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ON THE SUPERCUSPIDAL REPRESENTATIONS OF GL_N, N THE PRODUCT OF TWO PRIMES

BY PHILIP KUTZKO AND DAVID MANDERSCHEID

The purpose of this paper is the following result.

THEOREM 0. — *Let* F *be a nonarchimedean local field, let* N *be an integer which is the product of two primes and let* $G = GL_N(F)$. *Then every irreducible supercuspidal representation of* G *may be constructed by induction from some maximal open compact-modulo-center subgroup of* G.

Since the significance of — and progress made on — this problem for general N has been the subject of substantial discussion (*see*, e.g., [K4]), we will not dwell on these matters. However, it should be noted that our result is the first case to be settled since the cases N prime [Ca] and N not divisible by the residual characteristic of F [H], [Mo]; moreover, it is not unreasonable to expect that the general case will follow from a suitable generalization of our method.

Our paper is organized as follows. Section one is devoted to an exposition of the results of [B] and [HM1] on fundamental strata and those of [K3] on the consequences of the results of [B] and [HM1] for supercuspidal representations. These results enable us, for the rest of the paper, to assume that we are dealing with an irreducible supercuspidal representation of G which contains a proper alfalfa stratum; that is (*see* Definition 1.12), a representation of the form ψ_α on a subgroup $U^n(\mathscr{A})$, $n \geqq 1$, where $\mathscr{A}$ is a hereditary order in $A_F(V)$, the ring of endomorphisms of a F-vector space V of dimension N (identify G with $(A_F(V))^\times$); $U^n(\mathscr{A}) = 1 + \mathscr{P}^n$ ($\mathscr{P}$ the radical of $\mathscr{A}$); α is an element of $\mathscr{P}^{-n} - \mathscr{P}^{-n+1}$ such that, among other things, $E = F[\alpha]$ is a subfield of $A_F(V)$ and $1 < [E:F] < N$ (whence "proper"); and ψ_α is defined by $\psi_\alpha(x) = \psi(\mathrm{tr}(\alpha(x-1)))$ where ψ is a character of F^+ of conductor P_F. [One should note that, technically, it is the triple $(\mathscr{A}, n, \psi_\alpha)$ rather than the representation ψ_α to which the term "stratum" properly applies.]

In section two we recall and extend the definition and some properties of the map S_α first defined in [KM2]. Using S_α, we are then able to efficiently parametrize the complex

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dual of the group $U^r(\mathscr{A}_E)/U^{r+1}(\mathscr{A}_E)$ $r \geqq 1$ by elements of the abelian group $\mathscr{P}_E^{-r}/\mathscr{P}_E^{1-r}$; where $\mathscr{A}_E = \mathscr{A} \cap A_E(V)$ is a hereditary order in $A_E(V)$, the ring of E-endomorphisms of V [recall $A_F(V)$ contains E], with radical $\mathscr{P}_E = \mathscr{P} \cap A_E(V)$. All of this leads us to define certain extensions of ψ_α and then to select data which we refer to as *relative alfalfa strata.* We then make an appropriate definition of "fundmental" and "level" in this context and then state two of our major results: Theorem 2.20 and Theorem 2.21. These theorems imply, among other things, that an irreducible admissible representation which contains a proper alfalfa stratum also contains a fundamental relative alfalfa stratum. The proofs of these theorems involve the generalization and refinement of results in [B] and [HM1] and occupy the whole of section three.

In section four, by suitably generalizing the methods of [K3], we prove first that any irreducible supercuspidal representation π of G which contains a proper alfalfa stratum also contains either a relative alfalfa stratum which is, in a sense made precise there, itself "alfalfa" or a fundamental relative alfalfa stratum of relative level zero. Our second result is then that if N is the product of two primes and the first circumstance occurs then Theorem 0 holds for π. Finally, in section five we deal with the latter circumstance and thus complete our proof of Theorem 0.

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1. Fundamental strata and alfalfa strata

1.1. HEREDITARY ORDERS. — Let F be a nonarchimedean local field of residual characteristic p. Let $\mathcal{O} = \mathcal{O}_F$ be the ring of integers in F and let $\bar{\omega} = \bar{\omega}_F$ be a generator of the maximal ideal $P = P_F$ in $\mathcal{O}$. Let $k = k_F$ be the residue class field $\mathcal{O}/P$ and let $q = q_F$ be the cardinality of k. Finally, let $\nu(x) = \nu_F(x)$ denote the order of x in F.

Let V be a vector space over F of dimension N. Recall that an $\mathcal{O}_F$-*lattice* (or just a lattice) in V is a free, rank N, $\mathcal{O}_F$-submodule of V.

Let $A = A_F(V) = \mathrm{End}_F(V)$. We shall be concerned with hereditary orders in A. Recall that an *order* in A is a subring $\mathscr{A}$ of A which is also a lattice in A and that an order $\mathscr{A}$ in A is *hereditary* if any $\mathscr{A}$-lattice (*i.e.* $\mathscr{A}$-module which is also an $\mathcal{O}$-lattice) in any finitely generated A-module is $\mathscr{A}$-projective. We now collect some facts concerning hereditary orders. For further details and a more general treatment, *see* [BF].

The standard method for constructing hereditary orders in A is via lattice chains as follows. A *lattice chain* in V is a sequence $L = \{L_i\}_{i \in \mathbb{Z}}$ of $\mathcal{O}$-lattices in V such that:

(i) $L_i \supsetneqq L_{i+1}$ for all integers i.

(ii) There exists $e \geqq 1$ such that $L_{i+e} = PL_i$ for all integers i.

The uniquely determined integer $e=e(L)$ is called the *period* of the chain L. If L is a lattice chain in V, then denote by $\mathscr{A}=\mathscr{A}_F(L)$ the subring of A consisting of elements x for which $x L_i \subseteq L_i$ for all integers i. Then, $\mathscr{A}$ is a hereditary order in A and all hereditary orders in A arise in this manner. Further, if we define the obvious equivalence on the set of lattice chains in V, namely, that two chains are equivalent if one results from the other by a translation of index, then two lattice chains in V give rise to the same hereditary order in A if an only if they are equivalent. Thus one can define the *period* $e=e(\mathscr{A})$ of a hereditary order $\mathscr{A}$ to be the period of a lattice chain giving rise to $\mathscr{A}$.

If $\mathscr{A}=\mathscr{A}_F(L)$ is an order in A, then we let $\mathscr{P}=\mathscr{P}_F(L)$ denote the (Jacobson) radical of $\mathscr{A}$. If $\mathscr{A}$ is a hereditary order in A, then $\mathscr{P}$ is a invertible fractional ideal in $\mathscr{A}$. The powers $\mathscr{P}^n$, $n\in\mathbb{Z}$ of $\mathscr{P}$ are therefore also invertible where we define $\mathscr{P}^{-n}=(\mathscr{P}^{-1})^n$ is n is positive. Furthermore, $\mathscr{P}^n$ is the set of all elements x in A such $x L_i \subseteq L_{i+n}$ for all integers i. In particular, $P\mathscr{A}=\mathscr{P}^e$ where $e=e(\mathscr{A})$. In fact we have that $\mathscr{P}^n L_i = L_{i+n}$. It follows that we have a canonical map from $\mathscr{A}/\mathscr{P}$ onto $\mathrm{End}_k(L_i/L_{i+1})$ for each integer i. These maps give rise to a canonical isomorphism of k-algebras from $\mathscr{A}/\mathscr{P}$ onto $\prod_{i=1}^{e} \mathrm{End}_k(L_{i-1}/L_i)$ where $e=e(\mathscr{A})$. If we let $n_i=\dim_k(L_{i-1}/L_i)$, then $n_{i+e}=n_i$ for all i, $n_i \geqq 1$ and $\sum_{i=1}^{e} n_i = N$. Moreover, the map $\mathscr{A}\to(n_1, \ldots, n_e)$ gives a bijection between the $A^\times$-conjugacy classes of hereditary orders in A and the set of ordered partitions of N. Finally, for x a nonzero element of A, let $\nu(x)=\nu_{\mathscr{A}}(x)$ denote the largest integer m such that x is contained in $\mathscr{P}^m$ and set $\nu(0)=\infty$. Then $\nu(x+y)\geqq\min(\nu(x), \nu(y))$ for x and y in A and $\nu(xy)\geqq\nu(x)+\nu(y)$ for x and y in A with equality if x or y is in the normalizer in $A^\times$ of $\mathscr{A}^\times$.

1.2. Duality. — With notation as in the previous section, let $\mathrm{tr}=\mathrm{tr}_{A/F}$ denote the usual trace map from A to F. Let $\langle\ ,\ \rangle=\langle\ \ \rangle_{A/F}$ denote the bilinear form on A_F defined by $\langle x, y\rangle=\mathrm{tr}(xy)$. Now $\langle\ ,\ \rangle$ is nondegenerate and so may be used to identify A_F with its dual $\mathrm{Hom}_F(A, F)$ this identification assigning an element b in A_F to the functional $\tilde{b}$ given by $\tilde{b}(y)=\langle b, y\rangle$. As a matter of convenience, we modify the usual notion of complementary set by defining the complement T^* of a subset T of A to be the set of b in A for which $\tilde{b}(T)$ is contained in P (rather than $\mathcal{O}$). This notion of complementary sets enjoys the usual properties and, as is well known (*see*, e.g., [B], Remark following Corollary 1.13), we have

Proposition 1.1. — *If $\mathscr{A}$ is a hereditary order in* A *with radical $\mathscr{P}$, then $(\mathscr{P}^n)^*=\mathscr{P}^{1-n}$ for all integers n.*

Now we pass to topological duals. To this end, we first fix a (continuous) character ψ of F^+ of conductor P_F. Then the map $\psi_b\colon A\to\mathbb{C}$ defined by $\psi_b(x)=\psi(\tilde{b}(x))$ is in $A^\wedge$, the Pontryagin dual of A considered as an abelian group. Then as is well known (*see*, e.g., [W1], Chap. II, Thm. 3) we have

Proposition 1.2. — *The map $\psi\colon A\to A^\wedge$ defined by $b\to\psi_b$ is an isomorphism onto* $A^\wedge$.

We will often view the ψ_b as above as characters of various subgroups of A and quotients of subgroups B/C if b is in C*. With this in mind we have the following result (*see*, e.g., [B], 1.12).

PROPOSITION 1.3. — *Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathcal{O}$-lattices in A such that $\mathscr{M}$ contains $\mathscr{N}$. Then the map $b \to \psi_b$ induces an isomorphism from $\mathscr{N}^*/\mathscr{M}^*$ to $(\mathscr{M}/\mathscr{N})^\wedge$. In particular, if $m<n$ and $\mathscr{A}$ is a hereditary order in A, then the map $b \to \psi_b$ induces an ismorphism of $\mathscr{P}^{1-n}/\mathscr{P}^{1-m}$ onto $(\mathscr{P}^m/\mathscr{P}^n)^\wedge$.*

We now turn to the multiplicative aspects of duality for a hereditary order $\mathscr{A}$ in A. We first consider the multiplicative structures attached to $\mathscr{A}$. Let $\mathrm{G}=\mathrm{G}_\mathrm{F}=\mathrm{A}_\mathrm{F}^\times$ and $\mathrm{U}(\mathscr{A})=\mathrm{U}^0(\mathscr{A})=\mathscr{A}^\times$. The group $\mathrm{U}(\mathscr{A})$ has a filtration of compact open normal subgroups $\mathrm{U}^n(\mathscr{A})=1+\mathscr{P}^n$ where n is a positive integer.

By restricting to a smaller class of hereditary orders we can say more about multiplicative structure. A hereditary order $\mathscr{A}$ in A is called *principal* if $\mathscr{P}$ is principal as a left (or, equivalently, right) ideal in $\mathscr{A}$. If $(n_1, \ldots, n_e)$ is the ordered partition of N associated to $\mathscr{A}$, then $\mathscr{A}$ is principal if and only if all the n_i are equal. If $\mathscr{A}$ is a principal order in A, then the normalizer, $\mathrm{K}(\mathscr{A})$ say, of $\mathscr{A}^\times$ in $\mathrm{A}^\times$ is a maximal open compact-modulo-center subgroup of G and all such subgroups can be obtained in this manner (*see*, e.g., [BF], (1.3.2) (iv) and (1.3.4)).

We now turn to multiplicative duality. If $\mathscr{M}$ is an $\mathcal{O}$-lattice in A which is contained in the radical $\mathscr{P}$ of some hereditary order $\mathscr{A}$ in A and has the further property that $\mathscr{M}\supseteqq\mathscr{M}^2$, we set $\mathrm{M}(\mathscr{M})=1+\mathscr{M}$. Then $\mathrm{M}(\mathscr{M})$ is a subgroup of $\mathrm{A}^\times$. In particular, if $\mathscr{M}=\mathscr{P}^n$ with $n\geqq 1$, then $\mathrm{M}(\mathscr{M})$ is just $\mathrm{U}^n(\mathscr{A})$ as defined above.

Now if $\mathscr{M}$ and $\mathscr{N}$ are $\mathcal{O}$-lattices in A such that $\mathrm{M}(\mathscr{M})$ and $\mathrm{M}(\mathscr{N})$ are defined and $\mathscr{M}$ contains $\mathscr{N}$ while $\mathscr{N}$ contains $\mathscr{M}^2$, then $\mathrm{M}(\mathscr{N})$ is a normal subgroup of $\mathrm{M}(\mathscr{M})$ with abelian quotient. Moroever, the map $x \to 1+x$ induces an isomorphism from $\mathscr{M}/\mathscr{N}$ to $\mathrm{M}(\mathscr{M})/\mathrm{M}(\mathscr{N})$. In particular, if $\mathscr{A}$ is a hereditary order in A and m and n are integers such that $m<n$ and $2m\geqq n$, then $\mathscr{P}^m/\mathscr{P}^n$ and $\mathrm{U}^m(\mathscr{A})/\mathrm{U}^n(\mathscr{A})$ are isomorphic via the map induced by $x \to 1+x$. Also, if $(n_1, \ldots, n_e)$ is the ordered partition of N associated to $\mathscr{A}$ then $\mathrm{U}(\mathrm{A})/\mathrm{U}^1(\mathscr{A})$ is isomorphic to $\prod_{i=1}^{e} \mathrm{GL}_{n_i}(k)$. Thus, if we define a map $\psi_b^\times : \mathrm{A}^\times \to \mathbb{C}^\times$ by $\psi_b^\times(y)=\psi_b(y-1)$, Proposition 1.3 and the above remarks yield (*see*, e.g., [B], 1.15).

PROPOSITION 1.4. — *Let $\mathscr{A}$ be a hereditary order in* A *with associated ordered partition $(n_1, \ldots, n_e)$ of* N *and let $\mathscr{M}$ and $\mathscr{N}$ be as above.*

(i) The map $b \to \psi_b^\times$ gives rise to an isomorphism from $\mathscr{N}^*/\mathscr{M}^*$ to $(\mathrm{M}(\mathscr{M})/\mathrm{M}(\mathscr{N}))^\wedge$. In particular, if m and n are integers such that $m<n$ and $2m\geqq n$ this map gives rise to an isomorphism between $\mathscr{P}^{1-n}/\mathscr{P}^{1-m}$ and $(\mathrm{U}^m(\mathscr{A})/\mathrm{U}^n(\mathscr{A}))^\wedge$.

(ii) $(\mathrm{U}(\mathscr{A})/\mathrm{U}^1(\mathscr{A}))^\wedge$ is isomorphic to $\prod_{i=1}^{e} (\mathrm{GL}_{n_i}(k))^\wedge$.

In what follows it will be convenient to write $\psi_b=\psi_b^\times$. It will always be clear from context whether ψ_b is considered additively or multiplicatively.

In light of Proposition 1.4, an irreducible representation η of $U^m(\mathscr{A})/U^n(\mathscr{A})$, m and n as above can be parametrized by a coset $b(\eta)+\mathscr{P}^{1-m}$ in $\mathscr{P}^{1-n}$. We will write $b(\eta)$ or just b in place of this coset when no confusion can occur. We will employ similar notations for parametrization of representations of $M(\mathscr{M})/M(\mathscr{N})$ where $\mathscr{M}$ and $\mathscr{N}$ are as above. Finally, if η is an irreducible representation of $U(\mathscr{A})/U^1(\mathscr{A})$, then it can be parametrized by an e-tuple $(\sigma_1, \ldots, \sigma_e)$ where σ_i is a representation of $GL_{n_i}(k)$.

