, \phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} \omega e^{-i\omega \tau} a_{\mathcal{I}^+}(\omega, m, \ell) S_{m\ell}(a\omega, \cos \theta) e^{im\phi} d\omega.$$

To conclude the proof we simply recall that $\varphi_+|_{\mathcal{I}^+} = \varphi|_{\mathcal{I}^+}$. □

Now we turn to the horizon flux.

PROPOSITION 6.8.2. – *For all solutions ψ to (2) on \mathcal{D} arising from smooth compactly supported initial data along $\overline{\Sigma}$, let $a_{\mathcal{H}^+}(\omega, m, \ell)$ be the microlocal radiation field along \mathcal{H}^+ . Then $(\omega - \omega_+ m) a_{\mathcal{H}^+} \in L^2_{\omega} l^2_{m\ell}$ and*

(108)
$$\begin{aligned} &K\psi(t^*, r_+, \theta, \phi) \\ &= \frac{1}{\sqrt{4M\pi r_+}} \int_{-\infty}^{\infty} \sum_{m\ell} (\omega - \omega_+ m) e^{-i\omega t^*} a_{\mathcal{H}^+}(\omega, m, \ell) S_{m\ell}(a\omega, \cos \theta) e^{im\phi} d\omega. \end{aligned}$$

Proof. – First of all, as noted in Remark 5.1.2, ψ is sufficiently integrable in the sense of Definition 5.1.1 and thus the microlocal radiation field $a_{\mathcal{H}^+}$ is a well defined measurable function.

We first consider the case where the initial data for ψ are in fact compactly supported along $\mathring{\Sigma}$. We may then proceed in a completely analogous manner to the proof of Proposition 6.8.1. We note that the argument is in fact simpler since we will be able to rely directly on Theorem 3.7.1 instead of developing an analog of Theorem 3.8.1 near the horizon.

Define

$$\begin{aligned} \psi_+ &\doteq \chi(t)\psi, \\ \psi_- &\doteq (1 - \chi(t))\psi, \end{aligned}$$

where $\chi(x)$ is a cutoff function which is identically 0 for $x \leq 0$ and is identically 1 for $x \geq 1$.

The following facts are immediate consequences of ψ 's compact support away from the bifurcate sphere \mathcal{B} and the finite speed of propagation.

1. $\psi_+|_{\mathcal{H}^+} = \psi|_{\mathcal{H}^+}$.
2. $\psi_-|_{\mathcal{H}^-} = \psi|_{\mathcal{H}^-}$.
3. $\psi = \psi_+ + \psi_-$.
4. $\square_g \psi_+$ vanishes for small $r - r_+$.
5. $\square_g \psi_-$ vanishes for small $r - r_+$.

Recalling that the smooth extension of ∂_{r^*} to $\mathcal{H}^+ \cup \mathcal{H}^-$ satisfies $\partial_{r^*}|_{\mathcal{H}^+} = K$ and $\partial_{r^*}|_{\mathcal{H}^-} = -K$, we see that Theorem 3.7.2 immediately implies

$$(109) \quad \lim_{r \rightarrow r_+} (\partial_{r^*} - K) \psi_+ = 0 \text{ in } L^2_{t^*, \theta^*, \phi^*}(\sin \theta^* dt^* d\theta^* d\phi^*),$$

$$(110) \quad \lim_{r \rightarrow r_+} (\partial_{r^*} + K) \psi_- = 0 \text{ in } L^2_{t^*, \theta^*, \phi^*}(\sin \theta^* dt^* d\theta^* d\phi^*).$$

Appealing to Theorems 3.7.1 and 3.7.2, we may apply Carter's separation to ψ_+ and ψ_- and define u_+ and u_- . Since we clearly have $u = u_+ + u_-$, Proposition 5.2.1, (109), (110) and a similar argument as we used near \mathcal{I}^+ (note that the convergence of ψ_+ to its radiation field along the horizon in both $L^\infty_{\mathbb{R}_{\geq 0} \times \mathbb{S}^2}$ and $L^2_{\mathbb{R}_{\geq 0} \times \mathbb{S}^2}$ follows immediately from the fundamental theorem of calculus and Theorem 3.7.2) imply that for almost every ω and for each (m, ℓ) , we have

$$(111) \quad u_+ = a_{\mathcal{H}^+} e^{-i(\omega - \omega_+ m)r^*} + O(r - r_+) \text{ as } r \rightarrow r_+,$$

$$(112) \quad u_- = a_{\mathcal{H}^-} e^{i(\omega - \omega_+ m)r^*} + O(r - r_+) \text{ as } r \rightarrow r_+.$$

Now, we note that Theorem 3.7.1 is easily seen to imply that $\psi_+|_{r=s} \rightarrow \psi_+|_{r=r_+}$ as $s \rightarrow r_+$ in $L^2_{t^*, \theta^*, \phi^*}$. Arguing in a similar fashion as in the proof of Proposition 6.8.1 we conclude that $a_{\mathcal{H}^+}$ is in $L^2_{\omega} L^2_{m\ell}$ and

$$(113) \quad \psi(t^*, r_+, \theta, \phi) = \frac{1}{\sqrt{4M\pi r_+}} \int_{-\infty}^{\infty} \sum_{m\ell} e^{-i\omega t^*} a_{\mathcal{H}^+}(\omega, m, \ell) S_{m\ell}(a\omega, \cos \theta) e^{im\phi^*} d\omega.$$

Now we consider the case where the support of ψ may contain the bifurcate sphere \mathcal{B} . We begin by commuting (2) with K and conclude that $\square_g(K\psi) = 0$. Then we recall that $K\psi$ vanishes on the bifurcate sphere in view of (40). Now, let $\chi(x)$ be a smooth function which is identically 0 for $x \in (-\infty, 1]$ and identically 1 for $x \in [2, \infty)$. Set $\chi_\epsilon(x) \doteq \chi(\frac{x}{\epsilon})$,

and, recalling the coordinate system (U^+, V^+, θ, ϕ) near the bifurcate sphere which was introduced in Section 3.2, let $(K\psi)_\epsilon$ denote the solution to the wave equation with the initial data of $\chi_\epsilon(V^+)K\psi$. Using that $K\psi$ is smooth and vanishes at the bifurcate sphere, one may easily verify that

$$\lim_{\epsilon \rightarrow 0} \int_{\Sigma} \mathbf{J}_\mu^N [K\psi - (K\psi)_\epsilon] n_\Sigma^\mu = 0.$$

Theorem 3.7.1 then implies that

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{H}^+} \mathbf{J}_\mu^N [K\psi - (K\psi)_\epsilon] n_{\mathcal{H}^+}^\mu = 0.$$

Since $(K\psi)_\epsilon$ is compactly supported away from the bifurcate sphere,
(114)

$$(K\psi)_\epsilon(t^*, r_+, \theta, \phi) = \frac{1}{\sqrt{4M\pi r_+}} \int_{-\infty}^{\infty} \sum_{m\ell} e^{-i\omega t^*} a_{\epsilon, \mathcal{H}^+}^{(K)}(\omega, m, \ell) S_{m\ell}(a\omega, \cos \theta) e^{im\phi} d\omega,$$

where $a_{\epsilon, \mathcal{H}^+}^{(K)}$ is the microlocal radiation field along \mathcal{H}^+ for $(K\psi)_\epsilon$ (observe that $(K\psi)_\epsilon$ is easily seen to be sufficiently integrable in the sense of Definition 5.1.1).

In order to finish the proof, we just need to establish that $a_{\epsilon, \mathcal{H}^+}^{(K)} \rightarrow (\omega - \omega_+ m) a_{\mathcal{H}^+}$ in $L_\omega^2 l_{m\ell}^2$ as $\epsilon \rightarrow 0$. We begin by noting that the convergence of $(K\psi)_\epsilon$ to $K\psi$ and Plancherel imply that $\{a_{\epsilon, \mathcal{H}^+}^{(K)}\}$ has an $L_\omega^2 l_{m\ell}^2$ limit as $\epsilon \rightarrow 0$; hence, it suffices to check that $(\omega - \omega_+ m) a_{\epsilon, \mathcal{H}^+}^{(K)}$ converges to $(\omega - \omega_+ m) a_{\mathcal{H}^+}$ pointwise almost everywhere. In order to see this, we let $u_\epsilon^{(K)}$ denote the result of applying Carter's separation to $(K\psi)_\epsilon$, and observe that $a_{\epsilon, \mathcal{H}^+}^{(K)}$ is, up to an appropriate normalization, equal to the Wronskian of $u_\epsilon^{(K)}$ with U_{hor} :

$$a_{\epsilon, \mathcal{H}^+}^{(K)} = (-2i(\omega - \omega_+ m))^{-1} \left((u_\epsilon^{(K)})' \overline{U_{\text{hor}}} - u_\epsilon^{(K)} \overline{U_{\text{hor}}'} \right).$$

Since Theorem 3.7.1 may be easily used to show that for each (ω, m, ℓ) and r^* , $u_\epsilon^{(K)}(r^*, \omega, m, \ell)$ converges to $u_\epsilon(r^*, \omega, m, \ell)$ as $\epsilon \rightarrow 0$, we conclude that $(\omega - \omega_+ m) a_{\epsilon, \mathcal{H}^+}^{(K)}(\omega, m, \ell)$ converges to $(\omega - \omega_+ m) a_{\mathcal{H}^+}^{(K)}$ as $\epsilon \rightarrow 0$. \square

7. Boundedness revisited: A degenerate-energy boundedness statement

This section is dedicated to refining our recent proof from [30] of boundedness for the wave equation so as to apply for finite degenerate V -energy solutions.

We will collect all statements which we shall need for the remainder of the paper in Section 7.1. The key statement is Theorem 7.1 together with one immediate corollary. (In particular, after digesting these statements, the reader impatient to proceed to the scattering theory constructions can skip to Section 8.)

In the brief aside of Section 7.2, we shall also state the full degenerate-energy analog of Theorem 3.7.1 in Section 7.2 as Theorem 7.2. We shall not actually require the latter result in the paper and it in fact is more convenient to infer it a posteriori with the help of the backwards scattering maps which we shall construct in Section 9. Thus, the proof of Theorem 7.2 is in fact deferred till Section 9.4.

Section 7.3 gives the proof of Theorem 7.1. We note that the proof will crucially use Proposition 5.3.1, Proposition 6.2.1 and Proposition 6.3.1.

7.1. The main theorem and corollary

The main result which we shall require for later sections is the following.

THEOREM 7.1. – *For all solutions ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data on Σ_0^* which are compactly supported, we have*

$$(115) \quad \int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi]n_{\mathcal{I}^+}^\mu + \int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_\mu^K[\psi]n_{\mathcal{H}^+}^\mu \leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\psi]n_{\Sigma_0^*}^\mu.$$

REMARK 7.1.1. – *One can easily formulate and prove higher order versions of Theorem 7.1 but we will not pursue this here.*

Given that the restriction of the deformation tensor of V to $J^-(\Sigma_0^*) \cap J^+(\bar{\Sigma})$ is compactly supported away from $\mathcal{H}^+ \cup \mathcal{H}^- \cup \mathcal{B}$, a finite in time energy estimate, i.e., (24) with $X = V$, immediately implies

COROLLARY 7.1. – *For all solutions ψ to (2) on $J^+(\bar{\Sigma})$ arising from smooth compactly supported initial data along $\bar{\Sigma}$, we have*

$$(116) \quad \int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi]n_{\mathcal{I}^+}^\mu + \int_{\mathcal{H}^+} \mathbf{J}_\mu^K[\psi]n_{\mathcal{H}^+}^\mu \leq B \int_{\bar{\Sigma}} \mathbf{J}_\mu^V[\psi]n_{\bar{\Sigma}}^\mu.$$

7.2. Aside: the full degenerate boundedness and integrated decay statements

We note that we can in fact obtain the full analog of Theorem 3.7.1 where energy boundedness is given with respect to a spacelike foliation, and where integrated local energy decay is proven, both now involving the degenerate energy. We will not require this result in the rest of the paper and it is in fact convenient to obtain it a posteriori using our scattering theory.

THEOREM 7.2. – *For all solutions ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from smooth initial data on Σ_0^* which are compactly supported, we have*

$$(117) \quad \int_{\Sigma_s^*} \mathbf{J}_\mu^V[\psi]n_{\Sigma_s^*}^\mu \leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\psi]n_{\Sigma_0^*}^\mu, \quad \forall s \geq 0,$$

$$(118) \quad \int_{\mathcal{R}_{\geq 0}} \left(r^{-1}\zeta |\not{\nabla}\psi|^2 + r^{-1-\delta}\zeta |T\psi|^2 + (r-r_+)^2 r^{-3-\delta} |\tilde{Z}^*\psi|^2 + r^{-3-\delta} |\psi|^2 \right) \leq B(\delta) \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\psi]n_{\Sigma_0^*}^\mu,$$

where ζ is defined as in the statement of Theorem 3.7.1.

The proof is deferred till Section 9.4.

REMARK 7.2.1. – *Note the degeneration of the bulk integral at the horizon. One can easily formulate and prove higher order versions of Theorem 7.2 but we will not pursue this here.*

7.3. The proof of Theorem 7.1

Before we begin the discussion of the proof of Theorem 7.1, let us briefly indicate what would go wrong if we simply tried to repeat the proof of Theorem 3.7.1 as given in [30].

1. Anytime the redshift estimate of [27] and [23] is applied to ψ , one must put a term $\int_{\Sigma_0^*} \mathbf{J}_\mu^N[\psi] n_{\Sigma_0^*}^\mu$ on the right hand side of the resulting estimate.
2. In [30], when we proved the integrated energy decay statement for ψ we first proved an estimate for $\chi\psi$ where $\chi(t^*)$ was a cutoff function which was identically 0 in the past of Σ_0^* and identically 1 in the future of Σ_1^* . We then studied the inhomogeneous wave equation

$$\square_g(\chi\psi) = 2g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\psi + (\square_g\chi)\psi \doteq F.$$

The resulting estimate in [30] had, in particular, a term on the right hand side proportional to

$$\int_{\mathcal{R} \cap \{r \leq R\}} |F|^2,$$

for some constant $R > r_+$. Note that on the horizon, F will contain a term proportional to $Z^*\psi$. Unfortunately, this is exactly the derivative that the \mathbf{J}^V energy loses control of as $r \rightarrow r_+$.

In order to prove Theorem 7.1 we will first observe that without loss of generality, we can assume that the initial data for ψ is supported near the horizon. Applying a \mathbf{J}^K energy estimate for ψ and Plancherel then immediately reduce the problem to estimating the microlocal radiation fields for ψ_{∞} along \mathcal{I}^+ in the superradiant frequency regime $\mathcal{F}_\#^{(\epsilon)}$. Next, using the fundamental solution representation of Proposition 5.3.1 we will represent the microlocal radiation fields along \mathcal{I}^+ as an integral in r^* of the Fourier transform of F against U_{hor} . Following this, in the most subtle part of the proof, we will crucially exploit the fact that we are in a superradiant frequency regime where we can afford to lose a derivative, the fact we only need to estimate the flux to \mathcal{I}^+ , the fact that F is supported near the horizon and the oscillations of U_{hor} in r^* (as embodied in Proposition 6.3.1) in order to gain some degeneration in $r - r_+$. Somewhat surprisingly, this step does not use that $F = 2g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\psi + (\square_g\chi)\psi$; it treats F as an arbitrary inhomogeneity. Finally, the proof concludes with finite in time energy estimates and Hardy inequalities (of course, the fact that $F = 2g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\psi + (\square_g\chi)\psi$ is used in this step).

Proof of Theorem 7.1. – We start with an easy reduction; we may split ψ into ψ_1 and ψ_2 where ψ_1 has initial data supported near the horizon and ψ_2 has initial data supported away from the horizon. Of course, the estimate (115) for ψ_2 follows from Theorem 3.7.1. Thus, without loss of generality, we will assume that $\psi_2 = 0$ and that $\psi = \psi_1$ has initial data whose support is contained in $r \in [r_+, 10M]$.

We now define $\psi_{\infty} \doteq \chi\psi$ where χ is a function which is identically 1 in the future of Σ_1^* , and identically 0 in the past of Σ_0^* . This satisfies

$$\square_g\psi_{\infty} = 2g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\psi + (\square_g\chi)\psi \doteq F.$$

The functions u and H are then defined by applying Carter's separation to $\chi\psi$ and F respectively. This satisfies the radial o.d.e. (61) with a non-zero right hand side H . Let $a_{\mathcal{I}^+}$ denote the corresponding microlocal radiation field of u .

We begin by showing

$$(119) \quad \int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_\mu^K[\psi] n_{\mathcal{H}^+}^\mu + \int_{-\infty}^{\infty} \sum_{m\ell} \omega(\omega - \omega_+ m) |a_{\mathcal{I}^+}|^2 d\omega \leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\psi] n_{\Sigma_0^*}^\mu.$$

Note that part of the proof of this statement will be that the unsigned quantity $\int_{-\infty}^{\infty} \sum_{m\ell} \omega(\omega - \omega_+ m) |a_{\mathcal{I}^+}|^2 d\omega$ is absolutely convergent. (One should think of (119) as corresponding to the formal statement $\int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_\mu^K[\psi] n_{\mathcal{H}^+}^\mu + \int_{\mathcal{I}^+} \mathbf{J}_\mu^K[\psi] n_{\mathcal{I}^+}^\mu \leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\psi] n_{\Sigma_0^*}^\mu$. However, we will wish to avoid a discussion of the convergence of the unsigned integral $\int_{\mathcal{I}^+} \mathbf{J}_\mu^K[\psi] n_{\mathcal{I}^+}^\mu$.)

Let $s > 0$ and $r_0 > r_+$. We start with a \mathbf{J}^K energy estimate in the region bounded by $\mathcal{H}^+(0, s)$, $\Sigma_s^* \cap \{r \leq r_0\}$, $\{r = r_0\} \cap J^-(\Sigma_s^*)$, and Σ_0^* . We obtain

$$(120) \quad \int_{\mathcal{H}^+(0, s)} \mathbf{J}_\mu^K[\psi] n_{\mathcal{H}^+}^\mu + \int_{\Sigma_s^* \cap \{r \leq r_0\}} \mathbf{J}_\mu^K[\psi] n_{\Sigma_s^*}^\mu + \int_{\{r=r_0\} \cap J^-(\Sigma_s^*) \cap J^+(\Sigma_0^*)} \mathbf{J}_\mu^K[\psi] n_{\{r=r_0\}}^\mu \\ = \int_{\Sigma_0^* \cap \{r \leq r_0\}} \mathbf{J}_\mu^K[\psi] n_{\Sigma_0^*}^\mu.$$

It easily follows from Theorem 3.7.2 that for each r_0 , there exists a dyadic sequence $\{s_i\}_{i=1}^\infty$ such that

$$\lim_{i \rightarrow \infty} \int_{\Sigma_{s_i}^* \cap \{r \leq r_0\}} \mathbf{J}_\mu^K[\psi] n_{\Sigma_{s_i}^*}^\mu = 0.$$

We thus obtain

$$(121) \quad \int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_\mu^K[\psi] n_{\mathcal{H}^+}^\mu + \int_{\{r=r_0\} \cap J^+(\Sigma_0^*)} \mathbf{J}_\mu^K[\psi] n_{\{r=r_0\}}^\mu = \int_{\Sigma_0^* \cap \{r \leq r_0\}} \mathbf{J}_\mu^K[\psi] n_{\Sigma_0^*}^\mu.$$

Observe that Theorem 3.7.2 allows us to unambiguously assign a value to the unsigned quantity

$$\int_{\{r=r_0\} \cap J^+(\Sigma_0^*)} \mathbf{J}_\mu^K[\psi] n_{\{r=r_0\}}^\mu.$$

Next, recalling that $\psi|_{\Sigma_0^*}$ is supported with $[r_+, r_+ + 10M]$, we observe that if r_0 is sufficiently large, then (121) becomes

$$(122) \quad \int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_\mu^K[\psi] n_{\mathcal{H}^+}^\mu + \int_{\{r=r_0\}} \mathbf{J}_\mu^K[\psi] n_{\{r=r_0\}}^\mu \leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\psi] n_{\Sigma_0^*}^\mu.$$

Now we explicitly compute, apply Plancherel, and integrate by parts:

$$(123) \quad \int_{\{r=r_0\}} \mathbf{J}_\mu^K[\psi] n_{\{r=r_0\}}^\mu = \int_{\{r=r_0\}} ((T\psi_\infty + \omega_+ \Phi\psi_\infty) \partial_{r^*} \psi_\infty) (r^2 + a^2) \sin \theta dt d\theta d\phi \\ = \int_{\{r=r_0\}} ((T((r^2 + a^2)^{1/2} \psi_\infty) + \omega_+ \Phi((r^2 + a^2)^{1/2} \psi_\infty)) \\ \partial_{r^*} ((r^2 + a^2)^{1/2} \psi_\infty)) \sin \theta dt d\theta d\phi \\ = \int_{-\infty}^{\infty} \sum_{m\ell} (\omega - \omega_+ m) \operatorname{Im}(u' \bar{u})|_{r=r_0} d\omega.$$

Next, we consider the microlocal K -energy current (see Section 6.1):

$$Q^K[u] \doteq (\omega - \omega_+ m) \operatorname{Im}(u' \bar{u}).$$

This is conserved for $r \geq r_0$ for r_0 sufficiently large, i.e.,

$$(Q^K)' = 0.$$

Noting that the proof of Proposition 6.8.1 implies $(u' - i\omega u)|_{r=\infty} = 0$, we thus obtain

$$(124) \quad \int_{-\infty}^{\infty} \sum_{m\ell} (\omega - \omega_+ m) \operatorname{Im}(u'\bar{u})|_{r=r_0} d\omega = \int_{-\infty}^{\infty} \sum_{m\ell} \omega (\omega - \omega_+ m) |a_{\mathcal{I}^+}|^2 d\omega.$$

In particular, the right hand side of (124) is absolutely convergent. Combining (122), (123), and (124) yields (119).

Next, we observe that Propositions 6.8.1 and 4.2.2 together imply that

$$(125) \quad \int_{\mathcal{I}^+} \mathbf{J}_{\mu}^T[\psi] n_{\mathcal{I}^+}^{\mu} = \int_{-\infty}^{\infty} \sum_{m\ell} \omega^2 |a_{\mathcal{I}^+}|^2 d\omega.$$

Now, observing that $(\omega, m, \ell) \notin \mathcal{F}_{\#}^{(\epsilon)}$ imply that $\omega (\omega - \omega_+ m) \geq b(\epsilon)\omega^2$, it is clear that in order to finish the proof we need only show

$$(126) \quad \int_{-\infty}^{\infty} \sum_{\{m,\ell\}:(\omega,m,\ell) \in \mathcal{F}_{\#}^{(\epsilon)}\}} |\omega_+ \omega m| |a_{\mathcal{I}^+}|^2 d\omega \leq B(\epsilon) \int_{\Sigma_0^*} \mathbf{J}_{\mu}^V[\psi] n_{\Sigma_0^*}^{\mu},$$

for some sufficiently small $\epsilon > 0$.

We turn thus to the proof of (126).

First, note that Proposition 5.3.1 allows us to write

$$(127) \quad |a_{\mathcal{I}^+}|^2 = |\mathfrak{W}|^{-2} \left| \int_{-\infty}^{\infty} U_{\text{hor}}(x^*) H(x^*) dx^* \right|^2.$$

Keeping in mind that the set $\{(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)} : \Lambda \leq c\}$ is compact, standard o.d.e. theory implies

$$(128) \quad U_{\text{hor}}(r^*) = e^{-i(\omega - \omega_+ m)r^*} + E_{\text{hor}}(r^*),$$

where $(\omega, m, \ell) \in \mathcal{F}_{\#}^{(\epsilon)}$ and $\Lambda \leq c$ implies

$$(129) \quad |E_{\text{hor}}(r^*)| \leq B(\epsilon, c) |r - r_+|,$$

for $r - r_+$ sufficiently small. As $c \rightarrow \infty$, however, the dependence of $B(\epsilon, c)$ may be bad. Fortunately Proposition 6.3.1 shows that if $\epsilon > 0$ is sufficiently small and Λ is sufficiently large, then we have

$$|E_{\text{hor}}| \leq \frac{B(\epsilon) |\mathfrak{W}|}{\sqrt{\Lambda}} \sqrt{r - r_+},$$

for sufficiently small $r - r_+$.

