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ZEROS OF DIRICHLET L-FUNCTIONS

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Table of contents

Introduction

Let χ be a Dirichlet character and $L(s, \chi)$ the associated Dirichlet L-function. We are interested in the zeroes of $L(s, \chi)$ in the critical strip $0<\mathrm{Re}(s)<1$. In the past, most attention has focussed on this question near $s=1$. We shall be particularly interested in the situation near $s=1/2$.

It follows from classical results of Landau, Page and others (*see* Davenport [D] for example) that the number of real characters χ of conductor $\leqq x$ for which $L(s, \chi)$ has a real zero in the region $1-(1/\log x)\leqq\sigma\leqq 1$ is $O(\log \log x)$. On the other hand, the situation near $s=1/2$ is more delicate and not as well understood.

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Several authors have studied the frequency with which $L(1/2, \chi) \neq 0$. In [B] it is shown that there are at least $cq/(\log q)^{1000}$ characters $\chi \pmod q$ with $L(1/2, \chi) \neq 0$. (Here, and elsewhere, c is a positive constant, though not necessarily the same constant at different occurrences.) In another direction, we can allow both χ and q to vary while we fix the order of χ. A result of Jutila [J] implies that there are at least $cx/(\log x)$ *real* characters χ, of conductor at most x, for which $L(1/2, \chi) \neq 0$.

In both of these works, the method is to study the moments

$$\Sigma \, |L(1/2, \chi)|^k.$$

For example, by the Cauchy-Schwarz inequality,

$$\# \left\{ \chi \bmod q) : L\left(\frac{1}{2}, \chi\right) \neq 0 \right\} \gg \frac{(\sum |L(1/2, \chi)|)^2}{\sum |L(1/2, \chi)|^2}.$$

From this, we see that it would suffice to have a lower bound for $\sum |L(1/2, \chi)|$ and an upper bound for $\sum |L(1/2, \chi)| \sum |L(1/2, \chi)|^2$.

There are general conjectures which predict, in particular, the asymptotic growth of the above moments. However, even assuming these conjectures, it does not seem possible to use the Cauchy-Schwarz inequality to deduce that $L(1/2, \chi) \neq 0$ for a *positive proportion* of the characters χ to a given modulus, or that $L(1/2, \chi) \neq 0$ for a positive proportion of real characters χ. This result may be viewed as a (partial) q-analogue of theorems of Levinson-Selberg type.

On the other hand, no example is known of a character χ for which $L(1/2, \chi) = 0$. However, Siegel [S] has shown the fundamental result that any point on the line $\sigma = 1/2$ is a limit point of zeroes of the $L(s, \chi)$ as χ ranges over *all* Dirichlet characters.

In this paper, we take a different approach from [B] and [J]. We consider characters to a prime modulus q. Our first main result is the following.

THEOREM. — *Let q be a sufficiently large prime. Then, for a positive proportion of the characters $\chi \pmod q$, we have $L(1/2, \chi) \neq 0$.*

Our proof shows that the proportion is $\geqq .04$. (Using the explicit formula, Ram Murty [RM] has shown that this proportion can be improved to $\geqq .5$ if we assume the Riemann Hypothesis.) Our method actually produces a more general result (Theorem 11.1) which applies to any point $1/2 \leqq \sigma < 1$.

Our second main result (Theorem 12.1) gives a non-vanishing theorem which is uniform on a line segment.

THEOREM. — *Let q be a sufficiently large prime. For a positive proportion of the $\chi \pmod q$, there are no real zeroes of* $L(s, \chi)$ in the region $(1/2) + (c/\log q) \leqq \sigma < 1$. *Here, $c > 0$ is an absolute constant.*

In proving our results, our new idea is to count the desired characters directly, *without* the intermediary of moments of L-functions. Let χ be a non-trivial character. Using weights $\{\lambda(n)\}$ first defined by Barban and Vehov [BV], we consider a mollifier polynomial

$$M(s, \chi) = \sum_{n \leqq Z} \lambda(n)\chi(n) n^{-s}$$

where $Z = q^{1/2}$. The $\lambda(n)$ (which are closely related to Selberg's sieve) will be chosen with the property that if we set

$$a(n) = \sum_{d \mid n} \lambda(d),$$

then $a(1) = 1$ and $a(n) = 0$ for $1 < n < Y$ for some $1 \leqq Y < Z$. It turns out that to prove our non-vanishing result at a fixed point, the particular choice of Y is not so crucial and we could take $Y = 1$ if we wished. In the proof of the non-vanishing result on an interval, however, we need to take Y to be a power of q. We choose $Y = q^{1/4}$. Then, we consider the integral

$$\frac{1}{2\pi i}\int_{(2)} L(s+w, \chi) M(s+w, \chi) X^w \Gamma(w)\, dw$$

where we choose $X = q$. On the one hand it is equal to

$$S(s, \chi) = \sum \frac{a(n)\chi(n)}{n^s} e^{-n/X}$$

and on the other, it is

$$L(s, \chi) M(s, \chi) + \frac{1}{2\pi i}\int_{(-\eta)} L(s+w, \chi) M(s+w, \chi) X^w \Gamma(w)\, dw$$

where $\eta > 0$ is chosen appropriately. Now if χ is a primitive character, we can apply the functional equation to transform the integral into

$$\frac{1}{2\pi i}\int_{(-\eta)} L(1-s-w, \bar{\chi}) M(s+w, \chi) \gamma(s+w, \chi) X^w \Gamma(w)\, dw$$

where $\gamma(s, \chi)$ is an appropriate quotient of Γ-functions. Now if we have $\eta > \sigma$, we can expand $L(1-s-w, \bar{\chi})$ as a Dirichlet series. Splitting it into a Dirichlet polynomial of length Z and a tail, we get two integrals $I(s, \chi)$ and $J(s, \chi)$. Thus our basic equation is

$$S(s, \chi) = L(s, \chi) M(s, \chi) + I(s, \chi) + J(s, \chi).$$

If $L(s_0, \chi) = 0$ then $S(s_0, \chi)$ is equal to $I(s_0, \chi) + J(s_0, \chi)$. We show that this cannot happen too often by comparing mean-square estimates of $S(s_0, \chi)$, $I(s_0, \chi)$ and $J(s_0, \chi)$. Thus, we obtain a lower bound for the numer of $\chi \pmod q$ with $L(s_0, \chi) \neq 0$. We then extend this to a lower bound for the number of $\chi \pmod q$ for which $L(s, \chi) \neq 0$ in a circle of radius $(\log q)^{-1}$ about s_0. Equivalently, we obtain an upper bound for the number of $\chi \pmod q$ for which $L(s, \chi)$ does vanish in this circle. This bound decreases exponentially with $(\Re s_0)-(1/2)$. Choosing the point $s_0=(1/2)+j(\log q)^{-1}$ and summing over j produces our non-vanishing result on an interval.

The estimates for S and J are given in § 3 and § 4. The mean square of I is determined in § 10, after preparations in § 5-§ 9. The main results are proved in § 11 and § 12. For an exposition of some of the results and techniques of this paper, the reader may consult [KM].

It is a pleasure to thank J. Friedlander, M. Jutila, and R. Murty for encouraging and helpful discussions. We would also like to thank the referee for a careful reading of the manuscript.

NOTATION. — $\sum_{\chi \pmod q}$ denotes a sum over characters mod q. We denote by $\mathbf{d}(n)$ the number of positive divisors of n and for $r \in \mathbf{R}$, $\sigma_r(n)$ denotes the sum $\sum_{d \mid n} d^r$.

1. THE BARBAN-VEHOV WEIGHTS. — Let $1 \leqq z_1 \leqq z_2$. Following Barban and Vehov [BV], we introduce the functions

$$\Lambda_i(n)=\begin{cases} \mu(n) \log(z_i/n) & \text{if} \quad n \leqq z_i \\ 0 & \text{if} \quad n > z_i, \end{cases}$$

for $i=1, 2$. We also define

$$\lambda(n)=\frac{\Lambda_2(n)-\Lambda_1(n)}{\log(z_2/z_1)}$$

$$(1.1) \qquad =\begin{cases} \mu(n) & 1 \leqq n \leqq z_1 \\ \mu(n)\dfrac{\log(z_2/n)}{\log(z_2/z_1)} & z_1 \leqq n \leqq z_2 \\ 0 & n > z_2. \end{cases}$$

Let us define

$$a(n)=\sum_{d \mid n} \lambda(d).$$

Graham [Gr] has found asymptotic estimates for the mean square of the $a(n)$. We recall his main result.

PROPOSITION (1.1). — *We have*

$$\sum_{n \leqq N} |a(n)|^2=\begin{cases} \dfrac{N \log(N/z_1)}{\log^2(z_2/z_1)}+\mathbf{O}\left(\dfrac{N}{\log^2(z_2/z_1)}\right) & \text{if} \quad z_1 < N < z_2 \\ \dfrac{N}{\log(z_2/z_1)}+\mathbf{O}\left(\dfrac{N}{\log^2(z_2/z_1)}\right) & \text{if} \quad z_2 \leqq N. \end{cases}$$

Applying the Cauchy-Schwarz inequality and Proposition (1.1), we deduce the following.

PROPOSITION (1.2). — *Let* $r \leqq N$ *and* $(b, r)=1$. *We have*

$$\sum_{\substack{n \leqq N \\ n \equiv b \ (\mathrm{mod}\ r)}} |a(n)| \ll \frac{N}{\varphi(r)^{1/2} (\log z_2/z_1)^{1/2}}.$$

We next obtain an estimate for a shifted convolution.

PROPOSITION (1.3). — *Let* $1 \leqq k \in \mathbf{Z}$, $t \in \mathbf{R}$ *and* $k \leqq M < N$. *Then we have*

$$\sum_{M < n \leqq N} a(n) a(n-k) \left(\frac{n}{n-k}\right)^{it} \ll \frac{k}{\varphi(k)} \left(\frac{N+z_2^2}{(\log z_2/z_1)^2} |P(t)| + \frac{N |t|^4 (\log z_2)^4}{(\log z_2/z_1)^2}\right)$$

where $P(t)$ *is a polynomial in* t *(depending on* k*) with complex bounded coefficients and of degree* $\leqq 4$.

The proof will require two preliminary results. We begin by recalling a result from Graham [Gr, Lemma 2].

LEMMA (1.4). — *For any integer* r, *and any* $c>0$,

$$\sum_{\substack{n \leqq Q \\ (n, r)=1}} \frac{\mu(n)}{n} \log\left(\frac{Q}{n}\right) = \frac{r}{\varphi(r)} + \mathbf{O}_c(\sigma_{-1/2}(r) \log^{-c}(2Q)).$$

LEMMA (1.5). — *We have for* $1 \leqq d_1, d_2 \leqq z_2$ *and* $r_1, r_2 \geqq 1$ *that*

$$\sum_{\substack{1 \leqq j_1 \leqq z_1/d_1,\ 1 \leqq j_2 \leqq z_2/d_2 \\ (j_1, j_2)=(j_1, r_1)=(j_2, r_2)=1}} \frac{\Lambda_1(d_1 j_1) \Lambda_2(d_2 j_2)}{j_1 j_2}$$

$$\ll \left(\frac{d_1 r_1}{\varphi(d_1 r_1)} + \sigma_{-1/2}(d_1 r_1)\right)\left(\frac{d_2 r_2}{\varphi(d_2 r_2)} + \sigma_{-1/2}(d_2 r_2)\right).$$

The same estimate holds even if we drop the condition that $(j_1, j_2)=1$.

Proof. — The sum in question is

$$\sum \frac{\Lambda_1(d_1 j_1) \Lambda_2(d_2 j_2)}{j_1 j_2} \sum_{e \mid (j_1, j_2)} \mu(e) = \sum_{e \leqq z_1/d_1} \mu(e) \sum \frac{\Lambda_1(d_1 j_1) \Lambda_2(d_2 j_2)}{j_1 j_2},$$

the inner sum ranging over j_1, j_2 satisfying

$$1 \leqq j_1 \leqq z_1/d_1, \qquad 1 \leqq j_2 \leqq z_2/d_2$$
$$j_1 j_2 \equiv 0 \ (\mathrm{mod}\ e), \qquad (j_1\ r_1)=(j_2, r_2)=1.$$

Let us set $r=r_1 r_2$ and $d=d_1 d_2$. Then the sum is seen to be

$$\sum_{\substack{e \leqq z_1/d_1 \\ (e,r)=1}} \frac{\mu(e)}{e^2} \sum_{\substack{l_1 \leqq z_1/d_1 e,\ l_2 \leqq z_2/d_2 e \\ (l_1, r_1)=(l_2, r_2)=1}} \frac{\Lambda_1(d_1 e l_1) \Lambda_2(d_2 e l_2)}{l_1 l_2}$$

$$= \mu(d_1)\mu(d_2) \sum_{\substack{e \leqq z_1/d_1 \\ (e, dr)=1}} \frac{\mu(e)}{e^2} \left\{ \sum_{\substack{l_1 \leqq z_1/d_1 e \\ (l_1, d_1 e r_1)=1}} \frac{\mu(l_1) \log(z_1/d_1 e l_1)}{l_1} \right\}$$

$$\times \left\{ \sum_{\substack{l_2 \leqq z_2/d_2 e \\ (l_2, d_2 e r_2)=1}} \frac{\mu(l_2) \log(z_2/d_2 e l_2)}{l_2} \right\}$$

$$= \mu(d_1)\mu(d_2) \sum_{\substack{e \leqq z_1/d_1 \\ (e, dr)=1}} \frac{\mu(e)}{e^2} \prod_{k=1}^{2} \left\{ \frac{d_k e r_k}{\varphi(d_k e r_k)} + \mathbf{O}_c\left(\sigma_{-1/2}(d_k e r_k) \log^{-c}\left(\frac{2 z_k}{d_k e}\right)\right)\right\}$$

using Lemma (1.4).

The main terms contribute an amount

$$\frac{\mu(d_1)\mu(d_2)\, dr}{\varphi(d_1 r_1)\varphi(d_2 r_2)} \cdot \sum_{\substack{e \leqq z_1/d_1 \\ (e, dr)=1}} \frac{\mu(e)}{\varphi(e)^2} \ll \frac{dr}{\varphi(d_1 r_1)\varphi(d_2 r_2)}.$$

The product of the **O**-terms contributes an amount

$$\ll \sum \frac{1}{e^2} \sigma_{-1/2}(d_1 r_1)\, \sigma_{1/2}(d_2 r_2)\, \sigma_{-1/2}(e)^2 \ll \sigma_{-1/2}(d_1 r_1)\, \sigma_{1/2}(d_2 r_2).$$

The cross-terms contribute an amount

$$\ll \sum_{\substack{e \leqq z_1/d_1 \\ (e, dr)=1}} \frac{1}{e^2} \left\{ \frac{d_1 e r_1}{\varphi(d_1 e r_1)} \cdot \sigma_{-1/2}(d_2 e r_2) \log^{-c}\left(\frac{2 z_2}{d_2 e}\right)\right.$$

$$\left. + \frac{d_2 e r_2}{\varphi(d_2 e r_2)} \cdot \sigma_{-1/2}(d_1 e r_1) \log^{-c}\left(\frac{2 z_1}{d_1 e}\right)\right\}$$

$$\ll \left\{ \frac{d_1 r_1}{\varphi(d_1 r_1)}\, \sigma_{-1/2}(d_2 r_2) + \frac{d_2 r_2}{\varphi(d_2 r_2)}\, \sigma_{-1/2}(d_1 r_1) \right\}$$

since the series $\sum \sigma_{-1/2}(e)/e\,\varphi(e)$ converges. This proves the first statement. The second statement is easy to verify since there is now no condition relating j_1 and j_2. We argue as above setting $e=1$.

Now we are ready to prove the estimate of the shifted convolution.