To close this subsection we recall how the representations ψ_b behave under conjugation. In general, if (σ, W) is a representation of a subgroup H of a group G and g is an element of G, we define the *conjugate representation* (σ^g, W) of $H^g=gHg^{-1}$ on W by $\sigma^g(ghg^{-1})=\sigma(h)$. We say that g *intertwines* σ if $I(\sigma, \sigma^g)$, the set of intertwining operators from σ to σ^g, is nonempty. For an element x of $A=A_F(V)$ and g an element of $A^\times$, set $x^g=gxg^{-1}$. Then one can easily verify that if $\mathscr{M}$ and $\mathscr{N}$ are as above and ψ_b is a character of $M(\mathscr{M})/M(\mathscr{N})$, then for g an element of $G=A^\times$, $(\psi_b)^g=\psi_{b^g}$ as a character of $M(g\mathscr{M}g^{-1})/M(g\mathscr{N}g^{-1})$.

1.3. Fundamental strata. — In this section we recall some results of Bushnell, Howe, Moy and the first author. These results serve as a starting point for our proof of Theorem 0. We retain the notation of the previous subsections. We first define the notion of a stratum [B], 3.1 (or, equivalently, K-type [HM1], § 1).

Definition 1.5. — A *stratum* (or G-stratum) is a triple $(\mathscr{A}, n, \eta)$ consisting of a hereditary order $\mathscr{A}$ in A, an integer $n \geqq 0$ and an irreducible representation η of $U^n(\mathscr{A})/U^{n+1}(\mathscr{A})$; we define the *level* $l(((\mathscr{A}, n, \eta))$ of the stratum $(\mathscr{A}, n, \eta)$ to be $n/e(\mathscr{A})$.

Definition 1.6 ([B], 3.2, *see* also [HM1], § 1). — A stratum $(\mathscr{A}, n, \eta)$ is called *fundamental* if either:

(i) $n=0$ and $\eta=(\sigma_1, \ldots, \sigma_e)$ with each σ_i cuspidal (in the sense of finite Chevalley groups).

(ii) $n \geqq 1$ and the coset $b(\eta)$ contains no nilpotent element.

In discussing a stratum $(\mathscr{A}, n, \eta)$ we will often view η as a representation of $U^n(\mathscr{A})$ which is trivial on $U^{n+1}(\mathscr{A})$; we say that an admissible representation π of G *contains* the stratum $(\mathscr{A}, n, \eta)$ if it contains η upon restriction to $U^n(\mathscr{A})$. We define the *level* $l(\pi)$ of π to be the minimum of the levels of the strata contained in π. Then we have the following basic result.

Theorem 1.7 ([B], Theorem 2′ or [HM], Proof of Theorem 1.1). — *Let π be an irreducible admissible representation of* G.

(i) *If $l(\pi)=0$ and $(\mathscr{A}, n, \pi)$ is a stratum having maximal $e(\mathscr{A})$ among the strata of level 0 contained in π, then $(\mathscr{A}, n, \eta)$ is fundamental.*

(ii) *If $l(\pi)<0$ and $(\mathscr{A}, n, \eta)$ is a stratum contained in π of minimal level, then $(\mathscr{A}, n, \eta)$, is fundamental.*

Note that Theorem 1.7 implies that an irreductible admissible representation of G contains a fundamental stratum. There are some necessary conditions for two fundamental strata to occur in a given irreducible representation of G. We state these conditions as

PROPOSITION 1.8 ([HM], 6.1, 6.2). — *Suppose that* $(\mathscr{A}, n, \eta)$ *and* $(\mathscr{A}', n', \eta')$ *are fundamental strata contined in an irreducible admissible representation* π *of* G. *Then:*

(i) $l((\mathscr{A}, n, \eta)) = l(\mathscr{A}', n', \eta'))$.

(ii) *If* $n = n' = 0$, *then* $\mathrm{U}(\mathscr{A}/\mathrm{U}^1(\mathscr{A}) \cong \mathrm{U}(\mathscr{A}')/\mathrm{U}'(\mathscr{A}')$ *and* $\eta \cong \eta'$.

(iii) *If* $n, n' > 0$ *then there exist* g *in* G *such that*

$$(g(b(\eta) + \mathscr{I}^{1-n}(\mathscr{A}))g^{-1}) \cap (b(\eta') + \mathscr{I}^{1-n'}(\mathscr{A}'))$$

is nonempty.

1.4. ALFALFA STRATA. — If π is supercuspidal, then we may say more about the fundamental strata contained in π. To this end, let E be a finite dimensional extension field of F and let V be a finite dimensional vector space over E. Then we can also view V as an F-vector space so that $A_F(V)$ make sense. Then, since E is contained in A_E which is contained in A_F, left and right multiplication give A_F the structure of both an (A_E, A_E)-bimodule and an (E, E)-bimodule.

PROPOSITION 1.9. — *Let* L *be an* $\mathcal{O}_E$-*lattice chain in* V; *set* $\mathscr{A}_F = \mathscr{A}_F(L)$ *and similarly define* $\mathscr{A}_E$, $\mathscr{P}_E$ *and* $\mathscr{P}_F$. *Then:*

(i) L *is an* $\mathcal{O}_F$-*lattice chain of period* $e_F(L)e(E/F)$ *where* $e(E/F)$ *is the ramification degree of* E/F.

(ii) $\mathscr{P}_F^k \cap A_E = \mathscr{P}_E^k$ *for all integers* k.

(iii) *If* L *is uniform as on* $\mathcal{O}_E$-*chain and* z *generates* $\mathscr{P}_E$, *then* L *is uniform as an* $\mathcal{O}_F$-*chain and* z *also generates* $\mathscr{P}_F$.

Proof. — This is straightforward.

By the second part of the above proposition, if x is in A_E, then $\nu_{\mathscr{A}_E}(x) = \nu_{\mathscr{A}_F}(x)$. Thus, the notation $\nu_{\mathscr{A}}(x)$ is unambiguous and we will use this notation in what follows. It also follows from the above proposition that for all integers j, $\mathscr{P}_F^j$ has both the structure of an $(\mathscr{A}_E, \mathscr{A}_E)$-bimodule and an $(\mathcal{O}_E, \mathcal{O}_E)$-bimodule and that for all integers j, $\mathscr{P}_F^j/\mathscr{P}_F^{j+1}$ has both the structure of an $(\mathscr{A}_E/\mathscr{P}_E, \mathscr{A}_E/\mathscr{P}_E)$-bimodule and a (k_E, k_E)-bimodule. Finally, note that $E^\times$ is in the normalizer of $\mathscr{A}_F^\times$ since L is an $\mathcal{O}_E$-lattice chain. Thus, for x in $E^\times$ and y in A_F, $\nu_F(xy) = \nu_{\mathscr{A}}(x) + \nu_{\mathscr{A}}(y)$.

Example 1.10. — As we noted in [KM2], the following example is fundamental. Let $V = E$ as an E-vector space and Let $L_i = P_E^i$. Then, $L = \{L_i\}_{i \in \mathbb{Z}}$ is the unique, up to equivalence, $\mathcal{O}_E$-lattice chain in E. $\mathscr{A}_F(L)$ is a principal order in $A_F(E)$, $e(\mathscr{A}_F(L)) = e(E/F)$, $\mathscr{P}_E^k = P_E^k$ and $\mathscr{P}_F^k \cap E = P_E^k$. Since L is unique up to equivalence, in an abuse of notation, we will write $\mathscr{A}_F(E)$, $\mathscr{P}_F(E)$ and $\mathscr{P}_F^k(E)$ for $\mathscr{A}_F(L)$ $\mathscr{P}_F(L)$ and $\mathscr{P}_F^k(L)$ respectively.

Before introducing alfalfa strata we need to look briefly at finitely dimensional extensions of F. If α is algebraic over F, let $f_\alpha = f_{\alpha, F}$ denote the minimal polynomial of α over F. If E is a finite dimensional extension of F, let $e(E/F)$ and $f(E/F)$ denote the ramification degree and inertia degree, respectively, of E/F. Also let $N_{E/F}: E \to F$ and $\mathrm{Tr}_{E/F}: E \to F$ denote the norm map and trace map, respectively, from E to

F. Throughout the remainder of this paper we will use standard terminology and results from the theory of local fields. For unexplained terminology and results, *see* [S] or [W1].

DEFINITION 1.11 ([KM2], 1.6). — If E is a finite dimensional extension of F then an element α in E is E/F-*minimal* (or just minimal) if the following conditions hold:

(i) $E=F[\alpha]$.

(ii) $(\nu_E(\alpha), e(E/F))=1$.

(iii) $\mathcal{O}_F[N_{E/E_{nr}}(\alpha)/\bar{\omega}_F^{\nu_E(\alpha)}]=\mathcal{O}_{E_{nr}}$ where E_{nr}/F is the maximal unramified extension intermediate to E/F.

DEFINITION 1.12 ([K3], 3.1). — A stratum $(\mathcal{A}, n, \eta)$ is an *alfalfa stratum* (pure in the language of [HM2]) if $n\geqq 1$ and there exists an element α in $b(\eta)+\mathcal{P}^{1-n}$ which satisfies the following properties:

(i) $E=F[\alpha]$ is a subfield of A.

(ii) $\nu_{\mathcal{A}}(\alpha)=-n$.

(iii) α is E/F-minimal.

(iv) If $\mathcal{A}=A_F(L)$, then L is an $\mathcal{O}_E$-lattice chain.

We note that this definition is slightly more general than that of [K3]; in that definition $\mathcal{A}$ was restricted to be principal. The reader may check that alfalfa strata are fundamental. If $(\mathcal{A}, n, \eta)$ is an alfalfa stratum as above, we will often write $(\mathcal{A}, n, \alpha)$ for $(\mathcal{A}, n, \eta)$.

THEOREM 1.13 ([K3], 3.2 or [HM2], 5.4). — *If* π *is a supercuspidal representation of* G, *then either*

(i) π *contains a fundamental stratum* $(\mathcal{A}, 0, \eta)$ *with* $e(\mathcal{A})=1$ *or*

(ii) π *contains an alfalfa stratum.*

PROPOSITION 1.14. — (i) *If* $(\mathcal{A}, n, a)$ *is an alfalfa stratum, then*

$$l((\mathcal{A}, n, \alpha))=-\nu_E(\alpha)/e(E/F) \qquad \textit{where} \quad E=F[\alpha].$$

(*ii*) *If* $(\mathcal{A}, n, \alpha)$ *and* $(\mathcal{A}', n', \alpha')$ *are alfalfa strata contained in the same irreducible admissible representation of* G, *then* $l((\mathcal{A}, n, \alpha))=l((\mathcal{A}', n', \alpha'))$. Further, $[E:F]=[E':F]$, $e(E/F)=e(E'/F)$ *and* $\nu_E(\alpha)=\nu_{E'}(\alpha')$ *where* $E=F[\alpha]$ *and* $E'=F[\alpha']$.

Proof. – (i) This follows from Definition 1.2 (ii) since $\nu_{\mathcal{A}}(\alpha)=\nu_E(\alpha)e(\mathcal{A}_E)$ and $e(\mathcal{A}_E)e(E/F)$.

(ii) That $l((\mathcal{A}, n, \alpha))=l((\mathcal{A}', n',\alpha'))$ is a special case of Proposition 1.8 (i). Then it follows from the first part of this proposition that $\nu_E(\alpha)/e(E/F)=\nu_{E'}(\alpha')/e(E'/F)$. But then, since $(\nu_E(\alpha), e(E/F))=(\nu_{E'}(\alpha'), e(E'/F))=1$, it follows that $\nu_E(\alpha)=\nu_{E'}(\alpha')$ and $e(E/F)=e(E'/F)$. Let $k=\nu_E(\alpha)$ and $e=e(E/F)$.

Now, by Proposition 1.8 (iii), there exists T in $\mathcal{P}^{1-n}(\mathcal{A})$, T′ in $\mathcal{P}^{1-n'}(\mathcal{A}')$ and g in G such that

$$g(\alpha+T)g^{-1}=\alpha'+T'.$$

In particular, $\alpha+T$ and $\alpha'+T'$ have the same characteristic polynomials. Likewise $\beta=(\alpha+T)^e/\bar{\omega}_F^k$ and $\beta'=(\alpha'+T')^e/\bar{\omega}_F^k$ have the same characteristic polynomial. Moreover, this characteristic polynomial of these normalized elements has integral coefficients since β is in $\mathscr{A}$. Then, since

$$\beta\equiv\alpha^e/\bar{\omega}_F^k \bmod \mathscr{P}(\mathscr{A}),$$

the characteristic polynomials of β and $\gamma=\alpha^e/\bar{\omega}_F^k$ are the same modulo P_F. Similarly, the characteristic polynomials of β' and $\gamma'=(\alpha')^e/\bar{\omega}_F^k$ are the same modulo P_F. Thus the characteristic polynomials of γ and γ' have integral coefficients and are the same modulo P_F. But also, since α and α' are minimal, the residue classes of γ and γ' in k_E and $k_{E'}$ respectively are primitive elements [KM 2], Proof of Proposition 1.5. It then follows that $f(E/F)=f(E'/F)$ so that $[E:F]=[E':F]$ as desired.

Remark 1.15. — (i) As is well known, E and E′ in the above proposition need not be isomorphic even if $\mathscr{A}'=\mathscr{A}$ and $n'=n$ (*see*, e.g., [K 1] in the case $N=2$).

(ii) We define the degree of a alfalfa stratum $(\mathscr{A}, n, \alpha)$ to be $[F[\alpha]:F]$. By (ii) of the above proposition, this makes sense. We say that an alfalfa stratum is *proper* if its degree is strictly between 1 and N.

(iii) Let π be an irreducible supercuspidal representation of G. Then it is well known (*see*, e.g., [Ca], 4.2) that if π contains a fundamental stratum $(\mathscr{A}, 0, \eta)$ with $e(\mathscr{A})=1$ then π is induced (as in Theorem 0 of the introduction). Thus, in proving that π is induced, we may assume $l(\pi)>0$ and thus that π contains an alfalfa stratum. Suppose that π contains the alfalfa stratum $(\mathscr{A}, n, \alpha)$. If $[F[\alpha]:F]=N$ then it is a result of Carayol [Ca], 4.2 that π is induced. If $F[\alpha]=F$ so that α is in F then, as is well known (*see*, e.g., [Ca]), there exists a character θ of $F^\times$ such that $\theta\circ\det_{A/F}$ restricts to ψ_α on $U^n(\mathscr{A})$. Then $l(\pi\otimes(\theta^{-1}\circ\det_{A/F}))<l(\pi)$. Thus, since π is induced if and only if $\pi\otimes(\theta^{-1}\circ\det_{A/F})$ is induced, we have

PROPOSITION 1.16. — *It suffices to prove Theorem* 0 *for irreducible supercuspidal representations which contain a proper alfalfa stratum.*

2. Relative alfalfa strata

The purpose of this section is to define certain representations which refine alfalfa strata; that is to say, representations which contain alfalfa strata upon restriction to the appropriate subgroup. We also state an analog of Theorem 1.7 which will be proved in the third section of this paper and is a key result in our proof of Theorem 0. The map S_α which was defined in [KM 2] is crucial for this section and thus we begin with

2.1. THE MAP S_α. — Let V be a finite dimensional vector space over E where E/F is a finite dimensional extension. Set $R=\dim_E V$ and $S=[E:F]$. As before, $A=A_F=A_F(V)$ and $A_E=A_E(V)$.

Given a polynomial $f(X)$ in $F[X]$, we define in the usual manner the map $B\to f(B)$ from A to itself. Since $f: A\to A$ is polynomial in its coordinates, the (formal) total derivative of f at an element B of A makes sense. Let $D_B f$ denote this derivative. One can verify by direct calculation that

$$(2.0.1) \qquad (D_B(X^t))(y)=\sum_{j=0}^{t-1} B^j y B^{t-1-j}.$$

DEFINITION 2.1 [KM 2]. — For α E/F-minimal, define maps $A_\alpha = A_{\alpha, V, F}: A \to A$ and $S_\alpha = S_{\alpha, V, F}: A \to$ by $A_\alpha(y)=\alpha y \alpha^{-1} - y$ and $S_\alpha(y)=((D_\alpha f_\alpha)(y))\alpha^{1-S}$.