Applying (127), (128), (129), Proposition 6.2.1 and Proposition 6.3.1, we obtain

(130)

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \sum_{\{(m,\ell):(\omega,m,\ell)\in\mathcal{F}_{\#}^{(\epsilon)}\}} |\omega+\omega m| |a_{\mathcal{T}^+}|^2 d\omega \\
 &= \int_{-\infty}^{\infty} \sum_{\{(m,\ell):(\omega,m,\ell)\in\mathcal{F}_{\#}^{(\epsilon)}\}} \frac{|\omega+\omega m|}{|\mathfrak{W}|^2} \left| \int_{-\infty}^{\infty} U_{\text{hor}}(r^*) H(r^*) dr^* \right|^2 d\omega \\
 &\leq B \int_{-\infty}^{\infty} \sum_{\{(m,\ell):(\omega,m,\ell)\in\mathcal{F}_{\#}^{(\epsilon)}\}} \left[\frac{|\omega+\omega m|}{|\mathfrak{W}|^2} \left| \int_{-\infty}^{\infty} e^{-i(\omega-\omega_+ m)r^*} H(r^*) dr^* \right|^2 \right. \\
 &\quad \left. + \frac{|\omega+\omega m|}{|\mathfrak{W}|^2} \left| \int_{-\infty}^{\infty} e^{-i(\omega-\omega_+ m)r^*} E_{\text{hor}}(r^*) H(r^*) dr^* \right|^2 \right] d\omega \\
 &\leq B(\epsilon) \int_{-\infty}^{\infty} \sum_{\{(m,\ell):(\omega,m,\ell)\in\mathcal{F}_{\#}^{(\epsilon)}\}} \left[\left| \int_{-\infty}^{\infty} e^{-i(\omega-\omega_+ m)r^*} H(r^*) dr^* \right|^2 \right. \\
 &\quad \left. + \left| \int_{-\infty}^{\infty} \sqrt{r-r_+} |H(r^*)| dr^* \right|^2 \right] \\
 &\doteq B(\epsilon) \int_{-\infty}^{\infty} \sum_{\{(m,\ell):(\omega,m,\ell)\in\mathcal{F}_{\#}^{(\epsilon)}\}} [I + II].
 \end{aligned}$$

Let us now recall the explicit form of H :

$$(131) \quad H = \frac{\Delta}{(r^2 + a^2)^{3/2}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} e^{i\omega t} e^{-im\phi} S_{m\ell}(\theta, a\omega) (\rho^2 F) \sin \theta dt d\theta d\phi,$$

$$(132) \quad F = 2g^{\mu\nu} \nabla_{\mu} \chi \nabla_{\nu} \psi + (\square_g \chi) \psi.$$

We begin by showing that the term proportional to II can be dealt with easily. On the support of H we have $dr^* \sim (r - r_+)^{-1} dr$. Thus, changing variables from r^* to r , applying Cauchy-Schwarz, Fubini, Plancherel, a straightforward Hardy inequality, and a finite-in-time energy inequality yields

(133)

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \sum_{\{(m,\ell):(\omega,m,\ell)\in\mathcal{F}_{\#}^{(\epsilon)}\}} |II| \lesssim \int_{-\infty}^{\infty} \sum_{\{(m,\ell):(\omega,m,\ell)\in\mathcal{F}_{\#}^{(\epsilon)}\}} \left| \int_{r_+}^{\infty} (r - r_+)^{1/2} |\Delta^{-1} H(r)| dr \right|^2 \\
 &\lesssim \int_{-\infty}^{\infty} \sum_{\{(m,\ell):(\omega,m,\ell)\in\mathcal{F}_{\#}^{(\epsilon)}\}} \int_{r_+}^{\infty} (r - r_+) |\Delta^{-1} H(r)|^2 dr \\
 &\lesssim \int_{-\infty}^{\infty} \int_{r_+}^{\infty} \int_{\mathbb{S}^2} |F|^2 (r - r_+) \sin \theta dt dr d\theta d\phi \\
 &\lesssim \int_{\Sigma_0^*} \mathbf{J}_{\mu}^{\nu}[\psi] n_{\Sigma_0^*}^{\mu}
 \end{aligned}$$

For every $\gamma > 0$, the same direct application of Cauchy-Schwarz, Plancherel, a straightforward Hardy inequality, and a finite in time energy inequality to the term I will only give

$$(134) \quad \int_{-\infty}^{\infty} \sum_{\{(m,\ell):(\omega,m,\ell)\in\mathcal{F}_{\sharp}^{(\epsilon)}\}} |I| \leq B(\gamma) \int_{\Sigma_0^* \cap [r_+, r_+ + 10M]} (r - r_+)^{1-\gamma} |\tilde{Z}^* \psi|^2 + B \int_{\Sigma_0^*} \mathbf{J}_{\mu}^V[\psi] n_{\Sigma_0^*}^{\mu}.$$

Unfortunately, the first term on the right hand side is (barely) not controlled by $\int_{\Sigma_0^*} \mathbf{J}_{\mu}^V[\psi] n_{\Sigma_0^*}^{\mu}$.

We control the term I as follows (we will not lose anything by allowing the sum in m and ℓ to be over all of $\mathbb{Z} \times \mathbb{Z}_{\geq|m|}$):

$$(135) \quad \begin{aligned} \int_{-\infty}^{\infty} \sum_{m\ell} |I| \, d\omega &= \int_{-\infty}^{\infty} \sum_{m\ell} \left| \int_0^{\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\omega-\omega+m)r^*} e^{i\omega t} e^{-im\phi} S_{m\ell}(\theta, a\omega) \right. \\ &\quad \left. \frac{\Delta\rho^2}{(r^2+a^2)^{3/2}} F \sin\theta \, d\theta \, d\phi \, dt \, dr^* \right|^2 d\omega \\ &\leq \int_0^{\pi} \int_{-\infty}^{\infty} \sum_m \left| \int_{-\infty}^{\infty} \int_{\infty}^{\infty} \int_0^{2\pi} e^{-i\omega(r^*-t)} e^{im(\omega+r^*-\phi)} \right. \\ &\quad \left. \frac{\Delta\rho^2}{(r^2+a^2)^{3/2}} F \, d\phi \, dt \, dr^* \right|^2 \sin\theta \, d\omega \, d\theta. \end{aligned}$$

For each fixed m we have used the orthogonality of the $S_{m\ell}$ in the last inequality.

Now we introduce the variables $\tilde{v} \doteq t + r^*$ and $\tilde{u} \doteq t - r^*$ and keep in mind that F is only supported in a compact range of \tilde{v} . Then (135) becomes

$$(136) \quad \begin{aligned} &\int_0^{\pi} \sum_m \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \int_{\infty}^{\infty} \int_0^{2\pi} e^{i\omega\tilde{u}} e^{im(\omega+\frac{\tilde{v}-\tilde{u}}{2}-\phi)} \frac{\Delta\rho^2}{(r^2+a^2)^{3/2}} F \, d\phi \, d\tilde{u} \, d\tilde{v} \right|^2 \sin\theta \, d\omega \, d\theta \\ &\leq B \int_0^{\pi} \int_{-\infty}^{\infty} \sum_m \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \int_0^{2\pi} e^{i\omega\tilde{u}} e^{im(\omega+\frac{\tilde{v}-\tilde{u}}{2}-\phi)} \frac{\Delta\rho^2}{(r^2+a^2)^{3/2}} F \, d\phi \, d\tilde{u} \right|^2 \sin\theta \, d\omega \, d\tilde{v} \, d\theta. \\ &= (2\pi)B \int_0^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_m \left| \int_0^{2\pi} e^{im(\omega+\frac{\tilde{v}-\tilde{u}}{2}-\phi)} \frac{\Delta\rho^2}{(r^2+a^2)^{3/2}} F \, d\phi \right|^2 \sin\theta \, d\tilde{u} \, d\tilde{v} \, d\theta \\ &= (2\pi)B \int_0^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_m \left| \int_0^{2\pi} e^{-im\phi} \frac{\Delta\rho^2}{(r^2+a^2)^{3/2}} F \, d\phi \right|^2 \sin\theta \, d\tilde{u} \, d\tilde{v} \, d\theta \\ &= (2\pi)^2 B \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \left| \frac{\Delta\rho^2}{(r^2+a^2)^{3/2}} F \right|^2 \sin\theta \, d\tilde{u} \, d\tilde{v} \, d\theta \, d\phi \\ &\leq B \int_{\Sigma_0^*} \mathbf{J}_{\mu}^V[\psi] n_{\Sigma_0^*}^{\mu}. \end{aligned}$$

We have used Plancherel in the ω variable and the orthogonality of the $e^{im\phi}$. In the last line we used finite in time energy estimates and the Hardy inequality $\int_{r_+}^{\infty} f^2 \, dr \leq B \int_{r_+}^{\infty} (r - r_+)^2 (\partial_r f)^2 \, dr$, which holds for smooth functions f which vanish for large r .

Putting together (130), (133), (135) and (136), we have indeed obtained (126). The theorem is thus proven. \square

8. The forward maps

We now turn to our scattering theory proper.

The first order of business is to carefully set up the relevant spaces described in Section 2.1.4 of the introduction. This will be accomplished in Section 8.1 below.

We will then define in Section 8.2 the various forward maps \mathcal{F}_+ and infer their boundedness. The boundedness of the map with domain $\mathcal{E}_{\Sigma_0^*}^N$ (Theorem 8.2.2) is independent of Section 7. This will give Theorem 1 of Section 2.3.1.

The boundedness of the degenerate-energy theory maps with domain $\mathcal{E}_{\dot{\Sigma}}^V$ and $\mathcal{E}_{\bar{\Sigma}}^V$ (Theorems 8.2.3 and 8.2.4, respectively) indeed requires the statement of Theorem 7.1 just proven. This will give Theorem 3 of Section 2.3.3.

8.1. Function spaces

In this section we will define the function spaces for which we will formulate our scattering theory.

8.1.1. *Initial data on Σ_0^* , $\dot{\Sigma}$ and $\bar{\Sigma}$.* – Let us denote by ${}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*)$, ${}^2\mathcal{C}_{cp}^\infty(\dot{\Sigma})$, ${}^2\mathcal{C}_{cp}^\infty(\bar{\Sigma})$ the vector space of smooth compactly supported pairs of functions (ψ, ψ') defined on Σ_0^* , $\dot{\Sigma}$, $\bar{\Sigma}$, respectively. We will complete these vector spaces with respect to appropriate norms to define the Hilbert spaces of our scattering theory.

We start with the non-degenerate N -energy space. We shall only in fact consider this for initial data on Σ_0^* .

DEFINITION 8.1.1. – For $(\psi, \psi') \in {}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*)$ we set

$$\|(\psi, \psi')\|_{\mathcal{E}_{\Sigma_0^*}^N} \doteq \sqrt{\int_{\Sigma_0^*} \mathbf{J}_\mu^N[\Psi] n_{\Sigma_0^*}^\mu},$$

where Ψ is any extension of ψ to \mathcal{R} such that $n_{\Sigma_0^*} \Psi = \psi'$.

The above expression gives a norm on the vector space ${}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*)$, and we define the space

$$(\mathcal{E}_{\Sigma_0^*}^N, \|\cdot\|_{\mathcal{E}_{\Sigma_0^*}^N})$$

to be its completion.

Next, we define the degenerate V -energy spaces along Σ_0^* , $\dot{\Sigma}$, and $\bar{\Sigma}$, respectively.

DEFINITION 8.1.2. – For $(\psi, \psi') \in {}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*)$, ${}^2\mathcal{C}_{cp}^\infty(\dot{\Sigma})$, and ${}^2\mathcal{C}_{cp}^\infty(\bar{\Sigma})$, respectively, we set

$$\begin{aligned} \|(\psi, \psi')\|_{\mathcal{E}_{\Sigma_0^*}^V} &\doteq \sqrt{\int_{\Sigma_0^*} \mathbf{J}_\mu^V[\Psi] n_{\Sigma_0^*}^\mu}, & \|(\psi, \psi')\|_{\mathcal{E}_{\dot{\Sigma}}^V} &\doteq \sqrt{\int_{\dot{\Sigma}} \mathbf{J}_\mu^V[\Psi] n_{\dot{\Sigma}}^\mu}, \\ \|(\psi, \psi')\|_{\mathcal{E}_{\bar{\Sigma}}^V} &\doteq \sqrt{\int_{\bar{\Sigma}} \mathbf{J}_\mu^V[\Psi] n_{\bar{\Sigma}}^\mu}, \end{aligned}$$

where Ψ is any extension of ψ to \mathcal{D} such that $n_{\Sigma_0^*} \Psi = \psi'$, $n_{\dot{\Sigma}} \Psi = 0$, $n_{\bar{\Sigma}} \Psi = 0$ respectively.

The above expression gives norms on the vector spaces ${}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*)$, ${}^2\mathcal{C}_{cp}^\infty(\dot{\Sigma})$, and ${}^2\mathcal{C}_{cp}^\infty(\bar{\Sigma})$, respectively, and we define the spaces

$$(\mathcal{E}_{\Sigma_0^*}^V, \|\cdot\|_{\mathcal{E}_{\Sigma_0^*}^V}), \quad (\mathcal{E}_{\dot{\Sigma}}^V, \|\cdot\|_{\mathcal{E}_{\dot{\Sigma}}^V}), \quad (\mathcal{E}_{\bar{\Sigma}}^V, \|\cdot\|_{\mathcal{E}_{\bar{\Sigma}}^V}),$$

to be their respective completions.

REMARK 8.1.1. – Note that the energy density is pointwise degenerate because as $r \rightarrow r_+$ the vector field V becomes null. An explicit calculation gives

$$\mathbf{J}_\mu^V[\tilde{\Psi}]n_{\Sigma_0^*}^\mu \sim |\partial_{t^*}\tilde{\Psi}|^2 + (r - r_+) |Z^*\tilde{\Psi}|^2 + |\bar{\nabla}\tilde{\Psi}|^2 \text{ as } r \rightarrow r_+.$$

This does not however affect the positive definitivity of the above norms, which moreover are easily seen to arise from a positive definite inner product. Thus, $\mathcal{E}_{\Sigma_0^*}^N$, $\mathcal{E}_{\Sigma_0^*}^V$, $\mathcal{E}_{\dot{\Sigma}}^V$, $\mathcal{E}_{\bar{\Sigma}}^V$ are all in fact Hilbert spaces. Note moreover that both $\mathcal{E}_{\Sigma_0^*}^N$ and $\mathcal{E}_{\Sigma_0^*}^V$ may be identified with subsets of $L_{loc}^2(\Sigma_0^*) \times L_{loc}^2(\Sigma_0^*)$ and, after this identification is made, $\mathcal{E}_{\Sigma_0^*}^N$ is a proper subset of $\mathcal{E}_{\Sigma_0^*}^V$.

Finally, we note that one may easily check that a sufficient condition for a pair of smooth functions (ψ, ψ') to lie in $\mathcal{E}_{\dot{\Sigma}}^V$ is that $\|(\psi, \psi')\|_{\mathcal{E}_{\dot{\Sigma}}^V} < \infty$ and $\lim_{r \rightarrow r_+}(\psi, \psi') = \lim_{r \rightarrow \infty}(\psi, \psi') = 0$.

8.1.2. *Scattering data along $\mathcal{H}_{\geq 0}^+$, \mathcal{H}^+ and $\overline{\mathcal{H}^+}$.* – We now carry out similar constructions for data along $\mathcal{H}_{\geq 0}^+$, \mathcal{H}^+ and $\overline{\mathcal{H}^+}$. Let us denote by $\mathcal{C}_{cp}^\infty(\mathcal{H}_{\geq 0}^+)$, $\mathcal{C}_{cp}^\infty(\mathcal{H}^+)$, $\mathcal{C}_{cp}^\infty(\overline{\mathcal{H}^+})$ the vector space of smooth compactly supported functions ψ defined on $\mathcal{H}_{\geq 0}^+$, \mathcal{H}^+ , $\overline{\mathcal{H}^+}$, respectively.

We start with the case of finite non-degenerate energy data along $\mathcal{H}_{\geq 0}^+$.

DEFINITION 8.1.3. – For $\psi \in \mathcal{C}_{cp}^\infty(\mathcal{H}_{\geq 0}^+)$ we set

$$\|\psi\|_{\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N} \doteq \sqrt{\int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_\mu^N[\psi]n_{\mathcal{H}_{\geq 0}^+}^\mu}.$$

The above expression gives a norm on the vector space $\mathcal{C}_{cp}^\infty(\mathcal{H}_{\geq 0}^+)$, and we define the space

$$(\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N, \|\cdot\|_{\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N})$$

to be its completion.

Next, we define the K -energy spaces along $\mathcal{H}_{\geq 0}^+$, \mathcal{H}^+ , and $\overline{\mathcal{H}^+}$, respectively.

DEFINITION 8.1.4. – For $\psi \in \mathcal{C}_{cp}^\infty(\mathcal{H}_{\geq 0}^+)$, $\mathcal{C}_{cp}^\infty(\mathcal{H}^+)$, and $\mathcal{C}_{cp}^\infty(\overline{\mathcal{H}^+})$, respectively, we set

$$\begin{aligned} \|\psi\|_{\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K} &\doteq \sqrt{\int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_\mu^K[\psi]n_{\mathcal{H}_{\geq 0}^+}^\mu}, & \|\psi\|_{\mathcal{E}_{\mathcal{H}^+}^K} &\doteq \sqrt{\int_{\mathcal{H}^+} \mathbf{J}_\mu^K[\psi]n_{\mathcal{H}^+}^\mu}, \\ \|\psi\|_{\mathcal{E}_{\overline{\mathcal{H}^+}}^K} &\doteq \sqrt{\int_{\overline{\mathcal{H}^+}} \mathbf{J}_\mu^K[\psi]n_{\overline{\mathcal{H}^+}}^\mu}. \end{aligned}$$

The above expression gives norms on the vector spaces $\mathcal{C}_{cp}^\infty(\mathcal{H}_{\geq 0}^+)$, $\mathcal{C}_{cp}^\infty(\mathcal{H}^+)$, and $\mathcal{C}_{cp}^\infty(\overline{\mathcal{H}^+})$, respectively, and we define the spaces

$$(\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K, \|\cdot\|_{\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K}), \quad (\mathcal{E}_{\mathcal{H}^+}^K, \|\cdot\|_{\mathcal{E}_{\mathcal{H}^+}^K}), \quad (\mathcal{E}_{\overline{\mathcal{H}^+}}^K, \|\cdot\|_{\mathcal{E}_{\overline{\mathcal{H}^+}}^K}),$$

to be their respective completions.

REMARK 8.1.2. – Note that the K -based energy densities are pointwise degenerate in that the norms do not control $\partial_{\theta^*}\psi$ and $\partial_{\phi^*}\psi$. An explicit calculation gives

$$\mathbf{J}_\mu^K[\psi]n_{\mathcal{H}^+}^\mu \sim |K\psi|^2.$$

Again, this degeneration does not however affect the positive definitivity of the above norms, which moreover are again easily seen to arise from a positive definite inner product. Thus, $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N$, $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K$, $\mathcal{E}_{\mathcal{H}^+}^K$, $\mathcal{E}_{\mathcal{H}^+}^K$ are all in fact Hilbert spaces. Note moreover that both $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N$ and $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K$ may be identified with subsets of $L^2_{\text{loc}}(\mathcal{H}^+)$ and, after this identification is made, $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N$ is a proper subset of $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K$.

8.1.3. Scattering data along \mathcal{I}^+ . – Finally, we turn to null infinity. Let us denote by $\mathcal{C}_{cp}^\infty(\mathcal{I}^+)$ the vector space of smooth compactly supported functions ϕ defined on \mathcal{I}^+ .

The space of finite energy data along \mathcal{I}^+ is then defined as follows.

DEFINITION 8.1.5. – For $\phi \in \mathcal{C}_{cp}^\infty(\mathcal{I}^+)$ we set

$$\|\phi\|_{\mathcal{E}_{\mathcal{I}^+}^T} \doteq \sqrt{\int_{\mathcal{I}^+} |\partial_\tau \phi|^2}.$$

The above expression gives a norm on the vector space $\mathcal{C}_{cp}^\infty(\mathcal{I}^+)$, and we define the space

$$(\mathcal{E}_{\mathcal{I}^+}^T, \|\cdot\|_{\mathcal{E}_{\mathcal{I}^+}^T})$$

to be its completion.

REMARK 8.1.3. – Note that this energy density is pointwise degenerate in that it does not control $\partial_\theta\phi$ and $\partial_\phi\phi$. As before, this does not however affect the positive definitivity of the above norms, which moreover are easily seen to arise from a positive definite inner product. Thus, $\mathcal{E}_{\mathcal{I}^+}^T$ is in fact a Hilbert space.

8.2. Definition and boundedness of the forward maps

In this section we will define the various forward maps from Cauchy data to scattering data and infer their boundedness. However, we first need the following corollary of Theorems 3.7.2 and 4.2.1.

COROLLARY 8.2.1. – For all solutions ψ to (2) on $\mathcal{R}_{\geq 0}$ arising from initial data in ${}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*)$, we have that the radiation fields to $\mathcal{H}_{\geq 0}^+$ and \mathcal{I}^+ lie in the spaces $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N$ and $\mathcal{E}_{\mathcal{I}^+}^T$ respectively.

Proof. – Given Theorems 3.7.1 and 4.2.1, the only statement we need to check is that the radiation fields lie in the closure of compactly supported smooth functions.

In order to prove this, we start by giving a (standard) argument which upgrades Theorem 3.7.2 to the statement that

$$(137) \quad \lim_{s \rightarrow \infty} \int_{\Sigma_s^* \cap \{r \in [r_+, R]\}} \mathbf{J}_\mu^N[\psi]n_{\Sigma_s^*}^\mu = 0 \quad \forall R > r_+.$$

First we observe that the fundamental theorem of calculus and Theorem 3.7.2 immediately imply the following Lipschitz property:

$$(138) \quad \left| \int_{\Sigma_{s_2}^* \cap \{r \in [r_+, R]\}} \mathbf{J}_\mu^N[\psi] n_{\Sigma_{s_2}^*}^\mu - \int_{\Sigma_{s_1}^* \cap \{r \in [r_+, R]\}} \mathbf{J}_\mu^N[\psi] n_{\Sigma_{s_1}^*}^\mu \right| \leq B(\psi) |s_2 - s_1|.$$

Let $\epsilon > 0$. Using (138) we may obtain

$$(139) \quad \int_{\Sigma_s^* \cap \{r \in [r_+, R]\}} \mathbf{J}_\mu^N[\psi] n_{\Sigma_s^*}^\mu \leq B(\psi)\epsilon + \inf_{s' \in [s-\epsilon, s+\epsilon]} \int_{\Sigma_{s'}^* \cap \{r \in [r_+, R]\}} \mathbf{J}_\mu^N[\psi] n_{\Sigma_{s'}^*}^\mu.$$

Of course, Theorem 3.7.2 implies that

$$\lim_{s \rightarrow \infty} \inf_{s' \in [s-\epsilon, s+\epsilon]} \int_{\Sigma_{s'}^* \cap \{r \in [r_+, R]\}} \mathbf{J}_\mu^N[\psi] n_{\Sigma_{s'}^*}^\mu = 0.$$

Thus (139) implies that

$$\limsup_{s \rightarrow \infty} \int_{\Sigma_s^* \cap \{r \in [r_+, R]\}} \mathbf{J}_\mu^N[\psi] n_{\Sigma_s^*}^\mu \leq B(\psi)\epsilon.$$

Since ϵ was arbitrary, (137) follows.

Now, using Theorem 3.7.2 we immediately obtain higher order versions of (137). Sobolev inequalities then imply that

$$(140) \quad \lim_{s \rightarrow \infty} \sup_{\Sigma_s^* \cap \{r \in [r_+, R]\}} |\psi| = 0.$$

In particular, we may conclude that the radiation field along the horizon \mathcal{H}^+ lies in the closure of compactly supported functions.

For the radiation field along null infinity, we recall that in the proof of Proposition 6.8.1, we proved that $\partial_\tau \varphi|_{r=r_0}$ converges as $r_0 \rightarrow \infty$ to the ∂_τ derivative of the radiation field in $L^2_{\mathbb{R}_{\geq \tau_0} \times \mathbb{S}^2}$ for some sufficiently negative τ_0 . For each r_0 , (140) implies that $\varphi|_{r=r_0}$ lies in the closure of smooth compactly supported functions; completeness thus implies that the radiation field along null infinity also lies in this closure. \square

Similarly, we have the following two corollaries.