Proof of Proposition 1.3. — Again, we consider the sum

$$\sum_{M<n\leqq N} \Big(\sum_{d\mid n} \Lambda_1(d)\Big)\Big(\sum_{e\mid n-k} \Lambda_2(e)\Big)\left(\frac{n}{n-k}\right)^{it} \tag{1.2}$$

and we find that it is equal to

$$\sum_{d,e} \Lambda_1(d)\Lambda_2(e) \sum_{\substack{M<n\leqq N\\ n\equiv 0\ (\mathrm{mod}\ d)\\ n\equiv k\ (\mathrm{mod}\ e)}} \left(\frac{n}{n-k}\right)^{it}. \tag{1.3}$$

We see that the inner sum is zero unless $(d, e)\,|\,k$. Consider the identity

$$\left(\frac{n}{n-k}\right)^{it}=(1+k)^{it}\left(1-\frac{k}{1+k}\left(1-\frac{1}{n-k}\right)\right)^{it}.$$

We have an expansion

$$\left(\frac{n}{n-k}\right)^{it}=(1+k)^{it}\sum_{j=0}^{4} \mathrm{P}_j(t)(n-k)^{-j}+\mathbf{O}(|t|^4)$$

where $\mathrm{P}_j(t)$ is a polynomial in t of degree $\leqq 3$ with complex coefficients which are absolutely bounded and depend on k. Using this, we see that

$$\sum_{\substack{M<n\leqq N\\ n\equiv 0\ (\mathrm{mod}\ d)\\ n\equiv k\ (\mathrm{mod}\ e)}} \left(\frac{n}{n-k}\right)^{it}=(1+k)^{it}\sum_{j=0}^{4} \mathrm{P}_j(t) \sum_{\substack{M<n\leqq N\\ n\equiv 0\ (\mathrm{mod}\ d)\\ n\equiv k\ (\mathrm{mod}\ e)}} (n-k)^{-j}+\mathbf{O}\left(|t|^4 \sum_{\substack{M<n\leqq N\\ n\equiv 0\ (\mathrm{mod}\ d)\\ n\equiv k\ (\mathrm{mod}\ e)}} 1\right). \tag{1.4}$$

Inserting this into (1.3), we get a main term of

$$(1+k)^{it}\sum_{j=0}^{4} \mathrm{P}_j(t)\sum_{d,e} \Lambda_1(d)\Lambda_2(e) \sum_{\substack{M<n\leqq N\\ n\equiv 0\ (\mathrm{mod}\ d)\\ n\equiv k\ (\mathrm{mod}\ e)}} (n-k)^{-j}. \tag{1.5}$$

If $j=0$, the innermost sum is

$$\frac{N-M}{[d, e]}+\mathbf{O}(1)$$

and if $j=1$, it is

$$\frac{\log((N-k)/(M-k))+\mathbf{O}(1)}{[d, e]}.$$

For $j \geqq 2$ it is $\mathbf{O}(1)$. Thus, (1.5) is

$$(1.6)\quad (1+k)^{it}\left(P_0(t)(N-M)+P_1(t)\log\left(\frac{N-k}{M-k}\right)\right)\sum_{\substack{d,\, e\\ (d,\, e)\mid k}}\frac{\Lambda_1(d)\Lambda_2(e)}{[d,\, e]} + \mathbf{O}\Big((|t|+1)^3\sum_{\substack{d,\, e\\ (d,\, e)\mid k}}|\Lambda_1(d)\Lambda_2(e)|\Big)$$

The $\mathbf{O}$-term is easily seen to be

$$\ll z_1 z_2 (|t|+1)^3.$$

To evaluate the main term, we see that the sum over d, e is

$$(1.7)\qquad \sum_{\substack{d,\, e\\ (d,\, e)\mid k}}\frac{\Lambda_1(d)\Lambda_2(e)}{de}\sum_{\substack{m\mid d\\ m\mid e}}\varphi(m).$$

This is seen to be equal to

$$\sum_{m\mid k}\frac{\varphi(m)}{m^2}\sum_{d_0,\, e_0}\frac{\Lambda_1(m d_0)\Lambda_2(m e_0)}{d_0 e_0}.$$

Here, the inner sum ranges over pairs d_0, e_0 satisfying

$$1\leqq d_0\leqq \frac{z_1}{m},\qquad 1\leqq e_0\leqq\frac{z_2}{m}$$
$$(d_0,\, m)=(e_0,\, m)=1.$$

Also note that in the outer sum m must be squarefree for otherwise $\Lambda_1(m d_0)=\Lambda_2(m e_0)=0$. Thus, invoking Lemma (1.5), we find that the main term in (1.7) is

$$\ll\sum_{m\mid k}\frac{\mu^2(m)\varphi(m)}{m^2}\left(\frac{m}{\varphi(m)}+\sigma_{-1/2}(m)\right)^2$$
$$\ll\frac{k}{\varphi(k)}.$$

Hence the main term in (1.6) is

$$\ll\frac{k}{\varphi(k)}(|P_0(t)|N+|P_1(t)|\log N).$$

Summarizing, the main term of (1.4) contributes to (1.3) an amount

$$\ll\frac{k}{\varphi(k)}(|P_0(t)|N+|P_1(t)|\log N)+z_1 z_2(|t|+1)^3.$$

The error term in (1.4) contributes to (1.3) an amount

$$\ll \mathrm{N}\,|t|^4 \sum_{\substack{d,\, e \\ (d,\, e) \mid k}} \frac{|\Lambda_1(d)\Lambda_2(e)|}{[d,\, e]} + |t|^4 z_1 z_2.$$

The first term above is estimated by

$$\sum_{\substack{d,\, e \\ (d,\, e) \mid k}} \frac{|\Lambda_1(d)\Lambda_2(e)|}{[d,\, e]} \ll \sum_{m \mid k} \frac{\varphi(m)\mu(m)^2}{m^2} \left(\sum_{d_0} \frac{1}{d_0} \log \frac{z_1}{m d_0}\right)\left(\sum_{e_0} \frac{1}{e_0} \log \frac{z_2}{m e_0}\right)$$

$$\ll \sum_{m \mid k} \frac{\varphi(m)\mu(m)^2}{m^2} \left(\log \frac{z_1}{m}\right)^2 \left(\log \frac{z_2}{m}\right)^2$$

$$\ll \frac{k}{\varphi(k)} (\log z_1)^2 (\log z_2)^2.$$

Summarizing, the error term in (1.4) contributes to (1.3) an amount

$$\ll \frac{k}{\varphi(k)} |t|^4 \mathrm{N} (\log z_1)^2 (\log z_2)^2 + z_1 z_2 |t|^4.$$

The Proposition follows.

2. THE MOLLIFIER POLYNOMIAL. — We shall now introduce the following parameters. Let us set

$$\mathrm{Y} = (\log q)$$
$$\mathrm{Z} = q^{1/2}$$

Corresponding to the choices $z_1 = \mathrm{Y}$ and $z_2 = \mathrm{Z}$, we have from § 1 the weights

$$\lambda(n) = \frac{\Lambda_2(n) - \Lambda_1(n)}{\log(\mathrm{Z}/\mathrm{Y})}.$$

We define the Dirichlet polynomial

$$\mathrm{M}(s,\, \chi) = \sum_{n \leqq \mathrm{Z}} \frac{\lambda(n)\chi(n)}{n^s}$$

where χ is a Dirichlet character. Then, we have

$$\mathrm{L}(s,\, \chi)\,\mathrm{M}(s,\, \chi) = \sum_{n=1}^{\infty} \frac{a(n)\chi(n)}{n^s}$$

where

$$a(n) = \sum_{d \mid n} \lambda(d)$$

satisfies

$$a(1)=1$$
$$a(n)=0 \qquad \text{for} \quad 1<n \leqq Y.$$

We record the following estimate.

LEMMA (2.1). — *For* $|\sigma|<1/2$, *and* σ *bounded away from* $1/2$, *we have*

$$\sum_{\chi \,(\mathrm{mod}\, q)} |M(s, \chi)|^2 \ll \frac{(q+Z)}{(1-2\sigma)} \cdot \left(q^{1/2-\sigma} \cdot \frac{1}{(\log q)^2} + Y^{1-2\sigma}\right).$$

Proof. — We use the large sieve inequality [D] to get

$$\sum_{\chi \,(\mathrm{mod}\, r)} |M(s, \chi)|^2 \ll (Z+q) \sum_{n \leqq Z} \frac{|\lambda(n)|^2}{n^{2\sigma}}$$
$$\ll (q+Z)\left\{\sum_{n \leqq Y} \frac{1}{n^{2\sigma}} + \sum_{Y<n \leqq Z} \left(\frac{\log Z/n}{\log Z/Y}\right)^2 \cdot \frac{1}{n^{2\sigma}}\right\}$$
$$\ll (q+Z)\left\{\frac{Y^{1-2\sigma}}{1-2\sigma} + \frac{Z^{1-2\sigma}}{1-2\sigma} \cdot \frac{1}{(\log Z/Y)^2}\right\}.$$

The result follows from our choices of Y and Z.

3. THE BASIC EQUATION. — Let us define

$$S(s, \chi) = S(s, \chi, q) = \sum_{n=1}^{\infty} \frac{a(n)\chi(n)}{n^S} e^{-n/q}.$$

Let $s \in \mathbf{C}$ with $1>\sigma=\mathrm{Re}(s) \geqq 1/2$. Using the well-known identity

$$\frac{1}{2\pi i} \int_{(2)} X^w \Gamma(w)\, dw = e^{-1/X},$$

we find that for a character χ,

$$S(s, \chi) = \frac{1}{2\pi i} \int_{(2)} L(s+w, \chi) M(s+w, \chi) q^w \Gamma(w)\, dw.$$

Moving the line of integration to the left, we find that

$$(3.1) \qquad S(s, \chi) = L(s, \chi) M(s, \chi) + \frac{1}{2\pi i} \int_{(-\eta)} L(s+w, \chi) M(s+w, \chi) q^w \Gamma(w)\, dw$$

where $\sigma<\eta<1$.

We can decompose the integral along the line $-\eta$ into two parts as follows. Suppose that χ is non-trivial. We apply the functional equation

$$L(s, \chi)=\gamma(s, \chi) L(1-s, \bar{\chi})$$

where

$$\gamma(s, \chi)=\frac{\tau(\chi)}{i^a q^{1/2}}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{2\pi}{q}\right)^{s-(1/2)} \sin\left(\frac{\pi}{2}(a+s)\right)\Gamma(1-s).$$

[Here $\tau(\chi)$ is the Gauss sum, $a=0$, 1 and $\chi(-1)=(-1)^a$.] Then we truncate the Dirichlet series expansion of $L(1-s-w, \bar{\chi})$ at Z. Let us set

$$I(s, \chi)=\frac{1}{2\pi i}\int_{(-\eta)} \gamma(s+w, \chi)\left\{\sum_{n<Z}\frac{\bar{\chi}(n)}{n^{1-s-w}}\right\} M(s+w, \chi) q^w \Gamma(w)\, dw$$

and

$$J(s, \chi)=\frac{1}{2\pi i}\int_{(-\eta)} \gamma(s+w, \chi)\left\{\sum_{n\geqq Z}\frac{\bar{\chi}(n)}{n^{1-s-w}}\right\} M(s+w, \chi) q^w \Gamma(w)\, dw$$

Thus, we get

$$S(s, \chi)=L(s, \chi) M(s, \chi)+I(s, \chi)+J(s, \chi). \tag{3.2}$$

If $L(s, \chi)=0$, then $S(s, \chi)$ and $I(s, \chi)+J(s, \chi)$ are equal. We will therefore try to show that, in general, they are *not* equal and for this purpose we study their mean values. We begin with $J(s, \chi)$ which is the easiest of the three to estimate.

PROPOSITION (3.1). — *For* $|\text{Im}\, s|<1$, *and* $0\leqq\sigma\leqq 1$, *we have*

$$\sum_{1\neq\chi \,(\text{mod}\, q)} |J(s, \chi)| \ll_\varepsilon \frac{q^{(3/2)-\sigma}}{\log q}.$$

Proof. — From Stirling's formula, we know that

$$\gamma(s, \chi) \ll (q(|s|+1))^{(1/2)-\sigma}.$$

Using this and the definition, we find that

$$\sum_{1\neq\chi\,(\text{mod}\, q)} |J(s, \chi)| \ll q^{(1/2)-\sigma+\eta} q^{-\eta} \sum_{\chi\,(\text{mod}\, q)} \int_{(-\eta)} (|w|+1)^{(1/2)-\sigma+\eta}\left|\sum_{n\geqq z}\frac{\overline{\chi(\eta)}}{n^{1-s-w}}\right| |M(s+w, \chi)| |\Gamma(w)| |dw|$$

which by a double application of the Cauchy-Schwarz inequality is

$$\ll q^{(1/2)-\sigma} \sum_{\chi \,(\mathrm{mod}\, q)} \left(\int (|w|+1)^{1-2\sigma+2\eta} \left| \sum_{n \geqq Z} \frac{\bar{\chi}(n)}{n^{1-s-w}} \right|^2 |\Gamma(w)| \, |dw| \right)^{1/2}$$
$$\times \left(\int |M(s+w, \chi)|^2 |\Gamma(w)| \, |dw| \right)^{1/2}$$

$$\ll q^{(1/2)-\sigma} \left(\sum_{\chi \,(\mathrm{mod}\, q)} \int (|w|+1)^{1-2\sigma+2\eta} \left| \sum_{n \geqq Z} \frac{\bar{\chi}(n)}{n^{1-s-w}} \right|^2 |\Gamma(w)| \, |dw| \right)^{1/2}$$
$$\times \left(\sum_{\chi \,(\mathrm{mod}\, q)} \int |M(s+w, \chi)|^2 |\Gamma(w)| \, |dw| \right)^{1/2}.$$

Using the large sieve inequality and Lemma (2.1), we find that

$$\sum_{1 \neq \chi \,(\mathrm{mod}\, q)} |J(s, \chi)| \ll q^{(1/2)-\sigma} \left\{ \sum_{n \geqq Z} (q+n) n^{2(\sigma-\eta-1)} \right\}^{1/2}$$
$$\times \left\{ \frac{(q+Z)}{1-2(\sigma-\eta)} \cdot \left(\frac{q^{(1/2)-\sigma+\eta}}{(\log q)^2} + Y^{1-2(\sigma-\eta)} \right) \right\}$$
$$\ll q^{(1/2)-\sigma} Z^{\sigma-\eta} \left\{ \frac{q}{Z} \frac{1}{|2(\sigma-\eta)-1|} + \frac{1}{|\sigma-\eta|} \right\}^{1/2}$$
$$\times \left\{ \frac{(q+Z)^{1/2}}{|2(\sigma-\eta)-1|^{1/2}} \frac{q^{1/2((1/2)-\sigma+\eta)}}{\log q} \right\}.$$

Now, let us choose η so that it satisfies

$$\frac{1}{4} > |\eta - \sigma| > \frac{1}{8} \text{ (say)}$$

if $\sigma < 3/4$.

We would then have

$$\sum_{1 \neq \chi \,(\mathrm{mod}\, q)} |J(s, \chi))| \ll \frac{q^{(3/2)-\sigma}}{\log q} \tag{3.3}$$

which proves the result.

4. THE MEAN AND MEAN SQUARE OF $S(s, \chi)$.

PROPOSITION (4.1). — *For any* $\varepsilon > 0$, *we have*

$$\sum_{\chi \,(\mathrm{mod}\, q)} S(s, \chi) = \varphi(q) + \mathbf{O}_\varepsilon(q^{1-\sigma+\varepsilon}).$$

Moreover, the same estimate holds if we sum only over non-trivial characters.

Proof. — By definition, we have that

$$\sum_{\chi\,(\mathrm{mod}\,q)} \mathrm{S}(s,\chi) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e^{-n/q} \sum_{\chi\,(\mathrm{mod}\,q)} \chi(n)$$
$$= \varphi(q) \sum_{\substack{n=1 \\ n \equiv 1\,(\mathrm{mod}\,q)}}^{\infty} \frac{a(n)}{n^s} e^{-n/q}.$$

Using the bound $|a(n)| \leqq d(n) \ll_\varepsilon n^\varepsilon$, we find that the sum is

$$e^{-1/q} + \mathbf{O}_\varepsilon\left(\frac{1}{q^{\sigma-\varepsilon}} \sum_{t=1}^{\infty} t^{\varepsilon-\sigma} \exp(-t)\right).$$

The **O**-term is

$$\ll_\varepsilon q^{-\sigma+\varepsilon}.$$

It thus follows that

$$\sum_{\chi\,(\mathrm{mod}\,q)} \mathrm{S}(s,\chi) = \varphi(q) + \mathbf{O}_\varepsilon(q^{1-\sigma+\varepsilon}).$$

Finally,

$$\mathrm{S}(s,1) = \sum_{(n,\,q)=1} \frac{a(n)}{n^s} e^{-n/q} \ll q^{1-\sigma+\varepsilon}$$

as before. This proves the result.