DEFINITION 2.2 ([KM 2], 2.13). — If L is an $\mathcal{O}_E$-lattice chain in V, then by an $\mathcal{O}_E$-*basis* for L we mean a set of elements $v_i\, i=1, \ldots, R$ in V such that for each integer k there exists integers $n(i, k)\, i=1, \ldots, R$ such that

$$L_k = \bigoplus_{i=1}^{R} P_E^{n(i, k)} v_i$$

It is well known (*see*, e. g., [BF], 1.2.8) that any $\mathcal{O}_E$-lattice chain in V has an $\mathcal{O}_E$-basis.

DEFINITION 2.3. — Suppose that L is an $\mathcal{O}_E$-lattice chain in V with notation as in Section 1 and that $\mathcal{B}=\{v_i\}_{i=1}^{R}$ is an $\mathcal{O}_E$-basis for L. Let W be the F-span of $\mathcal{B}$ in V. Then $V=EW\equiv E\otimes_F W$. In what follows we identify V with $E\otimes_F W$. Set $V_j = E\otimes_F F v_j$. Let A_{ij} be the set of T in $A_F(V)$ satisfying $T(V_j)\subseteq V_i$ and $T(V_k)=0$ unless $k=j$. Similarly, define $(A_E)_{ij}$ in $A_E(V)$ and note that $(A_E)_{ij}=A_E\cap A_{ij}$. Then for any integer k, let $\mathcal{P}_{ij}^k=(\mathcal{P}^k\cap A_{ij})+\mathcal{P}^{k+1}$ and

$$(\mathcal{P}_E^k)_{ij}=(P_E^k\cap (A_E)_{ij})+\mathcal{P}_E^{k+1}=(\mathcal{P}_E^k\cap A_{ij})+\mathcal{P}_E^{k+1}.$$

We say that $\mathcal{P}_{ij}^k$ is the *(i, j)-coordinate* of $\mathcal{P}^k$ relative to $\mathcal{B}$. In what follows we will often omit mention of E and $\mathcal{B}$. We will also say that $(\mathcal{P}_E^k)_{ij}$ is the (i, j)-coordinate of $\mathcal{P}_E^k$ relative to $\mathcal{B}$ and likewise will often omit mention of $\mathcal{B}$.

Remark 2.4. – (i) When we speak of a sum of coordinates in $\mathcal{P}^k$ or $\mathcal{P}_E^k$, we will always be referring to coordinates with respect to a fixed basis. We will also, as a matter of convenience, refer to $\mathcal{P}^{m+1}$ as a sum (the trivial sum) of coordinates in $\mathcal{P}^m$.

(ii) In [KM 2], 1.16, we defined coordinates in $\mathcal{P}^k$ for a principal order $\mathcal{A}$ attached to an $\mathcal{O}_E$-chain. This definition coincides with the one above for coordinates in $\mathcal{P}^k$ if and only if $e_E(L)=R$. If $\mathcal{M}$ is a coordinate in $\mathcal{P}^k$ in the old sense, then $\mathcal{M}$ is a sum of coordinates in the new sense.

(iii) If $\mathcal{M}$ is a sum of coordinates in $\mathcal{P}^k$ for some k, then we set $\mathcal{M}_E=\mathcal{M}\cap A_E$. Note that if $\mathcal{M}=\sum_{l=1}^{r}\mathcal{P}_{i_l j_l}^k$, then $\mathcal{M}_E=\sum_{l=1}^{r}(\mathcal{P}_E^k)_{i_l j_l}$.

(iv) If $\mathscr{M}=\sum_{l=1}^{r} \mathscr{P}^k_{i_l j_l}$ is a sum of coordinates in $\mathscr{P}^k$, then one verifies by a straightforward calculation that

$$\mathscr{M}^*=\sum_{\substack{(m,\ n)\neq(j_l,\ i_l)\\ \forall l}} \mathscr{P}^{1-k}_{m,\ n}$$

and thus $\mathscr{M}^*$ is a sum of coordinates in $\mathscr{P}^{1-k}$. Similarly, the complement of $\mathscr{M}_E$ with respect to the bilinear form defined by $\mathrm{tr}_{A_E/E}$ is $(\mathscr{M}^*)_E$.

(v) $\mathscr{P}^k_{ij}$ and $(\mathscr{P}^k_E)_{ij}$ are $(\mathscr{O}_E, \mathscr{O}_E)$-submodules of A and A_E respectively.

(vi) $\mathscr{P}^k/\mathscr{P}^{k+1} \cong \bigoplus_{i,\ j=1}^{R} \mathscr{P}^k_{ij}/\mathscr{P}^{k+1}$ as (k_E, k_E)-bimodules.

(vii) Suppose $\mathscr{M}$ and $\mathscr{N}$ are sums of coordinates in $\mathscr{P}^k$ and $\mathscr{P}^l$ respectively which satisfy the hypotheses of Proposition 1.4 (i). Then one can check that a character ψ_b of $M(\mathscr{M})/M(\mathscr{N})$ has the property that b may be chosen in F is and only if there exists a character ρ of $F^\times$ such that $\rho\circ\det_{A/F}$ restricts to ψ_b on $M(\mathscr{M})$. If such a ρ exists we say that ψ_b factors through $\det_{A/F}$.

(viii) Coordinates are used in many of the results which follow, Often, quite general cases of these results can be proved without the choice of basis inherent to coordinates (*see*, e. g., [KM 2]).

DEFINITION 2.5 ([KM 2], 1.12). — Let T be an abelian group and let f and g be elements of End T. Then we say that (f, g) is a *pair* for T if $\ker f \supseteq \operatorname{Im} g$ and $\ker g \supseteq \operatorname{Im} f$. We say that (f, g) is an *exact pair* for T is these inclusions are equalities.

The following result is then fundamental.

THEOREM 2.6. — *With notation as above*, S_α *and* A_α *enjoy the following properties:*

(i) A_α *and* S_α *are* (A_E, A_E)*-bimodule endomorphisms of* A.

(ii) (S_α, A_α) *is an exact pair for* A.

(iii) *If* $\mathscr{M}$ *is a sum of coordinates in* $\mathscr{P}^k$, *then* $A_\alpha(\mathscr{M})\subseteq\mathscr{M}$ *and* $S_\alpha(\mathscr{M})\subseteq\mathscr{M}$. *In particular*, $A_\alpha(\mathscr{P}^k)\subseteq\mathscr{P}^k$ *and* $S_\alpha(\mathscr{P}^k)\subseteq\mathscr{P}^k$ *for all integers* k.

(iv) *Suppose that* $\mathscr{M}$ *is a sum of coordinates in* $\mathscr{P}^k$ *and that* $\mathscr{N}$ *is a sum of coordinates in* $\mathscr{P}^l$ *such that* $\mathscr{M}$ *contains* $\mathscr{N}$. *Then* $(\bar{S}_\alpha, \bar{A}_\alpha)$ *is an exact pair for* $\mathscr{M}/\mathscr{N}$. *In particular, if* $k\leqq l$, $(\bar{S}_\alpha, \bar{A}_\alpha)$ *is an exact pair for* $\mathscr{P}^k/\mathscr{P}^l$.

(v) *If* $\mathscr{M}$ *is a sum of coordinates in* $\mathscr{P}^k$, *then* (S_α, A_α) *is an exact pair for* $\mathscr{M}$. *In particular*, (S_α, A_α) *is an exact pair for* $\mathscr{P}^k$.

Proof. — (i) This is [KM 2], 1.10 (i).

(ii) This is [KM 2], 1.21 (i).

(iii) It suffices to show that $A_\alpha(\mathscr{P}^k_{ij})\subseteq\mathscr{P}^k_{ij}$ and $S_\alpha(\mathscr{P}^k_{ij})\subseteq\mathscr{P}^k_{ij}$ for all k and for $1\leqq i, j\leqq R$. This is equivalent to showing that $A_\alpha(A_{ij}\cap\mathscr{P}^k)\subseteq A_{ij}\cap\mathscr{P}^k$ and $S_\alpha(A_{ij}\cap\mathscr{P}^k_{ij})\subseteq A_{ij}\cap\mathscr{P}^k$ for all k and for $1\leqq i, j\leqq R$. It follows from (2.0.1), the definition of A_α and S_α and the definition of A_{ij}, that, since α is in E, we have that $A_\alpha(A_{ij}) \subseteq A_{ij}$ and $S_\alpha(A_{ij}) \subset A_{ij}$. Thus it suffices to show that $A_\alpha(\mathcal{P}^k) \subseteq \mathcal{P}^k$ and $S_\alpha(\mathcal{P}^k) \subseteq \mathcal{P}^k$. Since α is in E, $\nu_{\mathcal{A}}(\alpha x \alpha^{-1}) = \nu_{\mathcal{A}}(x)$ for all x in A so that $A_\alpha(\mathcal{P}^k) \subseteq \mathcal{P}^k$. Now consider S_α. We proceed in a manner similar to that of our proof of the corresponding statement for principal orders [KM 2], 1.11.

Write $f_\alpha(X) = \sum_{k=0}^{S} a_k X^k$. Then, since a_k is the $(S-k)$-symmetric polynomial in the roots of f_α,

$$\nu_{\mathcal{A}}(a_k) \geqq (S-k)\,\nu_{\mathcal{A}}(\alpha).$$

Thus, if y is in A, it follows from (2.0.1) that

$$\nu_{\mathcal{A}}(a_k(D_\alpha X^k)(y)) \geqq (S-1)\,\nu_{\mathcal{A}}(\alpha) + \nu_{\mathcal{A}}(y)$$

and the result follows.

(iv) and (v) Suppose that we were to show that $(\bar{S}_\alpha, \bar{A}_\alpha)$ is an exact pair for $\mathcal{P}^k_{ij}/\mathcal{P}^{k+1}$ for all k and $1 \leqq i, j \leqq R$. Then (iv) would follow in the case $l=k$ from Remark 2.4. Then (iv) and (v) would follow in general form from standard filtration arguments (*see*, e. g., [S], V, § 1, Lemma 2). Thus it suffices to show that $(\bar{S}_\alpha, \bar{A}_\alpha)$ is an exact pair for $\mathcal{P}^k_{ij}/\mathcal{P}^{k+1}$.

By the second isomorphism theorem,

$$\begin{aligned}\mathcal{P}^k_{ij}/\mathcal{P}^{k+1} &= ((A_{ij} \cap \mathcal{P}^k) + \mathcal{P}^{k+1})/\mathcal{P}^{k+1} \\ &\cong A_{ij} \cap \mathcal{P}^k/(A_{ij} \cap \mathcal{P}^k) \cap \mathcal{P}^{k+1} \\ &= A_{ij} \cap \mathcal{P}^k/A_{ij} \cap \mathcal{P}^{k+1}\end{aligned}$$

By the argument for (iii), $\bar{S}_\alpha$ and $\bar{A}_\alpha$ respect this isomorphism so that is suffices to show $(\bar{S}_\alpha, \bar{A}_\alpha)$ is an exact pair $A_{ij} \cap \mathcal{P}^k/A_{ij} \cap \mathcal{P}^{k+1}$. We may assume $A_{ij} \cap \mathcal{P}^k \neq A_{ij} \cap \mathcal{P}^{k+1}$ the other case being trivial.

Recalling that $V = \bigoplus_{l=1}^{R} V_l = \bigoplus_{l=1}^{R} E \otimes_F F v_l$, define $\psi_l : V_l \to E$ by $\psi_l(y \otimes v_l) = y$. Then the map $\psi : A_{ij} \to A_F(E)$ defined by $\psi(x) = \psi_i \circ x \circ (\psi_j)^{-1}$ is an (E, E)-bimodule isomorphism onto $A_F(E)$. Using that $\mathcal{B} = \{v_i\}$ is an $\mathcal{O}_E$-basis for L, one can then check that there exists $n = n(k)$ such that $\psi(A_{ij} \cap \mathcal{P}^k) = \mathcal{P}^n_F(E)$ and $\psi(A_{ij} \cap \mathcal{P}^{k+1}) = \mathcal{P}^{n+1}_F(E)$. Let $\bar{\psi}$ denote the induced (k_E, k_E)-bimodule isomorphism of $A_{ij} \cap \mathcal{P}^k/A_{ij} \cap \mathcal{P}^{k+1}$ onto $\mathcal{P}^n_F(E)/\mathcal{P}^{n+1}_F(E)$. Then, since $\bar{\psi} \circ \bar{S}_{\alpha,\,V} = \bar{S}_{\alpha,\,E} \circ \bar{\psi}$ and similarly for $\bar{A}_{\alpha,\,V}$, the result follows from the exactness of $(\bar{S}_{\alpha,\,E}, \bar{A}_{\alpha,\,E})$ for $\mathcal{P}^n_F(E)/\mathcal{P}^{n+1}_F(E)$ [KM 2], 1.14.

As was witnessed by the above proof, arguments which involve S_α can often be reduced to the one-dimensional case $V=E$. Along this line, we have the following lemma which will be used in Section 2.2.

LEMMA 2.7. — *With notation as above, let* W *be an* F *vector space such that* $V = E \otimes_F W$. *Then the following diagram commutes*

$$\begin{array}{ccc} A_F(E) \otimes A_F(W) & \xrightarrow{\varphi} & A_F(V) \\ {\scriptstyle S_{\alpha,E} \otimes 1} \downarrow & & \downarrow {\scriptstyle S_{\alpha,V}} \\ A_F(E) \otimes A_F(W) & \xrightarrow{\varphi} & A_F(V) \end{array}$$

where φ *is the natural* (E, E)-*bimodule isomorphism of* $A_F(E) \otimes A_F(W)$ *onto* $A_F(V)$.

Proof. — This is straightforward since the (E, E)-bimodule structure on $A_F(E) \otimes A_F(W)$ is given by left and right multiplication on the first coordinate.

2.2. ALFALFA DUALITY. — Let E/F be an extension of fields and let L be an $\mathcal{O}_E$-lattice chain in V where V is a finite dimensional vector space over E. Let $\mathcal{A} = \mathcal{A}_F(L)$ and suppose that $\mathcal{M}$ and $\mathcal{N}$ are sums of coordinates in $\mathcal{P}^k$ and $\mathcal{P}^l$ for some integers k and l with respect to some fixed $\mathcal{O}_E$-basis $\mathcal{B}$ for L. Suppose further that $M(\mathcal{M})$ and $M(\mathcal{N})$ are defined, $\mathcal{M}$ contains $\mathcal{N}$ and $\mathcal{N}$ contains $\mathcal{M}^2$ so that we are in the setting of Proposition 1.4. Later in this paper we will often restrict a character ψ_b of $M(\mathcal{M})/M(\mathcal{N})$ to $M(\mathcal{M}_E)/M(\mathcal{N}_E)$. Thus, in this subsection we will develop a parametrization (Proposition 2.14) of the dual of $M(\mathcal{M}_E)/M(\mathcal{N}_E)$ which differs from that of Section 1.2 but facilitates the study of these restrictions. At the heart of this parametrization is

LEMMA 2.8. — *With notation as above, if* α *is* E/F-*minimal, then there exists a unique* F-*linear functional* $\hat{\alpha} = \hat{\alpha}_{V/F}$ *on* A_E *such that the following diagram is commutative.*

$$\begin{array}{ccc} A_F & \xrightarrow{S_\alpha} & A_E \\ & {\scriptstyle tr_{A_F/F}} \searrow \quad \swarrow {\scriptstyle \hat{\alpha}} & \\ & F & \end{array}$$

Further, this map has the following properties:

(i) $\hat{\alpha}(\mathcal{A}_E) = \mathcal{O}_F$ *and* $\hat{\alpha}(\mathcal{P}_E) = P_F$.

(ii) $\hat{\alpha}_{V/F}(x) = \hat{\alpha}_{E/F}(tr_{A_E/E}(x))$ *for all* x *in* A_E.

(iii) *If* E/F *is separable, then*

$$\hat{\alpha}(x) = tr_{A_E/F}(x\,\alpha^{S-1}/f'_\alpha(\alpha))$$

for x *in* A_E *where* $S = [E:F]$ *and* f'_α *is the (formal) derivative* f_α.