COROLLARY 8.2.2. – *For all solutions ψ to (2) on \mathcal{R} arising from initial data in ${}^2\mathcal{C}_{cp}^\infty(\bar{\Sigma})$, we have that the radiation fields to \mathcal{H}^+ and \mathcal{I}^+ lie in the spaces $\mathcal{E}_{\mathcal{H}^+}^V$ and $\mathcal{E}_{\mathcal{I}^+}^T$ respectively.*

Proof. – It follows immediately from a K energy estimate near the bifurcate sphere that the radiation field of ψ along \mathcal{H}^+ vanishes for sufficiently negative t^* . Since a finite-in-time energy estimate implies that $(\psi|_{\Sigma_0^*}, n_{\Sigma_0^*} \psi|_{\Sigma_0^*}) \in {}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*)$, the rest of the proof may be concluded with an appeal to Corollary 8.2.1. \square

COROLLARY 8.2.3. – *For all solutions ψ to (2) on \mathcal{R} arising from initial data in ${}^2\mathcal{C}_{cp}^\infty(\bar{\Sigma})$, we have that the radiation fields to $\bar{\mathcal{H}}^+$ and \mathcal{I}^+ lie in the spaces $\mathcal{E}_{\bar{\mathcal{H}}^+}^V$ and $\mathcal{E}_{\mathcal{I}^+}^T$ respectively.*

Proof. – The proof is the same as the proof of Corollary 8.2.2. \square

The above three corollaries allow us to make the following definition.

DEFINITION 8.2.1. – We define the “forward maps”

$$\begin{aligned} \mathcal{F}_+ : {}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*) &\rightarrow \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N \oplus \mathcal{E}_{\mathcal{I}^+}^T, & \mathcal{F}_+ : {}^2\mathcal{C}_{cp}^\infty(\overset{\circ}{\Sigma}) &\rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T, \\ \mathcal{F}_+ : {}^2\mathcal{C}_{cp}^\infty(\overline{\Sigma}) &\rightarrow \mathcal{E}_{\overline{\mathcal{H}^+}}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T, \end{aligned}$$

to be the maps

$$(141) \quad (\psi|_{\Sigma_0^*, \overset{\circ}{\Sigma}, \text{ or } \overline{\Sigma}}, \psi'|_{\Sigma_0^*, \overset{\circ}{\Sigma}, \text{ or } \overline{\Sigma}}) \mapsto \psi$$

$$\mapsto (\psi|_{\mathcal{H}_{\geq 0}^+, \mathcal{H}^+, \text{ or } \overline{\mathcal{H}^+}} \doteq \psi|_{\mathcal{H}_{\geq 0}^+, \mathcal{H}^+, \text{ or } \overline{\mathcal{H}^+}}, \phi|_{\mathcal{I}^+} \doteq r\psi|_{\mathcal{I}^+})$$

which take smooth initial data in ${}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*)$, ${}^2\mathcal{C}_{cp}^\infty(\overset{\circ}{\Sigma})$ or ${}^2\mathcal{C}_{cp}^\infty(\overline{\Sigma})$, solve the wave equation to the future and then compute the radiation fields along $\mathcal{H}_{\geq 0}^+$, \mathcal{H}^+ or $\overline{\mathcal{H}^+}$, respectively, and \mathcal{I}^+ .

Theorem 4.2.1 and (46) now imply

THEOREM 8.2.1. – The forward map \mathcal{F}_+ uniquely extends by density to a bounded map

$$\mathcal{F}_+ : \mathcal{E}_{\Sigma_0^*}^N \rightarrow \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N \oplus \mathcal{E}_{\mathcal{I}^+}^T.$$

This gives Theorem 1 of Section 2.3.1.

Similarly, Theorem 7.1 implies the following theorem.

THEOREM 8.2.2. – The forward map \mathcal{F}_+ uniquely extends by density to a bounded map

$$\mathcal{F}_+ : \mathcal{E}_{\Sigma_0^*}^V \rightarrow \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T.$$

Lastly, Corollaries 7.1, 8.2.2 and 8.2.3 now imply the following two theorems.

THEOREM 8.2.3. – The forward map \mathcal{F}_+ uniquely extends by density to a bounded map

$$\mathcal{F}_+ : \mathcal{E}_{\overset{\circ}{\Sigma}}^V \rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T.$$

THEOREM 8.2.4. – The forward map \mathcal{F}_+ uniquely extends by density to a bounded map

$$\mathcal{F}_+ : \mathcal{E}_{\overline{\Sigma}}^V \rightarrow \mathcal{E}_{\overline{\mathcal{H}^+}}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T.$$

We have obtained thus Theorem 3 of the Section 2.3.3.

9. The backwards maps and the scattering matrix

This section represents the heart of the paper. We will here construct bounded maps

$$(142) \quad \mathcal{B}_- : \mathcal{E}_{\mathcal{H}^+_{\geq 0}}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\Sigma_0^*}^V, \quad \mathcal{B}_- : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\Sigma}^V, \quad \mathcal{B}_- : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\bar{\Sigma}}^V,$$

which will invert the maps \mathcal{F}_+ from Theorem 8.2.2, 8.2.3 and 8.2.4 and then we shall construct the scattering maps

$$(143) \quad \mathcal{S} : \mathcal{E}_{\mathcal{H}^-}^V \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\mathcal{H}^+}^V \oplus \mathcal{E}_{\mathcal{I}^+}^T, \quad \mathcal{S} : \mathcal{E}_{\mathcal{H}^-}^V \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\mathcal{H}^+}^V \oplus \mathcal{E}_{\mathcal{I}^+}^T.$$

It turns out that for technical reasons, it is easiest to first construct the middle map of (142) and show that it is a two-sided inverse of the corresponding forward map on \mathcal{E}_{Σ}^V . This will be the content of Section 9.1 where the main result is stated as Theorem 9.1.1. The remaining two backwards maps to Σ_0^* and $\bar{\Sigma}$ will then be easily constructed in Sections 9.2 and 9.3, and these will be shown in Theorems 9.2.1 and 9.3.1 to be two-sided inverses of the corresponding maps \mathcal{F}_+ . The above three theorems will give Theorem 4 of Section 2.3.4.

The scattering maps (143) and their boundedness will be deduced as Theorem 9.5.2 in Section 9.5 after introducing the past-analogs \mathcal{F}_- and \mathcal{B}_+ and inferring their boundedness (Theorem 9.5.1). This will give Theorem 5 of Section 2.3.5. We shall also represent \mathcal{S} in the frequency domain by Theorem 9.5.3, giving the relationship between the fixed-frequency and physical space theories. This will imply in particular Theorem 12 of Section 2.4.2.

Finally, this section contains two separate “asides,” Sections 9.4 and 9.6, either of which can be skipped, but both of which could have interest independent of the rest of the paper. In Section 9.4, we will use the maps (142) to complete the theory of boundedness and integrated decay for the degenerate V -energy by giving the proof of Theorem 7.2 from Section 7.2. In Section 9.6, we will give an alternative, self-contained discussion of the Schwarzschild $a = 0$ case using exclusively physical-space (i.e., “time-dependent”) methods.

9.1. The backwards map to $\mathring{\Sigma}$

We begin by constructing the map $\mathcal{B}_- : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\mathring{\Sigma}}^V$.

9.1.1. *A frequency-space definition of \mathcal{B}_- .* – First, we define what will turn out to be essentially the Fourier transform of our backwards map. We begin by recalling the coefficients $\mathfrak{A}_{\mathcal{I}^\pm}$, $\mathfrak{A}_{\mathcal{H}^\pm}$ and the Wronskian \mathfrak{W} from Definitions 5.3.1 and 5.3.3, as well as Theorem 5.3.1 which states that \mathfrak{W} never vanishes.

DEFINITION 9.1.1. – *For all smooth functions $a_{\mathcal{I}^+}(\omega, m, \ell)$ and $a_{\mathcal{H}^-}(\omega, m, \ell)$ which are only supported on a compact set of (ω, m, ℓ) , for all (ω, m, ℓ) with $\omega \neq 0$ and $\omega \neq \omega_+ m$, we define*

$$\hat{\mathcal{B}}_- (a_{\mathcal{H}^+}, a_{\mathcal{I}^+})|_{(r, \omega, m, \ell)} \doteq \left(a_{\mathcal{I}^+} \overline{\mathfrak{W}^{-1}(2i\omega)} \right) \overline{U_{\text{hor}}} + \left(a_{\mathcal{H}^+} \overline{\mathfrak{W}^{-1}(2i(\omega - \omega_+ m))} \right) \overline{U_{\text{inf}}}.$$

The next proposition explains the definition of $\hat{\mathcal{B}}_-$.

PROPOSITION 9.1.1. – For $\omega \neq 0$ and $\omega \neq \omega_+m$, $\hat{\mathcal{B}}_-(a_{\mathcal{H}^+}, a_{\mathcal{I}^+})$ is the unique solution u to the radial o.d.e. (61) with vanishing right hand side $H = 0$ such that there exist complex numbers $\alpha(\omega, m, \ell)$ and $\beta(\omega, m, \ell)$ satisfying

$$(144) \quad u = a_{\mathcal{H}^+} U_{\text{hor}} + \alpha(\omega, m, \ell) \overline{U_{\text{hor}}},$$

$$(145) \quad u = a_{\mathcal{I}^+} U_{\text{inf}} + \beta(\omega, m, \ell) \overline{U_{\text{inf}}}.$$

Proof. – We start with uniqueness. Suppose that we have two solutions u and \tilde{u} to the radial o.d.e. (61) with a vanishing right hand side H such that

$$\begin{aligned} u &= a_{\mathcal{H}^+} U_{\text{hor}} + \alpha(\omega, m, \ell) \overline{U_{\text{hor}}}, & \tilde{u} &= a_{\mathcal{H}^+} U_{\text{hor}} + \tilde{\alpha}(\omega, m, \ell) \overline{U_{\text{hor}}}, \\ u &= a_{\mathcal{I}^+} U_{\text{inf}} + \beta(\omega, m, \ell) \overline{U_{\text{inf}}}, & \tilde{u} &= a_{\mathcal{I}^+} U_{\text{inf}} + \tilde{\beta}(\omega, m, \ell) \overline{U_{\text{inf}}}. \end{aligned}$$

Then, for each (ω, m, ℓ) with $\omega \neq 0$ and $\omega \neq \omega_+m$, $\overline{u - \tilde{u}}$ would be a solution to the radial o.d.e. (61) with a vanishing right hand side such that

$$\begin{aligned} \overline{u - \tilde{u}} &\sim e^{i\omega r^*} \text{ as } r \rightarrow \infty, \\ \overline{u - \tilde{u}} &\sim e^{-i(\omega - \omega_+m)r^*} \text{ as } r \rightarrow r_+. \end{aligned}$$

These asymptotic conditions imply that $\overline{u - \tilde{u}}$ corresponds to a “mode solution” (see Definition 1.1 of [67]), and Theorem 1.6 of [67] proves that there are no non-zero mode solutions.

To see that $\hat{\mathcal{B}}_-$ verifies (144) and (145), it suffices to recall the relations

$$\begin{aligned} \overline{U_{\text{hor}}} &= \overline{a_{\mathcal{I}^+} U_{\text{inf}}} + \overline{(2i\omega)^{-1} \mathfrak{W} U_{\text{inf}}}, \\ \overline{U_{\text{inf}}} &= \overline{a_{\mathcal{H}^+} U_{\text{hor}}} + \overline{(2i(\omega - \omega_+m))^{-1} \mathfrak{W} U_{\text{hor}}}. \end{aligned} \quad \square$$

We now introduce a useful function space.

DEFINITION 9.1.2. – Let $\check{\mathcal{C}}_{cp}^\infty$ denote the set of functions $f : \mathbb{R} \times \mathbb{S}^2 \rightarrow \mathbb{C}$ such that

$$\hat{f}(\omega, m, \ell) \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} e^{-im\phi} S_{m\ell}(a\omega, \theta) f \sin \theta dt d\theta d\phi$$

is smooth in ω and vanishes for (ω, m, ℓ) outside a compact set of $\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$.

Next, observing that $\check{\mathcal{C}}_{cp}^\infty$ may be naturally identified as a subset of either $L_{\text{loc}}^2(\mathcal{H}^+)$ or $L_{\text{loc}}^2(\mathcal{I}^+)$, we let $\check{\mathcal{C}}_{cp}^\infty(\mathcal{H}^+)$ be the result of identifying $\check{\mathcal{C}}_{cp}^\infty$ with a subset of $\mathcal{E}_{\mathcal{H}^+}^K$, and let $\check{\mathcal{C}}_{cp}^\infty(\mathcal{I}^+)$ be the result of identifying $\check{\mathcal{C}}_{cp}^\infty$ with a subset of $\mathcal{E}_{\mathcal{H}^+}^T$.

REMARK 9.1.1. – One may easily check that $\check{\mathcal{C}}_{cp}^\infty(\mathcal{H}^+)$ is dense in $\mathcal{E}_{\mathcal{H}^+}^K$ and that $\check{\mathcal{C}}_{cp}^\infty(\mathcal{I}^+)$ is dense in $\mathcal{E}_{\mathcal{H}^+}^T$.

We now define the map \mathcal{B}_- on the space $\check{\mathcal{C}}_{cp}^\infty(\mathcal{H}^+) \oplus \check{\mathcal{C}}_{cp}^\infty(\mathcal{I}^+)$.

DEFINITION 9.1.3. – For all $(\psi, \phi) \in \check{C}_{cp}^\infty(\mathcal{H}^+) \oplus \check{C}_{cp}^\infty(\mathcal{I}^+)$, we define the function $\mathcal{B}_-(\psi, \phi) : \overset{\circ}{\mathcal{R}} \rightarrow \mathbb{C}$ by

$$\mathcal{B}_-(\psi, \phi)|_{(t,r,\theta,\phi)} \doteq \frac{1}{(r^2 + a^2)^{1/2} \sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} e^{-it\omega} e^{im\phi} S_{m\ell}(a\omega, \theta) \hat{\mathcal{B}}_-(\hat{\psi}, \hat{\phi})|_{(\omega,r,m,\ell)} d\omega.$$

Note that $\hat{\mathcal{B}}_-(\hat{\psi}, \hat{\phi})$ vanishes for all (ω, m, ℓ) outside a compact set; it immediately follows that $\mathcal{B}_-(\psi, \phi)$ is a smooth function of (t, r, θ, ϕ) .

9.1.2. *Boundedness.* – The following proposition will be used to show that the map $(\psi, \phi) \mapsto (\mathcal{B}_-(\psi, \phi)|_{\overset{\circ}{\Sigma}}, n_{\overset{\circ}{\Sigma}} \mathcal{B}_-(\psi, \phi)|_{\overset{\circ}{\Sigma}})$ is bounded.

PROPOSITION 9.1.2. – For all $(\psi, \phi) \in \check{C}_{cp}^\infty(\mathcal{H}^+) \oplus \check{C}_{cp}^\infty(\mathcal{I}^+)$, we have

$$\int_{\overset{\circ}{\Sigma}} \mathbf{J}_\mu^V [\mathcal{B}_-(\psi, \phi)] n_{\overset{\circ}{\Sigma}}^\mu \leq B \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} [|\partial_\tau \psi + \omega_+ \partial_\phi \psi|^2 + |\partial_\tau \phi|^2] \sin \theta d\tau d\theta d\phi.$$

Proof. – Set

$$\begin{aligned} u &\doteq \hat{\mathcal{B}}_-(\hat{\psi}, \hat{\phi})|_{(\omega,r,m,\ell)}, \\ \psi &\doteq \mathcal{B}_-(\psi, \phi)|_{(t,r,\theta,\phi)}. \end{aligned}$$

First of all, we observe that $\psi : \overset{\circ}{\mathcal{R}} \rightarrow \mathbb{C}$ is a smooth solution to $\square_g \psi = 0$, is easily seen to be sufficiently integrable in the sense of Definition 5.1.1, and that applying Carter's separation to ψ yields u .

Keeping the explicit formula for $\hat{\mathcal{B}}_-$ in mind, applying Theorem 6.2.1 to u implies that for each $-\infty < R_-^* < R_+^* < \infty$ we have

$$\begin{aligned} (146) \quad &(\omega - \omega_+ m)^2 |a_{\mathcal{H}^-}|^2 + \omega^2 |a_{\mathcal{I}^-}|^2 + \int_{R_-^*}^{R_+^*} [|u'|^2 + ((1 - r_{\text{trap}} r^{-1})^2 (\omega^2 + \Lambda) + 1) |u|^2] dr^* \\ &\leq B(R_-^*, R_+^*) [(\omega - \omega_+ m)^2 |\hat{\psi}|^2 + \omega^2 |\hat{\phi}|^2]. \end{aligned}$$

The rest of the proof will borrow some ideas from Section 13 of [30]. In order to work around the presence of the $(1 - r_{\text{trap}} r^{-1})^2$ term in (146), it will be useful to decompose ψ in pieces, each of which experience trapping near a specific value of r . We first define the following ranges of (ω, m, ℓ) :

DEFINITION 9.1.4. – Let $\epsilon > 0$ be a sufficiently small parameter to be fixed later. We define

$$\begin{aligned} \mathcal{F}_0 &\doteq \{(\omega, m, \ell) : r_{\text{trap}} = 0\}, \\ \mathcal{F}_i &\doteq \{(\omega, m, \ell) : r_{\text{trap}} \in [3M - s^- + \epsilon(i-1), 3M - s^- + \epsilon i)\}, \\ &\quad \forall i = 1, \dots, [\epsilon^{-1}(s^+ + s^-)]. \end{aligned}$$

Observe that each value of (ω, m, ℓ) lies in exactly one of the \mathcal{F}_i .

DEFINITION 9.1.5. – We define ψ_i by a phase space multiplication of ψ by $1_{\mathcal{F}_i}$, the indicator function of \mathcal{F}_i :

$$\psi_i \doteq \frac{1}{(r^2 + a^2)^{1/2} \sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} e^{-i\omega t} 1_{\mathcal{F}_i} S_{m\ell}(a\omega, \cos \theta) e^{im\phi} \hat{\mathcal{B}}_-(\hat{\psi}, \hat{\phi}) d\omega.$$

Note that each ψ_i is a smooth function from $\overset{\circ}{\mathcal{R}}$ to \mathbb{C} , satisfies $\square_g \psi_i = 0$ and is sufficiently integrable in the sense of Definition 5.1.1.

Next, keeping in mind that each $\hat{\psi}_i$ is compactly supported in (ω, m, ℓ) , Plancherel immediately implies that for each $r_+ < r_0 < r_1 < \infty$ we have

$$(147) \quad \int_{-\infty}^{\infty} \int_{\{t=s\} \cap \{r \in [r_0, r_1]\}} \mathbf{J}_{\mu}^V [\psi_i] n_{\{t=s\}}^{\mu} ds < \infty.$$

In particular, for each $r_+ < r_0 < r_1 < \infty$ and $i = 0, \dots, \lceil \epsilon^{-1} (s^+ + s^-) \rceil$ there exist a constant $C_i(r_0, r_1)$ and a dyadic sequence $\{s_n^{(i)}\}_{n=1}^{\infty}$ such that $s_n^{(i)} \rightarrow \infty$ as $n \rightarrow \infty$ and

$$(148) \quad \int_{\{t=s_n^{(i)}\} \cap \{r \in [r_0, r_1]\}} \mathbf{J}_{\mu}^N [\psi_i] n_{\{t=s_n^{(i)}\}}^{\mu} \leq \frac{C_i(r_0, r_1)}{s_n^{(i)}}.$$

Next, taking ϵ from Definition 9.1.4 sufficiently small (and then fixing ϵ), for each r_i we appeal to Corollary 3.4.1 and construct a T -invariant timelike vector field V_i on $\overset{\circ}{\mathcal{R}}$ which is Killing in the region

$$r \in [3M - s^- + (i - 1)\epsilon, 3M - s^- + i\epsilon],$$

and is equal to V for r sufficiently close to r_+ and r sufficiently large.

Finally, we are ready for our main estimate. For each $r_+ < r_0 < r_1 < \infty$ such that $r_0 - r_+$ is sufficiently small and r_1 is sufficiently large, we apply the energy identity associated to V_i in between the hypersurfaces $\overset{\circ}{\Sigma} \cap \{r \in [r_0, r_1]\}$, $\{r = r_0\} \cap J^+(\overset{\circ}{\Sigma}) \cap J^-(\{t = s_n^{(i)}\})$, $\{r = r_1\} \cap J^+(\overset{\circ}{\Sigma}) \cap J^-(\{t = s_n^{(i)}\})$, and $\{t = s_n^{(i)}\} \cap \{r \in [r_0, r_1]\}$. We obtain

$$(149) \quad \begin{aligned} & \int_{\overset{\circ}{\Sigma} \cap \{r \in [r_0, r_1]\}} \mathbf{J}_{\mu}^{V_i} [\psi_i] n_{\overset{\circ}{\Sigma}}^{\mu} \\ & \leq B \int_0^{s_n^{(i)}} \int_{\{t=s\} \cap \text{supp}(\mathbf{K}^{V_i})} \mathbf{J}_{\mu}^{V_i} [\psi_i] n_{\{t=s\}}^{\mu} ds + \int_{\{t=s_n^{(i)}\} \cap \{r \in [r_0, r_1]\}} \mathbf{J}_{\mu}^{V_i} [\psi_i] n_{\{t=s_n^{(i)}\}}^{\mu} \\ & \quad + B \left(\int_{\{r=r_0\}} + \int_{\{r=r_1\}} \right) \left| V \left((r^2 + a^2)^{1/2} \psi_i \right) \partial_{r^*} \left((r^2 + a^2)^{1/2} \psi_i \right) \right| \sin \theta dt d\theta d\phi, \end{aligned}$$

where we have used the calculation (123) and the fact that

$$|\mathbf{K}^{V_i}| \leq B \mathbf{J}^{V_i},$$

where we recall that $\mathbf{K}^{V_i} = (V_i)_{\pi_{\alpha\beta}} \mathbf{T}^{\alpha\beta}$ and $(V_i)_{\pi_{\alpha\beta}}$ denotes the deformation tensor of V_i .