PROPOSITION (4.2). — *We have*

$$\sum_{\chi\,(\mathrm{mod}\,q)} \left|\mathrm{S}\left(\frac{1}{2}+it,\chi\right)\right|^2 = \frac{5}{2}\varphi(q) + \mathbf{O}((1+|t|)^4 q(\log q)^{-1/2}) + \mathbf{O}(|t|^4 q(\log q)^{7/2}).$$

For $1/2 < \sigma \leqq 1$, *we have*

$$\sum_{\chi\,(\mathrm{mod}\,q)} |\mathrm{S}(\sigma+it,\chi)|^2 = \varphi(q) - \frac{4\varphi(q)q^{(1/2)-\sigma}}{(1-2\sigma)^2(\log q)^2} + \frac{4\varphi(q)\mathrm{Y}^{1-2\sigma}}{(\log q)^2(1-2\sigma)^2} + \frac{2\varphi(q)q^{1-2\sigma}}{(\log q)(1-2\sigma)}$$
$$+\mathbf{O}\left(\frac{\varphi(q)q^{(1/2)-\sigma}}{(\log q)^2(1-2\sigma)}\right) + \mathbf{O}\left(\frac{\varphi(q)q^{1-2\sigma}}{\log q}\right)$$
$$+\mathbf{O}\left(\frac{\varphi(q)q^{1-2\sigma}(\log q)^{-\sigma-1}}{1-\sigma}\{(1+|t|)^4 + (|t|\log q)^4\}\right)$$

where for $\sigma = 1$, *we interpret* $(1-\sigma)^{-1}$ *to be* $\log q$.

Proof. — We see that the sum is equal to

$$\sum_{n_1, n_2 = 1}^{\infty} \frac{a(n_1) a(n_2)}{(n_1 n_2)^{\sigma}} \left(\frac{n_2}{n_1}\right)^{it} \exp(-(n_1+n_2)/q) \sum_{\chi \,(\mathrm{mod}\, q)} \chi(n_1) \bar{\chi}(n_2)$$

which is seen to be

$$(4.1) \qquad \varphi(q) \sum_{n_1, n_2 = 1}^{\infty}{}' \frac{a(n_1) a(n_2)}{(n_1 n_2)^{\sigma}} \left(\frac{n_2}{n_1}\right)^{it} \exp(-(n_1+n_2)/q),$$

where the inner sum ranges over pairs (n_1, n_2) satisfying

$$n_1 \equiv n_2 \,(\mathrm{mod}\, q), \qquad (n_1, q) = (n_2, q) = 1.$$

We split the double sum into three pieces $\Sigma_1 + \Sigma_2 + \Sigma_3$. In Σ_1 we have $n_1 < n_2$, in Σ_2 we have $n_1 > n_2$, and in Σ_3 we have $n_1 = n_2$. The estimation of Σ_1 and Σ_2 is the same, so we only consider Σ_1. We have

$$(4.2) \qquad \Sigma_1 = \sum_{\substack{n_1 = 1 \\ (n_1, q) = 1}}^{\infty} \frac{a(n_1) \exp(-n_1/q)}{n_1^{\sigma}} \sum_{\substack{n_2 = 1 \\ n_2 \equiv n_1 \,(\mathrm{mod}\, q) \\ n_2 > n_1}}^{\infty} \frac{a(n_2) \exp(-n_2/q)}{n_2^{\sigma}} \left(\frac{n_2}{n_1}\right)^{it}.$$

We begin by considering the sum over n_2. We must necessarily have $n_2 > q$ for if $n_2 \leqq q$, then $n_1 \leqq q$ also and so the congruence $n_2 \equiv n_1 \,(\mathrm{mod}\, q)$ would force $n_1 = n_2$. We split Σ_1 into three subsums Σ_{11}, Σ_{12} and Σ_{13} where

in Σ_{11} we have $n_2 \geqq q \log q$

in Σ_{12} we have $q \leqq n_1 < q \log q$ and $n_1 < n_2 < q \log q$

in Σ_{13} we have $n_1 < q$ and $q < n_2 < q \log q$.

In Σ_{11}, we see, by partial summation, that the sum over n_2 is

$$\ll q^{-1} \int_{q \log q}^{\infty} \Bigg\{ \sum_{\substack{n \leqq u \\ n \equiv n_1 \,(\mathrm{mod}\, q)}} |a(n)| \Bigg\} u^{-\sigma} e^{-u/q} \, du.$$

We have from Proposition (1.2) that

$$\sum_{\substack{n \leqq u \\ n \equiv n_1 \,(\mathrm{mod}\, q)}} |a(n)| \ll \frac{u}{\varphi(q)^{1/2} (\log q)^{1/2}}.$$

Thus, we find that the integral is

$$\ll \frac{1}{q^{3/2} (\log q)^{1/2}} \int_{q \log q}^{\infty} u^{1-\sigma} e^{-u/q} \, du$$

and this is

$$\ll q^{1/2-\sigma}(\log q)^{-1/2}\int_{\log q}^{\infty} v^{1-\sigma}e^{-v}\,dv$$
$$\ll q^{1/2-\sigma}(\log q)^{-1/2-\sigma}.$$

Inserting this into the n_1 – sum, using Proposition (1.1), the Cauchy-Schwarz inequality and partial summation, we have

$$\Sigma_{11} \ll \frac{q^{1-\sigma}}{(1-\sigma)(\log q)^{1/2}}\,\frac{(\log q)^{1/2-\sigma}}{q^{1/2+\sigma}}$$
$$\ll \frac{q^{1/2-2\sigma}(\log q)^{-\sigma}}{1-\sigma}.$$

Now we consider the contribution of Σ_{12}. This is

$$\sum_{\substack{q\leqq n_1<q\log q\\(n_1,\,q)=1}} \frac{a(n_1)e^{-n_1/q}}{n_1^{\sigma}} \sum_{\substack{n_1<n_2<q\log q\\ n_2\equiv n_1\ (\mathrm{mod}\ q)}} \frac{a(n_2)e^{-n_2/q}}{n_2^{\sigma}}\left(\frac{n_2}{n_1}\right)^{it}.$$

We split the n_1 sum into $\mathbf{O}(\log\log q)$ sums of the form

$$\sum_{\substack{U<n_1\leqq 2U\\(n_1,\,q)=1}} \frac{a(n_1)e^{-n_1/q}}{n_1^{\sigma}} \sum_{\substack{n_1<n_2<q\log q\\ n_2\equiv n_1\ (\mathrm{mod}\ q)}} \frac{a(n_2)e^{-n_2/q}}{n_2^{\sigma}}\left(\frac{n_2}{n_1}\right)^{it}.$$

Let us write $n_2=n_1+jq$. The above double sum may therefore be written as

$$(4.3)\qquad \sum_{j<\log q} e^{-j} \sum_{\substack{U<n_1\leqq 2U\\(n_1,\,q)=1}} e^{-2n_1/q}\,\frac{a(n_1)\,a(n_1+jq)}{n_1^{\sigma}(n_1+jq)^{\sigma}}\left(\frac{n_1+jq}{n_1}\right)^{it}$$

If we drop the condition $(n_1, q)=1$, then we introduce an additional sum

$$(4.4)\qquad \sum_{j<\log q} e^{-j} \sum_{U<qk\leqq 2U} e^{-2k}\,\frac{a(kq)\,a((k+j)\,q)}{(kq)^{\sigma}((k+j)\,q)^{\sigma}}\left(\frac{k+j}{k}\right)^{it}.$$

Observe that as q is prime, and $\lambda(n)=0$ for $n>Z=q^{1/2}$, we have

$$a(kq)=\sum_{d\mid kq}\lambda(d)=\sum_{d\mid k}\lambda(d)=a(k).$$

Therefore, we have the estimate

$$|a(kq)|\leqq \mathbf{d}(k)\ll_{\varepsilon} k^{\varepsilon}.$$

A similar estimate hols for $a((k+j)q)$. Using this in (4.4), we see that it is

$$\ll q^{-2\sigma} \sum_{j<\log q} e^{-j} \sum_{U<qk\leqq 2U} \frac{e^{-2k}}{k^{\sigma-\varepsilon}(k+j)^{\sigma-\varepsilon}}$$

and this is

$$\ll q^{-2\sigma}.$$

The sum in (4.3) may thus be replaced by

$$\sum_{j<\log q} e^{-j} \sum_{U<n_1\leqq 2U} e^{-2n_1/q} \frac{a(n_1)a(n_1+jq)}{n_1^\sigma (n_1+jq)^\sigma} \left(\frac{n_1+jq}{n_1}\right)^{it} \tag{4.5}$$

Let us set

$$G(u)=\sum_{U<n_1\leqq u} a(n_1)a(n_1+jq)\left(\frac{n_1+jq}{n_1}\right)^{it}.$$

By Proposition (1.3), we see that for $U<u$,

$$G(u)\ll \frac{j}{\varphi(j)}\left(\frac{(u+(j+1)q)|P(t)|}{(\log q)^2}+(u+jq)|t|^4(\log q)^2\right).$$

The sum over n_1 in (4.5) can be estimated using partial summation. We find that it is equal to

$$\frac{G(u)e^{-2u/q}}{u^\sigma(u+jq)^\sigma}\Bigg|_U^{2U}+\int_U^{2U} G(u)\,d\left(\frac{e^{-2u/q}}{u^\sigma(u+jq)^\sigma}\right).$$

Using the estimate for $G(u)$ quoted above, we see that for $\sigma\neq 1$, this is

$$e^{-2U/q}\frac{j}{\varphi(j)}\frac{(U+jq)^{1-\sigma}}{U^\sigma}\frac{U}{q(1-\sigma)}(\log q)^{-2}(|P(t)|+(|t|\log q)^4)$$

If $\sigma=1$, then we can suppress the term $(1-\sigma)^{-1}$. Note that though the coefficients of $P(t)$ depend on j and q, they are absolutely bounded. Thus,

$$|P(t)|\ll(1+|t|)^4.$$

Incorporating these estimates into the sum over j, we find that (4.5) is for $\sigma\neq 1$

$$\ll \sum_{j<\log q} \frac{(U+jq)^{1-\sigma}U^{1-\sigma}}{q(1-\sigma)} e^{-2U/q}(\log q)^{-2}\frac{j}{\varphi(j)}e^{-j}(|P(t)|+(|t|\log q)^4)$$

which is

$$\ll \frac{U^{1-\sigma} e^{-2U/q}}{q(1-\sigma)(\log q)^2} (|P(t)|+(|t| \log q)^4) \sum_{j<\log q} e^{-j} \frac{j}{\varphi(j)} (U+jq)^{1-\sigma}$$

$$\ll q^{1-\sigma}(\log q)^{-1-\sigma} \frac{U^{1-\sigma}}{q(1-\sigma)} e^{-2U/q}(|P(t)|+(|t| \log q)^4).$$

Now summing this over U, we find it is

$$\ll q^{1-2\sigma}(\log q)^{-\sigma-1}(1-\sigma)^{-1}(|P(t)|+(|t| \log q)^4).$$

For $\sigma=1$, we can suppress the term $(1-\sigma)^{-1}$.

Now we discuss the contribution of Σ_{13}. By the Cauchy-Schwarz inequality, we see that

$$|\Sigma_{13}|^2 \ll \left(\sum_{n_1<q} \frac{a(n_1)^2}{n_1^{2\sigma}} \exp(-2n_1/q) \right) \left(\sum_{n_1 \leqq q} \left| \sum_{\substack{q<n_2<q \log q \\ n_2 \equiv n_1 \pmod q}} \frac{a(n_2) e^{-n_2/q}}{n_2^\sigma} n_2^{it} \right|^2 \right).$$

The first factor above is $\mathbf{O}(1)$ as can be seen from our discussion of Σ_3 below. As for the second factor, we see that it is equal to

$$\sum_{\substack{q<n_2, n_2'<q \log q \\ n_2 \equiv n_2' \,(\mathrm{moq}\ q)}} \frac{a(n_2) e^{-n_2/q}}{n_2^\sigma} \frac{a(n_2') e^{-n_2'/q}}{(n_2')^\sigma} \left(\frac{n_2}{n_2'}\right)^{it}.$$

Again, we split this sum into three sums according as $n_2<n_2'$, $n_2=n_2'$, and $n_2>n_2'$. The third is the same as the first. Also, we note that the first sum is just Σ_{12} which we have estimated above as being (for $\sigma \neq 1$)

$$\ll q^{1-2\sigma}(\log q)^{-\sigma-1}(1-\sigma)^{-1}(|P(t)|+(|t| \log q)^4).$$

If $\sigma=1$, then as before, we may suppress the $(1-\sigma)^{-1}$ term. As for the second, we see that it is equal to

$$\sum_{q \leqq n_2<q \log q} \frac{a(n_2)^2 e^{-2n_2/q}}{n_2^{2\sigma}}.$$

Using Proposition (1.1) and partial summation, this is

$$\ll \frac{q^{1-2\sigma}}{\log q}.$$

Inserting this into the above, we deduce that

$$\Sigma_{13} \ll q^{(1/2)-\sigma}(\log q)^{-(\sigma+1)/2}(|P(t)|+(|t| \log q)^4)^{1/2}.$$

Finally, we discuss the estimation of Σ_3, namely the terms with $n_1 = n_2$. Thus,

$$(4.6)\qquad \Sigma_3 = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{a(n)^2}{n^{2\sigma}} \exp(-2n/q) = \sum_{n \leqq Y} + \sum_{Y < n \leqq q} + \sum_{n > q}.$$

Since $a(n) = 0$ for $1 < n \leqq Y$, we have

$$(4.7)\qquad \sum_{n \leqq Y} = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, by partial summation and Proposition (1.1), we find that

$$(4.8)\qquad \sum_{n > q} \ll \frac{q^{1-2\sigma}}{\log(z_2/z_1)}.$$

Thus, we see from (4.6)-(4.8) that

$$\Sigma_3 = 1 + \sum_{\substack{Y < n \leqq q \\ (n,q)=1}} \frac{a(n)^2}{n^{2\sigma}} \exp(-2n/q).$$

Let us denote the sum on the right by S. We find that

$$S = \sum_{Y < n \leqq q} \frac{a(n)^2}{n^{2\sigma}} \left(1 + \mathbf{O}\left(\frac{n}{q}\right)\right).$$

Now, the $\mathbf{O}$-term is

$$\begin{aligned} &\ll \frac{1}{q} \sum_{\substack{Y < n \leqq q \\ (n,q)=1}} \frac{a(n)^2}{n^{2\sigma-1}} \\ &\ll \frac{1}{q} \frac{1}{\log(z_2/z_1)} \frac{q^{2-2\sigma}}{(1-\sigma)} \\ &\ll \frac{q^{1-2\sigma}}{(1-\sigma)\log q}. \end{aligned}$$

The main term is equal to

$$\sum_{Y < n < q} \frac{a(n)^2}{n^{2\sigma}}.$$

Finally, using Proposition (1.1),

$$\sum_{n < q} \frac{a(n)^2}{n^{2\sigma}} = \sum_{1 \leqq n \leqq Y} + \sum_{Y < n \leqq Z} + \sum_{Z < n < q} \frac{a(n)^2}{n^{2\sigma}}.$$

The first sum is equal to 1 since $a(n)=0$ for $1<n\leqq Y$. Using Proposition (1.1) and partial summation, we see that the second sum is

$$\sum_{Y<n\leqq Z} \frac{(\log n/Y)}{(\log Z/Y)^2}\cdot\frac{1}{n^{2\sigma}}+\mathbf{O}\left(\frac{1}{\log Z/Y}\right).$$

If $\sigma=1/2$ this is

$$\frac{1}{2}+\mathbf{O}\left(\frac{1}{\log q}\right)$$

and if $\sigma>1/2$, this is

$$\frac{2Z^{1-2\sigma}}{(1-2\sigma)(\log q)}\left(1-\frac{2}{(1-2\sigma)(\log q)}+\mathbf{O}\left(\frac{1}{\log q}\right)\right)+\frac{4Y^{1-2\sigma}}{(1-2\sigma)^2(\log q)^2}.$$

Similarly, the third sum is

$$\sum_{Z<n<q}\frac{1}{\log Z/Y}\,\frac{1}{n^{2\sigma}}+\mathbf{O}\left(\frac{1}{\log q}\right)$$

which is

$$=\begin{cases} 1+\mathbf{O}\left(\dfrac{1}{\log q}\right) & \text{if}\quad \sigma=\dfrac{1}{2} \\ \dfrac{1}{1-2\sigma}\cdot\dfrac{2}{(\log q)}(q^{1-2\sigma}-Z^{1-2\sigma})\left(1+\mathbf{O}\left(\dfrac{1}{\log q}\right)\right) & \text{if}\quad \sigma>\dfrac{1}{2}.\end{cases}$$

Putting these together we deduce that

$$\sum_{n<q}\frac{a(n)^2}{n}=\frac{5}{2}\left(1+\mathbf{O}\left(\frac{1}{\log q}\right)\right)$$

and for $\sigma>1/2$

$$\sum_{n<q}\frac{a(n)^2}{n^{2\sigma}}=1-\frac{4Z^{1-2\sigma}}{(\log q)^2(1-2\sigma)^2}+\frac{4Y^{1-2\sigma}}{(\log q)^2(1-2\sigma)^2}+\frac{2q^{1-2\sigma}}{(\log q)(1-2\sigma)}+\mathbf{O}\left(\frac{Z^{1-2\sigma}}{(\log q)^2(1-2\sigma)}\right).$$

This completes the proof of the proposition.