Remark 2.9. — Before proving Lemma 2.8, it is of interest to note one of its consequences in the case V = E. In this case, $A_E = E$ so that $\hat{\alpha}$ will be an F-linear functional on E which has the properties that $\hat{\alpha}(\mathcal{O}_E) = \mathcal{O}_F$ and $\hat{\alpha}(P_E) = P_F$. Thus, $\hat{\alpha}$ enjoys arithmetic properties similar to those enjoyed by $Tr_{E/F}$ in the case E/F tamely ramified.

Proof of Lemma 2.8. — Suppose that x is an element of A_E. Then, since x is in $\ker A_\alpha$ and (S_α, A_α) is an exact pair for A_F, x is also in $\operatorname{Im} S_\alpha$. Thus, set

$\hat{\alpha}(x)=\mathrm{tr}_{A_F/F}(x')$ where x' is some preimage of x in A_F under S_α. One can then check that, since (S_α, A_α) is an exact pair for A_F, $\hat{\alpha}$ is well-defined. One can also check that, by construction, $\hat{\alpha}$ is unique and F-linear and further that the indicated diagram commutes.

(i) Since (S_α, A_α) is an exact pair for $\mathscr{A}_F$,

$$\begin{aligned}\hat{\alpha}(\mathscr{A}_E)&=\hat{\alpha}(\ker A_\alpha(\mathscr{A}_F))\\ &=\hat{\alpha}(S_\alpha(\mathscr{A}_F))\\ &=\mathrm{tr}_{A_F/F}(\mathscr{A}_F)\\ &=\mathscr{O}_F.\end{aligned}$$

The corresponding statement for $\mathscr{P}_E$ is proved similarly.

(ii) Choose an F-vector space W such that $V\cong E\otimes_F W$ and then identify V with $E\otimes_F W$. Let φ denote the natural (E, E)-bimodule isomorphism of $A_F(E)\otimes A_F(W)$ onto $A_F(V)$. Note that the restriction of φ to $E\otimes A_F(W)$ gives an (E, E)-bimodule endomorphism of $E\otimes A_F(W)$ onto $A_E(V)$. Now let β be an element of E, x an element of $A_F(W)$ and β' a preimage of β in $A_F(E)$ under $S_{\alpha, E}$. Then by Lemma 2.7, $\varphi(\beta'\otimes x)$ is a preimage of $\beta\otimes x$ under $S_{\alpha, V}$. Thus,

$$\begin{aligned}\hat{\alpha}_{V/F}(\varphi(\beta\otimes x))&=\mathrm{tr}_{A_F(V)/F}(\varphi(\beta'\otimes x))\\ &=\mathrm{tr}_{A_E(F)/F}(\beta')\,\mathrm{tr}_{A_F(W)/F}(x)\\ &=\hat{\alpha}_{E/F}(\beta)\,\mathrm{tr}_{A_F(W)/F}(x)\\ &=\hat{\alpha}_{E/F}(\beta\,\mathrm{tr}_{A_F(W)/F}(x))\\ &=\hat{\alpha}_{E/F}(\mathrm{tr}_{A_E(V)/E}(\varphi(\beta\otimes x)))\end{aligned}$$

so that (ii) then follows by linearity.

(iii) In [KM 2], 1.23 we showed that if E/F is separable, then $S_\alpha(x)=(f'_\alpha(\alpha)/\alpha^{S-1})x$ for x in A_E whence the result.

We now parallel the constructions of Paragraph 1.2 as follows. Let $\langle\ ,\ \rangle_\alpha=\langle\ ,\ \rangle_{\alpha, A_E/F}$ denote the F-bilinear form on $A_E(V)$ defined by $\langle x, y\rangle_\alpha=\hat{\alpha}(xy)$. Then, since $\langle\ ,\ \rangle_{A_E/E}$ is a nondegenerate pairing, it follows from Lemma 2.8 (ii) that to show that $\langle\ ,\ \rangle_\alpha$ is nondegenerate it suffices to show that $\hat{\alpha}_{E/F}$ is nontrivial. But this follows from Lemma 2.8 (i). Thus $\langle\ ,\ \rangle_\alpha$ may be used to identify A_E with its dual $\mathrm{Hom}_F(A_E, F)$ this identification assigning to an element b in A_E the functional $\hat{b}_\alpha$ defined by $\hat{b}_\alpha(x)=\hat{\alpha}(bx)$.

Remark 2.10. — We note that in case $E=F$ α may be taken to be 1. Then S_1 is the identity map and A_1 is trivial. The unique F-linear functional $\hat{1}$ is just then $\mathrm{tr}_{A_F/F}$ so that $\hat{b}_1(x)=\mathrm{tr}_{A_F/F}(bx)=\langle b, x\rangle=\tilde{b}(x)$.

We define the *complement*, $(T)^*_\alpha$, of a subset T of A_E *with respect to* α to be the set of b in A_E for which $\hat{b}_\alpha(T)$ is contained in P_F. We have the following analog of Proposition 1.1 and Remark 2.4 (v).

PROPOSITION 2.11. — *With notation as above, if* $\mathcal{M}$ *is a sum of coordinates in* $\mathcal{P}^k$, $(\mathcal{M}_E)^*_\alpha = (\mathcal{M}^*)_E$. *In particular,* $(\mathcal{P}^k_E)^*_\alpha = \mathcal{P}^{1-k}_E$.

Proof. — It follows from Remark 2.4 (iv) and Lemma 2.8 that $(\mathcal{M}_E)^*_\alpha \supseteq (\mathcal{M}^*)_E$. Suppose, on the other hand, that x is in $(\mathcal{M}_E)^*_\alpha$ but not in $(\mathcal{M}^*)_E$. Then x is not in the complement of $\mathcal{M}_E$ with respect to $\mathrm{tr}_{A_E/E}$ by Remark 2.4 (iv). That is to say, $\mathrm{tr}_{A_E/E}(x\mathcal{M}_E)$ is not contained in P_E. But then, since $\mathrm{tr}_{A_E/E}(x\mathcal{M}_E)$ is a fractional ideal, $\mathrm{tr}_{A_E/E}(x\mathcal{M}_E) \supseteq \mathcal{O}_E$. Thus, by Lemma 2.8 (ii),

$$\hat{\alpha}(x\mathcal{M}_E) = \hat{\alpha}_{E/F}(\mathrm{tr}_{A_E/E}(x\mathcal{M}_E)) \supseteq \hat{\alpha}_{E/F}(\mathcal{O}_E) = \mathcal{O}_F - \text{ a contradiction.}$$

Now, as in Paragraph 1.2, we pass to topological duals. To this end, recall that ψ was fixed in Paragraph 1.2 to be a character of F^+ of conductor P_F. Then the map $\psi_{\alpha,b} : A_E \to \mathbb{C}^\times$ defined by $\psi_{\alpha,b}(x) = \psi(\hat{b}_\alpha(x))$ is in $A_E^\wedge$.

PROPOSITION 2.12. — (*i*) *Let* $\Psi_\alpha : A_E \to A_E^\wedge$ *be the map defined by* $b \to \psi_{\alpha,b}$ *and let* $\mathrm{res} = \mathrm{res}_{V,E/F} : A_F^\wedge \to A_E^\wedge$ *be the map defined by restriction. Then the following diagram is commutative where* $\Psi : A_F \to A_F^\wedge$ *is defined by* $b \to \psi_b$ *as in Paragraph* 1.2.

$$\begin{array}{ccc} A_F & \xrightarrow{\Psi} & A_F^\wedge \\ {\scriptstyle S_\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ A_E & \xrightarrow{\Psi_\alpha} & A_E^\wedge \end{array}$$

(ii) Ψ_α *is an isomorphism onto* $A_E^\wedge$.

Proof. — (i) Since S_α is a right A_E-module map and $\mathrm{tr}_{A_F/F}(x) = \hat{\alpha}(S_\alpha(x))$ for all x in A_F by Lemma 2.8,

$$\begin{aligned} \psi_{\alpha, S_\alpha(b)}(y) &= \psi(\hat{\alpha}(S_\alpha(b)y)) \\ &= \psi(\hat{\alpha}(S_\alpha(by)) \\ &= \psi(\mathrm{tr}_{A_F/F}(by)) \\ &= \psi_b(y) \end{aligned}$$

for all b in A_F and y in A_E.

(ii) The map Ψ is a homomorphism of the F-vector space A_E into the F-vector space $A_E^\wedge$. Then, since A_E and $A_E^\wedge$ have the same (finite) dimension as F-vector spaces it suffices to show that Ψ_α is injective. But this follows from Proposition 2.11.

PROPOSITION 2.13. — *Let* $\mathcal{M}$ *be a sum of coordinates in* $\mathcal{P}^k$, $\mathcal{N}$ *a sum of coordinates in* $\mathcal{P}^l$ *such that* $\mathcal{M}$ *contains* $\mathcal{N}$, *and let* Ψ *and* Ψ_α *be as in the previous proposition.*

(i) Ψ *and* Ψ_α *induce isomorphisms* Ψ *and* Ψ_α *from* $\mathcal{N}^*/\mathcal{M}^*$ *onto* $(\mathcal{M}/\mathcal{N})^\wedge$ *and from* $(\mathcal{N}_E)^*_\alpha/(\mathcal{M}_E)^*_\alpha$ *onto* $(\mathcal{M}_E/\mathcal{N}_E)^\wedge$.

(ii) *The* $(\mathcal{A}_E, \mathcal{A}_E)$ *endomorphism* $\bar{S}_\alpha$ *of* $\mathcal{N}^*/\mathcal{M}^*$ *induced by* S_α *maps onto* $(\mathcal{N}_E)^*_\alpha/(\mathcal{M}_E)^*_\alpha$. *Furthermore, the following diagram commutes where* $\overline{\mathrm{res}}$ *is the map induced* *by restriction from* $\mathscr{M}$ *to* $\mathscr{M}_E$.

$$\begin{array}{ccc} \mathscr{M}^*/\mathscr{M}^* & \xrightarrow{\bar{\Psi}} & (\mathscr{M}/\mathscr{N})^\wedge \\ \bar{S}_\alpha \downarrow & & \downarrow \overline{\text{res}} \\ (\mathscr{N}_E)^*_\alpha/(\mathscr{M}_E)^*_\alpha & \xrightarrow{\Psi_\alpha} & (\mathscr{M}_E/\mathscr{N}_E)^\wedge \end{array}$$

Proof. — (i) That $\bar{\Psi}$ is an isomorphism is a special case of Proposition 1.3. Since Ψ_α is an isomorphism from A_E to $A_E^\wedge$ it follows from the definition of the complement with respect to α that $\bar{\Psi}_\alpha$ is an isomorphism.

(ii) By Remark 2.4 (iv), $\mathscr{N}^*$ and $\mathscr{M}^*$ are sums of coordinates. Then, by Theorem 2.6 (v), $S_\alpha(\mathscr{N}^*)=(N^*)_E$ and $S_\alpha(\mathscr{M}^*)=(\mathscr{M}^*)_E$. But then the first statement follows from Proposition 2.11 and the second statement from Proposition 2.12.

We now turn to the multiplicative aspects of duality. With notation as above, for each b in A_E we define a map $\psi^\times_{\alpha,\,b}: A^\times \to \mathbb{C}^\times$ by $\psi^\times_{\alpha,\,b}(y)=\psi_{\alpha,\,b}(y-1)$. In what follows it will often be convenient to write $\psi_{\alpha,\,b}=\psi^\times_{\alpha,\,b}$. It will always be clear from context whether $\psi_{\alpha,\,b}$ is considered additively or multiplicatively. Then we have shown that the following proposition holds.

PROPOSITION 2.14. — *Let* L *be an* $\mathscr{O}_E$*-lattice chain with associated ordered partition* $(r_1, \ldots, r_e)$ *as an* $\mathscr{O}_E$*-lattice chain and associated hereditary order* $\mathscr{A}$ *as an* $\mathscr{O}_F$*-lattice chain. Then the following hold:*

(i) $(U(\mathscr{A}_E)/U^1(\mathscr{A}_E))^\wedge$ *is canonically isomorphic to* $\prod_{i=1}^{e}(GL_{r_i}(k_E))^\wedge$.

(ii) *If* $\mathscr{M}$ *is a sum of coordinates in* $\mathscr{P}^k$ *and* $\mathscr{N}$ *is a sum of coordinates in* $\mathscr{P}^l$ *such that* $M(\mathscr{M})$ *and* $M(\mathscr{N})$ *are defined and such that* $\mathscr{M}$ *contains* $\mathscr{N}$ *and* $\mathscr{N}$ *contains* $\mathscr{M}^2$, *then the maps* $b\to\psi_b$ *and* $b\to\psi_{\alpha,b}$ *induce isomorphisms from* $\mathscr{N}^*/\mathscr{M}^*$ *onto* $(M(\mathscr{M})/M(\mathscr{N}))^\wedge$ *and from* $(\mathscr{N}^*)_E/(\mathscr{M}^*)E$ *onto* $(M(\mathscr{M}_E)/M(\mathscr{N}_E))^\wedge$. *Moreover, for b in* $\mathscr{N}^*$, *the restriction of* ψ_b *from* $M(\mathscr{M})$ *to* $M(\mathscr{M}_E)$ *is* $\psi_{\alpha,\,S_\alpha(b)}$.

(iii) *In particular, if m and n are integers such that* $m<n$ *and* $2m\geqq n$, *the the map* $b\to\psi_{\alpha,\,b}$ *induces an isomorphism from* $\mathscr{P}_E^{1-n}/\mathscr{P}_E^{1-m}$ *onto* $(U^m(\mathscr{A}_E)/U^n(\mathscr{A}_E))^\wedge$. *Moreover, for b in* $\mathscr{P}^{1-n}$, *the restriction of* ψ_b *from* $U^m(\mathscr{A})$ *to* $U^m(\mathscr{A}_E)$ *is* $\psi_{\alpha,\,S_\alpha(b)}$.

We close this subsection by proving two properties of $\psi_{\alpha,\,b}$ analogous to properties enjoyed by ψ_b. In particular, we have

PROPOSITION 2.15. — *With* $\mathscr{M}$ *and* $\mathscr{N}$ *as in the previous proposition, the following hold:*

(i) *If* $\psi_{\alpha,\,b}$ *is an element of* $(M(\mathscr{M}_E)/M(\mathscr{N}_E))^\wedge$ *and g is an element of* G_E, *then*

$$(\psi_{\alpha,\,b})^g\equiv\psi_{\alpha,\,b^g}$$

as a representation of $M(g\mathscr{M}_E g^{-1})/M(g\mathscr{N}_E g^{-1})$.

(ii) *An element* $\psi_{\alpha,\,b}$ *of* $(M(\mathscr{M}_E)/M(\mathscr{N}_E))^\wedge$ *has the property that b can be chosen in* E *if and only if it factors through* $\det_{A_{A/E}}$.

Proof. — (i) If x is in $\mathrm{M}(g\mathscr{M}_{\mathrm{E}}g^{-1})$, then

$$\begin{aligned}(\psi_{\alpha,b})^{g}(x) &= \psi_{\alpha,b}(g^{-1}xg)\\ &= \psi(\hat{\alpha}_{\mathrm{V/F}}(b(g^{-1}xg-1)))\\ &= \psi(\hat{\alpha}_{\mathrm{E/F}}(\mathrm{tr}_{\mathrm{A_E/E}}\, b(g^{-1}xg-1)))\\ &= \psi(\hat{\alpha}_{\mathrm{V/F}}(b^{g}(x-1)))\\ &= \psi_{\alpha,b^{g}}(x).\end{aligned}$$

(ii) For x in $\mathrm{M}(\mathscr{M}_{\mathrm{E}})$ and b in $(\mathscr{N}^{*})_{\mathrm{E}}$,

$$\begin{aligned}\psi_{\alpha,\mathrm{b}}(x) &= \psi(\hat{\alpha}_{\mathrm{V/F}}(b(x-1)))\\ &= \psi(\hat{\alpha}_{\mathrm{E/F}}(\mathrm{tr}_{\mathrm{A_E/E}}(b(x-1)))).\end{aligned}$$

Now $\psi\circ\hat{\alpha}_{\mathrm{E/F}}$ is an additive character of E with conductor $\mathrm{P_E}$ so that (ii) now follows from Remark 2.4 (vii) with E in place of F.

In what follows, if ρ is a character of $\mathrm{F}^{\times}$, we will abbreviate $\rho\circ\det_{\mathrm{A/F}}$ by ρ. It will be clear from context whether ρ is considered as a character of $\mathrm{F}^{\times}$ or as a character of $\mathrm{A}^{\times}$. We will also use the analogous abbreviation for characters of $\mathrm{E}^{\times}$ where E/F is a finite dimensional extension.