Taking $n \rightarrow \infty$ and appealing to (148) then yields

(150)

$$\begin{aligned} & \int_{\Sigma \cap \{r \in [r_0, r_1]\}} \mathbf{J}_\mu^{V_i} [\psi_i] n_\Sigma^\mu \\ & \leq B \int_0^\infty \int_{\{t=s\} \cap \text{supp}(\mathbf{K}^{V_i})} \mathbf{J}_\mu^{V_i} [\psi_i] n_{\{t=s\}}^\mu ds \\ & \quad + B \left(\int_{\{r=r_0\}} + \int_{\{r=r_1\}} \right) \left| V \left((r^2 + a^2)^{1/2} \psi_i \right) \partial_{r^*} \left((r^2 + a^2)^{1/2} \psi_i \right) \right| \sin \theta dt d\theta d\phi, \end{aligned}$$

Next, since $(1 - r_{\text{trap}} r^{-1})^2 |\hat{\psi}_i|^2 \sim |\hat{\psi}_i|^2$ for values of r in the support of $\mathbf{K}^{(V_i)}$, we observe that applying Plancherel and (146) yields

(151)

$$\begin{aligned} & \int_0^\infty \int_{\{t=s\} \cap \text{supp}(\mathbf{K}^{V_i})} \mathbf{J}_\mu^{V_i} [\psi_i] n_{\{t=s\}}^\mu ds \\ & \leq \int_{-\infty}^\infty \int_{\{t=s\} \cap \text{supp}(\mathbf{K}^{V_i})} \left| \mathbf{J}_\mu^{V_i} [\psi_i] n_{\{t=s\}}^\mu \right| ds \\ & \leq B \int_{-\infty}^\infty \sum_{(\omega, m, \ell) \in \mathcal{F}_i} \int_{r_{\min}^*}^{r_{\max}^*} \left[|u'|^2 + \left((1 - r_{\text{trap}} r^{-1})^2 (\omega^2 + \Lambda) + 1 \right) |u|^2 \right] dr^* d\omega \\ & \leq B \int_{-\infty}^\infty \sum_{m\ell} \left[(\omega - \omega_+ m)^2 |\hat{\psi}|^2 + \omega^2 |\hat{\phi}|^2 \right] d\omega \\ & \leq B \int_{-\infty}^\infty \int_{\mathbb{S}^2} \left[|(\partial_\tau + \omega_+ \partial_\phi) \psi|^2 + |\partial_\tau \phi|^2 \right] \sin \theta d\tau d\theta d\phi. \end{aligned}$$

We conclude that

$$\begin{aligned} (152) \quad & \int_\Sigma \mathbf{J}_\mu^{V_i} [\psi_i] n_\Sigma^\mu \leq B \int_{-\infty}^\infty \int_{\mathbb{S}^2} \left[|(\partial_\tau + \omega_+ \partial_\phi) \psi|^2 + |\partial_\tau \phi|^2 \right] \sin \theta d\tau d\theta d\phi \\ & \quad + B \liminf_{r_0 \rightarrow r_+} \liminf_{r_1 \rightarrow \infty} \left(\int_{\{r=r_0\}} + \int_{\{r=r_1\}} \right) \\ & \quad \left| V \left((r^2 + a^2)^{1/2} \psi_i \right) \partial_{r^*} \left((r^2 + a^2)^{1/2} \psi_i \right) \right| \sin \theta dt d\theta d\phi. \end{aligned}$$

It immediately follows from Proposition 5.2.3, the compact support of $\hat{\psi}_i$ in (ω, m, ℓ) , and (146) that

(153)

$$\begin{aligned} & \liminf_{r_0 \rightarrow r_+} \int_{r=r_0} \left| V \left((r^2 + a^2)^{1/2} \psi_i \right) \partial_{r^*} \left((r^2 + a^2)^{1/2} \psi_i \right) \right| \sin \theta dt d\theta d\phi \\ & = \liminf_{r_0 \rightarrow r_+} \int_{r=r_0} \left| K \left((r^2 + a^2)^{1/2} \psi_i \right) \partial_{r^*} \left((r^2 + a^2)^{1/2} \psi_i \right) \right| \sin \theta dt d\theta d\phi \\ & \leq \liminf_{r_0 \rightarrow r_+} B \int_{-\infty}^\infty \sum_{m\ell} |(\omega - \omega_+ m) uu'| d\omega \end{aligned}$$

$$\begin{aligned} &\leq B \int_{-\infty}^{\infty} \sum_{m\ell} (\omega - \omega_+ m)^2 \left[|a_{\mathcal{H}^+}|^2 + |a_{\mathcal{H}^-}|^2 \right] d\omega \\ &\leq B \int_{-\infty}^{\infty} \sum_{m\ell} \left[(\omega - \omega_+ m)^2 |\hat{\psi}|^2 + \omega^2 |\hat{\phi}|^2 \right] d\omega \\ &\leq B \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \left[|(\partial_\tau + \omega_+ \partial_\phi) \psi|^2 + |\partial_\tau \phi|^2 \right] \sin \theta d\tau d\theta d\phi. \end{aligned}$$

REMARK 9.1.2. – Note that the passing of the limit through the integral and sum that implicitly occurs between lines 3 and 4 is justified by Proposition 5.2.3 and the compact support of $\hat{\psi}_i$ in (ω, m, ℓ) . Moreover, we emphasize that the compact support of $\hat{\psi}_i$ is only used qualitatively in this fashion; none of the constants depend on the size of the support of $\hat{\psi}_i$.

Similarly, it immediately follows from Proposition 6.7.1, Proposition 6.6.1, the compact support of $\hat{\psi}_i$, and (146) that

(154)

$$\begin{aligned} &\liminf_{r_1 \rightarrow \infty} \int_{r=r_1} \left| V \left((r^2 + a^2)^{1/2} \psi_i \right) \partial_{r^*} \left((r^2 + a^2)^{1/2} \psi_i \right) \right| \sin \theta dt d\theta d\phi \\ &= \liminf_{r_1 \rightarrow \infty} \int_{r=r_1} \left| T \left((r^2 + a^2)^{1/2} \psi_i \right) \partial_{r^*} \left((r^2 + a^2)^{1/2} \psi_i \right) \right| \sin \theta dt d\theta d\phi \\ &\leq \liminf_{r_1 \rightarrow \infty} B \int_{-\infty}^{\infty} \sum_{m\ell} |\omega u u'| \\ &\leq B \int_{-\infty}^{\infty} \sum_{m\ell} \omega^2 \left[|a_{\mathcal{I}^+}|^2 + |a_{\mathcal{I}^-}|^2 \right] d\omega \\ &\leq B \int_{-\infty}^{\infty} \sum_{m\ell} \left[(\omega - \omega_+ m)^2 |\hat{\psi}|^2 + \omega^2 |\hat{\phi}|^2 \right] d\omega \\ &\leq B \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \left[|(\partial_\tau + \omega_+ \partial_\phi) \psi|^2 + |\partial_\tau \phi|^2 \right] \sin \theta d\tau d\theta d\phi. \end{aligned}$$

Combining (152), (153), and (154) yields

$$(155) \quad \int_{\hat{\Sigma}} \mathbf{J}_\mu^{V_i} [\psi_i] n_{\hat{\Sigma}}^\mu \leq B \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \left[|(\partial_\tau + \omega_+ \partial_\phi) \psi|^2 + |\partial_\tau \phi|^2 \right] \sin \theta d\tau d\theta d\phi.$$

We conclude the proof with the (trivial) observation that

$$\int_{\hat{\Sigma}} \mathbf{J}_\mu^V [\psi] n_{\hat{\Sigma}}^\mu \leq B \sum_{i=1}^{\lceil \epsilon^{-1}(s^+ + s^-) \rceil} \int_{\{t=0\}} \mathbf{J}_\mu^{V_i} [\psi_i] n_{\{t=0\}}^\mu. \quad \square$$

The following proposition will be used to show that the range of $(\mathcal{B}_-|_{\hat{\Sigma}}, n_{\hat{\Sigma}} \mathcal{B}_-|_{\hat{\Sigma}})$ lies in $\mathcal{E}_{\hat{\Sigma}}^V$.

PROPOSITION 9.1.3. – For all $(\psi, \phi) \in \check{C}_{cp}^\infty(\mathcal{H}^+) \oplus \check{C}_{cp}^\infty(\mathcal{I}^+)$, we have

$$\begin{aligned} \limsup_{r \rightarrow r_+} \sup_{\mathbb{S}^2} |\mathcal{B}_-(\psi, \phi)|_{t=0} &= 0, \\ \limsup_{r \rightarrow \infty} \sup_{\mathbb{S}^2} |\mathcal{B}_-(\psi, \phi)|_{t=0} &= 0. \end{aligned}$$

Proof. – We start with the limit as $r \rightarrow r_+$. Since $\hat{\mathcal{B}}_-(\hat{\psi}, \hat{\phi})$ is compactly supported in (ω, m, ℓ) , and U_{hor} and \mathfrak{W} are smooth for $\omega \in \mathbb{R} \setminus \{0\}$, one may easily establish that for every $\delta > 0$

$$\min(|\omega|, |\omega - \omega_+ m|) \geq \delta \Rightarrow \hat{\mathcal{B}}_-(\hat{\psi}, \hat{\phi}) = a_{\mathcal{H}^+} e^{-i(\omega - \omega_+ m)r^*} + a_{\mathcal{H}^-} e^{i(\omega - \omega_+ m)r^*} + \text{Error},$$

where

$$|\text{Error}| \leq B(\delta, \psi, \phi)(r - r_+).$$

Let $\chi(x)$ be a cutoff function which is identically 1 in a neighborhood of 0 and identically 0 for $|x| > 1$. For every $\delta > 0$ we have

(156)

$$\begin{aligned} & \limsup_{r \rightarrow r_+} \left| (r^2 + a^2)^{1/2} \sqrt{2\pi} \mathcal{B}_-(0, r, \theta, \phi) \right| \\ &= \limsup_{r \rightarrow r_+} \left| \int_{-\infty}^{\infty} \sum_{m\ell} e^{im\phi} S_{m\ell}(a\omega, \theta) \hat{\mathcal{B}}_- d\omega \right| \\ &\leq \limsup_{r \rightarrow r_+} \left| \int_{-\infty}^{\infty} \sum_{m\ell} \chi(\omega\delta^{-1}) \chi((\omega - \omega_+ m)\delta^{-1}) e^{im\phi} S_{m\ell}(a\omega, \theta) \hat{\mathcal{B}}_- d\omega \right| \\ &\quad + \limsup_{r \rightarrow r_+} \left| \int_{-\infty}^{\infty} (1 - \chi(\omega\delta^{-1})) (1 - \chi((\omega - \omega_+ m)\delta^{-1})) \sum_{m\ell} e^{im\phi} S_{m\ell}(a\omega, \theta) \hat{\mathcal{B}}_- d\omega \right|. \end{aligned}$$

We estimate the first term simply with Cauchy-Schwarz and (146):

$$\begin{aligned} (157) \quad & \limsup_{r \rightarrow r_+} \left| \int_{-\infty}^{\infty} \sum_{m\ell} \chi(\omega\delta^{-1}) \chi((\omega - \omega_+ m)\delta^{-1}) e^{im\phi} S_{m\ell}(a\omega, \theta) \hat{\mathcal{B}}_- d\omega \right| \\ &\leq B(\psi, \phi) \limsup_{r \rightarrow r_+} \delta^{1/2} \sqrt{\int_{-\infty}^{\infty} \sum_{m\ell} |\hat{\mathcal{B}}_-|^2 d\omega} \\ &\leq B(\psi, \phi) \delta^{1/2} \sqrt{\int_{-\infty}^{\infty} \sum_{m\ell} [|a_{\mathcal{H}^+}|^2 + |a_{\mathcal{H}^-}|^2]} \\ &\leq B(\psi, \phi) \delta^{1/2}. \end{aligned}$$

Set $\tilde{\chi}_\delta \doteq (1 - \chi(\omega\delta^{-1}))(1 - \chi((\omega - \omega_+ m)\delta^{-1}))$. For the second term we use the oscillation in r^* :

(158)

$$\begin{aligned} & \limsup_{r \rightarrow r_+} \left| \int_{-\infty}^{\infty} \sum_{m\ell} e^{im\phi} \tilde{\chi}_\delta S_{m\ell}(a\omega, \theta) \hat{\mathcal{B}}_- d\omega \right| \\ & \leq B(\psi, \phi, \delta) \limsup_{r \rightarrow r_+} \left| \int_{-\infty}^{\infty} \sum_{m\ell} \tilde{\chi}_\delta e^{im\phi} S_{m\ell} \left(a_{\mathcal{H}^+} e^{-i(\omega - \omega_+ m)r^*} + a_{\mathcal{H}^-} e^{i(\omega - \omega_+ m)r^*} \right) d\omega \right| \\ & \leq B(\psi, \phi, \delta) \limsup_{r \rightarrow r_+} \delta^{-1} |r^*|^{-1} \\ & = 0. \end{aligned}$$

In the second to last line, the decay in r^* came from an integration by parts in ω .

Since δ may be taken arbitrary small, combining (156), (157), and (158) concludes the proof for the limit when $r \rightarrow r_+$. Moreover, it is easy to see that essentially the same proof works for the limit as $r \rightarrow \infty$. \square

The previous two propositions and Remark 9.1.1 immediately imply the following corollary.

COROLLARY 9.1.1. – *The map $(\mathcal{B}_-|_{\Sigma}, n_\Sigma \mathcal{B}_-|_{\Sigma})$, which we shall, by a mild abuse of notation, now denote by \mathcal{B}_- , extends by density to a bounded map*

$$\mathcal{B}_- : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_\Sigma^V.$$

Proof. – The key point is that a straightforward calculation (remember that V vanishes at the bifurcate sphere!) shows that $\lim_{r \rightarrow r_+} (\mathcal{B}_-|_{\Sigma}, n_\Sigma \mathcal{B}_-|_{\Sigma}) = 0$ and $\lim_{r \rightarrow \infty} (\mathcal{B}_-|_{\Sigma}, n_\Sigma \mathcal{B}_-|_{\Sigma}) = 0$ imply that $(\mathcal{B}_-|_{\Sigma}, n_\Sigma \mathcal{B}_-|_{\Sigma})$ lies in \mathcal{E}_Σ^V . \square

9.1.3. *Inverting the forward map.* – Finally, we are ready for the key result of the section.

THEOREM 9.1.1. – *Let \mathcal{B}_- and \mathcal{F}_+ be as in Corollary 9.1.1 and Theorem 8.2.3. Then \mathcal{B}_- and \mathcal{F}_+ are both bounded isomorphisms and satisfy $\mathcal{B}_- \circ \mathcal{F}_+ = \text{Id}$ and $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$.*

Proof. – Of course, it suffices to prove the assertions $\mathcal{B}_- \circ \mathcal{F}_+ = \text{Id}$ and $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$.

We start with establishing $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$. By density (Remark 9.1.1), it suffices to check that

$$(\mathcal{F}_+ \circ \mathcal{B}_-) |_{\check{\mathcal{C}}_{cp}^\infty(\mathcal{H}^+) \oplus \check{\mathcal{C}}_{cp}^\infty(\mathcal{I}^+)} = \text{Id}.$$

Let $(\psi, \phi) \in \check{\mathcal{C}}_{cp}^\infty(\mathcal{H}^+) \oplus \check{\mathcal{C}}_{cp}^\infty(\mathcal{I}^+)$. Proposition 9.1.1 implies that there exist functions $\alpha(\omega, m, \ell)$ and $\beta(\omega, m, \ell)$ such that

$$\begin{aligned} \hat{\mathcal{B}}_- (\hat{\psi}, \hat{\phi}) &= \hat{\psi} U_{\text{hor}} + \alpha \overline{U_{\text{hor}}}, \\ \hat{\mathcal{B}}_- (\hat{\psi}, \hat{\phi}) &= \hat{\phi} U_{\text{inf}} + \beta \overline{U_{\text{inf}}}. \end{aligned}$$

Now, using that $(\mathcal{B}_-(\psi, \phi)|_{\mathring{\Sigma}}, n_{\mathring{\Sigma}} \mathcal{B}_-(\psi, \phi)|_{\mathring{\Sigma}})$ lies $\mathcal{E}_{\mathring{\Sigma}}^V$, and Theorem 7.1, one may easily check that the same arguments used in the proofs of Propositions 6.8.1 and 6.8.2 immediately imply

$$(\mathcal{F}_+ \circ \mathcal{B}_-)(\psi, \phi) = (\psi, \phi).$$

We now turn to establishing $\mathcal{B}_- \circ \mathcal{F}_+ = \text{Id}$. By density, it suffices to study solutions arising from initial data $(\psi, \psi') \in {}^2\mathcal{C}_{cp}^\infty(\mathring{\Sigma})$. Let $a_{\mathcal{H}^+}$ and $a_{\mathcal{I}^+}$ denote the microlocal radiation fields. Then Propositions 6.8.1 and 6.8.2 yield

$$\mathcal{F}_+(\psi, \psi') = \left(\frac{1}{\sqrt{4M\pi r_+}} \int_{-\infty}^{\infty} \sum_{m\ell} e^{-it^*\omega} e^{im\phi} S_{m\ell} a_{\mathcal{H}^+} d\omega, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} e^{-i\tau\omega} e^{im\phi} S_{m\ell} a_{\mathcal{I}^+} d\omega \right).$$

It immediately follows from Proposition 9.1.1 that $(\mathcal{B}_- \circ \mathcal{F}_+)(\psi, \psi') = (\psi, \psi')$. \square

9.1.4. *A physical-space characterization of \mathcal{B}_- .* – Before we close the section it will be conceptually clarifying and technically useful to observe that the backwards map may also be characterized in physical space.

PROPOSITION 9.1.4. – *Let $(\psi_{\mathcal{H}^+}, \phi_{\mathcal{I}^+}) \in \mathcal{C}_{cp}^\infty(\mathcal{H}^+) \oplus \mathcal{C}_{cp}^\infty(\mathcal{I}^+)$. Pick $\tau_0 < \infty$ such that $\psi_{\mathcal{H}^+}$ is compactly supported in $\mathcal{H}^+(-\infty, \tau_0)$ and $\phi_{\mathcal{I}^+}$ is compactly supported in $\mathcal{I}_{\tau_0}^+$, and then let $\Phi_{\mathcal{I}^+}$ be any smooth extension of $\phi_{\mathcal{I}^+}$ to the manifold with boundary $\tilde{\mathcal{R}}$ (see Definition 4.2.1) such that $\Phi_{\mathcal{I}^+}$ vanishes in the neighborhood of S_{τ_0} .*

Next, using Proposition 3.6.4, for each $s > 0$ sufficiently large we may uniquely define a smooth solution ψ_s to (2) in the past of $\mathcal{H}_{\leq \tau_0}^+ \cup (S_{\tau_0} \cap \{r \leq r(\tau_0, s)\}) \cup (\{t = s\} \cap \{r \geq r(\tau_0, s)\})$ (see the diagram (43)) by requiring

$$\begin{aligned} \psi_s|_{\mathcal{H}_{\leq \tau_0}^+} &= \psi_{\mathcal{H}^+}, \\ (\psi_s|_{S_\tau \cap \{r \leq r(\tau, s)\}}, n_{S_\tau} \psi_s|_{S_\tau \cap \{r \leq r(\tau, s)\}}) &= (0, 0), \\ r \psi_s|_{\{t=s\} \cap \{r \geq r(\tau, s)\}} &= \Phi_{\mathcal{I}^+}|_{\{t=s\} \cap \{r \geq r(\tau, s)\}}. \end{aligned}$$

Let $\{s_i\}_{i=1}^\infty$ be a sequence satisfying $s_i \rightarrow \infty$ as $i \rightarrow \infty$. It follows that $\psi_{s_i}|_{J_+(\mathring{\Sigma})}$ and any finite number of derivatives form a bounded equicontinuous sequence. In particular, we may extract a smooth limit ψ which will be a solution to (2) in the region $D^-(S_{\tau_0}) \cap J^+(\mathring{\Sigma})$. Finally, we have

$$(159) \quad \mathcal{B}_-(\psi_{\mathcal{H}^+}, \phi_{\mathcal{I}^+}) = (\psi|_{\mathring{\Sigma}}, n_{\mathring{\Sigma}} \psi|_{\mathring{\Sigma}}).$$

Proof. – The boundedness and equicontinuity of any finite number of derivatives of $\{\psi_{s_i}\}$ follows immediately from (higher order) \mathbf{J}^V energy estimates (it may be useful for the reader to note that the intersection of $D^-(S_{\tau_0}) \cap J^+(\mathring{\Sigma})$ and the support of $({}^V)\pi$ is compact and contained in $\bigcup_{s \in [0, \tau_0]} S_s$).

Next, using Theorem 9.1.1, we note that (159) would follow from

$$(160) \quad \mathcal{F}_+(\psi|_{\mathring{\Sigma}}, n_{\mathring{\Sigma}} \psi|_{\mathring{\Sigma}}) = (\psi_{\mathcal{H}^+}, \phi_{\mathcal{I}^+}).$$

Now, the equality $\mathcal{F}_+(\psi|_{\tilde{\Sigma}}, n_{\tilde{\Sigma}} \psi_{\tilde{\Sigma}})|_{\mathcal{H}^+} = \psi_{\mathcal{H}^+}$ is a trivial consequence of the definition of the radiation field and Proposition 3.6.2.

Finally, the equality $\mathcal{F}_+(\psi|_{\tilde{\Sigma}}, n_{\tilde{\Sigma}} \psi_{\tilde{\Sigma}})|_{\mathcal{I}^+} = \phi_{\mathcal{I}^+}$ follows from Proposition 3.8.1 and a straightforward modification of the arguments given in the proof of Proposition 4.2.1. \square

9.2. The backwards map to Σ_0^*

In this section we will define the backwards map \mathcal{B}_- on $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T$.

DEFINITION 9.2.1. – *Let $\mathcal{E} : C_{cp}^\infty(\mathcal{H}_{\geq 0}^+) \rightarrow C_{cp}^\infty(\mathcal{H}^+)$ be any map satisfying*

1. $\mathcal{E}(f)|_{\mathcal{H}_{\geq 0}^+} = f$.
2. $\int_{\mathcal{H}^+} \mathbf{J}_\mu^K[\mathcal{E}(f)] n_{\mathcal{H}^+}^\mu \leq B \int_{\mathcal{H}_{\geq 0}^+} \mathbf{J}_\mu^K[f] n_{\mathcal{H}^+}^\mu$.

Note that such a map is easily constructed.

Then we define the backwards map

$$\mathcal{B}_- : C_{cp}^\infty(\mathcal{H}_{\geq 0}^+) \oplus C_{cp}^\infty(\mathcal{I}^+) \rightarrow \mathcal{E}_{\Sigma_0^*}^V,$$

by

$$(161) \quad \mathcal{B}_-(\psi_{\mathcal{H}_{\geq 0}^+}, \phi_{\mathcal{I}^+}) \doteq \left(\mathcal{B}_-(\mathcal{E}(\psi_{\mathcal{H}_{\geq 0}^+}), \phi_{\mathcal{I}^+})|_{\Sigma_0^*}, n_{\Sigma_0^*} \mathcal{B}_-(\mathcal{E}(\psi_{\mathcal{H}_{\geq 0}^+}), \phi_{\mathcal{I}^+})|_{\Sigma_0^*} \right).$$

The reader should keep in mind our standard recycling of the notation concerning the symbol \mathcal{B}_- . In particular, \mathcal{B}_- on the right hand side of (161) is as in Definition 9.1.3.

The next theorem establishes that the backwards map extends to $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T$ and inverts the forward map \mathcal{F}_+ .

THEOREM 9.2.1. – *The map \mathcal{B}_- defined above is a bounded map and thus uniquely extends to a map*

$$\mathcal{B}_- : \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\Sigma_0^*}^V.$$

Let \mathcal{F}_+ denote the forward map $\mathcal{F}_+ : \mathcal{E}_{\Sigma_0^}^V \rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T$. Then, $\mathcal{B}_- \circ \mathcal{F}_+ = \text{Id}$ and $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$ and thus \mathcal{B}_- and \mathcal{F}_+ are bounded isomorphisms.*

REMARK 9.2.1. – *Observe that one corollary of Theorem 9.2.1 is that \mathcal{B}_- does not depend on the choice of extension \mathcal{E} .*

Proof. – First of all, we observe that the boundedness of \mathcal{B}_- and the statement $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$ follow immediately from Theorem 9.1.1, Proposition 3.6.3 and finite in time energy estimates (cf. the proof of Corollary 7.1).

The equality $\mathcal{B}_- \circ \mathcal{F}_+ = \text{Id}$ is a bit more subtle. The key observation is that it suffices to check this on a dense subset and it pays to expend a little effort in creating a convenient one. We thus turn to the construction of a useful dense subset. First of all, $C_{cp}^\infty(\mathcal{H}^+) \oplus C_{cp}^\infty(\mathcal{I}^+)$ is a dense subset of $\mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T$, and thus Theorem 9.1.1 implies that $\mathcal{B}_-(C_{cp}^\infty(\mathcal{H}^+) \oplus C_{cp}^\infty(\mathcal{I}^+))$ is a dense subset of $\mathcal{E}_{\tilde{\Sigma}}^V$. Now, considering the elements of $\mathcal{B}_-(C_{cp}^\infty(\mathcal{H}^+) \oplus C_{cp}^\infty(\mathcal{I}^+))$ as Cauchy data along $\tilde{\Sigma}$, we may solve the wave equation to the future of $\tilde{\Sigma}$ with Proposition 3.6.2 and restrict the solutions to Σ_0^* . This defines a subset $\tilde{\mathcal{C}}_{\Sigma_0^*}^V$ of $\mathcal{E}_{\Sigma_0^*}^V$. It follows from Proposition 3.6.4

and finite in time energy estimates (cf. the proof of Corollary 7.1) that $\tilde{\mathcal{C}}_{\Sigma_0^*}$ is in fact a dense subset of $\mathcal{E}_{\Sigma_0^*}^V$.