In the next sections, we shall study the mean square of the integral $I(s,\chi)$.

5. THE INTEGRAL $R_a(s, \chi)$. — The purpose of the next few sections is to obtain an asymptotic formula for the mean square of $I(s, \chi)$. Recall that for $\chi \neq 1$, we have

$$I(s, \chi) = \frac{1}{2\pi i} \int_{(-\eta)} \gamma(s+w, \chi) \left\{ \sum_{n<Z} \frac{\overline{\chi(n)}}{n^{1-s-w}} \right\} M(s+w, \chi) q^w \Gamma(w) \, dw.$$

Thus

$$(5.1) \qquad I(s, \chi) = \sum_{m, n<Z} \frac{\overline{\chi(n)} \chi(m) \lambda(m)}{m^s n^{1-s}} \left(\frac{2\pi}{q}\right)^{s-(1/2)} \left(\frac{2}{\pi}\right)^{1/2} \frac{\tau(\chi)}{i^a q^{1/2}} R_a\left(s, \frac{2\pi n}{m}\right)$$

where

$$(5.2) \qquad R_a(s, y) = \frac{1}{2\pi i} \int_{(\delta)} y^w \sin\left(\frac{\pi}{2}(s+w+a)\right) \Gamma(1-s-w) \Gamma(w) \, dw.$$

Here $a=0, 1$, $\chi(-1) = (-1)^a$ and $-1 < \delta < 0$ is arbitrary, $y > 0$ and $0 < \mathrm{Re}(s) < 1$. Notice that

$$(5.3) \qquad \overline{R_a(s, y)} = R_a(\bar{s}, y).$$

The integrand has simple poles at $w = -k$ and $w = 1-s+k$ where $0 \leqq k \in \mathbf{Z}$. Since $1/2 \leqq \mathrm{Re}(s) < 1$, these are distinct points. We have the expansion

$$R_a(s, y) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} y^{-k} \sin\left(\frac{\pi}{2}(s-k+a)\right) \Gamma(1-s+k) \quad \text{for} \quad y \geqq 1.$$

Indeed, this is just the sum of the residues at the points $w = -k$, $0 \leqq k \in \mathbf{Z}$. The condition $y \geqq 1$ ensures that it converges. Indeed, we have the following asymptotic expansion.

LEMMA (5.1). — *For $s = \sigma + it$ with $1/2 \leqq \sigma < 1$, and $|t| < \sigma/10$, $y \geqq 1$, and $0 \leqq K \in \mathbf{Z}$, we have*

$$R_a(s, y) = \sum_{k=1}^{K} \frac{(-1)^k}{k!} y^{-k} \sin\left(\frac{\pi}{2}(s-k+a)\right) \Gamma(1-s+k) + \mathbf{O}\left(y^\delta \frac{1}{K^{\sigma/5}} |\Gamma(1-\delta-K-s) \Gamma(1+\delta+K)|\right)$$

for any $\delta \in (-K-1, -K)$.

Proof. — We need only estimate the integral defining R_a along a line $-K-1 < \delta < -K$. We write $w = -K - \eta$, $0 < \mathrm{Re}(\eta) < 1$. Write $s = \sigma + it$ *and* $\eta = \beta + i\gamma$. Then,

$$|\Gamma(1-s-w) \Gamma(w)| = \prod_{j=1}^{K} \left|1 - \frac{s}{j+\eta}\right| . |\Gamma(1+\eta-s) \Gamma(-\eta)|.$$

Now,

$$\left|1-\frac{s}{j+\eta}\right| \leqq \left|1-\frac{\sigma}{j+\eta}\right|+\frac{|t|}{|j+\eta|}$$

and

$$\left|1-\frac{\sigma}{j+\eta}\right|^2=\left(1-\frac{\sigma(j+\beta)}{(j+\beta)^2+\gamma^2}\right)^2+\frac{\sigma^2\gamma^2}{|j+\eta|^4}.$$

If $j+\beta>2\gamma$, we see that

$$\left|1-\frac{\sigma}{j+\eta}\right|^2 \leqq \left(1-\frac{4\sigma}{5(j+\beta)}\right)^2+\frac{\sigma^2}{4|j+\beta|^2}.$$

Therefore,

$$\left|1-\frac{s}{j+\eta}\right| \leqq 1-\frac{4\sigma}{5(j+\beta)}+\frac{(1/2)\sigma+|t|}{j+\beta}$$

which simplifies to

$$\left|1-\frac{s}{j+\eta}\right| \leqq 1-\frac{3\sigma/10-|t|}{j+\beta}.$$

Let us set

$$u=u(\eta)=\max([2\gamma-\beta], 0)+1$$

where $[x]$ denotes the greatest integer $\leqq x$. We deduce that

$$\prod_{j=u}^{K}\left|1-\frac{s}{j+\eta}\right| \leqq \prod_{j=u}^{K}\left(1-\frac{3\sigma/10-|t|}{j+\beta}\right) \ll \left(\frac{u}{K}\right)^{(3/10)\sigma-|t|}.$$

Moreover

$$\prod_{j\leqq u}\left|1-\frac{s}{j+\eta}\right| \leqq \left(1+\frac{3\sigma}{5\gamma}\right)^u \ll 1.$$

Note that the sine term in the integrand is bounded as a function of k.

There is a similar expression and estimate when $y \leqq 1$.

LEMMA (5.2). — *For* $s=\sigma+it$ *with* $1/2\leqq\sigma<1$, $|t|<\sigma/10$, $0<y\leqq 1$, $0\leqq K\in\mathbf{Z}$ and any $\delta\in(1-\sigma+K, 2-\sigma+K)$, we have

$$R_a(s, y)=-\sin\left(\frac{\pi}{2}(s+a)\right).\Gamma(1-s) - \sum_{k=1}^{K}\frac{(-1)^k}{k!}y^{1-s+k}\sin\left(\frac{\pi}{2}(a+k+1)\right)\Gamma(1-s+k) + \mathbf{O}(y^{\delta}K^{-\sigma/5}|\Gamma(2-s-\delta+K)\Gamma(\delta-K)|).$$

In both cases, we see that for s as above (that is, $s=\sigma+it$, and $1/2\leqq\sigma<1$ and $|t|<\sigma/10$),

$$R_a(s, y)\ll_c 1.$$

Finally, we define

$$\omega_k=\omega_k(s)=\frac{(-1)^k}{k!}\sin\left(\frac{\pi}{2}(a+k+1)\right)\Gamma(1-s+k). \tag{5.4}$$

The argument of Lemma (5.1) shows that for $s=\sigma+it$, with $1/2\leqq\sigma<1$, we have

$$\omega_k(s)\ll k^{-\sigma+|t|}. \tag{5.5}$$

6. AN EXPRESSION FOR THE MEAN SQUARE OF $I(s, \chi)$. — From (5.1) and (5.3), we see that for a fixed $a=0$ or 1, and an s, we have

$$\sum_{\substack{1\neq\chi\,(\mathrm{mod}\,q)\\ \chi(-1)=(-1)^a}}|I(s,\chi)|^2=\frac{2}{\pi}\left(\frac{2\pi}{q}\right)^{2\sigma-1}\sum_{\substack{m_1, m_2, n_1, n_2<Z\\ (m_1, q)=\ldots=(n_2, q)=1}}\frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} \times R_a\left(s, \frac{2\pi n_1}{m_1}\right)R_a\left(\bar{s}, \frac{2\pi n_2}{m_2}\right)\sum_{\substack{1\neq\chi \bmod q\\ \chi(-1)=(-1)^a}}\chi(m_1 n_2)\overline{\chi(n_1 m_2)}.$$

Notice that we can drop the condition $(m_1, q)=\ldots=(n_2, q)=1$ since $1 \leqq m_1, \ldots, n_2 \leqq Z < q$. Observe that for $(mn, q)=1$, and q odd, we have

$$\sum_{\substack{1 \neq \chi \,(\mathrm{mod}\, q) \\ \chi(-1)=(-1)^a}} \chi(m)\overline{\chi(n)} = \frac{1}{2}\sum_{1\neq\chi}\chi(m)\overline{\chi(n)} + \frac{1}{2}(-1)^a \sum_{1\neq\chi}\chi(-m)\overline{\chi(n)}$$

$$= \frac{1}{2}\sum_{\varepsilon=\pm 1}\varepsilon^a \sum_{1\neq\chi}\chi(\varepsilon m)\overline{\chi(n)}$$

$$= \begin{cases} \frac{1}{2}\varphi(q)-1 & \text{if } m\equiv \pm n \text{ and } a=0 \\ -1 & \text{if } m\not\equiv \pm n \text{ and } a=0 \\ \frac{1}{2}\varphi(q) & \text{if } m\equiv n \text{ and } a=1 \\ -\frac{1}{2}\varphi(q) & \text{if } m\equiv -n \text{ and } a=1 \\ 0 & \text{otherwise.} \end{cases}$$

Applying this to the innermost sum, we see that this is

$$\frac{1}{\pi}(2\pi)^{2\sigma-1}\varphi(q)q^{1-2\sigma}\sum_{\varepsilon}\varepsilon^a \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ m_1 n_2 \equiv \varepsilon n_1 m_2 \,(\mathrm{mod}\, q)}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} R_a\left(s, \frac{2\pi n_1}{m_1}\right) R_a\left(\bar{s}, \frac{2\pi n_2}{m_2}\right)$$

minus

$$\delta(a)\frac{2}{\pi}(2\pi)^{2\sigma-1}q^{1-2\sigma}\sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ (m_1 m_2 n_1 n_2, q)=1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} R_0\left(s, \frac{2\pi n_1}{m_1}\right) R_0\left(\bar{s}, \frac{2\pi n_2}{m_2}\right).$$

Here, $\delta(a)=1-a$. If we designate the second quantity as $|I(s, 1)|^2$, then setting $a=0, 1$ and adding, we deduce that

$$(6.1)\qquad \sum_{\chi \,(\mathrm{mod}\, q)} |I(s, \chi)|^2 = \frac{1}{\pi}(2\pi)^{2\sigma-1}\varphi(q)q^{1-2\sigma}(S^+(s, q)+S^-(s, q))$$

where

$$S^{\pm}(s, q) = \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ m_1 n_2 \equiv \pm n_1 m_2 \,(\mathrm{mod}\, q)}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}}$$

$$\times\left[R_0\left(s, \frac{2\pi n_1}{m_1}\right) R_0\left(\bar{s}, \frac{2\pi n_2}{m_2}\right) \pm R_1\left(s, \frac{2\pi n_1}{m_1}\right) R_1\left(\bar{s}, \frac{2\pi n_2}{m_2}\right)\right].$$

Note that if $s=1/2$, this can be rewritten as

$$S^{\pm}\left(\frac{1}{2}, q\right)=\sum_{\substack{m_1, m_2, n_1, n_2<Z\\(m_1 n_2, q)=(m_2 n_1, q)=1}} \frac{\lambda(m_1)\lambda(m_2)}{(m_1 m_2 n_1 n_2)^{1/2}} \times\left[R_0\left(\frac{1}{2}, \frac{2\pi n_1}{m_1}\right)R_0\left(\frac{1}{2}, \frac{2\pi n_2}{m_2}\right)\pm R_1\left(\frac{1}{2}, \frac{2\pi n_1}{m_1}\right)R_1\left(\frac{1}{2}, \frac{2\pi n_2}{m_2}\right)\right]$$

and

$$(6.2)\qquad \sum_{\chi\,(\mathrm{mod}\,q)}\left|I\left(\frac{1}{2}, \chi\right)\right|^2=\frac{1}{\pi}\varphi(q)\left(S^{+}\left(\frac{1}{2}, q\right)+S^{-}\left(\frac{1}{2}, q\right)\right).$$

Let us also define, for $a=0, 1$

$$S^{\pm}(s, q; a)=\sum_{\substack{m_1, m_2, n_1, n_2<Z\\ m_1 n_2\equiv \pm n_1 m_2\,(\mathrm{mod}\,q)}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} R_a\left(s, \frac{2\pi n_1}{m_1}\right)R_a\left(\bar{s}, \frac{2\pi n_2}{m_2}\right)$$

so that

$$S^{\pm}(s, q)=S^{\pm}(s, q, 0)\pm S^{\pm}(s, q, 1).$$

Our estimations are complicated by the unusual way in which the four indices of summation m_1, m_2, n_1, n_2 are interlaced. Our goal in the next sections will be to show that the main contribution comes from those terms where $m_1 n_2=n_1 m_2$ and $n_1\leqq(1/2\pi)m_1$, $n_2\leqq(1/2\pi)m_2$.

7. Estimate of the non-diagonal terms. — We wish to show that the terms in $S^{-}(s, q; a)$ contribute a negligible amount to the right hand side of (6.1). Since m_1, m_2, n_1, $n_2<Z$ and $m_1 n_2\equiv -n_1 m_2\,(\mathrm{mod}\,q)$, this means than $m_1 n_2=q-m_2 n_1$.) (Notice that for the same reason, the indices in $S^{+}(s, q; a)$ statisfy $m_1 n_2=m_2 n_1$).

Lemma (7.1). — *For* $1/2\leqq\sigma<1$, *we have*

$$S^{-}(s, q; a)\ll\frac{1}{(1-\sigma)^2(\log q)^2}+\frac{q^{\sigma-1}}{(1-\sigma)\log q}.$$

Proof. — We wish to estimate the sum

$$\sum_{\substack{1\leqq m_1, m_2, n_1, n_2<Z\\ m_1 n_2=q-m_2 n_1}} \frac{|\lambda(m_1)||\lambda(m_2)|}{m_1^{\sigma} m_2^{\sigma} n_1^{1-\sigma} n_2^{1-\sigma}}\left|R_a\left(s, \frac{2\pi n_1}{m_1}\right)R_a\left(s, \frac{2\pi n_2}{m_2}\right)\right|.$$

Without loss, we may suppose that $m_2 n_1<(1/2)q$. A consequence of this is that $m_1 n_2>(1/2)q$ and so

$$\frac{1}{2}Z<m_1, n_2<Z.$$

We may also suppose that m_1, m_2, n_1, n_2 are squarefree. Notice that we must have $(m_1, m_2)=1$. We consider two cases.

Case 1. – $m_2<n_2$.

In this case, we must have $n_2 \equiv \bar{m}_1 q \pmod{m_2}$ where $\bar{m}_1$ denotes the inverse of m_1 modulo m_2. Moreover,

$$\left| R_a\left(s, \frac{2\pi n_2}{m_2}\right)\right| \ll \frac{m_2}{n_2}.$$

Thus, we can rewrite our sum as

$$\sum_{m_2<Z} \frac{|\lambda(m_2)|}{m_2^\sigma} \sum_{(1/2)Z<m_1<Z} \frac{|\lambda(m_1)|}{m_1^\sigma} \sum_{\substack{n_2\equiv q\bar{m}_1 \pmod{m_2}\\ n_2>m_2\\ (1/2)Z<n_2<Z}} \frac{1}{n_2^{1-\sigma}}\left(\frac{q-m_1 n_2}{m_2}\right)^{\sigma-1}\frac{m_2}{n_2}$$

$$\ll Z^{\sigma-2}\sum_{m_2<Z}\frac{|\lambda(m_2)|}{m_2^{2\sigma-2}}\sum_{(1/2)Z<m_1<Z}\frac{|\lambda(m_1)|}{m_1^\sigma}\sum_{n_2}\left(\frac{1}{q-m_1 n_2}\right)^{1-\sigma}$$

$$\ll Z^{\sigma-2}\sum_{m_2<Z}\frac{|\lambda(m_2)|}{m_2^{2\sigma-1}}\sum_{(1/2)Z<m_1<Z}\frac{|\lambda(m_1)|}{m_1^{1+\sigma}}\left(q-\frac{m_1 Z}{2}\right)^\sigma$$

$$\ll Z^{2\sigma-2}\sum_{m_2<Z}\frac{|\lambda(m_2)|}{m_2^{2\sigma-1}}\sum_{(1/2)Z<m_1<Z}\frac{|\lambda(m_1)|}{m_1}$$

$$\ll Z^{2\sigma-2}\frac{1}{\log Z/Y}\left(\frac{\log Z}{2-2\sigma}+\frac{Z^{2-2\sigma}-1}{(2-2\sigma)^2}\right)\frac{1}{\log Z/Y}$$

$$\ll\left(\frac{q^{\sigma-1}}{\sigma-1}+\frac{1}{(\log q)(1-\sigma)^2}\right)\frac{1}{\log q}.$$

Case 2. – $m_2 \geqq n_2$.