2.3. Relative alfalfa strata. — Let $(\mathscr{A}, n, \alpha)$ be a proper alfalfa stratum and set $\mathrm{E}=\mathrm{F}[\alpha]$. In this subsection we will want to consider certain representations of subgroups H of G of the form $\mathrm{U}^{r}(\mathscr{A}_{\mathrm{E}})\,\mathrm{U}^{m}(\mathscr{A}_{\mathrm{F}})$ where *either* $n+1<m<[(n+2)/2]$ and $r=m-1$ *or* $m=[(n+2)/2]$ and $0\leqq r\leqq m-1$. (We note that m is determined by r; in fact $m=\max(r+1, [(n+2)/2])$.) In describing these representations, we note first that the group H stabilizes the representation ψ_{α} on $\mathrm{U}^{m}(\mathscr{A}_{\mathrm{F}})$ and that, further, the restriction of ψ_{α} to $\mathrm{U}^{m}(\mathscr{A}_{\mathrm{F}})\cap\mathrm{U}^{r}(\mathscr{A}_{\mathrm{E}})=\mathrm{U}^{m}(\mathscr{A}_{\mathrm{E}})$ factors through $\det_{\mathrm{A_E/E}}$. It follows that we may construct a representation ρ of $\mathrm{U}^{r}(\mathscr{A}_{\mathrm{E}})\,\mathrm{U}^{m}(\mathscr{A}_{\mathrm{F}})$ from the following data [in addition to the data $(\mathscr{A}, n, \alpha)$]:

(i) a character θ of $\mathrm{E}^{\times}$ for which θ and ψ_{α} agree upon restriction to $\mathrm{U}^{m}(\mathscr{A}_{\mathrm{E}})$ and

(ii) a $\mathrm{G_E}$-stratum $(\mathscr{A}_{\mathrm{E}}, r, \eta)$; this construction being effected by setting $\rho(h_1 h_2)=(\eta\otimes\theta)(h_1)\,\psi_{\alpha}(h_2)$, h_1 in $\mathrm{U}^{r}(\mathscr{A}_{\mathrm{E}})$, h_2 in $\mathrm{U}^{m}(\mathscr{A}_{\mathrm{F}})$. These considerations lead us to

Definition 2.16. — With notation as above, we call the seven-tuple $\Omega=(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ a *relative alfalfa stratum* and we let $(\eta\otimes\theta)\cdot\psi_{\alpha}$ denote the associated representation of $\mathrm{U}^{r}(\mathscr{A}_{\mathrm{E}})\,\mathrm{U}^{m}(\mathrm{A_F})$. We define the *level* $l(\Omega)$ of Ω and the *relative level* $l_r(\Omega)$ of Ω to be $n/e(\mathscr{A})$ and $r/e(\mathscr{A}_{\mathrm{E}})$ respectively. We say that an admissible representation of G *contains* Ω if it contains $(\eta\otimes\theta)\cdot\psi_{\alpha}$ upon restriction to $\mathrm{U}^{r}(\mathscr{A}_{\mathrm{E}})\,\mathrm{U}^{m}(\mathscr{A}_{\mathrm{F}})$.

Remark 2.17. — With notation as above, we note that an admissible representation of G which contains ψ_{α} upon restriction to $\mathrm{U}^{m}(\mathscr{A}_{\mathrm{E}})$ will, since $\mathrm{U}^{m-1}(\mathscr{A}_{\mathrm{E}})$ stabilizes ψ_{α}, contain a relative alfalfa stratum $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ with $r=m-1$. It follows that an admissible representation of G which contains a proper alfalfa stratum will contain a relative alfalfa stratum. If π is an irreducible admissible representation of G which contains a proper alfalfa stratum, then we define the *relative level* $l_r(\pi)$ of π to be the minimum of the relative levels of relative alfalfa strata contained in π.

We now turn to the problem of relativizing the notions of "fundamental" and "alfalfa" to the situation at hand. We note that the restriction to principal orders in Definition 2.19 below has been adopted merely as a matter of convenience in the two prime case.

DEFINITION 2.18. — We say that a relative alfalfa stratum $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ is *fundamental* if either $r=0$ or $r>0$ and the following conditions are satisfied:

(1) $\theta \otimes \eta$ does not factor through $\det_{A_E/E}$ $(E=F[\alpha])$.

(2) Let $\eta=\psi_{\alpha, b}$. Then $b+\mathscr{P}_E^{1-r}$ does not contain a nilpotent element.

DEFINITION 2.19. — We say that a relative alfalfa stratum $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ is a *second order alfalfa stratum* if $r>0$ *and the following conditions are satisfied*

(1) $\eta \otimes \theta$ does not factor through $\det_{A_E/E}$ $(E=F[\alpha])$.

(2) Let $\eta=\psi_{\alpha, b}$. Then there exists an element α' of the coset $b+\mathscr{P}_E^{1-r}$ such that:

(i) $E'=E[\alpha']$ is a subfield of A.

(ii) $\nu_{\mathscr{A}}(\alpha')=-r$.

(iii) α' is E'/E-minimal.

(iv) Let L be an $\mathcal{O}_F$-lattice chain such that $\mathscr{A}=\mathscr{A}(L)$. Then L is in fact a uniform $\mathcal{O}_{E'}$-lattice chain.

One can check that a second order alfalfa stratum is fundamental. We define the *degree* of a second order alfalfa stratum $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ to be $[E' : F]$ where E' is as above. By an argument similar to that for Proposition 1.14, this notion is well-defined. By Proposition 2.15, the degree of $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ is strictly greater than that of $(\mathscr{A}, n, \alpha)$.

In the next section we will prove the following two theorems.

THEOREM 2.20. — *Let π be an admissible representation of* G *which contains a relative alfalfa stratum $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ and suppose that $c \mid gcd(r, e(\mathscr{A}_E))$* $(E=F[\alpha])$. *Then π also contains a relative alfalfa stratum $(\mathscr{A}', m', n', \alpha, r', \eta', \theta)$ with $r'=r/c$ and $e(\mathscr{A}'_E)=e(\mathscr{A}_E)/c$.*

THEOREM 2.21. — *Let π be an irreducible admissible representation of* G *which contains a proper alfalfa stratum. Then a relative alfalfa strata of minimal relative level in π is fundamental.*

These theorems have the following corollary which will serve as the starting point for the fourth section of this paper.

COROLLARY 2.22. — *An irreducible admissible representation which contains a proper alfalfa stratum also contains a fundamental relative alfalfa stratum $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ of minimal relative level which has the property that $(r, e(\mathscr{A}_E))=1$* $(E=F[\alpha])$.

3. Fundamental relative alfalfa strata

In this section we will prove Theorem 2.20 and Theorem 2.21. The proofs of each of these two theorems will be broken into three parts, these parts forming the subsections of this section.

3.1. RELATIVELY LARGE RELATIVE LEVEL. — In this subsection we consider the case $m>[(n+2)/2]$ (*see* § 2.3). Before stating the first proposition of the subsection, it will be useful to first make some definitions and then prove a lemma.

Let L and L′ be $\mathcal{O}_F$-lattice chains in V. Then, as in [HM1], we that L′ in a *thinning* of L if for each integer i there exists an integer $j(i)$ such that $L'_i=L_{j(i)}$. We say that L′ is a *refinement* of L if L is a thinning of L′. We say that an $\mathcal{O}_F$-lattice chain is a *common thinning* of L and L′ if it thins both L and L′ and define *common refinement* similarly. If L is an $\mathcal{O}_F$-lattice chain and c is a positive integer dividing $e(L)$ then we single out the thinning L′ of L defined by $L'_i=L_{ic}$ and call L′ the *uniform thinning of* L *by a factor of c* while we say that L is a *uniform refinement of* L′ *by a factor of c.*

LEMMA 3.1. — *Let* L *be an* $\mathcal{O}_F$*-lattice chain, let c be a positive integer which divides* $e(L)$ *and let* L′ *be the uniform thinning of* L *by a factor of c; set* $\mathcal{P}=P(L)$ and $\mathcal{P}'=\mathcal{P}(L')$. *Then for all integers l the following hold:*

(i) $(\mathcal{P}')^l\supseteq\mathcal{P}^{lc}$.

(ii) $\mathcal{P}^l\supseteq(\mathcal{P}')^{[(l-2)/c]+2}$. *In particular, for l′ an integer,* $\mathcal{P}^{l'c+1}\supseteq(\mathcal{P}')^{l'+1}$.

Proof. — (i) If x is an element of $\mathcal{P}^{lc}$, then $x L'_i=x L_{ic}\subseteq L_{ic+lc}=L'_{i+l}$.

(ii) Suppose that x is an element of $(\mathcal{P}')^m$ with $m'=[(l-2)/c]+2$. Then it suffices to show that, for $0\leqq i<e(L)$, $x L_i\subseteq L_{i+l}$. Write $i=cj+k$ where j is a nonnegative integer and $0\leqq k<c$. Then $x L_i\subseteq x L'_j\subset L'_{j+m}=L_{c(j+m)}$. Thus it suffices to show that $c(j+m)\geqq i+l$ but this can be checked directly.

PROPOSITION 3.2. — *Let* π *be an admissible representation of* G *which contains a relative alfalfa stratum* $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$, let $E=F[\alpha]$ *and suppose that* $c\,|\,gcd(r, e(\mathcal{A}_E))$. *Suppose further that* $m>[(n+2)/2]$; *set* $n'=n/c$, $r'=r/c$, $e'=e(\mathcal{A}_E)/c$ and $m'=r'+1$. *Then* π *also contains a relative alfalfa stratum* $(\mathcal{A}', m', n'\alpha, r', \eta', \theta)$ *with* $e(\mathcal{A}'_E)=e'$.

Proof. — Suppose $\mathcal{A}=\mathcal{A}(L)$. Let L′ be the uniform thinning of L by a factor of c; set $\mathcal{A}'=\mathcal{A}(L')$ and set $\mathcal{P}'=\mathcal{P}(L')$. By the previous lemma $\mathcal{P}^{r+1}\supseteq(\mathcal{P}')^{r'+1}$. Thus π contains ψ_α upon restriction to $U^{m'}(\mathcal{A}'_F)$ and now the lemma follows from Remark 2.17.

In the remainder of this subsection we prove

PROPOSITION 3.3. — *Let* π *be an irreducible admissible representation of* G *which contains a proper alfalfa stratum and suppose that* $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ *is a relative alfalfa stratum contained in* π *of minimal relative level. Assume further that* $m>[(n+2)/2]$. *Then* $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ *is fundamental.*

A key ingreedient of our proof of Proposition 3.3 and the other propositions which will yield Theorem 2.21 will be

THEOREM 3.4 ([B], Theorem 1, [HM1], Theorem 4.1). — *If* $\mathcal{A}$ *is a hereditary order in* A, *with radical* $\mathcal{P}$, *and period e and the coset* $x+\mathcal{P}^{j+1}$ *in* $\mathcal{P}^j$ *contains a nilpotent element,* *then there exists a hereditary order $\mathscr{A}'$ in* A, *with radical $\mathscr{P}'$, and an integer j' such that*

$$x+\mathscr{P}^{j+1}\subseteq(\mathscr{P}')^{j'} \qquad \text{and} \qquad j'/e'>j/e$$

where $e'=e(\mathscr{A}')$.

In proving Theorem 2.21, we will actually use Proposition 3.5 below. As is shown below, Proposition 3.5 is a corollary to the proof of Theorem 1 in [B]. It also can be deduced from the proof of Theorem 4.1 in [HM1].

PROPOSITION 3.5. — *With notation and hypotheses as in Theorem* 3.4, *the hereditary order $\mathscr{A}'$ may be chosen with the following additional property: There exist lattice chains* L *and* L′ *having both a common refinement and a common thinning such that $\mathscr{A}=\mathscr{A}(\mathrm{L})$* and $\mathscr{A}'=\mathscr{A}(\mathrm{L}')$.

Proof. – As in [B], for any lattice M in V, write IM for the lattice generated by $(x+\mathscr{P}^{j+1})\mathrm{M}$. Thus $\mathrm{IM}=x\mathrm{M}+\mathscr{P}^{j+1}\mathrm{M}$. Set $\mathrm{I}^0\mathrm{M}=\mathrm{M}$ and inductively define $\mathrm{I}^n\mathrm{M}=\mathrm{I}(\mathrm{I}^{n-1}\mathrm{M})$ for $n\geqq 2$. Suppose that $\mathscr{A}=\mathscr{A}(\mathrm{L})$. We claim that the set of distinct lattices in the set

$$\{\mathrm{I}^k\mathrm{L}_i \,|\, k=0,\ \ldots,\ \infty,\ i\in\mathbb{Z}\} \tag{3.5.1}$$

forms a lattice chain. Since this set is closed under the operations $\mathrm{M}\rightarrow\mathrm{PM}$ and $\mathrm{M}\rightarrow\mathrm{P}^{-1}\mathrm{M}$, it suffices to show that it is linearly ordered. To this end, it suffices to show that for nonnegative integers b and c and integers k and l that either $\mathrm{I}^b\mathrm{L}_k\subseteq\mathrm{I}^c\mathrm{L}_l$ or $\mathrm{I}^b\mathrm{L}_k\supseteq\mathrm{I}^c\mathrm{L}_l$. We argue as in the proof of [B], 2.8. By symmetry, we may assume $b\leqq c$ and we proceed by induction on b. By [B], 2.3, there exists an integer m such that $\mathrm{L}_{m+1}\subseteq\mathrm{I}^c\mathrm{L}_l\subseteq\mathrm{L}_m$ whence the case $b=0$. Now the case of general b follows from either $\mathrm{I}^{b-1}\mathrm{L}_k\subseteq\mathrm{I}^{c-1}\mathrm{L}_l$ on $\mathrm{I}^{b-1}\mathrm{L}_k\supseteq\mathrm{I}^{c-1}\mathrm{L}_l$.

Let L″ be the chain determined by (3.5.1). Note that L″ is a refinement of L. In his proof of Theorem 3.3, Bushnell shows that there exists positive integers a and i_0 such that the hereditary order $\mathscr{A}'$ can be taken to be that associated to the thinning L′ of L″ determined by the set

$$\{\mathrm{I}^j\mathrm{L}_i \,|\, j=0,\ 1,\ \ldots,\ a-1,\ i\equiv i_0 \bmod e(\mathscr{A})\}. \tag{3.5.2}$$

Since L and L′ contain L_{i_0}, the proposition follows.

In addition to Proposition 3.5, the other key ingredient in our proof of Proposition 3.3 is the following technical lemma.

LEMMA 3.6. — *Suppose that $(\mathscr{A},\ n,\ \alpha)$ is an alfalfa stratum, that m is an integer satisfying $n\geqq m\geqq[(n+2)/2]+1$ and that $\mathscr{M}$ is a sum of coordinates in $\mathscr{P}^{m-1}$ with respect to some $\mathscr{O}_{\mathrm{E}}$-basis* $(\mathrm{E}=\mathrm{F}[\alpha])$ *for* L *where $\mathscr{A}=\mathscr{A}(\mathrm{L})$. Suppose further that θ is a character of $\mathrm{E}^\times$ which agrees with ψ_α upon restriction to $\mathrm{U}^m(\mathscr{A}_{\mathrm{E}})$. Suppose finally that π is an admissible representation of* G *which contains the representation $\theta\cdot\psi_\alpha$ upon restriction to $\mathrm{M}(\mathscr{M}_{\mathrm{E}})\,\mathrm{U}^m(\mathscr{A}_{\mathrm{F}})$. Then π contains a representation $\psi_{\alpha'}$ upon restriction to $\mathrm{M}(\mathscr{M}_{\mathrm{F}})$ where α' is an element of the coset $\alpha+\mathscr{P}^{1-n}$ such that ψ_α and $\psi_{\alpha'}$ agree upon restriction to*

$U^m(\mathscr{A}_F)$ *and the following properties are satisfied:*

(i) $E' = F[\alpha']$ *is a subfield of* A.

(ii) $\nu_{\mathscr{A}}(\alpha') = -n$.