We now turn to proving that $\mathcal{B}_- \circ \mathcal{F}_+|_{\tilde{\mathcal{C}}_{\Sigma_0^*}} = \text{Id}$. Let ψ be a solution to (2) in $\mathcal{R}_{\geq 0}$ whose initial data along Σ_0^* lie in $\tilde{\mathcal{C}}_{\Sigma_0^*}$. We then define a solution $\tilde{\psi}$ to (2) in $\mathcal{R}_{\geq 0}$ by applying Proposition 3.6.1 to solve the wave equation with initial data $(\mathcal{B}_- \circ \mathcal{F}_+)(\psi|_{\Sigma_0^*}, n_{\Sigma_0^*}\psi|_{\Sigma_0^*})$ along Σ_0^* . We need to prove that $\psi - \tilde{\psi} = 0$. Now, the key advantage to considering initial data in $\tilde{\mathcal{C}}_{\Sigma_0^*}$ is that it immediately follows from Proposition 9.1.4 that $\psi|_{\Sigma_0^*}, n_{\Sigma_0^*}\psi|_{\Sigma_0^*}, \tilde{\psi}|_{\Sigma_0^*}$ and $n_{\Sigma_0^*}\tilde{\psi}|_{\Sigma_0^*}$ are smooth functions and hence that ψ and $\tilde{\psi}$ extend smoothly to $\mathcal{H}_{\geq 0}^+$. Since $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$ we conclude in particular that $K(\psi - \tilde{\psi})|_{\mathcal{H}_{\geq 0}^+} = 0$.

Set $\psi^\dagger \doteq K(\psi - \tilde{\psi})$. Since the Cauchy data for ψ^\dagger along Σ_0^* vanishes at \mathcal{H}^+ , we may easily construct a sequence $\{\psi_i^\dagger\}_{i=1}^\infty$ of solutions to (2) whose initial data along Σ_0^* are smooth and compactly supported away from $\mathcal{H}^+ \cap \Sigma_0^*$ and spacelike infinity and which satisfy

$$\lim_{i \rightarrow \infty} \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\tilde{\psi} - \tilde{\psi}_i] n_{\Sigma_0^*}^\mu = 0.$$

Since the ψ_i^\dagger are compactly supported away from $\mathcal{H}^+ \cap \Sigma_0^*$, they may easily be extended as solutions to (2) to all of \mathcal{R} by applying Proposition 3.6.3 with vanishing initial data along $\mathcal{H}_{\leq 0}^+$. We will also denote the extension by ψ_i^\dagger . Since ψ_i^\dagger is easily seen to be sufficiently integrable in the sense of Definition 5.1.1, we may apply Carter's separation to ψ_i^\dagger to define u_i^\dagger and the corresponding microlocal fluxes $a_{i,\mathcal{H}^+}^\dagger$ and $a_{i,\mathcal{I}^+}^\dagger$. It follows immediately from the construction of ψ_i^\dagger , Corollary 7.1, Proposition 6.8.1 and Proposition 6.8.2 that

$$(162) \quad \lim_{i \rightarrow \infty} \int_{-\infty}^\infty \sum_{m\ell} \left[|a_{i,\mathcal{H}^+}^\dagger|^2 + |a_{i,\mathcal{I}^+}^\dagger|^2 \right] = 0.$$

Proposition 9.1.2 (and an easy density argument) then imply that

$$\lim_{i \rightarrow \infty} \int \mathbf{J}_\mu^V[\psi_i^\dagger] n_{\{t=0\}}^\mu = 0.$$

Then, finite in time energy estimates show that ψ^\dagger vanishes. Finally, using also (137) from the proof of Corollary 8.2.1, we conclude that $(\psi - \tilde{\psi})|_{\mathcal{R}_{\geq 0}}$ vanishes. \square

9.3. The backwards map to $\bar{\Sigma}$

With Theorem 9.2.1 proven, we can now revisit scattering to $\bar{\Sigma}$ and prove a version of Theorem 9.1.1 where \mathcal{E}_Σ^V is replaced by \mathcal{E}_Σ^V and $\mathcal{E}_{\mathcal{H}^+}^K$ is replaced by $\mathcal{E}_{\mathcal{H}^+}^K$.

THEOREM 9.3.1. – *Let \mathcal{F}_+ be the forward map $\mathcal{F}_+ : \mathcal{E}_\Sigma^V \rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T$. Then there exists a backwards map*

$$\mathcal{B}_- : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_\Sigma^V$$

such that \mathcal{B}_- is a bounded map, $\mathcal{B}_- \circ \mathcal{F}_+ = \text{Id}$ and $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$. Thus \mathcal{B}_- and \mathcal{F}_+ are both bounded isomorphisms.

Proof. – We begin by introducing the notation

$$\overline{\mathcal{H}}_{\leq 0}^+ \doteq \overline{\mathcal{H}^+} \cap J^-(\Sigma_0^*).$$

Then, we define the function space ${}^3\mathcal{C}_{cp}^\infty(\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*)$ to consist of triples $(\psi_{\overline{\mathcal{H}}_{\leq 0}^+}, \psi_{\Sigma_0^*}, \psi'_{\Sigma_0^*})$ such that $\psi_{\overline{\mathcal{H}}_{\leq 0}^+}$ is a smooth function on $\overline{\mathcal{H}}_{\leq 0}^+$, $\psi_{\Sigma_0^*}$ and $\psi'_{\Sigma_0^*}$ are smooth functions of compact support on Σ_0^* and there exists a smooth function $\tilde{\Psi}$ on \mathcal{D} such that $\tilde{\Psi}|_{\overline{\mathcal{H}}_{\leq 0}^+} = \psi_{\overline{\mathcal{H}}_{\leq 0}^+}$, $\tilde{\Psi}|_{\Sigma_0^*} = \psi_{\Sigma_0^*}$ and $n_{\Sigma_0^*}\tilde{\Psi}|_{\Sigma_0^*} = \psi'_{\Sigma_0^*}$.

Proposition 3.6.3 states that to each $(\psi_{\overline{\mathcal{H}}_{\leq 0}^+}, \psi_{\Sigma_0^*}, \psi'_{\Sigma_0^*}) \in {}^3\mathcal{C}_{cp}^\infty(\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*)$ there exists a unique smooth solution ψ to (2) in $J^-(\Sigma_0^*)$. Restricting these solutions to $\overline{\Sigma}$ thus defines a map

$$(163) \quad {}^3\mathcal{C}_{cp}^\infty(\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*) \mapsto {}^2\mathcal{C}_{cp}^\infty(\overline{\Sigma}).$$

Conversely, given any element of $(\psi, \psi') \in {}^2\mathcal{C}_{cp}^\infty(\overline{\Sigma})$, Proposition 3.6.2 yields a unique solution to (2) whose Cauchy data along $\overline{\Sigma}$ are given by (ψ, ψ') . Restricting these solutions to $\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*$ defines a map

$$(164) \quad {}^2\mathcal{C}_{cp}^\infty(\overline{\Sigma}) \mapsto {}^3\mathcal{C}_{cp}^\infty(\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*).$$

It immediately follows that the maps (163) and (164) are inverses of each other and hence that both are bijections.

Next, we let $\mathcal{E}_{\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*}^V$ denote the completion of $\mathcal{C}_{cp}^\infty(\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*)$ under the norm

$$\left\| (\psi_{\overline{\mathcal{H}}_{\leq 0}^+}, \psi_{\Sigma_0^*}, \psi'_{\Sigma_0^*}) \right\|_{\mathcal{E}_{\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*}^V} \doteq \sqrt{\int_{\overline{\mathcal{H}}_{\leq 0}^+} \mathbf{J}_\mu^K[\tilde{\Psi}]n_{\mathcal{H}^+}^\mu + \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\tilde{\Psi}]n_{\Sigma_0^*}^\mu},$$

where $\tilde{\Psi}$ is the smooth extension mentioned in the definition of the space $\mathcal{C}_{cp}^\infty(\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*)$. Finite in time \mathbf{J}^V energy estimates and the bijection (164) immediately yield a bounded isomorphism

$$(165) \quad \mathcal{E}_\Sigma^V \mapsto \mathcal{E}_{\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*}^V.$$

We conclude the proof by combining (165) with the easily observed fact that Theorem 9.2.1 implies that forward evolution yields a bounded isomorphism

$$\mathcal{E}_{\overline{\mathcal{H}}_{\leq 0}^+ \cup \Sigma_0^*}^V \mapsto \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T. \quad \square$$

9.4. Aside: Proof of Theorem 7.2

At this point, using the properties of the backwards map \mathcal{B}_- , we can now complete our study of boundedness and integrated local energy decay for the degenerate V -energy theory by giving the proof of Theorem 7.2 of Section 7.2.

Proof of Theorem 7.2. – Observe that for any $s \geq 0$, we could have defined a forward map $\mathcal{F}_+^{(s)} : \mathcal{E}_{\Sigma_s^*}^V \rightarrow \mathcal{E}_{\mathcal{H}_{\geq s}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T$ which, in the case of smooth compactly supported data, computes the radiation field of Cauchy data along Σ_s^* and, similarly, we could have defined a backwards map $\mathcal{B}_-^{(s)} : \mathcal{E}_{\mathcal{H}_{\geq s}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\Sigma_s^*}^V$. Just as before, we would obtain that $\mathcal{F}_+^{(s)}$ and $\mathcal{B}_-^{(s)}$ are both bounded (with a constant *independent of s*) and inverses of each other. In particular, since $\mathcal{F}_+^{(s)}(\psi|_{\Sigma_s^*}, n_{\Sigma_s^*}\psi|_{\Sigma_s^*}) = \mathcal{F}_+(\psi|_{\Sigma_0^*}, n_{\Sigma_0^*}\psi|_{\Sigma_0^*})$, we obtain

$$(166) \quad \begin{aligned} \int_{\Sigma_s^*} \mathbf{J}_\mu^V[\psi] n_{\Sigma_s^*}^\mu &\leq B \left\| \mathcal{F}_+^{(s)}(\psi|_{\Sigma_s^*}, n_{\Sigma_s^*}\psi|_{\Sigma_s^*}) \right\|_{\mathcal{E}_{\mathcal{H}_{\geq s}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T} \\ &\leq B \left\| \mathcal{F}_+(\psi|_{\Sigma_0^*}, n_{\Sigma_0^*}\psi|_{\Sigma_0^*}) \right\|_{\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T} \\ &\leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\psi] n_{\Sigma_0^*}^\mu. \end{aligned}$$

Next, we observe that during the proof of Proposition 9.1.2 an integrated estimate in r is in fact established. Using this, an easy density argument, a finite in time energy estimate and an application of Plancherel easily show that for any compact set $K \subset (r_+, \infty)$ we have

$$(167) \quad \begin{aligned} \int_{\mathcal{R}_{\geq 0} \cap \{r \in K\}} \left(\xi |\nabla \psi|^2 + \zeta |T\psi|^2 + |\tilde{Z}^* \psi|^2 + |\psi|^2 \right) \\ \leq B(K) \left\| \mathcal{F}_+(\psi|_{\Sigma_0^*}, n_{\Sigma_0^*}\psi|_{\Sigma_0^*}) \right\|_{\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T} \\ \leq B(K) \int_{\Sigma_0^*} \mathbf{J}_\mu^V[\psi] n_{\Sigma_0^*}^\mu. \end{aligned}$$

In order to finish the proof we need to exchange the restriction $\{r \in K\}$ in (167) for the appropriate weights in r and $r - r_+$. For r large, the desired estimate is a trivial consequence of the “large- r estimate” of Proposition 4.6.1 in [30] and the arguments of Section 9.4 in [30]. For r close to the horizon it is possible to apply a degenerate version of the redshift effect [57] to achieve the desired estimate. \square

9.5. The scattering matrix $\mathcal{S} = \mathcal{F}_+ \circ \mathcal{B}_+$

For notational convenience, we have so far restricted our attention to scattering data along \mathcal{H}^+ and \mathcal{I}^+ . However, in view of the discrete isometry (34) of \mathcal{D} , all of our theorems have exact analogs where \mathcal{H}^+ is replaced by \mathcal{H}^- and \mathcal{I}^+ is replaced by \mathcal{I}^- . In particular, we have the following version of Theorems 9.1.1 and 9.3.1.

THEOREM 9.5.1. – *Forward evolution (towards the past) uniquely extends to the bounded maps*

$$\mathcal{F}_- : \mathcal{E}_\Sigma^V \rightarrow \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T, \quad \mathcal{F}_- : \mathcal{E}_\Sigma^V \rightarrow \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T.$$

There exist bounded maps \mathcal{B}_+

$$\mathcal{B}_+ : \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_\Sigma^V, \quad \mathcal{B}_+ : \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_\Sigma^V,$$

such that $\mathcal{F}_- \circ \mathcal{B}_+ = \text{Id}$ and $\mathcal{B}_+ \circ \mathcal{F}_- = \text{Id}$.

Combining Theorems 9.1.1 and 9.5.1 allows us to define the maps between scattering data along $\mathcal{H}^- \cup \mathcal{I}^-$ and $\mathcal{H}^+ \cup \mathcal{I}^+$. We immediately obtain the following theorem.

THEOREM 9.5.2. – *We define the scattering map (or S-matrix)*

$$\mathcal{S} : \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T, \quad \mathcal{S} : \mathcal{E}_{\overline{\mathcal{H}}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T \rightarrow \mathcal{E}_{\overline{\mathcal{H}}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T,$$

by

$$(168) \quad \mathcal{S} \doteq \mathcal{F}_+ \circ \mathcal{B}_+.$$

The map \mathcal{S} is then a bounded isomorphism from $\mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T$ to $\mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T$ and $\mathcal{E}_{\overline{\mathcal{H}}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T$ to $\mathcal{E}_{\overline{\mathcal{H}}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T$

Furthermore, for every $(\psi_{\mathcal{H}^-}, \phi_{\mathcal{I}^-}) \in \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T$, there exists a unique set of initial data $(\psi, \psi') \in \mathcal{E}_{\Sigma}^V$ such that $\mathcal{F}_-(\psi, \psi') = (\psi_{\mathcal{H}^-}, \phi_{\mathcal{I}^-})$ and $\mathcal{F}_+(\psi, \psi') = \mathcal{S}(\psi_{\mathcal{H}^-}, \phi_{\mathcal{I}^-})$. An analogous statement holds for $(\psi_{\overline{\mathcal{H}}^-}, \phi_{\mathcal{I}^-}) \in \mathcal{E}_{\overline{\mathcal{H}}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T$

This is the precise statement of Theorem 5 of Section 2.3.5.

Next, we observe that our scattering map \mathcal{S} may be given by an explicit formula involving the reflection and transmission coefficients.

THEOREM 9.5.3. – *Let \mathcal{S} denote the scattering map from Theorem 9.5.2. Then, for $(\psi_{\mathcal{H}^-}, \phi_{\mathcal{I}^-})$ lying in either domain $\mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T$ or $\mathcal{E}_{\overline{\mathcal{H}}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T$, we have*

$$(169) \quad \begin{aligned} \mathcal{S}(\psi_{\mathcal{H}^-}, \phi_{\mathcal{I}^-}) &= \left(\frac{1}{\sqrt{4M\pi r_+}} \int_{-\infty}^{\infty} \sum_{m\ell} \left(- \left(\frac{\omega}{\omega - \omega_+} \right) a_{\mathcal{I}^-} \mathfrak{T} + a_{\mathcal{H}^-} \tilde{\mathfrak{R}} \right) e^{-it\omega} e^{im\phi} S_{m\ell} d\omega, \right. \\ &\quad \left. \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} \left(- \left(\frac{\omega - \omega_+ m}{\omega} \right) a_{\mathcal{H}^-} \tilde{\mathfrak{T}} + a_{\mathcal{I}^-} \mathfrak{R} \right) e^{-it\omega} e^{im\phi} S_{m\ell} d\omega \right). \end{aligned}$$

Here

$$\begin{aligned} -i(\omega - \omega_+ m) a_{\mathcal{H}^-} &\doteq \sqrt{\frac{Mr_+}{\pi}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} (\partial_t + \omega_+ \partial_\phi) \psi_{\mathcal{H}^-} e^{it^* \omega} e^{-im\phi} S_{m\ell} \sin \theta dt^* d\theta d\phi, \\ -i\omega a_{\mathcal{I}^-} &\doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \partial_t \phi_{\mathcal{I}^-} e^{it\omega} e^{-im\phi} S_{m\ell} \sin \theta dt d\theta d\phi, \end{aligned}$$

and we emphasize that in interpreting the formula (169), one must keep in mind that the $a_{\mathcal{I}^\pm}(\omega, m, \ell)$ are only defined as functions such that $\omega a_{\mathcal{I}^\pm} \in L_\omega^2 l_{m\ell}^2$ and that the $a_{\mathcal{H}^\pm}(\omega, m, \ell)$ are only defined as functions such that $(\omega - \omega_+ m) a_{\mathcal{H}^\pm} \in L_\omega^2 l_{m\ell}^2$.

Proof. – This follows immediately from the construction of \mathcal{S} , Propositions 6.8.1 and 6.8.2 and an easy density argument. \square

In particular, specializing to the case where $\psi_{\mathcal{H}^-} = 0$, this establishes Theorem 12 of Section 2.4.2.

REMARK 9.5.1. – We note that one can define the map \mathcal{S} by the expression (169) and prove directly by Theorem 6.2.2 that the map is a bounded isomorphism without relying on Theorem 9.1.1; in fact, the proof is a good deal easier because one need never establish the boundedness of the map \mathcal{B}_- . However, one would still have to prove the decomposition (168) so as to identify elements of $\mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T$ and $\mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T$ as radiation fields of solutions to the wave equation arising from finite energy Cauchy data.

9.6. Aside: A self-contained physical-space treatment of the Schwarzschild case

In the Schwarzschild case ($a = 0$) there is no superradiance and it is much easier to establish that \mathcal{F} is invertible using the unitarity property. Furthermore, the proofs may all be carried out in physical space, i.e., using “time-dependent methods”. In this section we will give a self-contained treatment of how this can be done in our set-up (cf. the related [59]).

The ease of the Schwarzschild scattering theory is all associated with the following unitarity property.

PROPOSITION 9.6.1. – Let $a = 0$, and observe that in this case $K = T = V$. Then the forward maps \mathcal{F}_+ of Theorems 8.2.2, 8.2.3 and 8.2.4 are unitary:

$$(170) \quad \|\mathcal{F}_+(\psi, \psi')\|_{\mathcal{E}_{\mathcal{H}^+}^T}^2 + \|\mathcal{F}_+(\psi, \psi')\|_{\mathcal{E}_{\mathcal{I}^+}^T}^2 = \|(\psi, \psi')\|_{\mathcal{E}_{\Sigma_0^*}^T}^2,$$

$$(171) \quad \|\mathcal{F}_+(\psi, \psi')\|_{\mathcal{E}_{\mathcal{H}^+}^T}^2 + \|\mathcal{F}_+(\psi, \psi')\|_{\mathcal{E}_{\mathcal{I}^+}^T}^2 = \|(\psi, \psi')\|_{\mathcal{E}_{\Sigma}^T}^2,$$

$$(172) \quad \|\mathcal{F}_+(\psi, \psi')\|_{\mathcal{E}_{\mathcal{H}^+}^T}^2 + \|\mathcal{F}_+(\psi, \psi')\|_{\mathcal{E}_{\mathcal{I}^+}^T}^2 = \|(\psi, \psi')\|_{\mathcal{E}_{\Sigma}^T}^2.$$

Proof. – We will only prove (170) as the proof of (171) and (172) is exactly the same. By density it suffices to prove (170) in the case when $(\psi, \psi') \in {}^2\mathcal{C}_{cp}^\infty(\Sigma_0^*)$. Let us assume this for the remainder of the proof. It follows now from Proposition 3.8.1 and Theorem 3.7.2 that we can find a dyadic sequence $\{\tau_i\}_{i=1}^\infty$ such that

$$\lim_{i \rightarrow \infty} \int_{S_{\tau_i}} \mathbf{J}_\mu^T[\mathcal{F}(\psi, \psi')] n_{S_{\tau_i}}^\mu = 0.$$

Next, for each τ_i a \mathbf{J}^T energy estimate yields

$$\begin{aligned} \int_{\mathcal{H}^+(0, \tau_i)} \mathbf{J}_\mu^T[\mathcal{F}_+(\psi, \psi')] n_{\mathcal{H}^+}^\mu + \int_{\mathcal{I}^+_{\leq \tau_i}} \mathbf{J}_\mu^T[\mathcal{F}_+(\psi, \psi')] n_{\mathcal{I}^+}^\mu + \int_{S_{\tau_i}} \mathbf{J}_\mu^T[\mathcal{F}_+(\psi, \psi')] n_{S_{\tau_i}}^\mu \\ = \|(\psi, \psi')\|_{\mathcal{E}_{\Sigma_0^*}^T}^2. \end{aligned}$$

We conclude the proof by taking $i \rightarrow \infty$. \square

REMARK 9.6.1. – Let us remark that in the Schwarzschild $a = 0$ case suitable versions of the statements of Theorems 3.7.1 and 3.7.2 can be obtained without phase space analysis with respect to either time or angular frequency decompositions. See [27] and [29]. Thus, not only the construction but also all relevant properties of \mathcal{F} are obtained purely with physical space (i.e., “time-dependent”) methods. Cf. with the Kerr $a \neq 0$ case where the construction of \mathcal{F} is still formulated in the time domain but requires the result of Theorem 3.7.1 which is itself based on frequency-analysis.

The injectivity of the forward map is an immediate corollary.

COROLLARY 9.6.1. – *Let $a = 0$. Then the forward map \mathcal{F}_+ is injective.*

We now construct the backwards map.

THEOREM 9.6.1. – *Let $a = 0$. Then the forward map \mathcal{F}_+ is a unitary isomorphism (with either domain \mathcal{E}_Σ^V or $\mathcal{E}_{\Sigma_0^*}^V$ or \mathcal{E}_Σ^V) with two-sided unitary inverse \mathcal{B}_- satisfying $\mathcal{B}_- \circ \mathcal{F}_+ = \text{Id}$, $\mathcal{F}_+ \circ \mathcal{B}_- = \text{Id}$.*

Proof. – We consider the case where the domain is \mathcal{E}_Σ^V , the cases \mathcal{E}_Σ^V or $\mathcal{E}_{\Sigma_0^*}^V$ are handled in an analogous fashion.

First of all, using the physical space construction from the proof of Proposition 9.1.4 we may *define* the backwards map on a dense set:

$$\mathcal{B}_- : C_{cp}^\infty(\mathcal{H}_{\geq 0}^+) \oplus C_{cp}^\infty(\mathcal{I}^+) \rightarrow \mathcal{E}_{\Sigma_0^*}^T.$$

Furthermore, the proof of Proposition 9.1.4 shows that $(\psi_{\mathcal{H}^+}, \phi_{\mathcal{I}^+}) \in C_{cp}^\infty(\mathcal{H}_{\geq 0}^+) \oplus C_{cp}^\infty(\mathcal{I}^+)$ implies

$$\mathcal{F}_+(\mathcal{B}_-(\psi_{\mathcal{H}^+}, \phi_{\mathcal{I}^+})) = (\psi_{\mathcal{H}^+}, \phi_{\mathcal{I}^+}).$$

We thus conclude that the forward map \mathcal{F}_+ has a dense image. Since the unitarity of \mathcal{F}_+ implies that the backwards map \mathcal{B}_- is bounded on its domain, it follows immediately that \mathcal{F}_+ is in fact surjective. The rest of theorem follows immediately. \square

REMARK 9.6.2. – *It is instructive to compare the above “time-dependent method” construction of \mathcal{B}_- to the stationary-method construction of Definition 9.1.1. Of course, one could have defined \mathcal{B}_- on a dense subset in the general Kerr case with Proposition 9.1.4, but one would still need to have used the representation of Definition 9.1.1 to estimate it so as to take the completion.*

Applying the discrete isometry $t \mapsto -t$ of Schwarzschild yields the analogs of the above statements for \mathcal{F}_- and \mathcal{B}_+ . As before, we then define the scattering map $\mathcal{S} = \mathcal{F}_+ \circ \mathcal{B}_+$. We immediately obtain the following corollary.

THEOREM 9.6.2. – *Let $a = 0$. Then the scattering maps $\mathcal{S} : \mathcal{E}_{\mathcal{H}^-}^T \oplus \mathcal{E}_{\mathcal{I}^-}^T \mapsto \mathcal{E}_{\mathcal{H}^+}^T \oplus \mathcal{E}_{\mathcal{I}^+}^T$ are $\mathcal{S} : \mathcal{E}_{\mathcal{H}^-}^T \oplus \mathcal{E}_{\mathcal{I}^-}^T \mapsto \mathcal{E}_{\mathcal{H}^+}^T \oplus \mathcal{E}_{\mathcal{I}^+}^T$ unitary isomorphisms.*

10. Further applications

We collect here some further applications of our scattering theory.