In this case, we write the congruence condition as $m_1 \equiv q\bar{n}_2 \pmod{m_2}$. Since $(1/2)Z<m_1<Z$, this implies that there are at most two possible values for m_1. Thus, we see that our sum is

$$\sum_{(1/2)Z<m_2<Z}\frac{|\lambda(m_2)|}{m_2^\sigma}\sum_{\substack{n_2\leqq m_2\\ (1/2)Z<n_2<Z}}\frac{1}{n_2^{1-\sigma}}\sum_{\substack{(1/2)Z<m_1<Z\\ m_1\equiv q\bar{n}_2 \pmod{m_2}}}\frac{|\lambda(m_1)|}{m_1^\sigma}\left(\frac{m_2}{q-m_1 n_2}\right)^{1-\sigma}$$

$$\ll\sum_{(1/2)Z<m_2<Z}\frac{|\lambda(m_2)|}{m_2^{2\sigma-1}}\sum_{(1/2)Z<n_2<Z}\frac{1}{n_2^{1-\sigma}}\frac{1}{\log Z/Y}\frac{1}{Z^\sigma}\frac{1}{(q-Zn_2)^{1-\sigma}}$$

$$\ll\frac{1}{Z^{2-\sigma}(\log Z/Y)}\sum_{(1/2)Z<m_2<Z}\frac{|\lambda(m_2)|}{m_2^{2\sigma-1}}\int_{(1/2)Z}^{m_2}\frac{dt}{(Z-t)^{1-\sigma}}$$

$$\ll \frac{1}{Z(\log Z/Y)} \sum_{(1/2)Z<m_2<Z} \frac{\log Z/m_2}{\log Z/Y}$$

$$\ll \frac{1}{(\log q)^2}.$$

This proves the result.

Finally in this section, we shall show that $|I(s, 1)|^2$ is negligible.

LEMMA (7.2). — *We have for* $1/2 \leqq \sigma < 1$

$$|I(s, 1)|^2 \ll \frac{q^{1-\sigma}}{(1-\sigma)^2}\left(1+\frac{q^{1-\sigma}}{(1-\sigma)^2(\log q)^2}\right).$$

If $\sigma=1$ *we have*

$$|I(s, 1)|^2 \ll (\log Z)^2.$$

Proof. — By definition, we have

$$|I(s, 1)|^2 = \frac{2}{\pi}(2\pi)^{2\sigma-1} q^{1-2\sigma} \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ (m_1 m_2 n_1 n_2, q)=1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} R_0\left(s, \frac{2\pi n_1}{m_1}\right) R_0\left(\bar{s}, \frac{2\pi n_2}{m_2}\right).$$

The n_1 and n_2 sums are estimated as $\ll Z^\sigma/\sigma$. To estimate the sum over m_1 and m_2 we observe that for $\sigma \neq 1$,

$$\sum_m \frac{|\lambda(m)|}{m^\sigma} \leqq \sum \frac{\log(Z/m)}{\log(Z/Y)} \cdot \frac{1}{m^\sigma} + O\left(\frac{Y^{1-\sigma}}{(1-\sigma)}\right) \ll \frac{1}{(\log Z/Y)(1-\sigma)}\left(\frac{Z^{1-\sigma}}{1-\sigma}+\log Z\right).$$

Using this estimate, we see that

$$|I(s, 1)|^2 \ll q^{1-2\sigma}\frac{Z^{2\sigma}}{\sigma^2}\left\{\frac{1}{(\log Z/Y)(1-\sigma)}\left(\frac{Z^{1-\sigma}}{1-\sigma}+\log Z\right)\right\}^2$$

and this simplifies to the stated expression, given our choices of Y and Z.

8. THE DIAGONAL TERMS. — We are now reduced to the study of the sum

$$(8.1)\quad \frac{1}{\pi}(2\pi)^{2\sigma-1}\varphi(q) q^{1-2\sigma} \times \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ (m_1 n_2, q)=1 \\ m_1 n_2 = m_2 n_1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}}\left(\left|R_0\left(s, \frac{2\pi n_1}{m_1}\right)\right|^2+\left|R_1\left(s, \frac{2\pi n_1}{m_1}\right)\right|^2\right).$$

Let us define

$$(8.2)\qquad D_a(s)=\sum_{\substack{m_1\, m_2,\, n_1,\, n_2<Z\\ m_1 n_2=m_2 n_1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}}\left|R_a\left(s, \frac{2\pi n_1}{m_1}\right)\right|^2.$$

For future reference let us denote by s the set of quadruples (m_1, m_2, n_1, n_2) included in the above sum. Then, the sum in (8.1) can be written as

$$\frac{1}{\pi}(2\pi)^{2\sigma-1}\frac{\varphi(q)}{q^{2\sigma-1}}(D_0(s)+D_1(s)).$$

We define a splitting

$$D_a(s)=M_a(s)+E_a(s)$$

where in $M_a(s)$, we range over those quadruples (m_1, m_2, n_1, n_2) in (8.2) which in addition satisfy $n_1 \leqq (1/2\pi)m_1$. (Note that as $n_1/m_1=n_2/m_2$, we will then also have $n_2 \leqq (1/2\pi)m_2$.)

We shall henceforth assume that $1/2 \leqq \sigma \leqq 1$. The case $\sigma=1$ can also be handled, but it will not be necessary for us. Moreover, we suppose that $|t|$ is sufficiently small in the strong sense that

$$|t| \ll \frac{1}{\log q}.$$

We begin our study $M_a(s)$ by replacing $R_a(s, (2\pi n_1/m_1))$ with the Taylor expansion of Lemma (5.2). We find that

$$(8.3)\quad M_a(s)=\sum \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}}\left|\sin\left(\frac{\pi}{2}(s+a)\right)\Gamma(1-s)+\sum_{k=1}^{K}\left(\frac{2\pi n_1}{m_1}\right)^{1-s+k}\omega_k\right.$$
$$\left.+\mathbf{O}\left(\left(\frac{2\pi n_1}{m_1}\right)^{\delta} K^{-\sigma}\Gamma(2-\delta+K-\sigma)\Gamma(\delta-K)\right)\right|^2$$

where we recall (from (5.4)) that

$$\omega_k=\frac{(-1)^k}{k!}\sin\left(\frac{\pi}{2}(a+k+1)\right)\Gamma(1-s+k)$$

and $0 \leqq K \in \mathbf{Z}$ and $1-\sigma+K<\delta<2-\sigma+K$. Expanding, we find that $M_a(s)$ splits into a main term and an **O**-term. We shall now analyze the **O**-term with the help of the following lemma.

LEMMA (8.1). — *For any $\beta>0$ we have*

$$\sum_{m_1, m_2, n_1, n_2 \in s} \frac{|\lambda(m_1)|\,|\lambda(m_2)|}{(m_1 m_2)^{\sigma}(n_1 n_2)^{1-\sigma}}\left(\frac{n_1}{m_1}\right)^{\beta} \ll \frac{1}{(2\sigma+\beta-1)}\cdot\frac{1}{(2\pi)^{2\sigma+\beta-1}}(\log Z)^3.$$

Proof. — We use the fact that

$$m_1 n_2 = m_2 n_1$$

$$n_1 \leqq \frac{1}{2\pi} m_1, \qquad n_2 \leqq \frac{1}{2\pi} m_2$$

and

$$|\lambda(m)| \leqq 1 \quad \text{for all } m.$$

Note that $2\sigma+\beta>1$. Then, denoting $m_1 n_2$ by j, we see that the sum is bounded by

$$\sum_{m_1, m_2} \frac{1}{(m_1 m_2)^{2\sigma+\beta-1}} \sum_j j^{2\sigma+\beta-2} \tag{8.4}$$

where the inner sum ranges over integers j satisfying

$$1 \leqq j \leqq \frac{1}{2\pi} m_1 m_2$$

$$j \equiv 0 \bmod [m_1, m_2].$$

Let us set

$$i = (m_1, m_2).$$

Then $[m_1, m_2] = m_1 m_2 / i$ and (8.4) is

$$\ll \sum \frac{[m_1, m_2]^{2\sigma+\beta-2}}{(m_1 m_2)^{2\sigma+\beta-1}} \left(\frac{1}{2\pi} \frac{m_1 m_2}{[m_1, m_2]} \right)^{2\sigma+\beta-1} \cdot \frac{1}{2\sigma+\beta-1}$$

$$\ll \frac{1}{2\sigma+\beta-1} \cdot \frac{1}{(2\pi)^{2\sigma+\beta-1}} \cdot \sum \frac{i}{m_1 m_2}.$$

Moreover,

$$\sum \frac{i}{m_1 m_2} \ll \sum_{m_1 \leqq Z} \frac{1}{m_1} \sum_{i \mid m_1} \sum_{m \leqq (Z/i)} \frac{1}{m}$$

$$\ll (\log Z)^3.$$

This proves the lemma.

Now, the **O**-term in (8.3) is, for any $0 \leqq K \in \mathbf{Z}$ and any $1-\sigma+K<\delta<2-\sigma+K$,

$$\ll \sum \frac{|\lambda(m_1)||\lambda(m_2)|}{(m_1 m_2)^{\sigma}(n_1 n_2)^{1-\sigma}} \left(\frac{2\pi n_1}{m_1}\right)^{\delta} K^{-\sigma/5} \Gamma(2-\sigma-\delta+K)\Gamma(\delta-K).$$

$$\left(\left|\sin\left(\frac{\pi}{2}(s+a)\right)\Gamma(1-s)\right| + \sum_{k=1}^{K} \left(\frac{2\pi n_1}{m_1}\right)^{1-\sigma+k} |\omega_k| + \left(\frac{2\pi n_1}{m_1}\right)^{\delta} K^{-\sigma/5} \Gamma(2-\sigma-\delta+K)\Gamma(\delta-K)\right).$$

Now using (5.5), we find that the above is

$$\ll K^{-\sigma/5}.(\log Z)^3.|\Gamma(2-\sigma-\delta+K)\Gamma(\delta-K)|(2\pi)^{\delta}$$
$$\times\left\{\left|\sin\left(\frac{\pi}{2}(s+a)\right)\Gamma(1-s)\right| \frac{1}{2\sigma+\delta-1}\,\frac{1}{(2\pi)^{2\sigma+\delta-1}} + \sum_{k=1}^{K}|\omega_k|(2\pi)^{1-\sigma+k}.\frac{1}{\sigma+\delta+k}\,\frac{1}{(2\pi)^{\sigma+\delta+k}}\right.$$
$$\left. +|\Gamma(2-\sigma-\delta+K)\Gamma(\delta-K)|(2\pi)^{\delta}K^{-\sigma/5}(2\sigma+2\delta-1)^{-1}(2\pi)^{1-2\sigma-2\delta}\right\}.$$

Choosing $\delta=(3/2)-\sigma+K$, this is

$$\ll K^{-1-\sigma/5}(\log Z)^3\left\{\left|\sin\left(\frac{\pi}{2}(s+a)\right)\Gamma(1-s)\right| + K^{1-\sigma/5}+K^{-\sigma/5}\right\}.$$

Finally, choosing

$$K=(\log q)^{20}$$

shows that the **O**-term in (8.3) is

$$\ll |\Gamma(1-s)|.(\log q)^{-1}. \tag{8.5}$$

Now we analyze the main term of (8.3), namely

$$\sum \frac{\lambda(m_1)\lambda(m_2)}{m_1^{s} m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}}\left|\sin\left(\frac{\pi}{2}(s+a)\right)\Gamma(1-s)+\sum_{k=1}^{K}\left(\frac{2\pi n_1}{m_1}\right)^{1-s+k}\omega_k\right|^2. \tag{8.6}$$

For this purpose we utilise a more refined version of Lemma (8.1).

LEMMA (8.2). — *We have for any* w with $\beta=\operatorname{Re} w>0$

$$\sum_{m_1, m_2, n_1, n_2 \in \mathcal{S}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^{\sigma} m_2^{\sigma} n_1^{1-\sigma} n_2^{1-\sigma}}.\left(\frac{n_1}{m_1}\right)^{w} \ll \frac{(2\pi)^{1-2\sigma-\beta}}{1-2\sigma-\beta}.\frac{1}{\log q}.$$

Proof. — We see that the sum is

$$T \stackrel{\text{def}}{=} \sum_{m_1, m_2, n_1, n_2 \in \mathscr{S}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^\sigma m_2^\sigma n_1^{1-\sigma} n_2^{1-\sigma}} \left(\frac{n_1}{m_1}\right)^w = \sum_{1 \leqq m_1, m_2 < Z} \frac{\lambda(m_1)\lambda(m_2)}{(m_1 m_2)^{2\sigma+1+w}} \sum j^{2\sigma-2+w}$$

where the inner sum ranges over integers j satisfying

$$j \equiv 0 \ (\mathrm{mod}\ [m_1, m_2])$$

$$1 \leqq j \leqq \frac{1}{2\pi} m_1 m_2.$$

Setting $j = j_0 [m_1, m_2]$, and $i = (m_1, m_2)$ as before, we see that the sum is

$$T = \sum_{1 \leqq m_1, m_2 < Z} \frac{\lambda(m_1)\lambda(m_2)[m_1, m_2]^{w+2\sigma-2}}{(m_1 m_2)^{2\sigma-1+w}} \cdot \sum_{1 \leqq j_0 \leqq i/2\pi} j_0^{2\sigma-2+w}.$$

Since $[m_1, m_2] = m_1 m_2 / i$, this may be rewritten as

$$(8.7) \qquad T = \sum_{1 \leqq j_0 \leqq (1/2\pi) Z} j_0^{2\sigma-2+w} \sum_{2\pi j_0 \leqq i \leqq Z} \frac{1}{i^{2\sigma-2+w}} \sum_{\substack{1 \leqq m_1, m_2 \leqq Z \\ (m_1, m_2) = i}} \frac{\lambda(m_1)\lambda(m_2)}{m_1 m_2}.$$

The innermost sum can be written

$$(8.8) \qquad S \stackrel{\text{def}}{=} \sum \frac{\lambda(ij_1)\lambda(ij_2)}{i^2 j_1 j_2}$$

where the sumation ranges over pairs (j_1, j_2) satisfying

$$1 \leqq j_1 \leqq \frac{Z}{i}, \qquad 1 \leqq j_2 \leqq \frac{Z}{i}$$

$$(j_1, j_2) = 1$$

We may suppose that ij_1, ij_2 are squarefree (else $\lambda(ij_1)\lambda(ij_2)$ will be zero). In particular, this implies that $(j_1, i) = 1$ and $(j_2, i) = 1$. Applying Lemma (1.5) to (8.8), we find that

$$(8.9) \qquad S \ll \left(\frac{i}{\varphi(i)} + \sigma_{-1/2}(i^2)\right)^2 \cdot \frac{1}{\log^2(Z/Y)} \frac{1}{i^2}.$$

Substituting this estimate into (8.7), we find that

$$T \ll \frac{1}{\log^2(Z/Y)} \cdot \sum_{1 \leq j_0 \leq (1/2\pi) Z} j_0^{2\sigma-2+\beta} \sum_{2\pi j_0 \leq i \leq Z} \frac{1}{i^{2\sigma-2+\beta}} \frac{1}{i^2}\left(\frac{i}{\varphi(i)}+\sigma_{-1/2}(i^2)\right)^2$$

$$\ll \frac{1}{(\log q)^2} \sum_{1 \leq j_0 \leq (1/2\pi) Z} j_0^{2\sigma-2+\beta} \sum_{2\pi j_0 \leq i \leq Z} \frac{1}{i^{2\sigma+\beta}}$$

$$\ll \frac{1}{(1-2\sigma-\beta)(\log q)^2}(2\pi)^{1-2\sigma-\beta} \sum_{1 \leq j_0 \leq (1/2\pi) Z} \frac{1}{j_0}$$

$$\ll \frac{1}{\log q} \frac{(2\pi)^{1-2\sigma-\beta}}{1-2\sigma-\beta}.$$

Here, we have used the fact that

$$\left(\frac{i}{\varphi(i)}+\sigma_{-1/2}(i^2)\right)^2$$

is bounded on average. This proves the Lemma.