(iii) α' *is* E'/F-*minimal.*

(iv) L *is an* $\mathcal{O}_{E'}$-*lattice chain.*

Proof. — Before beginning the proof we note that (ii) is redundant since it is implied by the requirement that ψ_α and $\psi_{\alpha'}$ agree on $U^m(\mathscr{A}_F)$. We only include it for emphasis. To begin the proof, let v be a nonzero vector in the space of π which transforms according to $\theta\cdot\psi_\alpha$ under the action of $M(\mathscr{M}_E)\,U^m(\mathscr{A}_F)$. Since $U^{m-1}(\mathscr{A}_F)$ contains $M(\mathscr{M}_F)$ and $m-1 \geqq [(n+2)/2]$, the $M(\mathscr{M}_F)$-span of v decomposes as a sum of characters which extend $\theta\cdot\psi_\alpha$. Thus, changing v if necessary, we may assume that, under the action of $M(\mathscr{M}_F)$, v transforms according to ψ_b where ψ_b is a character of $M(\mathscr{M}_F)$ extending $\theta\cdot\psi_\alpha$. Then, by Proposition 2.14 (iii), $\psi_{\alpha, S_\alpha(b)}$ and θ agree on $M(\mathscr{M}_E)$. Thus by Proposition 2.15 (ii), $S_\alpha(b)+\mathscr{M}_E^*$ contains an element of E^*. Now it follows from the definition of S_α and (2.0.1) that $S_\alpha(\alpha)$ is in E. Thus there exists an element γ of E which is also in $S_\alpha(b-\alpha)+\mathscr{M}_E^*$.

We now claim that there exists α' such that ψ_α and $\psi_{\alpha'}$, agree on $U^m(\mathscr{A}_F)$, (i) through (iv) are satisfied and $S_\alpha(b-\alpha')$ is in $\mathscr{M}_E^*$. To this end, let $\mathscr{B}$ be an $\mathcal{O}_E$-basis for L. Then, as in Definition 2.3, we may identify V with $E\otimes_F W$ where W is the F-span of $\mathscr{B}$ in V. Then, as in Lemma 2.7, there exists an (E, E)-bimodule isomorphism, φ say, from $A_F(E)\otimes \mathrm{End}_F(W)$ onto $A_F(V)$.

Now, if γ is in $\mathscr{M}_E^*$, our claim follows from letting $\alpha'=\alpha$. Thus suppose γ is not in $\mathscr{M}_E^*$, Since ψ_b and ψ_α agree on $U^m(\mathscr{A}_F)$, $b-\alpha$ is in $\mathscr{P}_F^{1-m}$ and thus $\nu_{\mathscr{A}}(\gamma)=1-m$. By Theorem 2.6, there exists β in $A_F(E)$ such that $S_{\alpha, E}(\beta)=\gamma$ and $\nu_{\mathscr{A}(E)}(\beta)=\nu_{\mathscr{A}(E)}(\gamma)$. Let $\alpha'=\alpha+\varphi(\beta\otimes 1)$. Then

$$\begin{aligned} S_{\alpha, V}(\alpha') &= S_{\alpha, V}(\alpha)+S_{\alpha, V}(\varphi(\beta)\otimes 1)\\ &= S_{\alpha, V}(\alpha)+\varphi(S_{\alpha, E}(\beta)\otimes 1)\\ &= S_{\alpha, V}(\alpha)+\varphi(\gamma\otimes 1)\\ &= S_{\alpha, V}(\alpha)+\gamma. \end{aligned}$$

Thus, writing $S_\alpha = S_{\alpha, V}$ once again, we have that $S_\alpha(b-\alpha')$ is in $\mathscr{M}_E^*$. Then, since $\mathscr{B}$ is an $\mathcal{O}_E$-basis for L and $\nu_{\mathscr{A}(E)}(\beta)=\nu_{\mathscr{A}(E)}(\gamma)$, one can check that $\nu_{\mathscr{A}}(\varphi(\beta\otimes 1))=1-m$ so that $\nu_{\mathscr{A}}(\alpha')=-n$ and $\alpha-\alpha'$ is in $\mathscr{P}^{1-m}$. Thus, to complete the proof of the claim, it suffices to show that (i), (iii) and (iv) hold.

Since φ is an algebra homomorphism and since $\alpha'=\alpha+\varphi(\beta\otimes 1)=\varphi((\alpha+\beta)\otimes 1)$, to show that $E'=F[\alpha']$ is a subfield of $A_F(V)$ and that α' is E'/F'-minimal, it suffices to show that $E_1=F[\alpha_1]$, where $\alpha_1=\alpha+\beta$, is a subfield of $A_F(E)$ with α_1 E_1/F-minimal. This however, is a result of Carayol [Ca], 3.2 since $\nu_{\mathscr{A}(E)}(\beta)>\nu_{\mathscr{A}(E)}(\alpha)$ and α is E/F-minimal. It then also follows from a result of Carayol [Ca], 3.4 that $\mathcal{O}_{E_1}$ is contained in $\mathscr{A}_F(E)$ and that P_{E_1} is contained in $\mathscr{P}_F(E)$. Then, since $\mathscr{B}$ is an $\mathcal{O}_E$-basis for L, it follows that $\mathcal{O}_{E'}$ is contained in $\mathscr{A}_F$ and $P_{E'}$ is contained in $P_E\mathscr{A}_F$. Thus, each lattice L_i in L is an $\mathcal{O}_{E'}$-lattice and $P_{E'}\, L_i \subseteqq L_{i+e}$ for all i where e is the period of L as an $\mathcal{O}_E$-chain. Suppose $P_{E'}\, L_i$ is properly contained in L_{i+e} for some i. Then $(P_{E'})^{e\,(E'/F)}\, L_i$ is properly contained in $L_{i+ee\,(E'/F)}$. But $(P_{E'})^{e\,(E/F)} = P_F$ so that then $P_F\, L_i$ is properly contained in $L_{i+ee\,(E'/F)}$ a contradiction. Thus $P_{E'}\, L_i = L_{i+e}$ for all i so that L is an $\mathcal{O}_{E'}$-lattice chain as desired and the claim follows.

Since $S_\alpha(\alpha' - b)$ is in $\mathcal{M}_E^*$ and $\alpha' - b$ is in $\mathcal{P}^{1-m}$, there exists, by Theorem 2.6, an h in $\mathcal{P}^{1-m}$ such that $A_\alpha(h) - (b - \alpha')$ is in $\mathcal{M}^*$. Now, to prove the lemma, it suffices to prove that, $v_1 = \pi((1 + h\alpha^{-1}))\, v$ transforms according to $\psi_{\alpha'}$ under the action of $M(\mathcal{M}_F)$. To this end, let $\alpha + T = b$. Then,

$$
\begin{aligned}
(1+h\alpha^{-1})\, b\, (1+h\alpha^{-1})^{-1} &\equiv (1+h\alpha^{-1})\, b\, (1-h\alpha^{-1}) \bmod \mathcal{M}^* \\
&\equiv (\alpha + h + T)(1 - h\alpha^{-1}) \bmod \mathcal{M}^* \\
&\equiv \alpha + T - A_\alpha(h) \bmod \mathcal{M}^* \\
&\equiv \alpha' \bmod \mathcal{M}^*
\end{aligned}
\tag{3.6.1}
$$

by definition of h. Thus, since $1 + h\alpha^{-1}$ stabilizes $\mathcal{M}$, v_1 transforms according to $\psi_{\alpha'}$ as desired.

With Proposition 3.5 and Lemma 3.6 now in place we close this subsection with the

Proof of Proposition 3.3. — With notation as in the statement of the proposition, assume that $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ is not fundamental. If $\eta \otimes \theta$ factors through the determinant, then Lemma 3.6 (with $\mathcal{M} = \mathcal{P}^{m-1}$ and $\eta \otimes \theta$ in place of θ) and Remark 2.17 imply that π contains a relative alfalfa stratum $(\mathcal{A}, m-1, n, \alpha', r-1, \eta', \theta')$. This however contradicts the assumption that $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ is of minimal level. Thus we may assume that $\eta = \psi_{\alpha, b}$ where $b + \mathcal{P}_E^{2-m}$ contains a nilpotent element. Then, by Proposition 3.5, there exists an $\mathcal{O}_E$-lattice chain L' and an integer m' such that the following hold ($\mathcal{A}' = \mathcal{A}(L')$ and $\mathcal{P}' = \mathcal{P}(L')$):

$$b + \mathcal{P}_E^{2-m} \subseteqq (\mathcal{P}'_E)^{1-m'}. \tag{3.3.1}$$

$$(m-1)/e(\mathcal{A}_E) > (m'-1)/e(\mathcal{A}'_E). \tag{3.3.2}$$

(3.3.3) L and L' have a common refinement.

We claim that $\mathcal{M} = \mathcal{P}^m + (\mathcal{P}')^{m'}$ is a sum of coordinates in $\mathcal{P}^{m-1}$ with respect to an $\mathcal{O}_E$-basis for a common refinement of L and L'. To this end, (3.3.1) implies that $\mathcal{P}_E^{2-m} \subseteqq (\mathcal{P}'_E)^{1-m'}$ so that $\mathcal{P}^{2-m} \subseteqq (\mathcal{P}')^{1-m'}$. Thus

$$
\begin{aligned}
\mathcal{P}^{m-1} &= (\mathcal{P}^{2-m})^* \\
&\supseteqq ((\mathcal{P}')^{1-m})^* \\
&= (\mathcal{P}')^{m'}
\end{aligned}
\tag{3.3.4}
$$

so that $\mathcal{P}^{m-1} \supseteqq \mathcal{M} \supseteqq \mathcal{P}^m$. Then, since an $\mathcal{O}_E$-basis for a common refinement of L and L' is also an $\mathcal{O}_E$-basis for both L and L', the claim follows.

By (3.3.1), for $1+h$ in $U^{m'}(\mathscr{A}'_E)$ we have that

$$\begin{aligned}(3.3.5) \qquad \psi_{\alpha, b}(1+h) &= \psi(\hat{\alpha}(bh)) \\ &= 1.\end{aligned}$$

Thus π contains the representation $\theta \cdot \psi_\alpha$ upon restriction to $M(\mathscr{M}_E) U^m(\mathscr{A}_F)$. Now, by Lemma 3.6, there exists α' as in the lemma such that π restricted to $M(\mathscr{M}_F)$ contains $\psi_{\alpha'}$. Hence π restricted to $U^{m'}(\mathscr{A}'_F)$ contains $\psi_{\alpha'}$. Now, with $n' = -\nu_{\mathscr{A}'}(\alpha')$, $(\mathscr{A}', n', \alpha')$ is an alfalfa stratum. Moreover, with $E' = F[\alpha']$, $e = e(\mathscr{A}_E)$ and $e' = e(\mathscr{A}_{E'})$, Proposition 1.14 and (3.3.2) imply that

$$\begin{aligned} n' &= -\nu_{E'}(\alpha') e' \\ &= -\nu_E(\alpha) e' \\ &= ne'/e \\ &> (m-1) e'/e \\ &> m'-1 \end{aligned}$$

But now, by Remark 2.17, $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ is not of minimal level — a contradiction.

3.2. Relatively small relative level: the even case. — In this subsection and the next we prove Theorem 2.20 and Theorem 2.21 in the case $m = [(n+2)/2]$. We divide these proofs into two subcases with the distinction between these subcases being effected by the following definition.

Definition 3.7. — We say that an alfalfa stratum $(\mathscr{A}, n, \alpha)$ or a relative alfalfa stratum $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ is *even* if $\nu_E(\alpha)$ is even ($E = F[\alpha]$). We say that such a stratum is *odd* if $\nu_E(\alpha)$ is odd. Note that if π is an irreducible admissible representation of G which contains an alfalfa stratum, then, by Proposition 1.14 all the alfalfa strata and all the relative alfalfa strata (if π contains a proper alfalfa stratum) contained in π have the same parity. Note also that if an alfalfa stratum $(\mathscr{A}, n, \alpha)$ is even, then n is even so that $[(n+2)/2] = (n+2)/2$.

Proposition 3.8. — *Let π be an admissible representation of* G *which contains an even relative alfalfa stratum $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$; set $E = F[\alpha]$, $e = e(\mathscr{A}_E)$ and suppose that $c \mid gcd(r, e)$. Assume further that $m = (n+2)/2$; set $n' = n/c$, $r' = r/c$, $e' = e/c$ and $m' = (n'+2)/2$. Then π also contains a relative alfalfa stratum $(\mathscr{A}', m', n', \alpha, r', \theta)$ with $e(\mathscr{A}'_E) = e'$.*

Proof. — Suppose that $\mathscr{A} = \mathscr{A}(L)$ and let L′ be the uniform thinning of L by a factor of c. Set $k = -\nu_E(\alpha)$, $\mathscr{A}' = \mathscr{A}(L')$, $\mathscr{P}' = \mathscr{P}(L')$ and note that $e(\mathscr{A}'_E) = e'$.

By Lemma 3.1, we have that

$$(3.8.1) \qquad \mathscr{P}^{r+1} \supseteq (\mathscr{P}')^{r'+1}.$$

Also,

$$\begin{aligned} r' &= r/c \\ (3.8.2) \qquad &\leqq (m-1)/c \end{aligned}$$

$$= ke'/2$$
$$= m' - 1.$$

In addition, since L′ is a thinning of L,

$$\begin{aligned} \mathcal{P}^m &= \mathcal{P}^{ke/2+1} \\ &= \mathrm{P}_{\mathrm{E}}^{k/2}\,\mathcal{P} \\ &\supseteqq \mathrm{P}_{\mathrm{E}}^{k/2}\,\mathcal{P}' \\ &= (\mathcal{P}')^{m'}. \end{aligned} \tag{3.8.3}$$

By (3.8.1) and (3.8.3), π contains $\theta \cdot \psi_\alpha$ upon restriction to $\mathrm{U}^{r'+1}(\mathcal{A}'_{\mathrm{E}})\,\mathrm{U}^{m'}(\mathcal{A}'_{\mathrm{F}})$. Now Lemma 3.8 follows from Remark 2.17 if $r' = m' - 1$ and from the following lemma if $r' < m' - 1$.

LEMMA 3.9. — *If π is an admissible representation of* G *which contains a relative alfalfa stratum* $(\mathcal{A}, m, n, \alpha, r+1, \mathrm{Id}, \theta)$ *with* $m = [(n+2)/2]$ *and* $r \geqq 0$ *where* Id *denotes the identity character, then π also contains a relative alfalfa stratum* $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ *for some* η.

Proof. — This is straightforward (*see* Remark 2.17).

Our goal for the remainder of this section is

PROPOSITION 3.10. — *Let π be an irreductible supercuspidal representation of* G *which contains an even proper alfalfa stratum and let* $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ *be a relative alfalfa stratum contained in π of minimal relative level. Assume further that* $m = (n+2)/2$. *Then* $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ *is fundamental.*

Proof. — If $r = 0$, then $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ is fundamental by definition. Thus suppose $r > 0$, set $\mathrm{E} = \mathrm{F}[\alpha]$, $\mathcal{A} = \mathcal{A}(\mathrm{L})$ and write $\eta = \psi_{\alpha, b}$. If $\eta \otimes \theta$ factors through the determinant, then it follows from Lemma 3.9 that $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ is not of minimal relative level — a contradiction. Thus, if $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$ is not fundamental, we may assume $b + \mathcal{P}_{\mathrm{E}}^{1-r}$ contains a nilpotent element. Then, by Proposition 3.5, there exists an $\mathcal{O}_{\mathrm{E}}$-lattice chain L′ and an integer r' such that the following hold:

$$b + \mathcal{P}_{\mathrm{E}}^{-r+1} \subseteqq \mathcal{P}_{\mathrm{E}}(\mathrm{L}')^{-r'}. \tag{3.10.1}$$

$$r'/e_{\mathrm{E}}(\mathrm{L}') < r/e_{\mathrm{E}}(\mathrm{L}). \tag{3.10.2}$$

(3.10.3) L and L′ have a common thinning and a common refinement.

Before proceeding further with the proof of Proposition 3.10, we state the following technical lemma which is much in the spirit of Lemma 3.6. We will use this lemma frequently in what follows.