In Section 10.1, we will construct a physical-space (time-domain) theory of superradiant reflection. Theorem 10.1.1 will give the results of Theorem 6 and Theorem 7 of Section 2.3.6.

We will also formulate and prove an analogous amplification statement in terms of compactly supported smooth Cauchy data (Theorem 10.1.2).

We will then show in Section 10.2 a “pseudo-unitary” property (Theorem 10.2.1) of our scattering map \mathcal{S} restricted to past scattering data supported only on \mathcal{I}^- , as well as a genuine unitarity property of \mathcal{S} restricted to an appropriate Hilbert space of non-superradiant data (Theorem 10.2.2). This will give Theorem 8 and Theorem 9 of Section 2.3.7.

Finally, in Section 10.3 we will establish the injectivity result theorem 10.3.1, which corresponds to “uniqueness of scattering states” for improperly posed scattering problems (for which there is no existence). This will give Theorem 10 of Section 2.3.8.

10.1. A physical space theory of superradiant reflection

First we define the physical-space reflection and transmission maps referred to already in Section 2.3.6.

DEFINITION 10.1.1. – Define the reflection map \mathcal{R} and the transmission map \mathcal{T} by

$$\mathcal{R} \doteq \pi_{\mathcal{E}_{\mathcal{I}^+}^T} \circ \mathcal{S}|_{\{0\} \oplus \mathcal{E}_{\mathcal{I}^-}^T}, \quad \mathcal{T} \doteq \pi_{\mathcal{E}_{\mathcal{H}^+}^K} \circ \mathcal{S}|_{\{0\} \oplus \mathcal{E}_{\mathcal{I}^-}^T}$$

where

$$\pi_{\mathcal{E}_{\mathcal{I}^+}^T} : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\mathcal{I}^+}^T, \quad \pi_{\mathcal{E}_{\mathcal{H}^+}^K} : \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T \rightarrow \mathcal{E}_{\mathcal{H}^+}^K$$

are the natural projections.

We can view

$$\mathcal{S}|_{\{0\} \oplus \mathcal{E}_{\mathcal{I}^-}^T} = \mathcal{R} \oplus \mathcal{T}.$$

We are now ready for the following theorem.

THEOREM 10.1.1. – The operator norms of \mathcal{T} and \mathcal{R} are bounded

$$\|\mathcal{T}\| \leq B, \quad \|\mathcal{R}\| \leq B.$$

If $a = 0$, then $\|\mathcal{R}\| = 1$, whereas if $a \neq 0$ then

$$\|\mathcal{R}\| > 1.$$

Proof. – The maps \mathcal{T} and \mathcal{R} are compositions of the bounded maps $\pi_{\mathcal{E}_{\mathcal{I}^+}^T}$, $\pi_{\mathcal{E}_{\mathcal{H}^+}^K}$ and \mathcal{S} and hence are bounded.

Next, it follows immediately from the formula (169) that

$$\|\mathcal{R}\| = \sup_{(\omega, m, \ell)} \mathfrak{R}(\omega, m, \ell).$$

Thus, when $a \neq 0$, Corollary 5.3.1 shows that $\|\mathcal{R}\| > 1$, and when $a = 0$, Corollary 6.4.1 shows that $\|\mathcal{R}\| = 1$. \square

We have now established Theorems 6 and 7 from Section 2.3.6.

With a little more work, we can upgrade the above result to the following statement.

THEOREM 10.1.2. – *Let $a \neq 0$. There exists a smooth solution ψ on \mathcal{D} such that the initial data for ψ along $\mathring{\Sigma}$ is supported away from the bifurcate sphere \mathcal{B} (though not necessarily of compact support), ψ has finite V -energy along $\mathring{\Sigma}$ and we have*

$$\int_{\mathcal{H}^-} \mathbf{J}_\mu^K[\psi]n_{\mathcal{H}^-}^\mu = 0,$$

$$\int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi]n_{\mathcal{I}^+}^\mu > \int_{\mathcal{I}^-} \mathbf{J}_\mu^T[\psi]n_{\mathcal{I}^-}^\mu.$$

Also, for all $R < \infty$ there exists a solution ψ_R to (2) on \mathcal{R} such that the initial data for ψ_R along $\mathring{\Sigma}$ are compactly supported within $r \in [R, \infty)$, and ψ_R exhibits superradiance in the sense that

$$\int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi_R]n_{\mathcal{I}^+}^\mu > \int_{\mathring{\Sigma}} \mathbf{J}_\mu^T[\psi_R]n_{\mathring{\Sigma}}^\mu.$$

Proof. – We start by letting $a_{\mathcal{I}^-}(\omega, m, \ell)$ be a non-zero smooth function which is compactly supported in the set of (ω, m, ℓ) which satisfy

$$\omega > 0,$$

$$(\omega - \omega_+ m) < 0.$$

We define

$$u \doteq \frac{\mathfrak{I}}{-i(\omega - \omega_+ m)} a_{\mathcal{I}^-} U_{\text{hor}} = \frac{\mathfrak{R}}{i\omega} a_{\mathcal{I}^-} U_{\text{inf}} + \frac{1}{i\omega} a_{\mathcal{I}^-} \overline{U_{\text{inf}}},$$

$$\psi_0 \doteq \frac{1}{(r^2 + a^2)^{1/2} \sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} e^{-i\omega t} e^{im\phi} S_{m\ell}(a\omega, \theta) u \, d\omega.$$

Note that Proposition 6.8.1, Theorem 7.1, and Proposition 9.1.2 imply

$$\int_{\mathcal{I}^-} \mathbf{J}_\mu^T[\psi_0]n_{\mathcal{I}^-}^\mu = \sqrt{\int_{-\infty}^{\infty} \sum_{m\ell} |a_{\mathcal{I}^-}|^2}.$$

Now, Corollary 5.3.1 implies

$$|\mathfrak{R}|^2 |a_{\mathcal{I}^-}|^2 \geq |a_{\mathcal{I}^-}|^2 + \epsilon,$$

for some sufficiently small $\epsilon > 0$ on a compact set of frequencies. Integrating and summing and applying Proposition 6.8.1, Theorem 7.1, and Proposition 9.1.2 yields

(173)

$$\int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi_0]n_{\mathcal{I}^+}^\mu = \sqrt{\int_{-\infty}^{\infty} \sum_{m\ell} |\mathfrak{R}|^2 |a_{\mathcal{I}^-}|^2 \, d\omega} > \sqrt{\int_{-\infty}^{\infty} \sum_{m\ell} |a_{\mathcal{I}^-}|^2 \, d\omega} = \int_{\mathcal{I}^-} \mathbf{J}_\mu^T[\psi_0]n_{\mathcal{I}^-}^\mu.$$

Finally, applying Proposition 6.8.2, Theorem 7.1, and Proposition 9.1.2 yields

$$\int_{\mathcal{H}^-} \mathbf{J}_\mu^K[\psi_0]n_{\mathcal{H}^-}^\mu = 0.$$

Thus, we may multiply ψ_0 by an appropriate constant to define a solution ψ_1 which will satisfy

$$1. \int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi_1]n_{\mathcal{I}^+}^\mu > 1.$$

2. $\int_{\mathcal{I}^-} \mathbf{J}_\mu^T [\psi_1] n_{\mathcal{I}^-}^\mu = 1.$
3. $\int_{\mathcal{H}^-} \mathbf{J}_\mu^K [\psi_1] n_{\mathcal{H}^-}^\mu = 0.$

Let $\tilde{\varphi}$ denote the radiation field for ψ_1 along \mathcal{I}^- . Let $\chi(\tau)$ be a bump function, $\epsilon > 0$ be sufficiently small, and let ψ_2 be the unique solution to (2) whose radiation field vanishes along \mathcal{H}^- and has the radiation field

$$\frac{\chi(\tau\epsilon^{-1})\tilde{\varphi}}{\sqrt{\int_{\mathcal{I}^-} \mathbf{J}_\mu^T [\chi(\tau\epsilon^{-1})\tilde{\varphi}] n_{\mathcal{I}^-}^\mu}}$$

along \mathcal{I}^- . Using the boundedness of the map \mathcal{S} from Definition 9.5.2, it is clear that taking ϵ sufficiently small (and then fixing ϵ) will imply that

1. $\int_{\mathcal{I}^+} \mathbf{J}_\mu^T [\psi_2] n_{\mathcal{I}^+}^\mu > 1.$
2. $\int_{\mathcal{I}^-} \mathbf{J}_\mu^T [\psi_2] n_{\mathcal{I}^-}^\mu = 1.$
3. $\int_{\mathcal{H}^-} \mathbf{J}_\mu^K [\psi_2] n_{\mathcal{H}^-}^\mu = 0.$

Note that an easy domain of dependence argument shows that the initial data for ψ_2 along $\mathring{\Sigma}$ is compactly supported; thus we may set $\psi \doteq \psi_2$.

In order to construct ψ_R we need to do a little more work. We begin by recalling the estimate (137), the proof of which (being invariant under time reversal) implies

$$(174) \quad \lim_{s \rightarrow -\infty} \int_{\{t=s\} \cap [r_+, r_+ + A]} \mathbf{J}_\mu^N [\psi_2] n_{\{t=s\}}^\mu = 0 \quad \forall 0 < \epsilon' < A < \infty.$$

Let $\chi(x)$ be cut-off which is 0 for $x \in [0, 1]$ and identically 1 for $x \in [2, \infty)$. Letting ϵ' be small enough so that K is timelike for $r \in [r_+, r_+ + 2\epsilon']$, applying a \mathbf{J}^K energy estimate to $(1 - \chi(r(\epsilon')^{-1}))\psi_\epsilon$ easily implies

$$(175) \quad \int_{\{t=s\} \cap [r_+, r_+ + \epsilon']} \mathbf{J}_\mu^K [\psi_2] n_{\{t=s\}}^\mu \\ \leq \int_{\mathcal{H}^-} \mathbf{J}_\mu^K [\psi_2] n_{\mathcal{H}^-}^\mu + B(\epsilon') \int_s^\infty \int_{\{t=s\} \cap [r_+, r_+ + 2\epsilon']} [\mathbf{J}_\mu^K [\psi_2] n_{\{t=s\}}^\mu + |\psi_2|^2] ds.$$

Theorem 3.7.1 implies that the second term on the right hand side of this estimate converges to 0 as $s \rightarrow \infty$. Since the first term on the right hand side vanishes, we conclude that

$$(176) \quad \limsup_{s \rightarrow -\infty} \int_{\{t=s\} \cap [r_+, r_+ + \epsilon']} \mathbf{J}_\mu^K [\psi_2] n_{\{t=s\}}^\mu = 0.$$

Taking R suitably large, and applying a similar argument in the region $r \geq R$, one may easily deduce that

$$(177) \quad \limsup_{s \rightarrow -\infty} \int_{\{t=s\} \cap \{r \in [R/10, \infty)\}} \mathbf{J}_\mu^T [\psi_2] n_{\{t=s\}}^\mu \leq 1.$$

Let $\epsilon'' > 0$ be a small constant to be fixed later. Now we choose R sufficiently large and $s = s(\epsilon'', R)$ sufficiently large and negative so that

$$(178) \quad \int_{\{t=s\}} \mathbf{J}_\mu^V [\psi_2 - \chi(rR^{-1})\psi_2] n_{\{t=s\}}^\mu < \epsilon'',$$

$$(179) \quad \int_{\{t=s\}} \mathbf{J}_\mu^V [\chi(rR^{-1})\psi_2] n_{\{t=s\}}^\mu \leq 1 + \epsilon'',$$

Let ψ_3 be the solution to (2) whose initial data along $\{t = s\}$ are given by $\chi(rR^{-1})\psi_2$. Now set $\psi_4(t, r, \theta, \phi) \doteq \tilde{\psi}_3(t - s, r, \theta, \phi)$. It is clear that if we choose ϵ'' small enough, then Theorem 7.1 will imply that

$$\int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi_4] n_{\mathcal{I}^+}^\mu > \int_{\{t=0\}} \mathbf{J}_\mu^T[\psi_4] n_{\{t=0\}}^\mu.$$

Finally, appealing to Theorem 7.1 one more time, we may define ψ_R to be the unique solution to (2) whose initial data along $\{t = 0\}$ is given by $(1 - \chi(rS^{-1}))\psi_4$ for some sufficiently large S . \square

10.2. Pseudo-unitarity and non-superradiant unitarity

The next sequence of results expresses the conservation of the \mathbf{J}^T flux. Since this flux is unsigned along \mathcal{H}^+ we may interpret this as a statement of “pseudo-unitarity”.

PROPOSITION 10.2.1. – *Let ψ be a solution to (2) whose initial data lie in $\mathcal{E}_{\Sigma_0^*}^N$. Observe that Theorem 3.7.1 implies that*

$$\int_{\mathcal{H}_0^+} \left| \mathbf{J}_\mu^T[\psi] n_{\mathcal{H}^+}^\mu \right| \leq B \int_{\mathcal{H}_0^+} \mathbf{J}_\mu^N[\psi] n_{\mathcal{H}^+}^\mu \leq B \int_{\Sigma_0^*} \mathbf{J}_\mu^N[\psi] n_{\Sigma_0^*}^\mu.$$

In particular, even though the integrand is unsigned, the integral

$$\int_{\mathcal{H}_0^+} \mathbf{J}_\mu^T[\psi] n_{\mathcal{H}^+}^\mu$$

is well defined and finite.

We then have

$$\int_{\mathcal{H}_0^+} \mathbf{J}_\mu^T[\psi] n_{\mathcal{H}^+}^\mu = \int_{\Sigma_0^*} \mathbf{J}_\mu^T[\psi] n_{\Sigma_0^*}^\mu - \int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\psi] n_{\mathcal{I}^+}^\mu.$$

Proof. – By density considerations, we may assume that ψ lies in $\mathcal{C}_{cp}^\infty(\Sigma_0^*)$. As we have already argued a few times before, Proposition 3.8.1 and Theorem 3.7.2 then allow us to find a dyadic sequence $\{\tau_i\}$ such that $\int_{S_{\tau_i}} \mathbf{J}_\mu^N[\psi] n_{S_{\tau_i}}^\mu \rightarrow 0$ as $i \rightarrow \infty$. For each τ_i , a \mathbf{J}^T energy estimate yields

$$\int_{\mathcal{H}^+(0, \tau_i)} \mathbf{J}_\mu^T[\psi] n_{\mathcal{H}^+}^\mu + \int_{S_{\tau_i}} \mathbf{J}_\mu^T[\psi] n_{S_{\tau_i}}^\mu + \int_{\mathcal{I}^+_{\leq \tau_i}} \mathbf{J}_\mu^T[\psi] n_{\mathcal{I}^+}^\mu = \int_{\Sigma_0^*} \mathbf{J}_\mu^T[\psi] n_{\Sigma_0^*}^\mu.$$

Now we simply take $\tau_i \rightarrow \infty$ and observe that $\left| \mathbf{J}_\mu^T[\psi] n_{S_{\tau_i}}^\mu \right| \leq B \mathbf{J}_\mu^N[\psi] n_{S_{\tau_i}}^\mu$. \square

REMARK 10.2.1. – *Of course, one may prove a version of Proposition 10.2.1 where the hypersurface Σ_0^* is replaced by $\bar{\Sigma}$.*

THEOREM 10.2.1. – *For any $\phi \in \mathcal{C}_{cp}^\infty(\mathcal{I}^-)$ we have*

$$(180) \quad \int_{\mathcal{H}^+} \mathbf{J}_\mu^T[\mathcal{T}\phi] n_{\mathcal{H}^+}^\mu + \int_{\mathcal{I}^+} \mathbf{J}_\mu^T[\mathcal{R}\phi] n_{\mathcal{I}^+}^\mu = \int_{\mathcal{I}^-} \mathbf{J}_\mu^T[\phi] n_{\mathcal{I}^-}^\mu,$$

$$(181) \quad \int_{\mathcal{H}^+} \left| \mathbf{J}_\mu^T[\mathcal{T}\phi] n_{\mathcal{H}^+}^\mu \right| \leq B \int_{\mathcal{I}^-} \mathbf{J}_\mu^T[\phi] n_{\mathcal{I}^-}^\mu.$$

Then, an easy density argument shows that (180) and (181) hold for arbitrary $\phi \in \mathcal{E}_{\mathcal{I}^-}^T$.

Proof. – The equality (180) follows immediately from Remark 10.2.1 and the fact that $\phi \in C_{cp}^\infty(\mathcal{I}^-)$ implies that

$$\int_{\Sigma} \mathbf{J}_\mu^N[\mathcal{B}_+(0, \phi)] n_\Sigma^\mu < \infty.$$

The inequality (181) follows immediately from Plancherel, Theorem 9.5.3, Theorem 10.1.1 and the fact that combining Theorem 10.1.1 and Corollary 5.3.1 implies that $\left| \frac{\omega}{\omega - \omega_+} \mathfrak{T} \right|$ is uniformly bounded. \square

This gives Theorem 8 of Section 2.3.7.

REMARK 10.2.2. – Note that we cannot consider the case of general initial data in $\mathcal{E}_{\mathcal{H}^-}^K$ as $\psi \in \mathcal{E}_{\mathcal{H}^-}^K$ does not imply that

$$\int_{\mathcal{H}^-} \mathbf{J}_\mu^T[\psi] n_{\mathcal{H}^-}^\mu < \infty.$$

Finally, we observe that if we restrict the initial data along \mathcal{H}^- and \mathcal{I}^- to be non-superradiant, then the map \mathcal{S} will be unitary in the standard sense. First we introduce the relevant function spaces.

DEFINITION 10.2.1. – We define $\mathcal{E}_{\mathcal{I}^\pm}^{T, \mathfrak{h}}$ to be the Hilbert space consisting of functions $f(\tau, \theta, \phi) : \mathcal{I}^\pm \rightarrow \mathbb{C}$ such that

$$\hat{f}(\omega, m, \ell) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} e^{i\omega t} e^{-im\phi} S_{m\ell} f \sin \theta dt d\theta d\phi,$$

lies in the closure of functions compactly supported in $\{(\omega, m, \ell) : \omega(\omega - \omega_+ m) > 0\}$ under the inner product

$$\int_{-\infty}^{\infty} \sum_{m\ell} \omega^2 \operatorname{Re}(f_1 \overline{f_2}).$$

DEFINITION 10.2.2. – We define $\mathcal{E}_{\mathcal{H}^\pm}^{T, \mathfrak{h}}$ to be the Hilbert space consisting of functions $f(\tau, \theta, \phi) : \mathcal{H}^\pm \rightarrow \mathbb{C}$ such that

$$\hat{f}(\omega, m, \ell) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} e^{i\omega t} e^{-im\phi} S_{m\ell} f \sin \theta dt d\theta d\phi,$$

lies in the closure of functions compactly supported in

$$(182) \quad \{(\omega, m, \ell) : \omega(\omega - \omega_+ m) > 0\}$$

under the inner product

$$\int_{-\infty}^{\infty} \sum_{m\ell} \omega(\omega - \omega_+ m) \operatorname{Re}(f_1 \overline{f_2}).$$

The following theorem is an immediate consequence of the microlocal energy identity of Proposition 5.2.2.

THEOREM 10.2.2. – *The restriction of the map \mathcal{S} to functions whose Fourier transforms are compactly supported in (182) extends by density to a map $\mathcal{S} : \mathcal{E}_{\mathcal{H}^-}^{T,\mathfrak{h}} \oplus \mathcal{E}_{\mathcal{I}^-}^{T,\mathfrak{h}} \rightarrow \mathcal{E}_{\mathcal{H}^-}^{T,\mathfrak{h}} \oplus \mathcal{E}_{\mathcal{I}^-}^{T,\mathfrak{h}}$ which is a unitary isomorphism with respect to the positive definite inner product*

$$\langle (\psi_1, \phi_1), (\psi_2, \phi_2) \rangle = \int_{-\infty}^{\infty} \sum_{m\ell} \left[\omega (\omega - \omega_+ m) \operatorname{Re} \left(\widehat{\psi}_1 \overline{\widehat{\psi}_2} \right) + \omega^2 \operatorname{Re} \left(\widehat{\phi}_1 \overline{\widehat{\phi}_2} \right) \right].$$

This gives Theorem 9 of Section 2.3.7. Note that the above reduces again to Theorem 9.6.2 in the case $a = 0$, where $\mathcal{E}_{\mathcal{H}^\pm}^{T,\mathfrak{h}} \oplus \mathcal{E}_{\mathcal{I}^\pm}^{T,\mathfrak{h}}$ coincide with $\mathcal{E}_{\mathcal{H}^\pm}^T \oplus \mathcal{E}_{\mathcal{I}^\pm}^T$. It also yields in particular that \mathcal{S} restricted to axisymmetric scattering data is unitary.

REMARK 10.2.3. – *One may also formally consider the inner product*

$$\langle (\psi_1, \phi_1), (\psi_2, \phi_2) \rangle \doteq \int_{-\infty}^{\infty} \sum_{m\ell} \left[\omega (\omega - \omega_+ m) \operatorname{Re} \left(\widehat{\psi}_1 \overline{\widehat{\psi}_2} \right) + \omega \operatorname{Re} \left(\widehat{\phi}_1 \overline{\widehat{\phi}_2} \right) \right].$$

This corresponds to the particle current which is relevant in quantum field theory; see [71]. In this setting, one may establish the analog of the pseudo-unitarity statement of Theorem 10.2.1.

10.3. Uniqueness of ill-posed scattering states

We turn finally to the “ill-posed case,” where one attempts to pose scattering data on $\mathcal{H}^+ \cup \mathcal{H}^-$, $\mathcal{I}^+ \cup \mathcal{I}^-$, $\mathcal{H}^- \cup \mathcal{I}^+$ or $\mathcal{H}^+ \cup \mathcal{I}^-$.

To state our theorems, let us note first that we may define the forward maps

$$(183) \quad \begin{aligned} \mathcal{F} : \mathcal{E}_{\Sigma}^V &\rightarrow \mathcal{E}_{\mathcal{I}^+}^T \oplus \mathcal{E}_{\mathcal{I}^-}^T, & \mathcal{F} : \mathcal{E}_{\Sigma}^V &\rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{H}^-}^K, \\ \mathcal{F} : \mathcal{E}_{\Sigma}^V &\rightarrow \mathcal{E}_{\mathcal{H}^-}^K \oplus \mathcal{E}_{\mathcal{I}^+}^T, & \mathcal{F} : \mathcal{E}_{\Sigma}^V &\rightarrow \mathcal{E}_{\mathcal{H}^+}^K \oplus \mathcal{E}_{\mathcal{I}^-}^T, \end{aligned}$$

by completion of

$$(184) \quad (\psi, \psi') \mapsto \psi \mapsto (\phi|_{\mathcal{I}^+}, \phi|_{\mathcal{I}^-}) \text{ or } (\psi|_{\mathcal{H}^+}, \psi|_{\mathcal{H}^-}) \text{ or } (\psi|_{\mathcal{H}^-}, \phi|_{\mathcal{I}^+}) \text{ or } (\psi|_{\mathcal{H}^+}, \phi|_{\mathcal{I}^-}),$$

and these are again bounded maps by our previous results. We have the following statement of uniqueness (but not existence!) of “improper” scattering states:

THEOREM 10.3.1. – *The maps \mathcal{F} of (183) are all injective.*

Proof. – We start with the case of the first two maps of (183).

First of all, the proof is conceptually clearer in the case of smooth compactly supported initial data, and we thus begin with this case. Consider $(\psi, \psi') \in {}^2\mathcal{C}_{cp}^\infty(\overline{\Sigma})$, let ψ be solution of (184) and assume (ψ, ψ') is in the kernel of the first or second map of (183). Then, upon an application of Carter’s separation to ψ we have that for almost every (ω, m, ℓ) , the resulting u is a smooth solution to the radial o.d.e. (61) such that when $\omega \neq 0$, $|a_{\mathcal{I}^+}|^2 + |a_{\mathcal{I}^-}|^2 = 0$ or $|a_{\mathcal{H}^+}|^2 + |a_{\mathcal{H}^-}|^2 = 0$, respectively. It follows immediately from the local existence theory for these o.d.e.’s that u is identically 0 (see [60]) whenever $\omega \neq 0$, and thus ψ is 0. It follows that $(\psi, \psi') = (0, 0)$.