We now apply Lemma (8.2) to analyze (8.6). We find that it is equal to

$$\sum \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^s n_1^{1-\bar{s}} n_2^{1-\bar{s}}}\left\{\left|\sin\left(\frac{\pi}{2}(s+a)\right)\Gamma(1-s)\right|^2\right.$$

$$+2\sum_{k=1}^{K} \operatorname{Re}\left(\sin\frac{\pi}{2}(s+a).\Gamma(1-s).\left(\frac{2\pi n_1}{m_1}\right)^{1-s+k}\bar{\omega}_k\right.$$

$$\left.+\sum_{k_1,k_2=1}^{K}\left(\frac{2\pi n_1}{m_1}\right)^{1-\sigma+k_1}\omega_{k_1}\left(\frac{2\pi n_1}{m_1}\right)^{1-\sigma+k_2}\bar{\omega}_{k_2}\right\}$$

$$=\left|\sin\left(\frac{\pi}{2}(s+a)\right)\Gamma(1-s)\right|^2 \sum \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^s n_1^{1-\bar{s}} n_2^{1-\bar{s}}}$$

$$+\mathbf{O}\left(\sum_{k=1}^{K} \frac{1}{k^{\sigma-|t|}}(2\pi)^{1-\sigma+k}\left|\sin\left(\frac{\pi}{2}(s+a)\right)\Gamma(1-s)\right|\frac{1}{\log q}\frac{(2\pi)^{1-2\sigma-(1-\sigma+k)}}{1-2\sigma-(1-\sigma+k)}\right)$$

$$+\mathbf{O}\left(\sum_{k_1,k_2=1}^{K} \frac{1}{(k_1 k_2)^{\sigma-|t|}}\cdot(2\pi)^{2-2\sigma+k_1+k_2}\frac{1}{\log q}\frac{(2\pi)^{1-2\sigma-(2-2\sigma+k_1+k_2)}}{1-2\sigma-(2-2\sigma+k_1+k_2)}\right).$$

By our assumption that $|t| \ll (\log q)^{-1}$ we may ignore $|t|$ in the estimations below. We observe that the sum over k in the first error term is

$$\sum_{k=1}^{K} \frac{1}{k^\sigma}\frac{(2\pi)^{1-2\sigma}}{(k+\sigma)} \ll 1.$$

Since $\sigma \geqq (1/2)$, the double sum over k_1, k_2 is

$$\sum_{k_1, k_2 = 1}^{K} \frac{(2\pi)^{1-2\sigma}}{(k_1 k_2)^{\sigma} (k_1 + k_2 + 1)} \ll \begin{cases} (\log K) & \text{always} \\ (2\sigma - 1)^{-1} & \text{if } \sigma > (1/2) \end{cases}.$$

We also note that if σ is close to 1/2, it is sometimes more convenient to use the first estimate. Recalling that $K = (\log q)^{20}$, we deduce that

$$(8.10) \quad M_a(s) = \sum_{\substack{(m_1, m_2, n_1, n_2) \in \mathcal{S} \\ n_1 \leqq (1/2\pi) m_1}} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right|^2 + \mathbf{O}\left(\frac{1}{(2\sigma - 1)\log q}\right) + \mathbf{O}\left(\frac{1}{\log q} \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right| \right)$$

where $2\sigma - 1$ is to be interpreted as $(\log\log q)^{-1}$ when $\sigma = 1/2$. By an entirely analogous argument, it can be shown that

$$(8.11) \qquad E_a(s) = \mathbf{O}\left(\frac{1}{(2\sigma - 1)\log q}\right) + \mathbf{O}\left(\frac{1}{\log q} \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right| \right)$$

with the same interpretation of $2\sigma - 1$ as above.

To summarize, we deduce from (6.1), Lemma (7.1), Lemma (7.2), (8.1), (8.9) and (8.10), (8.11) that

$$\sum_{1 \neq \chi \,(\mathrm{mod}\, q)} |I(s, \chi)|^2 = \frac{1}{\pi} (2\pi)^{2\sigma - 1} \frac{\varphi(q)}{q^{2\sigma - 1}} \left(\left| \sin\left(\frac{\pi}{2} s\right) \right|^2 + \left| \sin\left(\frac{\pi}{2}(s+1)\right) \right|^2 \right) |\Gamma(1-s)|^2 \sum_{\substack{(m_1, m_2, n_1, n_2) \in \mathcal{S} \\ n_1 \leqq (1/2\pi) m_1}} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}}$$
$$+ \mathbf{O}\left(\frac{\varphi(q)}{q^{2\sigma-1} \log q} \min\left(\frac{1}{(2\sigma - 1)}, \log\log q\right)\right) + \mathbf{O}\left(\frac{\varphi(q) |\Gamma(1-s)|}{q^{2\sigma - 1} (\log q)}\right) + \mathbf{O}\left(\frac{q^{1-\sigma}}{(1-\sigma)\log q}\right) + \mathbf{O}\left(\frac{q^{1-\sigma}}{(1-\sigma)^2}\right) + \mathbf{O}\left(\frac{\varphi(q)}{q^{2\sigma - 1}} \frac{1}{(1-\sigma)^4 (\log q)^2}\right).$$

9. Analysis of the main term. — We shall now analyze the sum in the main term, namely,

$$N(\sigma) \stackrel{\text{def}}{=} \sum \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}}$$

where the sum ranges over quadruples (m_1, m_2, n_1, n_2) satisfying

$$1 \leqq m_1, m_2, n_1, n_2 < Z$$
$$m_1 n_2 = m_2 n_1$$
$$n_1 \leqq \frac{1}{2\pi} m_1.$$

We note that $N(\sigma)$ is well defined since the relation $m_1 n_2 = m_2 n_1$ makes the right hand side independent of the imaginary part of s.

As before, we set $j = m_1 n_2 = m_2 n_1$, $i = (m_1, m_2)$. Note that given m_1, m_2 and j, n_1 and n_2 are uniquely determined. We may thus rewrite $N(\sigma)$ as

$$N(\sigma) = \sum_{1 \leqq m_1, m_2 < Z} \frac{\lambda(m_1)\lambda(m_2)}{(m_1 m_2)^{2\sigma - 1}} \sum j^{2(\sigma - 1)}$$

where the inner sum ranges over integers j satisfying

$$1 \leqq j \leqq \frac{1}{2\pi} m_1 m_2$$
$$j \equiv 0 \qquad (\operatorname{mod}[m_1, m_2]).$$

We can rewrite $N(\sigma)$ as in (8.7). Thus, setting $j = j_0 [m_1, m_2]$ and $i = (m_1, m_2)$, we get

$$N(\sigma) = \sum_{1 \leqq j_0 \leqq (1/2\pi) Z} j_0^{2\sigma - 2} \sum_i i^{-2\sigma} \sum \frac{\lambda(i j_1)\lambda(i j_2)}{j_1 j_2} \tag{9.1}$$

where the sum over i ranges over

$$2\pi j_0 \leqq i \leqq Z$$

and the inner sum ranges over pairs (j_1, j_2) satisfying

$$1 \leqq j_1 \leqq \frac{Z}{i}, \qquad 1 \leqq j_2 \leqq \frac{Z}{i} \tag{9.2}$$
$$(j_1, j_2) = 1.$$

Notice that we can also stipulate that

$$i \leqq \min(m_1, m_2) \leqq Z \tag{9.3}$$

and that

$$(j_1, i) = (j_2, i) = 1.$$

We write the innermost sum of (9.1) as

$$(9.4)\qquad \sum \frac{\lambda(ij_1)\lambda(ij_2)}{j_1 j_2} \sum_{\substack{e \mid j_1 \\ e \mid j_2}} \mu(e) = \sum_e \frac{\mu(e)}{e^2} \sum_{l_1, l_2} \frac{\lambda(iel_1)\lambda(iel_2)}{l_1 l_2}$$

where on the right, e ranges over

$$1 \leqq e \leqq \frac{Z}{i}$$

and l_1, l_2 range over

$$1 \leqq l_1 \leqq \frac{Z}{ie}, \qquad 1 \leqq l_2 \leqq \frac{Z}{ie}$$

$$(el_1, i) = (el_2, i) = 1.$$

Writing

$$\lambda(m) = \frac{\Lambda_1(m) - \Lambda_2(m)}{(\log Z/Y)},$$

we find that (9.4) breaks up into four subsums of the form

$$(9.5)\qquad \frac{1}{(\log Z/Y)^2} \sum \frac{\mu(e)}{e^2} \left\{ \sum_{l_1} \frac{\Lambda_g(iel_1)}{l_1} \right\} \left\{ \sum_{l_2} \frac{\Lambda_h(iel_2)}{l_2} \right\}$$

where $g, h \in \{1, 2\}$, $z_1 = Y$, $z_2 = Z$.

LEMMA (9.1). — *Define*

$$X_{g,h} = \sum \frac{\mu(e)}{e^2} \left\{ \sum_{l_1} \frac{\Lambda_g(iel_1)}{l_1} \right\} \left\{ \sum_{l_2} \frac{\Lambda_h(iel_2)}{l_2} \right\}$$

and let $z = \min(z_g, z_h)$. *Then,*

$$X = 0 \qquad \textit{if} \quad i > z.$$

If $i \leqq z$, *then*

$$(9.6)\quad X = \sum \frac{\mu(e)}{e^2} \mu(ie)^2 \cdot \left(\frac{ie}{\varphi(ie)} \right)^2 + O_c\left(\frac{\sigma_{-1/2}(i)\, i}{\varphi(i)(\log(2Z/i))^c} \right) + O_c\left(\frac{\sigma_{-1/2}(i)^2}{(\log(2Z/i))^{2c}} \right).$$

Proof. — The first assertion is obvious from the definition of Λ_g and Λ_h. Therefore, suppose that

$$1 \leqq e \leqq \frac{Z}{i}.$$

We have

$$X_{g,h}=\sum\frac{\mu(e)}{e^2}\left\{\sum_{l_1\leqq z_g/ie}\frac{\mu(iel_1)}{l_1}\log\left(\frac{z_g}{iel_1}\right)\right\}\left\{\sum_{l_2\leqq z_h/ie}\frac{\mu(iel_2)}{l_2}\log\left(\frac{z_h}{iel_2}\right)\right\}$$

$$=\sum\frac{\mu(e)}{e^2}\mu(ie)^2\left\{\sum_{\substack{l_1\leqq z_g/ie\\(l_1,ie)=1}}\frac{\mu(l_1)}{l_1}\log\left(\frac{z_g}{iel_1}\right)\right\}\left\{\sum_{\substack{l_2\leqq z_h/ie\\(l_2,ie)=1}}\frac{\mu(l_2)}{l_2}\log\left(\frac{z_h}{iel_2}\right)\right\}.$$

Using Lemma (1.4), we have for any $c>0$,

$$X_{g,h}=\sum\frac{\mu(e)}{e^2}\mu(ie)^2\left\{\frac{ie}{\varphi(ie)}+\mathbf{O}_c\left(\sigma_{-1/2}(ie)\log^{-c}\left(\frac{2z}{ie}\right)\right)\right\}^2.$$

There are two error terms $\mathscr{E}_1$, $\mathscr{E}_2$ (say). The first is

$$\mathscr{E}_1\ll_c\sum\frac{1}{e^2}\frac{ie}{\varphi(ie)}.\sigma_{-1/2}(ie)\log^{-c}\left(\frac{2z}{ie}\right)=\Sigma_1+\Sigma_2$$

where in Σ_1, $e<\sqrt{z/i}$ and in Σ_2, $\sqrt{z/i}\leqq e<z/i$. We have

$$\Sigma_1\ll_c\sum\frac{i}{\varphi(i)}\sigma_{-1/2}(i)\frac{1}{(\log(2z/i))^c}\frac{\sigma_{-1/2}(e)}{e\varphi(e)}$$

$$\ll_c\frac{\sigma_{-1/2}(i)\,i}{\varphi(i)(\log(2z/i))^c}.$$

Also,

$$\Sigma_2\ll_c\sum\frac{i}{\varphi(i)}\sigma_{-1/2}(i).\frac{\sigma_{-1/2}(e)}{\varphi(e)\,e}$$

$$\ll_c\sigma_{-1/2}(i)\frac{i}{\varphi(i)}\frac{\sqrt{i}}{\sqrt{z}}$$

$$\ll_c\frac{\sigma_{-1/2}(i)\,i}{\varphi(i)(\log(2z/i))^c}$$

for any $c>0$. The second error term is

$$\mathscr{E}_2\ll_c\sum\frac{1}{e^2}\sigma_{-1/2}(ie)^2\log^{-2c}\left(\frac{2z}{ie}\right)$$

$$\ll_c\sigma_{-1/2}(i)^2\sum\frac{1}{e^2}\sigma_{-1/2}(e)^2.\log^{-2c}\left(\frac{2z}{ie}\right)$$

$$\ll_c\sigma_{-1/2}(i)^2\left(\log\frac{2z}{i}\right)^{-2c}$$

for any $c>0$. The last estimate is obtained by proceeding as with $\mathscr{E}_1$. This proves the lemma.

Now, notice that if $i \leqq Y$, then

$$X_{1,1}-2X_{1,2}+X_{2,2}=\sum_{Y/i<e\leqq z/i}\frac{\mu(e)}{e^2}\mu(ie)^2\left(\frac{ie}{\varphi(ie)}\right)^2 + O_c\left(\frac{\sigma_{-1/2}(i)\,i}{\varphi(i)(\log(2Y/i))^c}\right)+O_c\left(\frac{\sigma_{-1/2}(i)^2}{(\log(2Y/i))^c}\right).$$

The contribution of such terms to $N(\sigma)$ is

$$\ll \frac{1}{(\log Z/Y)^2}\sum_{1\leqq j_0\leqq Z/2\pi} j_0^{2\sigma-2}\sum_{2\pi j_0\leqq i\leqq Y} i^{-2\sigma}\left(\frac{i^3}{\varphi(i)^2 Y}+\frac{\sigma_{-1/2}(i)\,i}{\varphi(i)(\log(2Y/i))^c}+\frac{\sigma_{-1/2}(i)^2}{(\log(2Y/i))^c}\right)$$

and this is

$$\ll \frac{\log Z}{(\log Z/Y)^2} \quad \text{if} \quad \sigma=1/2$$

$$\ll \frac{1}{(\log Z/Y)^2(2\sigma-1)} \quad \text{if} \quad \sigma>1/2.$$

On the other hand, if $i>Y$, then $X_{1,1}=X_{1,2}=0$. Since

$$N(\sigma)=\frac{1}{\log^2(Z/Y)}\sum_{1\leqq j_0\leqq(1/2\pi)Z} j_0^{2\sigma-2}\sum_{2\pi j_0\leqq i\leqq Z} i^{-2\sigma}(X_{1,1}-2X_{1,2}+X_{2,2})$$

we deduce that

$$\text{(9.7)}\quad N(\sigma)=\frac{1}{\log^2(Z/Y)}\sum_{1\leqq j_0\leqq(1/2\pi)Z} j_0^{2\sigma-2}\sum_{\substack{2\pi j_0\leqq i\leqq z\\ i>Y}} i^{-2\sigma}\sum_{1\leqq e\leqq Z/i}\frac{\mu(e)}{e^2}\mu(ie)^2\left(\frac{ie}{\varphi(ie)}\right)^2$$
$$+O\left(\frac{1}{\log^2(Z/Y)}\left\{\frac{1}{1-2\sigma}\text{ or }\log Z\right\}\right)$$
$$+O\left(\frac{1}{\log^2(Z/Y)}\sum_{1\leqq j_0\leqq(1/2\pi)Z} j_0^{2\sigma-2}\sum i^{-2\sigma}\left\{\frac{\sigma_{-1/2}(i)}{(\log(2Z/i))^c}+\frac{i}{\varphi(i)}\right\}\frac{\sigma_{-1/2}(i)}{(\log(2Z/i))^c}\right).$$

The first **O**-term is

$$\ll \frac{1}{|1-2\sigma|(\log q)^2} \quad \text{if} \quad \sigma>\frac{1}{2}$$

and

$$\ll \frac{1}{(\log q)} \quad \text{if} \quad \sigma=\frac{1}{2}.$$

Let us simplify the second **O**-term. If $\sigma > 1/2$, we interchange the i and the j_0 sums and we find that this **O**-term is

$$\ll \frac{1}{(\log q)^2} \sum i^{-2\sigma} \left(\log \frac{2Z}{i}\right)^{-c} \sum_{1 \leq j_0 \leq i} j_0^{2\sigma-2}$$

$$\ll \frac{1}{(\log q)^c} \frac{1}{2\sigma - 1}.$$

If $\sigma = 1/2$, then the **O**-term is

$$\ll \frac{1}{(\log 2Z)^c}.$$

(The value of c is not the same at each occurrence.) Summarizing, we have proved that

$$(9.8) \quad N(\sigma) = \frac{1}{\log^2(Z/Y)} \sum_{1 \leq j_0 \leq (1/2\pi) Z} j_0^{2\sigma-2} \sum_{\substack{2\pi j_0 \leq i \leq z \\ i > Y}} i^{-2\sigma} \sum_{1 \leq e \leq Z/i} \frac{\mu(e)}{e^2} \mu(ie)^2 \left(\frac{ie}{\varphi(ie)}\right)^2 + \begin{cases} \mathbf{O}\left(\dfrac{1}{\log q}\right) & \text{if } \sigma = \dfrac{1}{2} \\ \mathbf{O}\left(\dfrac{1}{(\log q)^2} \dfrac{1}{2\sigma - 1}\right) & \text{if } \sigma > \dfrac{1}{2}. \end{cases}$$

Note that in the above sum, we may suppose that e and i are squarefree.