LEMMA 3.11. — *Let* $(\mathcal{A}, n, \alpha)$ *be an alfalfa stratum, let* L *be an* $\mathcal{O}_{\mathrm{E}}$*-lattice chain* $(\mathrm{E} = \mathrm{E}[\alpha])$ *such that* $\mathcal{A} = \mathcal{A}(\mathrm{L})$ *and let* $\mathcal{B}$ *be an* $\mathcal{O}_{\mathrm{E}}$*-basis for* L. *Suppose that* $\mathcal{M}$ *and* $\mathcal{N}$ *are sums of coordinates in* $\mathcal{P}^a$ *and* $\mathcal{P}^b$ *respectively with respect to* $\mathcal{B}$ *such that* α^{-1} *is in* $\mathcal{M}$, $\mathrm{M}(\mathcal{M})$ *and* $\mathrm{M}(\mathcal{N})$ *are defined,* $\mathcal{N} \supseteqq \mathcal{M}$ *and ker* $\psi_\alpha \supseteqq \mathcal{N}^2$. *Let* θ *be a character of* $\mathrm{E}^\times$ *which agrees with* ψ_α *upon restriction to* $\mathrm{M}(\mathcal{M}_{\mathrm{E}})$. *Let* L′ *be an* $\mathcal{O}_{\mathrm{E}}$*-lattice chain, set*

$\mathscr{A}'=\mathscr{A}(\mathrm{L}')$, $\mathscr{P}'=\mathscr{P}(\mathrm{L}')$ *and suppose that* r' *is a nonnegative integer such that* $(\mathscr{P}')^{r'+1}\supseteqq\mathscr{N}$ *and* $\mathrm{U}^{r'+1}(\mathscr{A}'_{\mathrm{E}})$ *normalizes both* $\mathrm{M}(\mathscr{M})$ *and* $\mathrm{M}(\mathscr{N})$. *Suppose finally that the following conditions hold:*

(i) $\mathscr{M}^*\cap\mathrm{A}_{ii}=\mathscr{N}^*\cap\mathrm{A}_{ii}$ *for all* i *where the* A_{ii} *are defined with respect to* $\mathscr{B}$.

(ii) $\alpha^{-1}(\mathscr{P}')^{r'+1}\mathscr{M}^*\subseteqq\mathscr{M}$.

Then an admissible representation π *of* G *which contains the representation* $\theta\cdot\psi_\alpha$ *of* $\mathrm{U}_{\mathrm{E}}^{r'+1}(\mathscr{A}')\mathrm{M}(\mathscr{M})$ upon restriction also contains $\theta\cdot\psi_\alpha$ *upon restriction* $\mathrm{U}^{r'+1}(\mathscr{A}'_{\mathrm{E}})\mathrm{M}(\mathscr{N})$.

Remark 3.12. — Before proving Lemma 3.11, we show in this remark that the lemma allows us to conclude the proof of Proposition 3.10. To see this, with notation as in the proof of Proposition 3.10, let $\mathrm{L}^\vee$ and $\mathrm{L}^\wedge$ be $\mathscr{O}_{\mathrm{E}}$-lattice chains which are, respectively, a common thinning and a common refinement of L and L′. Set $\mathscr{A}^\vee=\mathscr{A}(\mathrm{L}^\vee)$, $\mathscr{P}^\vee=\mathscr{P}(\mathrm{L}^\vee)$, $\mathscr{A}^\wedge=\mathscr{A}(\mathrm{L}^\wedge)$, $\mathscr{P}^\wedge=\mathscr{P}(\mathrm{L}^\wedge)$, $k=-\nu_{\mathrm{E}}(\alpha)$, $e=e(\mathscr{A}_{\mathrm{E}})$, $e'=e(\mathscr{A}'_{\mathrm{E}})$, $n'=ke'$ and $m'=(n'+2)/2$; similarly define $e^\vee\,.\,e^\wedge\,.\,n^\vee$, $n^\wedge$, $m^\vee$ and $m^\wedge$.

Note that (3.10.2) implies that

$$\begin{aligned} r'&<re'/e\\ &\leqq(m-1)e'/e \qquad (3.12.1)\\ &=m'-1.\end{aligned}$$

Note also that (3.10.1) implies that

$$\mathscr{P}_{\mathrm{E}}^{r}\supseteqq(\mathscr{P}'_{\mathrm{E}})^{r'+1} \qquad (3.12.2)$$

and that $\psi_{\alpha,b}$ is trivial upon restriction to $\mathrm{U}^{r'+1}(\mathscr{A}'_{\mathrm{E}})$. Let $\mathscr{B}$ be an θ_{E}-basis for $\mathrm{L}^\wedge$. Let $\mathscr{M}=\mathscr{P}^m\cap(\mathscr{P}')^{r'+1}$ and let $\mathscr{N}=(\mathscr{P}^\wedge)^{m^\wedge}\cap(\mathscr{P}')^{r'+1}$. Now we claim that $\mathscr{M}$ and $\mathscr{N}$ are sums of coordinates $(\mathscr{P}^\vee)^{m^\vee-1}$ with respect to $\mathscr{B}$. To this end, we first not that $\mathscr{P}^\vee\subseteqq\mathscr{P}\subseteqq\mathscr{A}^\vee$. Thus, multiplying by $\mathrm{P}_{\mathrm{E}}^{k/2}$ we obtain that $\mathrm{P}_{\mathrm{E}}^{k/2}\mathscr{P}^\vee\subseteqq\mathrm{P}_{\mathrm{E}}^{k/2}\mathscr{P}\subseteqq\mathscr{P}_{\mathrm{E}}^{k/2}\mathscr{A}^\vee$ which we may rewrite as

$$(\mathscr{P}^\vee)^{m^\vee}\subseteqq\mathscr{P}^m\subseteqq(\mathscr{P}^\vee)^{m^\vee-1}. \qquad (3.12.3)$$

Next, since $(\mathscr{P}')^{r'+1}\supseteqq(\mathscr{P}')^{m'}$, an argument similar to that for (3.12.3) yields that $(\mathscr{P}')^{r'+1}\supseteqq(\mathscr{P}^\vee)^{m^\vee}$. Thus, since an $\mathscr{O}_{\mathrm{E}}$-basis for $\mathrm{L}^\wedge$ is also an $\mathscr{O}_{\mathrm{E}}$-basis for L, L′ and $\mathrm{L}^\vee$, it follows that $\mathscr{M}$ is a sum of coordinates in $(\mathscr{P}^\vee)^{m^\vee-1}$. The argument for $\mathscr{N}$ is similar.

Since α^{-1} is in $(\mathscr{P}^\wedge)^{n^\wedge}$ and $(\mathscr{P}')^{n'}$ it follows from (3.12.1) that α^{-1} is in $\mathscr{M}$. An argument similar to (3.12.3) shows that $\mathscr{N}\supseteqq\mathscr{M}$. Since $(\mathscr{P}^\wedge)^{m^\wedge}\supseteqq\mathscr{N}$, it follows that $\mathrm{M}(\mathscr{N})$ is defined and $\ker\psi_\alpha\supseteqq\mathscr{N}^2$. Similarly $\mathrm{M}(\mathscr{M})$ is defined. Note that $(\mathscr{P}')^{r'+1}\supseteqq\mathscr{N}$. To show that $\mathrm{U}^{r'+1}(\mathscr{A}'_{\mathrm{E}})$ normalizes $\mathrm{M}(\mathscr{M})$ and $\mathrm{M}(\mathscr{N})$ it suffices to show that $\mathrm{U}^{r'+1}(\mathscr{A}'_{\mathrm{E}})$ is contained in $\mathscr{A}^\times$ and $(\mathscr{A}^\wedge)^\times$. The first containment follows from (3.12.2) and the second from (3.12.2) and the fact that $(\mathscr{A}^\wedge)^\times\supseteqq\mathrm{U}^1(\mathscr{A})$. Thus all the hypotheses of Lemma 3.11 are satisfied (with $\mathscr{A}^\vee$ in place of $\mathscr{A}$) with the possible exceptions of (i) and (ii). We now show that these hypotheses also hold.

To show that (i) holds, it suffices to show that

$$\mathscr{P}^{1-m}\cap\mathscr{A}_{ii}=(\mathscr{P}^\wedge)^{1-m^\wedge}\cap\mathrm{A}_{ii}\quad\text{for all } i. \qquad (3.12.4)$$

But since $\mathscr{P}^{1-m}=P_E^{-k/2}\mathscr{A}$, $(\mathscr{P}^\wedge)^{1-m^\wedge}=P_E^{-k/2}\mathscr{A}^\wedge$ and $L^\wedge$ is a refinement of L this is clear. Now consider (ii). Here we have that

$$(3.12.5)\quad \alpha^{-1}(\mathscr{P}')^{r'+1}\mathscr{M}^*=\alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}^{1-m}+(\mathscr{P}')^{-r'})=(\mathscr{P}')^{r'+1}\mathscr{P}^{m-1}+(\mathscr{P}')^{n'+1}.$$

By (3.12.2),

$$(3.12.6)\qquad (\mathscr{P}')^{r'+1}\mathscr{P}^{m-1}\subseteq\mathscr{P}^m.$$

Since $k\geqq 2$, we have that $m-1\geqq e$ so that

$$(3.12.7)\qquad \begin{aligned}(\mathscr{P}')^{r'+1}\mathscr{P}^{m-1}&\subseteq(\mathscr{P}')^{r'+1}\mathscr{P}^{e}\\ &\subseteq(\mathscr{P}')^{r'+1}\mathscr{P}^{\vee}\\ &\subseteq(\mathscr{P}')^{r'+2}.\end{aligned}$$

Finally, one checks that $(\mathscr{P}')^{n'+1}$ is contained in $(\mathscr{P}^\vee)^{m^\vee}$ and thus in $\mathscr{M}$ so that (ii) follows from (3.12.5), (3.12.6) and (3.12.7).

Now, since $\mathscr{P}^m\supseteq\mathscr{M}$, $(\mathscr{P}')^{r'+1}\subseteq\mathscr{P}^r$ and $\psi_{\alpha,b}$ is trivial on $U^{r'+1}(\mathscr{A}'_E)$, π contains $\theta\cdot\psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}'_E)M(\mathscr{M})$. Thus, if Lemma 3.11 holds, the above argument implies that π contains $\theta\cdot\psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}'_E)M(\mathscr{N})$. Then, since

$$\begin{aligned}(\mathscr{P}^\wedge)^{m^\wedge}&=P_E^{k/2}\mathscr{P}^\wedge\\ &\supseteq P_E^{k/2}\mathscr{P}'\\ &=(\mathscr{P}')^{m'},\end{aligned}$$

(3.12.1) implies that π contains the relative alfalfa stratum $(\mathscr{A}', m', n', \alpha, r'+1, \mathrm{Id}, \theta)$. But now (3.10.2) and Lemma 3.9 yield a contradiction. Thus, to conclude the proof of Proposition 3.10, it suffices to prove Lemma 3.11.

Proof of Lemma 3.11. — With notation as in Lemma 3.11, let v be a nonzero vector in the space of π which transforms according to $\theta\cdot\psi_\alpha$ under the action of $U^{r'+1}(\mathscr{A}'_E)M(\mathscr{M})$. Let W denote the $U^{r'+1}(\mathscr{A}'_E)M(\mathscr{N})$ span of v. We claim that W decomposes as a sum of characters of the form $\theta\cdot\psi_b$ where ψ_b extends ψ_α. To this end, it suffices to show that $\ker(\theta\cdot\psi_\alpha)$ (with $\theta\cdot\psi_\alpha$ viewed as a representation of $U^{r'+1}(\mathscr{A}'_E)M(\mathscr{M})$) is a normal subgroup of $U^{r'+1}(\mathscr{A}'_E)M(\mathscr{N})$ with abelian quotient. Since $U^{r'+1}(\mathscr{A}'_E)$ stabilizes ψ_α this is straightforward once we have the following:

(3.11.1) If g is in $U^{r'+1}(\mathscr{A}'_E)$ and h is in $M(\mathscr{N})$, then $hgh^{-1}g^{-1}$ is in $\ker(\theta\cdot\psi_\alpha)$.

To prove (3.11.1), write $g=1+\beta$, $h=1+x$, $g^{-1}=1+\bar\beta$ and $h^{-1}=1+\bar x$. Then since $x+\bar x+x\bar x=\beta+\bar\beta+\beta\bar\beta=0$ we have that

$$(3.11.2)\qquad hgh^{-1}g^{-1}=1+\beta\bar x+\bar x\bar\beta+x\beta\bar x+x\bar x\beta+\beta\bar x\bar\beta+x\beta\bar x\bar\beta.$$

Now $\mathrm{tr}(\alpha^{-1}(\mathscr{P}')^{r'+1}\mathscr{M}^*\mathscr{M}^*)=\mathrm{tr}(\alpha^{-1}\mathscr{M}^*(\mathscr{P}')^{r'+1}\mathscr{M}^*)$ so that (ii) implies that

$$(3.11.3)\qquad \alpha^{-1}\mathscr{M}^*(\mathscr{P}')^{r'+1}\subseteq\mathscr{M}.$$

Then, since $\mathcal{N} \supseteqq \mathcal{M}$ and $\ker \psi_\alpha \supseteqq \mathcal{N}$, it follows that $\alpha \mathcal{N} \subseteqq \mathcal{M}^*$ and thus that $(\mathcal{P}')^{r'+1} \mathcal{N}$ and $\mathcal{N}(\mathcal{P}')^{r'+1}$ are contained in $\mathcal{M}$. Then since (ii) implies that $(\mathcal{P}')^{r'+1} \mathcal{M} \subseteqq \mathcal{M}$ it follows that $hgh^{-1}g^{-1}$ is in $\mathcal{M}$ and that

$$(3.11.4) \qquad \theta \cdot \psi_\alpha (hgh^{-1}g^{-1}) = \psi_\alpha (1 + \beta \bar{x} + \bar{x} \bar{\beta} + x \beta \bar{x} + x \bar{x} \beta + \beta \bar{x} \bar{\beta})$$

Now $\psi_\alpha (1 + \beta \bar{x} + \bar{x} \bar{\beta} + \beta \bar{x} \bar{\beta}) = 1$ since α commutes with β. Thus (3.11.4) becomes

$$(3.11.5) \qquad \theta \cdot \psi_\alpha (hgh^{-1}g^{-1}) = \psi_\alpha (1 + x \beta \bar{x} + x \bar{x} \beta)$$

But then since x and $\bar{x}$ are in $\alpha^{-1} \mathcal{M}^*$ it follows from (ii) that $\theta \cdot \psi_\alpha (hgh^{-1}g^{-1}) = 1$ as desired.

Changing v if necessary, we may now assume that v transforms according to $\theta \cdot \psi_b$ under the action of $U^{r'+1}(\mathcal{A}'_E) M(\mathcal{N})$ where ψ_α and ψ_b agree upon restriction to $M(\mathcal{M})$. Then, as in the proof of Lemma 3.6, there exists an element γ in E which is also in $S_\alpha(b-\alpha) + \mathcal{N}^*_E$. Since $S_\alpha(b-a)$ is in $\mathcal{M}^*_E$ and γ is in E and thus in $\bigoplus_i A_{ii}$, (i) implies that $S_\alpha(b-\alpha)$ is in $\mathcal{N}^*_E$.

Since $S_\alpha(b-\alpha)$ is in $\mathcal{N}^*_E$ and $b-\alpha$ is in $\mathcal{M}^*$, there exists, by Theorem 2.6, an x in $\mathcal{M}^*$ such that $A_\alpha(x) - (b-\alpha)$ is in $\mathcal{N}^*$. Let $y = 1 + x\alpha^{-1}$. Since α^{-1} is in $\mathcal{M}$ by assumption and $\mathcal{M}$ is a sum of coordinates, it follows that $\alpha^{-1}\mathcal{M}^*$ is contained in the radical of some hereditary order. Moreover one can check that $(\alpha^{-1}\mathcal{M}^*)(\alpha^{-1}\mathcal{M}^*) \subseteqq \alpha^{-1}\mathcal{M}^*$ whence $M(\alpha^{-1}\mathcal{M}^*)$ is defined. Let $v_1 = \pi(y) v$. Then one checks that (ii) implies that

$$(3.11.6) \qquad \begin{aligned} (1 + x\alpha^{-1}) b (1 + x\alpha^{-1})^{-1} &\equiv (1 + x\alpha^{-1}) b (1 - x\alpha^{-1}) \bmod \mathcal{N}^* \\ &\equiv (\alpha + x + (b-\alpha))(1 - x\alpha^{-1}) \bmod \mathcal{N}^* \\ &\equiv \alpha + (b-\alpha) - A_\alpha(x) \bmod \mathcal{N}^* \\ &\equiv \alpha \bmod \mathcal{N}^*. \end{aligned}$$

Thus, since y stabilizes $\mathcal{N}$, v_1 transforms according to ψ_α under the action of $M(\mathcal{N})$. Finally a computation similar to that used to prove (3.11.1) (we only used that $\mathcal{N} \subseteqq \alpha^{-1}\mathcal{M}^*$) yields that v_1 transforms according to $\theta \cdot \psi_\alpha$ under the action of $U^{r'+1}(\mathcal{A}_E) M(\mathcal{M})$ as desired.