For general $(\psi, \psi') \in \mathcal{E}_{\Sigma}^V$, let us first consider the case only of the first map of (183), i.e., let $(\psi, \psi') \in \ker \mathcal{F} : \mathcal{E}_{\Sigma}^V \rightarrow \mathcal{E}_{\mathcal{I}^+}^T \oplus \mathcal{E}_{\mathcal{I}^-}^T$. Let ψ denote the solution of the wave equation (2) arising from (ψ, ψ') . Theorem 3.7.1 implies that ψ lies in $L_{loc,r}^2 L_{t,r,\theta,\phi}^2$. In particular, we can take the Fourier transform of ψ and define the Carter separated function $u(r, \omega, m, \ell)$ which will lie in $L_{loc,r}^2 L_{\omega,m,\ell}^2$.

Let $\epsilon > 0$, let \mathcal{F}_b denote an arbitrary compact set of (ω, m, ℓ) and let \mathcal{K} denote an arbitrary compact set in (r_+, ∞) . Now, by regularizing the initial data for ψ , we can produce a solution ψ_ϵ to (2) with smooth compactly supported initial data such that

$$\int_{\bar{\Sigma}} \mathbf{J}_\mu^V [\psi - \psi_\epsilon] n_\Sigma^\mu \leq \epsilon.$$

It follows immediately from the fact that the forward map is well defined, that

$$\int_{\mathcal{I}^\pm} \mathbf{J}_\mu^T [\psi_\epsilon] n_{\mathcal{I}^\pm}^\mu \leq B\epsilon.$$

In particular, if we let $a_{\epsilon, \mathcal{I}^\pm}$ denote the microlocal radiation fields for ψ_ϵ , Propositions 6.8.1 and 4.2.2 imply that

$$\int_{-\infty}^{\infty} \sum_{m\ell} [|a_{\epsilon, \mathcal{I}^+}|^2 + |a_{\epsilon, \mathcal{I}^-}|^2] d\omega \leq B\epsilon.$$

Letting u_ϵ denote the result of applying Carter's separation to ψ_ϵ , it now follows immediately from standard o.d.e. theory that

$$\int_{\mathcal{K}} \int_{(\omega, m, \ell) \in \mathcal{F}_b} |u_\epsilon|^2 \leq B(\mathcal{K}, \mathcal{F}_b)\epsilon.$$

Finally, an application of Theorem 3.7.1 to the $\psi - \psi_\epsilon$ followed by an application of Plancherel implies

$$\int_{\mathcal{K}} \int_{(\omega, m, \ell) \in \mathcal{F}_b} |u|^2 \leq B(\mathcal{K}, \mathcal{F}_b)\epsilon,$$

where $u(r, \omega, m, \ell)$ is the result of applying Carter's separation to ψ . Since ϵ , \mathcal{K} , and \mathcal{F}_b were arbitrary, we conclude that u and hence ψ vanish.

The case where (ψ, ψ') lies in the kernel of the second map of (183) is treated in exactly the same way.

We turn now to the case when (ψ, ψ') lies in the kernel of the third and fourth map of (183). Since ψ is not necessarily sufficiently integrable, we cannot use Definition 5.4.1 to define the microlocal radiation fields; instead we define $a_{\mathcal{I}^\pm}(\omega, m, \ell)$ and $a_{\mathcal{H}^\pm}(\omega, m, \ell)$ by applying Carter's separation to the functions $\mathcal{F}_\pm(\psi, \psi')|_{\mathcal{I}^\pm}$ and $\mathcal{F}_\pm(\psi, \psi')|_{\mathcal{H}^\pm}$:

$$\begin{aligned} a_{\mathcal{I}^\pm} &\doteq \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} e^{it\omega} e^{-im\phi} S_{m\ell} \mathcal{F}_\pm(\psi, \psi')|_{\mathcal{I}^\pm} \sin \theta dt d\theta d\phi, \\ a_{\mathcal{H}^\pm} &\doteq \sqrt{2Mr_+} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} e^{it^*\omega} e^{-im\phi} S_{m\ell} \mathcal{F}_\pm(\psi, \psi')|_{\mathcal{H}^\pm} \sin \theta dt^* d\theta^* d\phi^*. \end{aligned}$$

Now, we may apply Theorem 9.5.3 (and its complex conjugated version) to conclude that

(185)

$$\mathcal{F}_+(\psi, \psi')|_{\mathcal{I}^+} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} \left(-\left(\frac{\omega - \omega_+ m}{\omega}\right) a_{\mathcal{H}^-} \tilde{\mathfrak{X}} + a_{\mathcal{I}^-} \mathfrak{R} \right) e^{-it\omega} e^{im\phi} S_{m\ell} d\omega,$$

(186)

$$\mathcal{F}_-(\psi, \psi')|_{\mathcal{I}^-} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} \left(-\left(\frac{\omega - \omega_+ m}{\omega}\right) a_{\mathcal{H}^+} \bar{\tilde{\mathfrak{X}}} + a_{\mathcal{I}^+} \bar{\mathfrak{R}} \right) e^{-it\omega} e^{im\phi} S_{m\ell} d\omega,$$

(187)

$$\mathcal{F}_+(\psi, \psi')|_{\mathcal{H}^+} = \frac{1}{\sqrt{4M\pi r_+}} \int_{-\infty}^{\infty} \sum_{m\ell} \left(-\left(\frac{\omega}{\omega - \omega_+}\right) a_{\mathcal{I}^-} \mathfrak{X} + a_{\mathcal{H}^-} \tilde{\mathfrak{R}} \right) e^{-it\omega} e^{im\phi} S_{m\ell} d\omega,$$

(188)

$$\mathcal{F}_-(\psi, \psi')|_{\mathcal{H}^-} = \frac{1}{\sqrt{4M\pi r_+}} \int_{-\infty}^{\infty} \sum_{m\ell} \left(-\left(\frac{\omega}{\omega - \omega_+}\right) a_{\mathcal{I}^+} \bar{\mathfrak{X}} + a_{\mathcal{H}^+} \bar{\tilde{\mathfrak{R}}} \right) e^{-it\omega} e^{im\phi} S_{m\ell} d\omega.$$

Observe that if (ψ, ψ') lies in the kernel of the third map of (183), then $a_{\mathcal{H}^-}$ and $a_{\mathcal{I}^+}$ will vanish almost everywhere. Then (185) and (188) imply that $a_{\mathcal{I}^-} \mathfrak{R}$ and $a_{\mathcal{H}^+} \bar{\tilde{\mathfrak{R}}}$ both vanish almost everywhere. However, Corollary 6.5.1 implies that \mathfrak{R} and $\tilde{\mathfrak{R}}$ can only vanish at isolated points in ω . We conclude that $a_{\mathcal{H}^+}$ and $a_{\mathcal{I}^-}$ can only be non-zero at isolated points and hence that $a_{\mathcal{H}^+}$ and $a_{\mathcal{I}^-}$ vanish almost everywhere. We conclude that $\mathcal{F}_{\pm}(\psi, \psi') = (0, 0)$ and thus that ψ vanishes.

The case where (ψ, ψ') lies in the kernel of the fourth map of (183) is treated in a similar fashion. □

We have thus obtained now Theorem 10 of Section 2.3.8.

REMARK 10.3.1. – *In regard to the first two maps of (183), we note that it is possible to prove localized versions of the above via the techniques of “unique continuation,” where ψ is only assumed to vanish on certain portions of $\mathcal{H}^+ \cup \mathcal{H}^-$ or portions of $\mathcal{I}^+ \cup \mathcal{H}^+$, but with stronger regularity assumptions and decay at infinity. See [14] for such results in the Schwarzschild case, [1] for such results on general asymptotically flat spacetimes and [2] for such results for (among other things) certain non-linear wave equations on Minkowski space.*

11. The backwards blue-shift instability and horizon-singular solutions

In this final section, we shall show that any solution of the wave equation (2) on Schwarzschild assumed to have a particular choice of radiation field necessarily would have infinite N -energy on the hypersurface Σ_0^* . Our theorem can be stated as follows:

THEOREM 11.1. – *Let $a = 0$ and let ψ be a smooth spherically symmetric solution of the wave equation in the region $\mathring{\mathcal{R}}_{\geq 0}$ such that*

1. *The initial data for ψ lies in the closure of compactly supported initial data under the norm*

$$\int_{\Sigma_0^*} [\mathbf{J}_\mu^T[\psi] + \mathbf{J}_\mu^T[T\psi]] n_{\Sigma_0^*}^\mu.$$

2. $\partial_{\tilde{v}}\psi$ extends continuously to the function $(t^* + 1)^{-p}$ on $\mathcal{H}_{\geq 0}^+$ for some $p > 2$.
3. $\partial_{\tilde{v}}(T\psi)$ extends continuously to the function $-p(t^* + 1)^{-p-1}$ on $\mathcal{H}_{\geq 0}^+$ for the same p as above.
4. There exists τ_0 such that $\tau_1 > \tau_0$ implies

$$\lim_{r \rightarrow \infty} r\psi|_{\tau=\tau_1} = 0.$$

Then

$$\int_{\Sigma_0^*} \mathbf{J}_\mu^N[\psi]n_{\Sigma_0^*}^\mu = \infty.$$

We will prove Theorem 11.1 in Sections 11.1–11.2 below. We have stated our theorem in the above form so as to be independent of the existence of the scattering theory maps \mathcal{F}_+ , \mathcal{B}_- , etc., proven in this paper. Thus, the proof of Theorem 11.1 can be read independently of the rest of our paper. The argument exploits the blue-shift factor of the horizon together with a simple monotonicity property of the spherically symmetric wave equation.

In combination with the results of our paper, Theorem 11.1 can be reinterpreted in the context of both our N -energy and our T -energy theories. First, applying Theorem 9.2.1, we shall construct solutions ψ satisfying the assumptions of Theorem 11.1 such that their induced data lie in $\mathcal{E}_{\Sigma_0^*}^T$ and give a short discussion of the significance of the existence of such solutions. Finally, in Section 11.4, we shall reinterpret Theorem 11.3 as a statement of the non-surjectivity of the map $\mathcal{F}_+ : \mathcal{E}_{\Sigma_0^*}^N \rightarrow \mathcal{E}_{\mathcal{H}^+}^N \oplus \mathcal{E}_{\mathcal{I}^+}^N$ of Theorem 8.2.1. This will thus give Theorem 2 of Section 2.3.2.

11.1. Schwarzschild computations

Setting $a = 0$ in (29), the Schwarzschild metric in Boyer-Lindquist coordinates takes the form

$$(189) \quad g_{\text{Schw}} = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

To get an explicitly regular expression for the metric near the event horizon \mathcal{H}^+ we introduce the $(\tilde{v}, r, \theta, \phi)$ coordinate system defined by

$$\frac{dr^*}{dr} \doteq \left(1 - \frac{2M}{r}\right)^{-1}, \quad \tilde{v} \doteq t + r^*.$$

The metric then takes the form

$$(190) \quad g_{\text{Schw}} = -\left(1 - \frac{2M}{r}\right) d\tilde{v}^2 + 2d\tilde{v}dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

Note that we have $T = \partial_{\tilde{v}}$ in the $(\tilde{v}, r, \theta, \phi)$ coordinate system. Let us also agree to set $Y \doteq \partial_r$.

It will also be useful to introduce a $(\tilde{t}, r, \theta, \phi)$ coordinate system in the following fashion. Let $\chi(r)$ be a cut-off which is identically 0 for $r \in [2M, 3M]$ and identically 1 for $r \in [4M, \infty)$. We then set

$$\tilde{t}(t, r) \doteq t + r^* - 2\chi(r)r^*.$$

Note that $\partial_{\tilde{t}} = T$ is Killing.

Finally, it turns out that it is often convenient to work in the null coordinate system $(\tilde{u}, \tilde{v}, \theta, \phi)$ where \tilde{v} is defined as before and

$$\tilde{u} \doteq t - r^*.$$

The metric then takes the form

$$(191) \quad g_{\text{Schw}} = -2 \left(1 - \frac{2M}{r}\right) d\tilde{u}d\tilde{v} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

REMARK 11.1.1. – *These coordinates break down at the horizon, where $\tilde{u} = \infty$. Nevertheless we can still use these coordinates in an effective manner near \mathcal{H}^+ as long as we remember that the $(1 - \frac{2M}{r})^{-1} \partial_{\tilde{u}} = Y$ is a regular vector field on \mathcal{H}^+ .*

If not explicitly noted otherwise, $\partial_{\tilde{v}}$ and $\partial_{\tilde{u}}$ will also be defined in the $(\tilde{u}, \tilde{v}, \theta, \phi)$ coordinate system. Let us agree to set $L \doteq \partial_{\tilde{v}}$.

In null coordinates, the wave equation (2) applied to a spherically symmetric function ψ takes the form

$$(192) \quad \begin{aligned} \partial_{\tilde{v}} (r^2 \partial_{\tilde{u}} \psi) + \partial_{\tilde{u}} (r^2 \partial_{\tilde{v}} \psi) &= 0 \\ \Leftrightarrow \partial_{\tilde{v}, \tilde{u}}^2 \psi + \frac{(\partial_{\tilde{u}} r) \partial_{\tilde{v}} \psi}{r} + \frac{(\partial_{\tilde{v}} r) \partial_{\tilde{u}} \psi}{r} &= 0. \end{aligned}$$

The equation (192) is equivalent to the following coupled transport equations for $r \partial_{\tilde{u}} \psi$ and $r \partial_{\tilde{v}} \psi$:

$$(193) \quad \partial_{\tilde{u}} (r \partial_{\tilde{v}} \psi) = -\frac{(\partial_{\tilde{v}} r)(r \partial_{\tilde{u}} \psi)}{r} = -\frac{(1 - \frac{2M}{r}) r \partial_{\tilde{u}} \psi}{r},$$

$$(194) \quad \partial_{\tilde{v}} (r \partial_{\tilde{u}} \psi) = -\frac{(\partial_{\tilde{u}} r)(r \partial_{\tilde{v}} \psi)}{r} = \frac{(1 - \frac{2M}{r}) r \partial_{\tilde{v}} \psi}{r}.$$

Near the event horizon it will be useful to work with transport equations for $r (1 - \frac{2M}{r})^{-1} \partial_{\tilde{u}} \psi$ and $r \partial_{\tilde{v}} \psi$:

$$(195) \quad \left(1 - \frac{2M}{r}\right)^{-1} \partial_{\tilde{u}} (r \partial_{\tilde{v}} \psi) = -\frac{(1 - \frac{2M}{r}) \left[r \left(1 - \frac{2M}{r}\right)^{-1} \partial_{\tilde{u}} \psi \right]}{r},$$

$$(196) \quad \partial_{\tilde{v}} \left(r \left(1 - \frac{2M}{r}\right)^{-1} \partial_{\tilde{u}} \psi \right) + \frac{2M}{r^2} \left(r \left(1 - \frac{2M}{r}\right)^{-1} \partial_{\tilde{u}} \psi \right) = \frac{r \partial_{\tilde{v}} \psi}{r}.$$

REMARK 11.1.2. – *The fact that $\frac{2M}{r^2}|_{\mathcal{H}^+} = \frac{1}{2M} > 0$ represents the positivity of surface gravity and is intimately tied to the (local) redshift effect. See the discussion in [27].*

11.2. Proof of Theorem 11.1

We are now ready for the proof of Theorem 11.1. We will proceed in four steps.

1. Letting ψ be as in Theorem 11.1, we begin by establishing a local energy decay statement with a sharp rate:

$$\int_{S_\tau \cap \{r \leq R\}} \left[(T\psi)^2 + (\partial_{r^*} \psi)^2 \right] \leq B(R)(1 + \tau)^{-2p} \quad \forall R > r_+.$$

2. Using the decay from the previous step, for all sufficiently small $\epsilon > 0$ we will propagate the $(1 + \tilde{v})^{-p}$ lower bound for $\partial_{\tilde{v}}\psi$ along \mathcal{H}^+ to a $(1 + v)^{-p}$ lower bound on the hypersurfaces $\{r = 2M + \epsilon\}$. Cf. [18, 22].
3. Using the $(\tilde{t}, r, \theta, \phi)$ coordinate system, define $\tilde{\psi}(\tilde{t}, r) \doteq -\int_{\tilde{t}}^{\infty} \psi(\tilde{s}, r) d\tilde{s}$. Using the equation (196) and the previous steps, we will prove that unless $\int_{\Sigma_0^*} \mathbf{J}_{\mu}^N[\psi]n_{\Sigma_0^*}^{\mu} = \infty$, then $(1 - \frac{2M}{r})^{-1} \partial_{\tilde{u}}\tilde{\psi}$ and $\partial_{\tilde{v}}\tilde{\psi}$ both are positive along one of the hypersurfaces $\{r = 2M + \epsilon\}$.
4. Under the assumption that $\int_{\Sigma_0^*} \mathbf{J}_{\mu}^N[\psi]n_{\Sigma_0^*}^{\mu} < \infty$ we will use some monotonicity hidden in the system (193) and (194), and show that the positivity of $\partial_{\tilde{u}}\tilde{\psi}$ and $\partial_{\tilde{v}}\tilde{\psi}$ along $\{r = 2M + \epsilon\}$ propagates along outgoing null curves. Finally, we will see that this positivity of $r\partial_{\tilde{u}}\tilde{\psi}$ and $r\partial_{\tilde{v}}\tilde{\psi}$ implies that $r\psi$ cannot vanish along \mathcal{I}^+ , yielding a contradiction to the assumption $\int_{\Sigma_0^*} \mathbf{J}_{\mu}^N[\psi]n_{\Sigma_0^*}^{\mu} < \infty$.

11.2.1. *Local energy decay.* – We begin with the following proposition.

PROPOSITION 11.2.1. – *Let ψ and p be as in the statement of Theorem 11.1. Then, for all $R < \infty$ we have*

$$\int_{S_{\tau} \cap \{r \leq R\}} \left[(T\psi)^2 + (\partial_{r^*}\psi)^2 \right] \leq B(R)(1 + \tau)^{-2p}.$$

Proof. – We begin by arguing that

$$(197) \quad \int_{S_{\tau}} \mathbf{J}_{\mu}^T[\psi]n_{S_{\tau}}^{\mu} \leq B \left[\int_{\mathcal{H}_{\geq \tau}^+} \mathbf{J}_{\mu}^T[\psi]n_{\mathcal{H}^+}^{\mu} + \int_{\mathcal{I}_{\geq \tau}^+} \mathbf{J}_{\mu}^T[\psi]n_{\mathcal{I}^+}^{\mu} \right] \leq B(1 + \tau)^{-2p+1}.$$

Let $\tau < \infty$. Since $\int_{\Sigma_0^*} \mathbf{J}_{\mu}^T[\psi]n_{\Sigma_0^*}^{\mu} < \infty$ we may find a sequence of solutions $\{\psi_i\}$ to (2) whose initial data lie in $\tilde{\mathcal{E}}_{\Sigma_0^*}$ and which satisfy $\lim_{i \rightarrow \infty} \int_{S_{\tau}} \mathbf{J}_{\mu}^T[\psi_i]n_{S_{\tau}}^{\mu} = \int_{S_{\tau}} \mathbf{J}_{\mu}^T[\psi]n_{S_{\tau}}^{\mu}$. As we have already observed multiple times, Theorem 3.7.2 and Proposition 3.8.1 imply that we may find a dyadic sequence $\{\tau_j^{(i)}\}_{j=1}^{\infty}$ such that $\lim_{j \rightarrow \infty} \int_{S_{\tau_j^{(i)}}} \mathbf{J}_{\mu}^T[\psi_i]n_{S_{\tau_j^{(i)}}}^{\mu} = 0$. Then we may apply a \mathbf{J}^T energy estimate to each ψ_i and conclude

$$\int_{S_{\tau}} \mathbf{J}_{\mu}^T[\psi_i]n_{S_{\tau}}^{\mu} = \int_{\mathcal{H}^+(\tau, \tau_j^{(i)})} \mathbf{J}_{\mu}^T[\psi_i]n_{\mathcal{H}^+}^{\mu} + \int_{\mathcal{I}^+(\tau, \tau_j^{(i)})} \mathbf{J}_{\mu}^T[\psi_i]n_{\mathcal{I}^+}^{\mu} + \int_{S_{\tau_j^{(i)}}} \mathbf{J}_{\mu}^T[\psi_i]n_{S_{\tau_j^{(i)}}}^{\mu}.$$

Taking j to infinity and then i to infinity yields

$$\int_{S_{\tau}} \mathbf{J}_{\mu}^T[\psi]n_{S_{\tau}}^{\mu} = \int_{\mathcal{H}_{\geq \tau}^+} \mathbf{J}_{\mu}^T[\psi]n_{\mathcal{H}^+}^{\mu} + \int_{\mathcal{I}_{\geq \tau}^+} \mathbf{J}_{\mu}^T[\psi]n_{\mathcal{I}^+}^{\mu}.$$

Finally, using that $\partial_{\tilde{v}}\psi$ extends continuously to $(1 + t^*)^{-p}$ one may easily show that $\int_{\mathcal{H}_{\geq \tau}^+} \mathbf{J}_{\mu}^T[\psi]n_{\mathcal{H}^+}^{\mu} \leq B \int_{\tau}^{\infty} (1 + t^*)^{-2p} dt^*$, and hence establish (197).

Next, we commute with the Killing vector field T and consider the solution $T\psi$. Repeating the above procedure (using in particular that $T\psi$ is assumed to have a finite \mathbf{J}^T energy along Σ_0^* and the assumption on the limit of $\partial_{\tilde{v}}(T\psi)$ to $\mathcal{H}_{\geq 0}^+$) another \mathbf{J}^T energy estimate implies

$$(198) \quad \int_{S_{\tau}} \mathbf{J}_{\mu}^T[T\psi]n_{S_{\tau}}^{\mu} \leq B \left[\int_{\mathcal{H}_{\geq \tau}^+} \mathbf{J}_{\mu}^T[T\psi]n_{\mathcal{H}^+}^{\mu} + \int_{\mathcal{I}_{\geq \tau}^+} \mathbf{J}_{\mu}^T[T\psi]n_{\mathcal{I}^+}^{\mu} \right] \leq B(1 + \tau)^{-2p-1}.$$

The final ingredient is an integrated local energy decay estimate. Setting $X \doteq f(r^*)\partial_{r^*}$ for a function f to be fixed later, a straightforward calculation yields the following general formula:

(199)

$$\nabla^\mu \mathbf{J}_\mu^X[\psi] = \left(\frac{f'}{2} \left(1 - \frac{2M}{r} \right)^{-1} + fr^{-1} \right) (T\psi)^2 + \left(\frac{f'}{2} \left(1 - \frac{2M}{r} \right)^{-1} - fr^{-1} \right) (\partial_{r^*}\psi)^2.$$

We set $f \doteq -r^{-3}$ and obtain

(200)
$$\nabla^\mu \mathbf{J}_\mu^X[\psi] = \frac{1}{2}r^{-4} (T\psi)^2 + \frac{5}{2}r^{-4} (\partial_{r^*}\psi)^2.$$

Keeping in mind that $X|_{\mathcal{H}^+} = -(2M)^{-3}T$ and $\left| \int_{S_\tau} \mathbf{J}_\mu^X[\psi] n_{S_\tau}^\mu \right| \leq B \int_{S_\tau} \mathbf{J}_\mu^T[\psi] n_{S_\tau}^\mu$, combining (200) with (197) and (198) yields the following two estimates:

(201)
$$\int_\tau^\infty \int_{S_\tau \cap \{r \leq R\}} \left[(T\psi)^2 + (\partial_{r^*}\psi)^2 \right] \leq B(R) (1 + \tau)^{-2p+1} \quad \forall R > r_+,$$

(202)
$$\int_\tau^\infty \int_{S_\tau \cap \{r \leq R\}} \left[(T^2\psi)^2 + (\partial_{r^*}T\psi)^2 \right] \leq B(R) (1 + \tau)^{-2p-1} \quad \forall R > r_+.$$

We will now interpolate between these four estimates in a straightforward fashion.⁽¹⁵⁾

For every $k \geq 1$, using the fact that

$$\int_{2^k}^{2^{k+1}} \int_{S_\tau \cap \{r \leq R\}} \left[(T\psi)^2 + (\partial_{r^*}\psi)^2 \right] \leq B(R) (2^k)^{-2p+1},$$

we may find a $\tau_k \in [2^k, 2^{k+1}]$ such that

(203)
$$\int_{S_{\tau_k} \cap \{r \leq R\}} \left[(T\psi)^2 + (\partial_{r^*}\psi)^2 \right] \leq B(R) \tau_k^{-2p}.$$

Now consider $\tau \in [\tau_k, \tau_{k+1}]$. The fundamental theorem of calculus and the estimates (201), (202), and (203) imply

(204)

$$\begin{aligned} & \int_{S_\tau \cap \{r \leq R\}} \left[(T\psi)^2 + (\partial_{r^*}\psi)^2 \right] \\ & \leq \int_{S_{\tau_{k+1}} \cap \{r \leq R\}} \left[(T\psi)^2 + (\partial_{r^*}\psi)^2 \right] \\ & \quad + B \int_{\tau_k}^{\tau_{k+1}} \int_{S_s \cap \{r \leq R\}} \left[\tau_k^{-1} (T\psi)^2 + \tau_k^{-1} (\partial_{r^*}\psi)^2 + \tau_k (T^2\psi)^2 + \tau_k (\partial_{r^*}T\psi)^2 \right] \\ & \leq B(R) \left[\tau_{k+1}^{-2p} + \tau_k^{-2p} + \tau_k^{-2p} \right] \leq B(R) \tau^{-2p}. \quad \square \end{aligned}$$

REMARK 11.2.1. – *If in the process of carrying out a contradiction argument one adds the assumption that $\int_{\Sigma_0^*} \mathbf{J}_\mu^N[\psi] n_{\Sigma_0^*}^\mu < \infty$, then one could establish the local energy decay statement using only transport equations in the region $\{r \leq R\}$.*

⁽¹⁵⁾ For example, see [65] for an application of such an interpolation argument to interior decay for the wave equation.