10. THE MAIN TERM: CONTINUED. — Let us define the constant

$$C_2 = \frac{2}{3\zeta(2)} \prod_{p>2} \left(1 + \frac{2}{(p-2)(p+1)}\right) \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \approx .45.$$

The main result of this section is the following.

PROPOSITION (10.1). — *We have*

$$\sum_{1 \neq \chi \ (\mathrm{mod}\ q)} |I(s, \chi)|^2 = c(s, q)\,\varphi(q) + \mathscr{E}(\sigma).$$

Here,

$$c\left(\frac{1}{2} + it, q\right) = \frac{C_2}{\pi} \left|\Gamma\left(\frac{1}{2} - it\right)\right|^2 (\cosh \pi t)$$

and for $1 > \sigma > 1/2$,

$$c(\sigma + it, q) = \frac{2C_2}{\pi} \frac{1}{2\sigma - 1} |\Gamma(1 - \sigma - it)|^2 (\cosh \pi t) \frac{q^{1-2\sigma}}{\log q}.$$

Also,

$$\mathscr{E}\left(\frac{1}{2}\right) \ll \frac{q(\log\log q)}{\log q}$$

and

$$\mathscr{E}(\sigma) \ll \frac{q^{2-2\sigma}}{\log q} \min\left(\frac{1}{(2\sigma-1)}, \log\log q\right)$$

if $1/2<\sigma\leqq 3/4$ (*say*), *while*

$$\mathscr{E}(\sigma) \ll q^{2-2\sigma}(\log q)^2$$

if $3/4\leqq\sigma<1-(1/\log q)$.

Proof. — We saw in (6.1), § 7, and § 8 that

$$(10.1)\qquad \sum_{1\neq\chi \,(\mathrm{mod}\, q)} |\mathrm{I}(s,\chi)|^2$$
$$= \frac{1}{\pi}(2\pi)^{2\sigma-1}|\Gamma(1-s)|^2 \mathrm{N}(\sigma)\varphi(q)q^{1-2\sigma}\left(\left|\sin\left(\frac{\pi}{2}s\right)\right|^2+\left|\sin\left(\frac{\pi}{2}(s+1)\right)\right|^2\right)$$
$$+\mathbf{O}\left(\frac{\varphi(q)}{q^{2\sigma-1}\log q}\min\left(\frac{1}{(2\sigma-1)}, \log\log q\right)\right)+\mathbf{O}\left(\frac{\varphi(q)|\Gamma(1-s)|}{q^{2\sigma-1}(\log q)}\right)$$
$$+\mathbf{O}\left(\frac{q^{1-\sigma}}{(1-\sigma)\log q}\right)+\mathbf{O}\left(\frac{q^{1-\sigma}}{(1-\sigma)^2}\right)+\mathbf{O}\left(\frac{\varphi(q)}{q^{2\sigma-1}}\,\frac{1}{(1-\sigma)^4(\log q)^2}\right).$$

We shall now study the main term of $\mathrm{N}(\sigma)$. From (9.8), we see that it is

$$(10.2)\qquad = \frac{1}{\log^2(\mathrm{Z}/\mathrm{Y})}\sum_{1\leqq j_0\leqq(1/2\pi)\mathrm{Z}} j_0^{2\sigma-2}\sum_{\substack{2\pi j_0\leqq i\leqq \mathrm{Z}\\ i>\mathrm{Y}}}\frac{i^{2-2\sigma}\mu(i)^2}{\varphi(i)^2}\sum_{\substack{e\leqq \mathrm{Z}/i\\ (e,i)=1}}\frac{\mu(e)}{\varphi(e)^2}$$
$$= \frac{4}{(\log q)^2}\cdot\sum_{\substack{2\pi\leqq i\leqq \mathrm{Z}\\ i>\mathrm{Y}}}\frac{i^{2-2\sigma}\mu(i)^2}{\varphi(i)^2}\left\{\sum_{1\leqq j_0\leqq i/2\pi} j_0^{2\sigma-2}\right\}\left\{\sum_{\substack{e\leqq \mathrm{Z}/i\\ (e,i)=1}}\frac{\mu(e)}{\varphi(e)^2}\right\}.$$

We note that

$$\sum_{1\leqq j_0\leqq i/2\pi} j_0^{2\sigma-2} = \begin{cases}\left(\dfrac{i}{2\pi}\right)^{2\sigma-1}\cdot\dfrac{1}{2\sigma-1}+\mathbf{O}(i^{2\sigma-2}) & \text{if}\quad \sigma\neq 1/2\\[2ex] \log\left(\dfrac{i}{2\pi}\right)+\mathbf{O}(1) & \text{if}\quad \sigma=1/2.\end{cases}$$

We easily check that the contribution of the **O**-terms is

$$\ll \frac{1}{(\log q)}$$

which is negligible.

If we replace in (10.2) the sum over e with

$$\sum_{\substack{e=1\\(e,\,i)=1}}^{\infty} \frac{\mu(e)}{\varphi(e)^2},$$

we introduce an error of

$$\ll \frac{1}{(\log q)^2}\,\frac{1}{2\sigma-1} \quad \text{if} \quad \sigma \neq \frac{1}{2}$$

and of

$$\ll \frac{1}{\log q} \quad \text{if} \quad \sigma = \frac{1}{2}.$$

In any case, it is negligible.

Notice that

$$\sum_{\substack{e=1\\(e,\,i)=1}}^{\infty} \frac{\mu(e)}{\varphi(e)^2} = \begin{cases} \prod_{p \nmid i}\left(1-\frac{1}{(p-1)^2}\right) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is old.} \end{cases}$$

We see that for $\sigma > 1/2$,

$$\mathrm{N}(\sigma) = \frac{1}{(2\pi)^{2\sigma-1}}\,\frac{1}{2\sigma-1}\,\frac{4}{(\log q)^2} \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) \sum_{i \text{ even}} \frac{\mu(i)^2\, i}{\varphi(i)^2} \prod_{p \mid (i/2)}\left(1-\frac{1}{(p-1)^2}\right)^{-1} + \mathbf{O}\left(\frac{1}{\log q}\right) + \mathbf{O}\left(\frac{1}{(\log q)^2}\,\frac{1}{2\sigma-1}\right).$$

On the other hand, for the case $\sigma = 1/2$ we have

$$\mathrm{N}\left(\frac{1}{2}\right) = \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\frac{4}{(\log q)^2} \sum_{i \text{ even}} \frac{i\,\mu^2(i)}{\varphi(i)^2}\log\left(\frac{i}{2\pi}\right) \prod_{p \mid (i/2)}\left(1-\frac{1}{(p-1)^2}\right)^{-1} + \mathbf{O}\left(\frac{1}{\log q}\right). \tag{10.3}$$

Here, the sum over i has range

$$\max(\mathrm{Y},\, 2\pi) \leqq i \leqq \mathrm{Z}, \qquad i \text{ even}.$$

Also, note that as i may be assumed squarefree, $i/2$ is odd. We observe that

$$\frac{i\mu^2(i)}{\varphi(i)^2}\cdot\prod_{p\mid(i/2)}\left(1-\frac{1}{(p-1)^2}\right)^{-1}=4\frac{\mu^2(i)}{i}\cdot\prod_{p\mid(i/2)}\frac{p^2}{(p-1)^2}\cdot\frac{(p-1)^2}{((p-1)^2-1)}=2\frac{\mu^2(i)}{\varphi_1(i/2)},$$

where $\varphi_1(n)$ is defined by

$$\varphi(n)=\sum_{d\mid n}\varphi_1(n).$$

Thus, the sum over i is

$$2\sum\frac{\mu^2(i)}{\varphi_1(i/2)}\log\left(\frac{i}{2\pi}\right)\quad\text{if}\quad\sigma=\frac{1}{2}$$

$$2\sum\frac{\mu^2(i)}{\varphi_1(i/2)}\quad\text{if}\quad\sigma>\frac{1}{2}.$$

This sum can be estimated as follows. We first observe that for $\mathrm{Re}(s)>1$

$$\sum_{\substack{i=1\\ i\text{ odd}}}^{\infty}\frac{\mu^2(i)\,i}{\varphi_1(i)\,i^s}=\prod_{p>2}\left(1+\frac{p}{(p-2)p^s}\right).$$

Now,

$$\left(1+\frac{p}{(p-2)p^s}\right)=\left(1-\frac{1}{p^{2s}}\right)\left(1-\frac{1}{p^s}\right)^{-1}\left(1+\frac{2}{(p-2)(p^s+1)}\right).$$

Thus,

$$\sum_{\substack{i\leq x\\ i\text{ odd}}}\frac{\mu^2(i)\,i}{\varphi_1(i)}=\mathrm{C}_1\,x+\mathbf{O}(x^{3/4})$$

where

$$\mathrm{C}_1=\frac{2}{3\zeta(2)}\prod_{p>2}\left(1+\frac{2}{(p-2)(p+1)}\right).$$

By partial summation, it follows that

$$2\sum_{\substack{i=1\\ i\text{ odd}}}^{\mathrm{Z}/2}\frac{\mu^2(i)\log i}{\varphi_1(i)}=2\int_{1^-}^{\mathrm{Z}/2}\left(\frac{\log u}{u}\right)d\left(\sum_{\substack{n\leq u,\\ n\text{ odd}}}\frac{\mu^2(n)\,n}{\varphi_1(n)}\right)$$

$$=\mathrm{C}_1(\log\mathrm{Z})^2+\mathbf{O}(\log\mathrm{Z}).$$

Similarly,

$$2 \sum_{\substack{i \leqq Z/2 \\ i \text{ odd}}} \frac{\mu^2(i)}{\varphi_1(i)} = 2\,C_1 \log(Z/2) + \mathbf{O}(1).$$

Substituting this information into (10.3), we find that

$$N\left(\frac{1}{2}\right) = C_2 + \mathbf{O}\left(\frac{1}{\log q}\right)$$

where

$$C_2 = C_1 \cdot \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \approx .45.$$

Similarly, if $\sigma > 1/2$,

$$N(\sigma) = 4\,C_2 \cdot \frac{(2\pi)^{1-2\sigma}}{2\sigma-1} \cdot \frac{1}{(\log q)^2} (\log q + \mathbf{O}(1)) + \mathbf{O}\left(\frac{1}{(\log q)^2 (2\sigma-1)}\right).$$

Inserting this into (10.1), and observing that

$$|\sin(x+iy)|^2 + |\cos(x+iy)|^2 = \cosh 2y$$

we deduce that

$$\sum_{1 \neq \chi \,(\mathrm{mod}\, q)} |I(s,\chi)|^2 = c(s,q)\,\varphi(q) + \mathscr{E}$$

where

$$c(\sigma+it, q) = \begin{cases} \dfrac{C_2}{\pi} \left|\Gamma\left(\dfrac{1}{2} - it\right)\right|^2 (\cosh \pi t) & \text{if } \sigma = \dfrac{1}{2} \\[2ex] \dfrac{4\,C_2}{\pi} \dfrac{1}{2\sigma-1} |\Gamma(1-\sigma-it)|^2 (\cosh \pi t) \dfrac{q^{1-2\sigma}}{\log q} & \text{if } \sigma > \dfrac{1}{2} \end{cases}$$

and

$$\mathscr{E} = \mathbf{O}\left(\frac{q^{2-2\sigma}}{\log q} \min\left(\frac{1}{(2\sigma-1)}, \log\log q\right)\right) + \mathbf{O}\left(\frac{q^{2-2\sigma}}{(1-\sigma)(\log q)}\right) + \mathbf{O}\left(\frac{q^{2-2\sigma}}{(\log q)^2 (1-\sigma)^4}\right) + \mathbf{O}\left(\frac{q^{1-\sigma}}{(1-\sigma)^2}\right) + \mathscr{E}_1(\sigma)$$

and

$$\mathscr{E}_1\left(\frac{1}{2}\right) \ll \frac{q}{\log q}$$

and for $1/2<\sigma<1$,

$$\mathscr{E}_1(\sigma) \ll \frac{q^{2-2\sigma}}{(1-\sigma)^2(2\sigma-1)(\log q)^2}.$$

In particular, we see that if $\sigma=1/2$

$$\mathscr{E} \ll \frac{q(\log\log q)}{\log q}.$$

If $1/2<\sigma\leqq 3/4$ (say),

$$\mathscr{E} \ll \frac{q^{2-2\sigma}}{\log q}\min\left(\frac{1}{(2\sigma-1)}, \log\log q\right)$$

and if $3/4\leqq\sigma<1-(1/\log q)$

$$\mathscr{E} \ll q^{2-2\sigma}(\log q)^2.$$

This proves the result.

11. Non-vanishing at a fixed point. — The main result of this section is the following.

Theorem (11.1). — *Fix a* σ *in the interval* $1/2\leqq\sigma<1$. *Then, for all sufficiently large primes* q,

$$L(\sigma, \chi)\neq 0$$

for a positive proportion of the characters $\chi\ (\mathrm{mod}\ q)$.

Remark. — The proof will produce a lower bound for this proportion. Notice that how large q must be taken will depend on σ.

Proof. — Let us fix $s_0\in\mathbf{C}$ with $1/2\leqq\mathrm{Re}\,s_0<1-(1/\log q)$. We return now to (3.2). For $\chi\neq 1$, we have

$$S(s_0, \chi)=L(s_0, \chi)M(s_0, \chi)+I(s_0, \chi)+J(s_0, \chi).$$

Thus,

$$\sum_{\chi\neq 1} S(s_0, \chi)={\sum}'(I(s_0, \chi)+J(s_0, \chi))+{\sum}''S(s_0, \chi)$$

where $\sum'$ ranges over $\chi\neq 1$ such that $L(s_0, \chi)=0$ and $\sum''$ over the remaining non-trivial $\chi\ (\mathrm{mod}\ q)$. By Proposition (4.1), we have

$$\sum_{\chi\neq 1} S(s_0, \chi)=\varphi(q)+\mathbf{O}_\varepsilon(q^{1-\sigma+\varepsilon}).$$

Thus, we have

$${\sum}''S(s_0, \chi)=\varphi(q)-{\sum}'(I(s_0, \chi)+J(s_0, \chi))+\mathbf{O}_\varepsilon(q^{1-\sigma_0+\varepsilon})$$

and consequently,

$$\sum'' S(s_0, \chi) \geqq \varphi(q) - \left|\sum' (I(s_0, \chi) + J(s_0, \chi))\right| + O_\varepsilon(q^{1-\sigma+\varepsilon}).$$

Now, if we assume that $|\operatorname{Im} s_0| < 1$ (say), then

$$\left|\sum' (I(s_0, \chi) + J(s_0, \chi))\right| \leqq \sum |I(s_0, \chi)| + \sum |J(s_0, \chi)|$$

$$\leqq \varphi(q)^{1/2} \left(\sum |I(s_0, \chi)|^2\right)^{1/2} + O\left(\frac{q^{(3/2)-\sigma}}{\log q}\right)$$

by Proposition (3.1). Now using the main result of § 10, namely

$$\sum |I(s_0, \chi)|^2 = c(s_0, q) \cdot \varphi(q) + \mathscr{E}(\sigma)$$

we have

$$\left|\sum' (I(s_0, \chi) + J(s_0, \chi))\right| \leqq \sqrt{c(s_0, q)}\, \varphi(q) + O\left(\sqrt{\varphi(q)\mathscr{E}(\sigma)}\right) + O(q^{3/2-\sigma}(\log q)^{-1}).$$

Thus,

$$\left|\sum'' S(s, \chi)\right| \geqq \left(1 - \sqrt{c(s_0, q)}\right) \varphi(q) + O(q^{1-\sigma+\varepsilon}) + O\left(\sqrt{\varphi(q)\mathscr{E}(\sigma)}\right) + O(q^{3/2-\sigma}(\log q)^{-1}).$$

On the other hand, by the Cauchy-Schwarz inequality, setting $\mathscr{N}(s_0, q)$ to be the number of $\chi \pmod q$ with $L(s_0, \chi) \neq 0$, we get

$$\left|\sum'' S(s_0, \chi)\right|^2 \leqq \mathscr{N}(s_0, q) \left(\sum |S(s_0, \chi)|^2\right).$$

From now on, we shall assume that $t = \operatorname{Im}(s_0)$ satisfies $|t| \ll (\log q)^{-1}$. Suppose first that $\sigma_0 = 1/2$. We have from Proposition (4.2)

$$\sum_{\chi \pmod q} \left| S\left(\frac{1}{2} + it, \chi\right)\right|^2 = \frac{5}{2} \varphi(q) + O(q(\log q)^{-1/2}).$$

We deduce that

$$\frac{2}{5} \varphi(q) \left(1 - \sqrt{c(s_0, q)}\right)^2 + O\left(q \sqrt{\frac{\log \log q}{\log q}}\right) \leqq \mathscr{N}(s_0, q)(1 + O((\log q)^{-1/2})).$$

Thus,

$$\mathscr{N}(s_0, q) \geqq \frac{2}{5} \varphi(q) \left(1 - \sqrt{C_2}\right)^2 + O\left(q \sqrt{\frac{\log \log q}{\log q}}\right).$$

Now let us set

$$j = \left[\left(\sigma_0 - \frac{1}{2}\right) \log q\right] + 1.$$

Thus,

$$\frac{1}{2}+\frac{j-1}{\log q}<\sigma_0 \leqq \frac{1}{2}+\frac{j}{\log q}.$$

We will suppose that q is sufficiently large that $j \geqq 2$. Then Proposition (4.2) gives

$$(11.1) \quad \sum_{\chi \,(\mathrm{mod}\, q)} |S(\sigma_0+it_0,\chi)|^2=\varphi(q)\left\{1-\frac{e^{-j+1}}{(j-1)^2}+\frac{Y^{1-2\sigma_0}}{(j-1)^2}-\frac{e^{-2(j-1)}}{j-1}\right.$$
$$\left.+\mathbf{O}\left(\frac{e^{-2j}}{\log q}\left(1+\frac{1}{j}+\frac{1}{(\log q)^{\sigma_0}(1-\sigma_0)}\right)\right)\right\}$$
$$=\varphi(q)\left\{1-\frac{e^{-j+1}}{(j-1)^2}+\frac{Y^{1-2\sigma_0}}{(j-1)^2}-\frac{e^{-2(j-1)}}{j-1}+\mathbf{o}(e^{-j})\right\}.$$

Also, if σ_0 is bounded away from 1 (say $\sigma_0 \leqq 3/4$) then

$$\sum_{\chi \neq 1} |I(s_0,\chi)|^2=c(s_0,q)\varphi(q)+\mathbf{O}\left(qe^{-2j}\min\left(\frac{1}{j-1},\frac{\log\log q}{\log q}\right)\right).$$

If $3/4 \leqq \sigma_0<1-(\log q)^{-1}$, Then

$$\sum_{\chi \neq 1} |I(s_0,\chi)|^2=c(s_0,q)\varphi(q)+\mathbf{O}(qe^{-2j}(\log q)^2).$$

We see that under our assumption on $|t|$, we have

$$c(s_0,q)=\frac{2C_2}{\pi j}|\Gamma(1-\sigma_0)|^2 e^{-2j}+\mathbf{O}\left(\frac{1}{\log q}\right).$$

Putting all these estimates together, we deduce that

$$\mathcal{N}(s_0,q) \geqq (\alpha_j+\mathbf{o}(1))\varphi(q)$$

Where

$$\alpha_j=\frac{(1-e^{-j}\sqrt{2C_2/\pi j}|\Gamma(1-\sigma_0)|)^2}{(1-(j-1)^{-2}(e^{-j+1}-1)-(j-1)^{-1}e^{-2j+2})}.$$

12. Non-Vanishing at a variable point. — In the previous section, we showed that a positive proportion of the $L(s,\chi)$ are non-zero at a given real value σ of s in the critical strip. Now we shall refine this to a statement uniform on a line. Up to this point, we have made no significant use of the parameter Y. We shall now choose it to be $Y=q^{1/4}$.

Theorem (12.1). — *Suppose that q is a sufficiently large prime. For a positive proportion of the χ (mod q), $L(s,\chi)$ does not have a real zero in the region $1/2+c/\log q \leqq \sigma<1$. Here, $c>0$ is an absolute constant.*

Proof. By the functional equation, it suffices to concentrate attention on the region $\sigma \geqq 1/2$. It is well known that there is at most one χ with a real zero in the range

$$1-(\log q)^{-1} \leqq \sigma \leqq 1.$$

Thus we consider

$$\frac{1}{2}+\frac{2}{\log q} \leqq \sigma < 1-(\log q)^{-1}$$

and split it into intervals

$$\mathrm{I}_j:\ \frac{1}{2}+\frac{j}{\log q} < \sigma \leqq \frac{1}{2}+\frac{j+1}{\log q},$$

of length $1/\log q$. Here $2 \leqq j \leqq (1/2)\log q - 2$. We count the number $\mathrm{Z}(j, q)$ of $\chi \pmod q$ for which $\mathrm{L}(s, \chi)$ has a zero in I_j. For each χ, let $\sigma(\chi)$ denote a point in I_j, and let $\sigma=\sigma_j=(1/2)+(j/\log q)$. Let $\mathrm{C}=\mathrm{C}_j$ denote the circle of radius $r=r_j=2/\log q$ about σ. We have by Cauchy's theorem

$$\sum_{n>\mathrm{Y}} \frac{a(n)\chi(n)}{n^{\sigma(\chi)}} e^{-n/q} - \sum_{n>\mathrm{Y}} \frac{a(n)\chi(n)}{n^{\sigma}} e^{-n/q}$$
$$= \frac{1}{2\pi i}\int_{\mathrm{C}} \left\{ \sum_{n>\mathrm{Y}} \frac{a(n)\chi(n)}{n^{w}} e^{-n/q} \right\} \left(\frac{1}{w-\sigma(\chi)} - \frac{1}{w-\sigma} \right) dw.$$

Let us denote the left hand side by $\mathrm{S}_{\mathrm{diff}}(\sigma, \sigma_\chi, \chi)$ and let us write $w=u+iv$. By (a variant of) Proposition (4.2).

$$(12.1)\quad \sum_{\chi} \left| \sum_{n>\mathrm{Y}} a(n)\chi(n) n^{-w} \exp\left(-\frac{n}{q}\right) \right|^2$$
$$= \varphi(q) \sum_{\mathrm{Y} \leqq n \leqq q} \frac{a(n)^2}{n^{2u}} + \mathbf{O}\left(\frac{q^{(3/2)-u}}{(1-2u)(\log q)^2} \right)$$
$$+ \mathbf{O}(q^{1-u}(\log q)^{3/2}) + \mathbf{O}\left(q^{2-2u}(\log q)^{3u-2} \frac{1}{1-u} \right)$$

for $1 > \sigma > (1/2)-(1/\log q)$. Now

$$\sum_{\chi} |\mathrm{S}_{\mathrm{diff}}(\sigma, \sigma_\chi, \chi)|^2 \leqq \sum_{\chi} \frac{1}{4\pi^2} \left(\int_{\mathrm{C}} \left| \sum_{n>\mathrm{Y}} \frac{a(n)\chi(n)}{n^{w}} e^{-n/q} \right|^2 |dw| \right)$$
$$\times \left(\int_{\mathrm{C}} \left| \frac{1}{w-\sigma(\chi)} - \frac{1}{w-\sigma} \right|^2 |dw| \right)$$

and

$$\int_C \left| \frac{1}{w-\sigma(\chi)} - \frac{1}{w-\sigma} \right|^2 |dw| = \int_C \left| \frac{\sigma(\chi)-\sigma}{(w-\sigma(\chi))(w-\sigma)} \right|^2 |dw|$$
$$\leqq \frac{(r/2)^2}{r^2 (r/2)^2} . 2\pi r$$
$$= \frac{2\pi}{r}.$$

Therefore, for $j \geqq 1$, we have by (12.1) (*see* also (11.1)) that

$$\sum_\chi |S_{\text{diff}}(\sigma, \sigma_\chi, \chi)|^2 \leqq \frac{1}{2\pi r} \int_C \sum_\chi \left| \sum_{n>Y} \frac{a(n)\chi(n)}{n^w} e^{-n/q} \right|^2 |dw|$$
$$\leqq \varphi(q) \left(\sum_{Y \leqq n \leqq q} \frac{a(n)^2}{n^{2((1/2)+((j-2)/\log q))}} + \mathbf{o}(e^{-j}) \right).$$

We observe that

$$\sum_{Y \leqq n \leqq q} \frac{a(n)^2}{n^{2((1/2)+((j-2)/\log q))}}$$
$$\sim \sum_{Y \leqq n \leqq Z} \frac{\log n/Y}{(\log Z/Y)^2} \frac{1}{n^{1+((2j-4)/\log q)}} + \sum_{Z \leqq n \leqq q} \frac{1}{(\log Z/Y)} \frac{1}{n^{1+((2j-4)/\log q)}}$$

and this is seen to be

$$\sim \frac{4}{(j-2)^2} \{ Y^{-(2j-4)/\log q} - Z^{-(2j-4)/\log q} \} - \frac{2}{j-2} q^{-(2j-4)/\log q}$$

and this is

$$= \frac{4}{(j-2)^2} (e^{-1/2(j-2)} - e^{-(j-2)}) - \frac{2}{j-2} e^{-(2j-4)}.$$

(Here, we have used the fact that $Y = q^{1/4}$.) Let us denote the above expression by $f(j-2)$. If $j=2$ we have to replace the above by

$$\frac{5}{2} + \mathbf{O}((\log q)^{-1}).$$

Then,

$$(12.2)\quad \sum_\chi \left| \sum_{n>Y} \frac{a(n)\chi(n)}{n^{\sigma(\chi)}} e^{-n/q} \right|^2 \leqq 2\left\{ \sum_\chi \left| \sum_{n>Y} \frac{a(n)\chi(n)}{n^{\sigma}} e^{-n/q} \right|^2 + \sum_\chi |S_{\text{diff}}(\sigma, \sigma_\chi, \chi)|^2 \right\} \leqq 8\varphi(q)(f(j)+f(j-2)+\mathbf{o}(e^{-j})).$$

It is convenient to introduce here the notation

$$S^*(s, \chi) = \sum_{n>Y} \frac{a(n)\chi(n)}{n^s} e^{-n/q}.$$

Clearly, it is equal to $S(x, \chi)-1$.

Similarly, we have

$$\sum_\chi |I(\sigma_\chi, \chi)|^2 \leqq 2\{\sum_\chi |I(\sigma, \chi)|^2 + \sum_\chi |I_{\text{diff}}(\sigma, \sigma_\chi, \chi)|^2\}$$

where now,

$$I_{\text{diff}}(\sigma, \sigma_\chi, \chi) = \frac{1}{2\pi i}\int_C I(w, \chi)\left(\frac{1}{w-\sigma(\chi)} - \frac{1}{w-\sigma}\right) dw.$$

As before, if $j \leqq \log q/\log\log q$, then by Proposition (10.1), we see that

$$\sum_\chi |I_{\text{diff}}(\sigma, \sigma_\chi, \chi)|^2 \leqq \frac{1}{2\pi r}\int_C \sum_\chi |I(w, \chi)|^2 |dw|$$
$$\leqq \frac{1}{2\pi r}\varphi(q)\left(c(s_0, q) + \mathbf{O}\left(qe^{-2j}\min\left(\frac{1}{j}, \frac{\log\log q}{\log q}\right)\right)\right)2\pi r$$
$$= \varphi(q)\left(\frac{C_2}{\pi j}|\Gamma(1-\sigma_0)|^2 e^{-2j} + \mathbf{O}\left(e^{-2j}\frac{\log\log q}{\log q}\right)\right).$$

If $j \geqq \log q/\log\log q$, the last estimate above is replaced by

$$\leqq \varphi(q)\left(\frac{C_2}{\pi j}|\Gamma(1-\sigma_0)|^2 e^{-2j} + \mathbf{O}(e^{-2j}(\log q)^2)\right).$$

The same estimate holds for

$$\sum_\chi |I(\sigma, \chi)|^2.$$

Hence we deduce that

$$\frac{1}{\varphi(q)}\sum_\chi |I(\sigma_\chi, \chi)|^2 \leqq \frac{4C_2}{\pi j}|\Gamma(1-\sigma_0)|^2 e^{-2j} + \begin{cases} \mathbf{O}(e^{-2j(\log\log q/\log q)}) & \text{if } j \leqq 3/4\log q, \\ \mathbf{O}(e^{-2j}(\log q)^2) & \text{otherwise.} \end{cases}$$

Finally, a calculation similar to the one above and in Proposition (3.1) shows that

$$\sum_{\chi} |J(\sigma_\chi, \chi)|^2 \ll q^{2-2\sigma_0}.$$

With that established, we return to our basic equation

$$S(\sigma_\chi, \chi) = L(\sigma_\chi, \chi) M(\sigma_\chi, \chi) + I(\sigma_\chi, \chi) + J(\sigma_\chi, \chi)$$

and deduce that

$$(12.3)\quad \sum_{\chi} |L(\sigma_\chi, \chi) M(\sigma_\chi, \chi) - 1|^2 \leqq 3 \sum_{\chi} (|S^*(\sigma_\chi, \chi)|^2 + |I(\sigma_\chi, \chi)|^2 + |J(\sigma_\chi, \chi)|^2).$$

Using the estimates established above, we see that the right hand side is

$$(12.4)\qquad \leqq 3\varphi(q)\left(8(f(j)+f(j-2)) + \frac{4C_2}{\pi j}|\Gamma(1-\sigma_0)|^2 e^{-2j} + \mathbf{o}(e^{-j})\right).$$

Now let us set $Z(j, q)$ to be the number of characters $\chi \pmod q$ such that $L(s, \chi)$ has a real zero in the circle C_j. It follows immediately from (12.3) and (12.4) that

$$Z(j, q) \leqq 3\varphi(q)\left(8(f(j)+f(j-2)) + \frac{4C_2}{\pi j}\left|\Gamma\left(\frac{1}{2} - \frac{j+(1/2)}{\log q}\right)\right|^2 e^{-2j} + \mathbf{o}(e^{-j})\right).$$

If we sum this over $j \geqq j_0$, for some absolute constant j_0 we see that we have

$$\frac{1}{\varphi(q)} \sum_{j \geqq j_0} Z(j, q) \leqq 3 \sum_{j \geqq j_0} \left(\frac{32}{(j-2)^2}(e^{-(1/2)(j-2)} - e^{-(j-2)}) + \frac{32}{j^2}(e^{-(1/2)j} - e^{-j}) + \mathbf{o}(e^{-j})\right).$$

If we choose j_0 sufficiently large, we see that the right hand side is <1. This completes the proof.

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