COROLLARY 11.2.1. – *Of course, the use of the particular foliation $\{S_\tau\}$ is not important for the above proposition. By modifying S_τ to equal $\{\tilde{v} = \tau\}$ in the region $\{r \leq R\}$ and repeating the above proof, one immediately obtains*

$$\int_{\{\tilde{v}=\tau\} \cap \{r \leq R\}} \left[(T\psi)^2 + (\partial_{r^*}\psi)^2 \right] \leq B(R)(1 + \tau)^{-2p}.$$

Analogously, for $(r_0, r_1) \subset (r_+, \infty)$ one may show

$$\int_{\{\tilde{u}=\tau\} \cap \{r \in [r_0, r_1]\}} \left[(T\psi)^2 + (\partial_{r^*}\psi)^2 \right] \leq B(r_0, r_1)(1 + \tau)^{-2p}.$$

11.2.2. *Pushing the tail off the horizon.* – We now turn to the proof of

PROPOSITION 11.2.2. – *Let ψ be as in Theorem 11.1. Then, for all $\epsilon > 0$ sufficiently small,*

$$b(1 + \tilde{v})^{-p} \leq (r \partial_{\tilde{v}} \psi) |_{\{r \leq 2M + \epsilon\}} \leq B(1 + \tilde{v})^{-p}.$$

Proof. – Keeping in mind that $(1 - \frac{2M}{r})^{-1} \partial_{\tilde{u}} = Y$ is equal to ∂_r in $(\tilde{v}, r, \theta, \phi)$ coordinates, we integrate the transport equation (193) and obtain

$$(205) \quad (r \partial_{\tilde{v}} \psi) |_{(\tilde{v}, r) = (\tau, 2M + \epsilon)} = (r \partial_v \psi) |_{(\tilde{v}, r) = (\tau, 2M)} + \int_{\{\tilde{v}=\tau\} \cap \{r \leq 2M + \epsilon\}} \partial_{\tilde{u}} \psi \, dr.$$

Cauchy-Schwarz and Corollary 11.2.1 then yield

$$(206) \quad \left((r \partial_{\tilde{v}} \psi) |_{(\tilde{v}, r) = (\tau, 2M + \epsilon)} - (1 + \tau)^{-p} \right)^2 \leq \epsilon \int_{\{\tilde{v}=\tau\} \cap \{r \leq 2M + \epsilon\}} \left[(T\psi)^2 + (\partial_{r^*}\psi)^2 \right] \leq B\epsilon(1 + \tau)^{-2p}. \quad \square$$

11.2.3. *Positivity of $\tilde{\psi}$.* – As we have already indicated in the outline, it will be useful to introduce the function

$$(207) \quad \tilde{\psi}(\tilde{t}, r) \doteq \int_{\tilde{t}}^{\infty} \psi(\tilde{s}, r) \, d\tilde{s}.$$

Using the fact that T is Killing, and the fact that $|\psi(\tau, r)| \leq B(r)(1 + \tau)^{-p+1}$, one may easily check that $\tilde{\psi}$ is a smooth solution to (2) in $\mathring{\mathcal{R}}_0$ and that $T\tilde{\psi} = -\psi$. The goal of this section is to use the transport equation (196) to show that $r(1 - \frac{2M}{r})^{-1} \partial_{\tilde{u}} \tilde{\psi}$ inherits some of $r \partial_{\tilde{v}} \psi$'s positivity.

We begin by studying $r(1 - \frac{2M}{r})^{-1} \partial_{\tilde{u}} \tilde{\psi}$.

PROPOSITION 11.2.3. – *Let ψ be as in the statement of Theorem 11.1, \tilde{v}_0 be a fixed sufficiently large constant, and $(\tilde{u}, \tilde{v}) \in \{r = 2M + \epsilon\}$ for $\tilde{v} \geq \tilde{v}_0$ and $\epsilon > 0$ sufficiently small. Then*

$$\begin{aligned} & \left(r \left(1 - \frac{2M}{r} \right)^{-1} \partial_{\tilde{u}} \tilde{\psi} \right) \Big|_{(\tilde{u}, \tilde{v})} \\ & \geq B \exp \left(- \int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}' \right) \left| \left(r \left(1 - \frac{2M}{r} \right)^{-1} \partial_{\tilde{u}} \tilde{\psi} \right) \Big|_{(\tilde{u}, \tilde{v}_0)} \right| + b\tilde{v}^{-p}. \end{aligned}$$

Proof. – We may write Equation (196) as

$$(208) \quad \partial_{\tilde{v}} \left[\exp \left(\int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}' \right) r \left(1 - \frac{2M}{r} \right)^{-1} \partial_{\tilde{u}} \psi \right] = \exp \left(\int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}' \right) \partial_{\tilde{v}} \psi.$$

We conclude that

$$(209) \quad \left(r \left(1 - \frac{2M}{r} \right)^{-1} \partial_{\tilde{u}} \psi \right) \Big|_{(\tilde{u}, \tilde{v})} = \exp \left(- \int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}' \right) \left[\left(r \left(1 - \frac{2M}{r} \right)^{-1} \partial_{\tilde{u}} \psi \right) \Big|_{(\tilde{u}, \tilde{v}_0)} + \int_{\tilde{v}_0}^{\tilde{v}} \left(\exp \left(\int_{\tilde{v}_0}^{\tilde{v}'} \frac{2M}{r^2} d\tilde{v}'' \right) \partial_{\tilde{v}'} \psi d\tilde{v}' \right) \right].$$

Next, using Proposition 11.2.2, we observe that

$$(210) \quad \exp \left(- \int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}' \right) \int_{\tilde{v}_0}^{\tilde{v}} \left(\exp \left(\int_{\tilde{v}_0}^{\tilde{v}'} \frac{2M}{r^2} d\tilde{v}'' \right) \partial_{\tilde{v}'} \psi d\tilde{v}' \right) \geq b \exp \left(- \int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}' \right) \int_{\tilde{v}_0}^{\tilde{v}} \left(\exp \left(\int_{\tilde{v}_0}^{\tilde{v}'} \frac{2M}{r^2} d\tilde{v}'' \right) (\tilde{v}')^{-p} d\tilde{v}' \right).$$

Using that $|\partial_{\tilde{v}} r| \leq B\epsilon$, a straightforward series of integration by parts yields

$$(211) \quad \begin{aligned} & \exp \left(- \int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}' \right) \int_{\tilde{v}_0}^{\tilde{v}} \left(\exp \left(\int_{\tilde{v}_0}^{\tilde{v}'} \frac{2M}{r^2} d\tilde{v}'' \right) (\tilde{v}')^{-p} d\tilde{v}' \right) \\ & \geq \left(\frac{r^2}{2M} \right) \Big|_{(\tilde{v}, \tilde{u})} \tilde{v}^{-p} - \left(\frac{r^2}{2M} \right) \Big|_{(\tilde{v}_0, \tilde{u})} (\tilde{v}_0)^{-p} \exp \left(- (1 - B\epsilon) (\tilde{v} - \tilde{v}_0) \right) \\ & \quad - B\epsilon \exp \left(- \int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}' \right) \int_{\tilde{v}_0}^{\tilde{v}} \left(\exp \left(\int_{\tilde{v}_0}^{\tilde{v}'} \frac{2M}{r^2} d\tilde{v}'' \right) (\tilde{v}')^{-p} d\tilde{v}' \right) \\ & \quad - B \left(\frac{r^2}{2M} \right) \Big|_{(\tilde{v}, \tilde{u})} \tilde{v}^{-p-1} + b \left(\frac{r^2}{2M} \right) \Big|_{(\tilde{v}_0, \tilde{u})} (\tilde{v}_0)^{-p-1} \exp \left(- (1 - B\epsilon) (\tilde{v} - \tilde{v}_0) \right) - B\tilde{v}^{-p-1}. \end{aligned}$$

We conclude that

$$(212) \quad \exp \left(- \int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}' \right) \int_{\tilde{v}_0}^{\tilde{v}} \left(\exp \left(\int_{\tilde{v}_0}^{\tilde{v}'} \frac{2M}{r^2} d\tilde{v}'' \right) (\tilde{v}')^{-p} d\tilde{v}' \right) \geq b\tilde{v}^{-p}.$$

Combining (212) with (209) finishes the proof. \square

REMARK 11.2.2. – *If we added the assumption that $(1 - \frac{2M}{r})^{-1} \partial_{\tilde{u}} \psi$ was uniformly bounded, then for sufficiently large \tilde{v} this proposition would prove that $(1 - \frac{2M}{r})^{-1} \partial_{\tilde{u}} \psi \geq b\tilde{v}^{-p}$. Cf. [18].*

We now have

PROPOSITION 11.2.4. – Let ψ be as in Theorem 11.1, $\tilde{\psi}$ defined by (207), and \tilde{v}_0 be a sufficiently large constant. Then $(\tilde{u}, \tilde{v}) \in \{r = 2M + \epsilon\}$, $\epsilon > 0$ sufficiently small, and $\tilde{v} \geq \tilde{v}_0$ imply

$$r \left(1 - \frac{2M}{r}\right)^{-1} \partial_{\tilde{u}} \tilde{\psi} \Big|_{(\tilde{u}, \tilde{v})} \geq -B e^{-\frac{(1-B\epsilon)\tilde{v}}{4M}} \sqrt{\int_{\{\tilde{v}=\tilde{v}_0\} \cap \{r \leq 2M+\epsilon\}} \mathbf{J}_\mu^N[\psi] n_{\{\tilde{v}=\tilde{v}_0\}}^\mu} + b \tilde{v}^{-p+1}.$$

Proof. – Using that $(\tilde{u}, \tilde{v}) \in \{r = 2M + \epsilon\}$ implies that $\tilde{u} = \tilde{v} - (4M + 2\epsilon)^*$, applying Proposition 11.2.3 to ψ and integrating implies that

$$(213) \quad r \left(1 - \frac{2M}{r}\right)^{-1} \partial_{\tilde{u}} \tilde{\psi} \\ \geq -B \int_{\tilde{v}}^{\infty} \left| \exp\left(-\int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}'\right) \left(1 - \frac{2M}{r}\right)^{-1} \partial_{\tilde{u}} \psi \Big|_{(\tilde{v}-(4M+2\epsilon)^*, \tilde{v}_0)} \right| d\tilde{v} + b \tilde{v}^{-p+1}.$$

Now, we observe that a change of variables yields

$$(214) \quad \int_{\tilde{v}}^{\infty} \left| \exp\left(-\int_{\tilde{v}_0}^{\tilde{v}} \frac{2M}{r^2} d\tilde{v}'\right) \left(1 - \frac{2M}{r}\right)^{-1} \partial_{\tilde{u}} \psi \Big|_{(\tilde{v}-2(2M+\epsilon), \tilde{v}_0)} \right| d\tilde{v} \\ \leq \int_{2M}^{2M+Be^{-\frac{\tilde{v}}{2M}}} (r-2M)^{-B\epsilon} |Y\psi| \Big|_{\tilde{v}=\tilde{v}_0} dr.$$

Cauchy-Schwarz then gives us

$$(215) \quad \int_{2M}^{2M+Be^{-\frac{\tilde{v}}{2M}}} (r-2M)^{-B\epsilon} |Y\psi| \Big|_{\tilde{v}=\tilde{v}_0} dr \leq B e^{-\frac{(1-B\epsilon)\tilde{v}}{4M}} \sqrt{\int_{\{\tilde{v}=\tilde{v}_0\} \cap \{r \leq 2M+\epsilon\}} \mathbf{J}_\mu^N[\psi] n_{\{\tilde{v}=\tilde{v}_0\}}^\mu}.$$

□

11.2.4. *Positivity on \mathcal{I}^+ and the contradiction.* – Finally, we will show that if $\partial_{\tilde{v}} \tilde{\psi}$ and $\partial_{\tilde{u}} \tilde{\psi}$ are eventually positive along $\{r = 2M + \epsilon\}$, then the null derivatives of $\tilde{\psi}$ must eventually be positive in a neighbourhood of \mathcal{I}^+ .

PROPOSITION 11.2.5. – Let ψ be as in Theorem 11.1 and define $\tilde{\psi}$ by (207). Additionally, let us assume that $\int_{\{\tilde{v}=\tilde{v}_0\} \cap \{r \leq 2M+\epsilon\}} \mathbf{J}_\mu^N[\psi] n_{\{\tilde{v}=\tilde{v}_0\}}^\mu < \infty$. Then, there exists a constant c such that

$$(216) \quad r \partial_{\tilde{v}} \tilde{\psi} \geq c \tilde{v}^{-p+1}, \quad r \partial_{\tilde{u}} \tilde{\psi} \geq c \tilde{u}^{-p+1},$$

for all sufficiently large r and \tilde{v} .

Proof. – Propositions 11.2.2 and 11.2.4 imply that we may find $r_0 > 2M$ and $\tilde{v}_0 < \infty$ such that $\tilde{v} \geq \tilde{v}_0$ and $(\tilde{u}, \tilde{v}) \in \{r^* = r_0^*\}$ implies that there exists a constant $c > 0$ such that

$$(217) \quad r \partial_{\tilde{v}} \tilde{\psi} \geq c \tilde{v}^{-p+1}, \quad r \partial_{\tilde{u}} \tilde{\psi} \geq c \tilde{u}^{-p+1}.$$

Now, we define

$$\mathcal{A} \doteq \{s \in [0, \infty] : \tilde{v} \geq \tilde{v}_0 \text{ and } r^*(\tilde{u}, \tilde{v}) \in [r_0^*, r_0^* + s) \Rightarrow r \partial_{\tilde{v}} \tilde{\psi} \geq c \tilde{v}^{-p+1} \text{ and } r \partial_{\tilde{u}} \tilde{\psi} \geq c \tilde{u}^{-p+1}\},$$

where we emphasize that c is the constant from (217). The proof will be finished if we can prove that $\mathcal{A} = [0, \infty)$.

It is clear that \mathcal{A} is a closed and non-empty subset of $[0, \infty)$, so it suffices to prove that \mathcal{A} is open. Suppose that $s_0 \in \mathcal{A}$. We show that $s_0 + \epsilon \in \mathcal{A}$ for $\epsilon > 0$ sufficiently small.

It immediately follows from the transport equations (193) and (194) and Corollary 11.2.1 that ϵ sufficiently small implies that for all $r^* \in [r_0^* + s_0, r_0^* + s_0 + \epsilon]$ and $\tilde{v} \geq \tilde{v}_0$ we have

$$(218) \quad r \partial_{\tilde{v}} \tilde{\psi} \geq \frac{1}{2} c \tilde{v}^{-p+1}, \quad r \partial_{\tilde{u}} \tilde{\psi} \geq \frac{1}{2} c \tilde{u}^{-p+1}.$$

Given these estimates, we integrate again the transport equations (193) and (194) and now use (218) to determine that in the region $r^* \in [r_0^* + s_0, r_0^* + s_0 + \epsilon]$ and $\tilde{v} \geq \tilde{v}_0$, $r \partial_{\tilde{v}} \tilde{\psi}$ is monotonically increasing in $-\tilde{u}$ and $r \partial_{\tilde{u}} \tilde{\psi}$ is monotonically increasing in \tilde{v} . We conclude that $r^* \in [r_0^* + s_0, r_0^* + s_0 + \epsilon]$ and $\tilde{v} \geq \tilde{v}_0$ imply (216). \square

The next corollary establishes the desired contradiction and thus concludes the proof of Theorem 11.1.

COROLLARY 11.2.2. – *Let ψ be as in Theorem 11.1 and define $\tilde{\psi}$ by (207). Then, for each sufficiently large τ_1 ,*

$$\lim_{r \rightarrow \infty} r \psi(r, \tau_1) < 0.$$

Proof. – Using Proposition 11.2.5 and the facts $\partial_{\tilde{u}} + \partial_{\tilde{v}} = T$ and $T \tilde{\psi} = -\psi$ we find that

$$-r \psi = r T \tilde{\psi} = r \partial_{\tilde{u}} \tilde{\psi} + r \partial_{\tilde{v}} \tilde{\psi} \geq c (\tilde{u}^{-p+1} + \tilde{v}^{-p+1}).$$

The result follows since $\lim_{r \rightarrow \infty} \tilde{u}(r, \tau_1)$ is bounded above. \square

11.3. Construction of ψ using the degenerate T -scattering theory

We now apply our degenerate scattering theory of Theorem 9.2.1 (see also Section 9.6) to indeed construct solutions ψ as in the statement of Theorem 11.1. Let $\psi_{\mathcal{H}_{\geq 0}^+} : \mathcal{H}_{\geq 0}^+ \rightarrow \mathbb{R}$ denote the function

$$\psi_{\mathcal{H}_{\geq 0}^+}(t^*) = \frac{(t^* + 1)^{-p+1}}{-p + 1}$$

for $p > 2$.

PROPOSITION 11.3.1. – *Let $a = 0$. For $\psi_{\mathcal{H}_{\geq 0}^+} \oplus 0 \in \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N \oplus \mathcal{E}_{\mathcal{I}^+}^T \subset \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^T \oplus \mathcal{E}_{\mathcal{I}^+}^T$ above, the solution $\mathcal{B}_-(\psi_{\mathcal{H}_{\geq 0}^+}, 0)$ satisfies all of the hypothesis of Theorem 11.1.*

Proof. – Let us set $\psi \doteq \mathcal{B}_-(\psi_{\mathcal{H}_{\geq 0}^+}, 0)$. We first note that the spherical symmetry of Schwarzschild, Theorem 9.2.1, and commutations with T and Ω^α are easily seen to imply that ψ is a smooth spherically symmetric solution in $\tilde{\mathcal{R}}$ and that

$$\int_{\Sigma_0^*} (\mathbf{J}_\mu^T[\psi] + \mathbf{J}_\mu^T[T\psi]) n_{\Sigma_0^*}^\mu < \infty.$$

Next, we observe that (193), the fundamental theorem of calculus, Cauchy Schwarz and an easy density argument show that $\partial_{\tilde{v}} \psi$ and $\partial_{\tilde{v}} (T\psi)$ extend continuously to the functions $(1 + t^*)^{-p}$ and $-p(1 + t^*)^{-p-1}$ respectively along $\mathcal{H}_{\geq 0}^+$.

In order to establish that $\lim_{r \rightarrow \infty} r\psi|_{S_\tau} = 0$ we first observe the unitarity property of Theorem 9.6.1 yields

$$(219) \quad \int_{\Sigma_t^*} \mathbf{J}_\mu^T[\psi] n_{\Sigma_t^*}^\mu = \int_{\mathcal{H}_{\geq t}^+} \mathbf{J}_\mu^T[\psi] n_{\mathcal{H}_{\geq t}^+}^\mu \leq B(1+t)^{-2p+1}.$$

Next, for r sufficiently large, the fundamental theorem of calculus implies

$$(220) \quad |\psi(t, r)| \leq \int_r^\infty |\partial_r \psi| dr \leq r^{-1/2} \sqrt{\int_r^\infty (\partial_r \psi)^2 r^2 dr} \\ \leq r^{-1/2} \sqrt{\int_{\Sigma_t^*} \mathbf{J}_\mu^T[\psi]} \leq Br^{-1/2}(1+t)^{-p+1/2}.$$

Since r is comparable to t along any fixed hypersurface S_τ , and $p > 2$, the estimate (220) immediately implies that

$$\lim_{r \rightarrow \infty} r\psi|_{S_\tau} = 0. \quad \square$$

We immediately obtain the following corollary.

COROLLARY 11.1. – *Let $a = 0$. For $\psi_{\mathcal{H}_{\geq 0}^+} \oplus 0 \in \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N \oplus \mathcal{E}_{\mathcal{I}^+}^T \subset \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^T \oplus \mathcal{E}_{\mathcal{I}^+}^T$ above, then the map \mathcal{B}_- of Theorem 9.2.1 maps*

$$\mathcal{B}_-(\psi_{\mathcal{H}_{\geq 0}^+}, 0) \in \mathcal{E}_{\Sigma_0^*}^T \setminus \mathcal{E}_{\Sigma_0^*}^N.$$

More pedestrianly,

COROLLARY 11.2. – *There indeed exists ψ as in Theorem 11.1.*

We note that by what we have shown in Proposition 11.2.1, ψ has several nice additional properties. In particular, we have the following decay result.

COROLLARY 11.3. – *Let $\psi_{\mathcal{H}_{\geq 0}^+}$ be as in Corollary 11.1. Then, for every $R < \infty$ we have*

$$\int_{S_\tau \cap \{r \leq R\}} \left[\left| T_{\mathcal{B}_-}(\psi_{\mathcal{H}_{\geq 0}^+}, 0) \right|^2 + \left| \partial_{r^*} \mathcal{B}_-(\psi_{\mathcal{H}_{\geq 0}^+}, 0) \right|^2 \right] \leq B(R)(1+\tau)^{-2p} \quad \forall \tau \geq 0.$$

These strong decay properties lend further support to Conjecture 2.5.

11.4. Non-surjectivity of the N -energy forward map

Lastly, we can immediately reinterpret Corollary 11.1 as a non-surjectivity result (cf. the discussion in Section 2.3.2).

COROLLARY 11.4. – *Let $a = 0$. The asymptotic state $\psi_{\mathcal{H}_{\geq 0}^+} \oplus 0$ is not in the image of the map $\mathcal{F}_+ : \mathcal{E}_{\Sigma_0^*}^N \rightarrow \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N \oplus \mathcal{E}_{\mathcal{I}^+}^T$. Thus, the map of Theorem 8.2.1 is not surjective, in fact, the image $\mathcal{F}_+(\mathcal{E}_{\Sigma_0^*}^N)$ has infinite co-dimension in $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N \oplus \mathcal{E}_{\mathcal{I}^+}^T$ and infinite codimension when intersected with $\mathcal{E}_{\mathcal{H}_{\geq 0}^+}^{i_n N} \oplus 0$ for any $n \geq 0$.*

Proof. – If $\psi_{\mathcal{H}_{\geq 0}^+} \oplus 0$ was in the image of the map $\mathcal{F}_+ : \mathcal{E}_{\Sigma_0^*}^N \rightarrow \mathcal{E}_{\mathcal{H}_{\geq 0}^+}^N \oplus \mathcal{E}_{\mathcal{I}^+}^T$, then Theorem 9.2.1 and Corollary 11.1 would immediately yield a contradiction. \square

We have thus obtained the final remaining Theorem 2 of Section 2.3.2.

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