3.3. Relative small relative level: the odd case. — We will begin this subsection by completing the proof of Theorem 2.20. We will close the subsection by completing the proof of Theorem 2.21. In light of the previous sections, the proof of Theorem 2.20 will be complete once we prove

Proposition 3.13. — *Let π be an admissible representation of* G *which contains an odd relative alfalfa stratum $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$; set $e = e(\mathcal{A}_E)$ $(E = F[\alpha])$ and suppose that $c \mid gcd(r, e)$. Suppose further that $m = [(n+2)/2]$; set $n' = n/c$, $r' = r/c$, $e' = e/c$ and $m' = [(n'+2)/2]$. Then π also contains a relative alfalfa stratum $(\mathcal{A}', m', n', \alpha, r', \eta', \theta)$ with $e(\mathcal{A}'_E) = e'$.*

Proof. — To begin, we assume $c>1$ — the other case being trivial. Suppose that $\mathscr{A}=\mathscr{A}(L)$ and let L' denote the uniform thinning of L by a factor of c; set $\mathscr{A}'=\mathscr{A}(L')$, let $\mathscr{P}'$ denote the radical of $\mathscr{A}'$ and note that $e(\mathscr{A}'_E)=e'$. Set $k=-\nu_E(\alpha)$. Note that ψ_α is trivial of $U^{n'+1}(\mathscr{A}_F)$ and that, by Lemma 3.1,

$$\mathscr{P}^{r+1}\supseteq(\mathscr{P}')^{r'+1}. \tag{3.13.1}$$

Note also that

$$\begin{aligned} r' &= re'/e \\ &\leqq [(m-1)\,e'/e] \\ &= [[ke/2]\,e'/e] \\ &\leqq [ke'/2] \\ &= m'-1. \end{aligned} \tag{3.13.2}$$

Now, if e' is even, then, by Lemma 3.1, we have that

$$\begin{aligned} \mathscr{P}^m &= P_E^{k/2}\,\mathscr{P}^{(e+2)/2} \\ &\supseteq P_E^{k/2}\,(\mathscr{P}')^{(e'+2)/2} \\ &= (\mathscr{P}')^{m'}. \end{aligned} \tag{3.13.3}$$

Thus, π contains $\theta\cdot\psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}'_E)\,U_F^{m'}(\mathscr{A}')$ whence the proposition follows from (3.13.2) and either Remark 2.17 or Lemma 3.9.

Now consider the case e' odd. Here, by Lemma 3.1, we have that

$$\mathscr{P}^m\supseteq(\mathscr{P}')^{m''} \tag{3.13.4}$$

with

$$\begin{aligned} m'' &= [(m-2)/c]+2 \\ &= (k-1)\,e'/2+[([e/2]-1)/c]+2 \\ &= (k-1)\,e'/2+[((e'-1)\,c/2+[c/2]-1)/c]+2 \\ &= (k-1)\,e'/2+(e'-1)/2+[[c/2]-1/c]+2 \\ &= (k-1)\,e'/2+(e'-1)/2+2 \\ &= m'+1. \end{aligned} \tag{3.13.5}$$

Thus we have that

$$\mathscr{P}^m\supseteq(\mathscr{P}')^{m'+1}. \tag{3.13.6}$$

It follows from (3.13.6) that $\mathscr{M}=\mathscr{P}^m\cap(\mathscr{P}')^{m'}$ is a sum of coordinates in $(\mathscr{P}')^{m'}$ with respect to an $\mathscr{O}_E$-basis $\mathscr{B}$ for L. It also follows from the above calculations that π contains $\theta\cdot\psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}'_E)\,M(\mathscr{M})$. Let $\mathscr{N}=(\mathscr{P}')^{m'}$. Then one can check that all the hypotheses of Lemma 3.11 are satisfied with the possible exceptions of the following:

(3.13.7) $$\mathscr{M}^* \cap \mathrm{A}_{ii} = \mathscr{N}^* \cap \mathrm{A}_{ii} \quad \text{for all } i.$$

(3.13.8) $$\alpha^{-1}(\mathscr{P}')^{r'+1}\mathscr{M}^* \subseteqq \mathscr{M}.$$

Thus, to complete the proof of the proposition, it suffices to prove (3.13.7) and (3.13.8).

We first consider (3.13.7). Since $(\mathscr{P}')^{1-m'} \supseteqq \mathscr{P}^{(1-m')c}$ and $\mathscr{M}^* = \mathscr{P}^{1-m} + (\mathscr{P}')^{1-m'}$ while $\mathscr{N}^* = (\mathscr{P}')^{1-m'}$ it suffices to show that e does not divide $(1-m)+l$ for $l=0, 1, \ldots, (1-m')c+(m-2)$. Now

(3.13.9) $$\begin{aligned} 1-m &= -[ke/2] \\ &= -(k-1)e/2-[e/2] \end{aligned}$$

and

(3.13.10) $$\begin{aligned} (1-m')c &= -(ke'-1)c/2 \\ &= -(k-1)e/2-(e-c)/2. \end{aligned}$$

Thus, $(1-m')c+(m-2)=[e/2]-1-(e-c)/2$. Thus (3.13.7) holds if

(3.3.11) $$[e/2]-1-(e-c)/2<[e/2].$$

But this always holds.

To prove (3.13.8) we need to show that

(3.13.12) $$\alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}^{1-m}+(\mathscr{P}')^{1-m'}) \subseteqq \mathscr{P}^m \cap (\mathscr{P}')^{m'}.$$

Since α^{-1} is in both $(\mathscr{P}')^{n'}$ and $\mathscr{P}^n$ it follows from (3.13.6) and (3.13.1) that to prove (3.13.9) it suffices to prove that

(3.13.13) $$\alpha^{-1}(\mathscr{P}')^{r'+1}\mathscr{P}^{1-m} \subseteqq (\mathscr{P}')^{m'} \quad \text{and} \quad \alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}')^{1-m'} \subseteqq \mathscr{P}^m.$$

Thus it suffices to show that

$$\mathscr{P}^{1-m} \subseteqq (\mathscr{P}')^{1-m'} \quad \text{and} \quad (\mathscr{P}')^{m'+1} \subseteqq \mathscr{P}^m$$

but these are just (3.13.6) and it dual.

Our proof of Theorem 2.20 is now complete. The next proposition completes the proof of Theorem 2.21.

PROPOSITION 3.14. — *Let* π *be an irreducible representation of* G *which contains an odd proper alfalfa stratum and suppose that* $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ *is a relative alfalfa stratum contained in* π *of minimal relative level. Assume further that* $m=[(n+2)/2]$. *Then* $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ *is fundamental.*

Proof. — If $r=0$, then $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ is fundamental by definition. Thus suppose $r>0$, write $\eta=\psi_{\alpha, b}$ and set $\mathrm{E}=\mathrm{F}[\alpha]$. If $\eta \otimes \theta$ factors through the determinant, then it follows from Lemma 3.9 that $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ is not of minimal relative level – a contradiction. Thus, if $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ is not fundamental, we may assume $b+\mathscr{P}_E^{1-r}$ contains a nilpotent element. Then, by Proposition 3.5, there exists an $\mathcal{O}_E$-lattice chain L′ and an integer r' such that the following hold $[\mathscr{P}'=\mathscr{P}(L'), \mathscr{A}'=\mathscr{A}(L')]$:

(3.14.1) $$b+\mathscr{P}_E^{-r+1}\subseteqq(\mathscr{P}'_E)^{-r'}.$$

(3.14.2) $$r'/e'<r/e \quad \text{where} \quad e=e_E(L) \quad \text{and} \quad e'=e_E(L').$$

(3.14.3) L and L′ have a common refinement and a common thinning as $\mathcal{O}_E$-lattice chains.

Now (3.14.1) implies that

(3.14.4) $$(\mathscr{P}'_E)^{r'+1}\subseteqq\ker\psi_{\alpha,b}\subseteqq\mathscr{P}_E^r.$$

Thus π contains the representation $\theta\cdot\psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}'_E)\,U^m(\mathscr{A}_F)$. Let $k=-\nu_E(\alpha)$, $n'=ke'$ and $m'=[(n'+2)/2]$. Note that ψ_α is trivial on $U^{n'+1}(\mathscr{A}'_F)$. Also, by an argument similar to (3.13.1), we have that

(3.14.5) $$r'\leqq m'-1$$

Thus, as above, Proposition 3.14 will follow from

PROPOSITION 3.15. — *With notation and hypotheses as above, π contains $\theta\cdot\psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}'_E)\,U^{m'}(\mathscr{A}'_F)$.*

We prove this proposition in the following five lemmas.

LEMMA 3.16. — *Proposition 3.15 holds if $r'\geqq e'$.*

Proof. – Let $L^\vee$ denote a period one common thinning of L and L′ as $\mathcal{O}_E$-chains; set $\mathscr{A}^\vee=\mathscr{A}(L^\vee)$, $\mathscr{P}^\vee=\mathscr{P}(L^\vee)$, $n^\vee=k$ and $m^\vee=(k+1)/2$. By an argument similar to that for (3.13.6), we have that

(3.16.1) $$\mathscr{P}^m\supseteqq(\mathscr{P}^\vee)^{m^\vee+1} \quad \text{and} \quad (\mathscr{P}')^{m'}\supseteqq(\mathscr{P}^\vee)^{m^\vee+1}.$$

Thus $\mathscr{M}=\mathscr{P}^m\cap(\mathscr{P}^\vee)^{m^\vee}\cap(\mathscr{P}')^{r'+1}$ is a sum of coordinates in $(\mathscr{P}^\vee)^{m^\vee}$ as is $\mathscr{N}=(\mathscr{P}^\vee)^{m^\vee}\cap(\mathscr{P}')^{r'+1}$. Since $r'\geqq e'$,

(3.16.2) $$\begin{aligned}(\mathscr{P}')^{r'+1}&\subseteqq(\mathscr{P}')^{e'+1}\\ &=P_E\mathscr{P}'\\ &\subseteqq P_E\mathscr{A}^\vee\\ &=\mathscr{P}^\vee.\end{aligned}$$

Now one can check that Lemma 3.11 is applicable if the following hold:

(3.16.3) $$\mathscr{M}^*\cap A_{ii}=\mathscr{N}^*\cap A_{ii} \quad \text{for all } i.$$

(3.16.4) $$\alpha^{-1}(\mathscr{P}')^{r'+1}\mathscr{M}^*\subseteqq\mathscr{M}.$$

Since $\mathscr{M}^*=\mathscr{P}^{1-m}+(\mathscr{P}^\vee)^{1-m^\vee}+(\mathscr{P}')^{-r'}$ and $\mathscr{N}^*=(\mathscr{P}^\vee)^{1-m^\vee}+(\mathscr{P}')^{-r'}$ to show that (3.16.3) holds it suffices to check that $(\mathscr{P}^{1-m}+(\mathscr{P}^\vee)^{1-m^\vee})\cap A_{ii}=(\mathscr{P}^\vee)^{1-m^\vee}\cap A_{ii}$. This is checked as was (3.13.7). To show that (3.16.4) holds, we need to show that

$$(3.16.5)\quad \alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}^{1-m}+(\mathscr{P}^{\vee})^{1-m^{\vee}}+(\mathscr{P}')^{-r'})\subseteqq \mathscr{P}^{m}\cap(\mathscr{P}^{\vee})^{m^{\vee}}\cap(\mathscr{P}')^{r'+1}.$$

From (3.16.2) it follows that $\alpha^{-1}(\mathscr{P}')^{r'+1}\mathscr{P}^{1-m}\subseteqq\mathscr{P}^{m}$, $\alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}^{\vee})^{1-m^{\vee}}\subseteqq(\mathscr{P}^{\vee})^{m^{\vee}}$ and $\alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}')^{-r'}\subseteqq(\mathscr{P}')^{r'}$. By (3.16.1), we have that

$$(3.16.6)\qquad \mathscr{P}^{1-m}\subseteqq(\mathscr{P}^{\vee})^{-m^{\vee}}\quad\text{and}\quad(\mathscr{P}')^{1-m'}\subseteqq(\mathscr{P}^{\vee})^{-m^{\vee}}.$$

Thus it follows from (3.16.2) that $\alpha^{-1}(\mathscr{P}')^{r'+1}\mathscr{P}^{1-m}\subseteqq(\mathscr{P}^{\vee})^{m^{\vee}}$ and $\alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}')^{1-m'}\subseteqq(\mathscr{P}^{\vee})^{m^{\vee}}$. Now $\alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}')^{-r'}=\alpha^{-1}\mathscr{P}'$ and thus $\alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}')^{-r'}\subseteqq(\mathscr{P})^{m^{\vee}}\cap\mathscr{P}^{m}$. Finally, $\alpha^{-1}\mathscr{P}^{1-m}\subseteqq\mathscr{P}^{m-1}\subseteqq\mathscr{P}\subseteqq\mathscr{A}'$ and similarly $\alpha^{-1}(\mathscr{P}^{\vee})^{1-m^{\vee}}\subseteqq\mathscr{A}'$ so that $\alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}^{1-m}+(\mathscr{P}^{\vee})^{1-m^{\vee}})\subseteqq(\mathscr{P}')^{r'+1}$ whence (3.16.4).

Now, by Lemma 3.11, π contains $\theta\cdot\psi_{\alpha}$ upon restriction to $\mathrm{U}^{r'+1}(\mathscr{A}'_{\mathrm{E}})\mathrm{M}(\mathscr{N})$. By Lemma 3.1,

$$(3.16.7)\qquad (\mathscr{P}^{\vee})^{m^{\vee}}\supseteqq(\mathscr{P}')^{(k+1)e'/2}.$$

Thus π contains $\theta\cdot\psi_{\alpha}$ upon restriction to $\mathrm{U}^{r'+1}(\mathscr{A}'_{\mathrm{E}})\mathrm{U}^{m''}(\mathscr{A}'_{\mathrm{F}})$ where $m''=(k+1)e'/2$. Then one checks that since $r'\geqq e'$, Lemma 3.11 implies the lemma.

If $r'<e'$, the argument used in proving Lemma 3.16 does not work since Lemma 3.11 is not immediately applicable. A more intricate agrument is necessary.

LEMMA 3.17. — *Proposition* 3.15 *holds if* $r'<e'$ *and* $k>1$.

Proof. — Let $\mathrm{L}=\mathrm{L}^{0}$ and let $\mathrm{L}^{0}, \mathrm{L}^{1}, \ldots, \mathrm{L}^{l}$ be a sequence of $\mathscr{O}_{\mathrm{E}}$-lattice chains such that L^{i} refines L^{i-1}, $e_{\mathrm{E}}(\mathrm{L}^{i})=e_{\mathrm{E}}(\mathrm{L}^{0})+i$ and L^{l} is a common refinement of L and L′ as $\mathscr{O}_{\mathrm{E}}$-lattice chains. Let $\mathscr{P}_{i,\mathrm{F}}=\mathscr{P}_{i}$ denote the radical of $\mathscr{A}_{i,\mathrm{F}}=\mathscr{A}_{i}=\mathscr{A}_{\mathrm{F}}(\mathrm{L}^{i})$, set $\mathscr{A}_{i,\mathrm{E}}=\mathscr{A}_{i}\cap\mathrm{A}_{\mathrm{E}}$ and similarly define $\mathscr{P}_{i,\mathrm{E}}$. Set $k=-\nu_{\mathrm{E}}(\alpha)$, $e_{i}=e_{\mathrm{E}}(\mathrm{L}^{i})$, $n_{i}=ke_{i}$ and $m_{i}=[(n_{i}+2)/2]$. Note that since $r>1$ we have that $\mathscr{A}_{i}\supseteqq\mathscr{P}^{r}\supseteqq(\mathscr{P}')^{r'+1}$. Thus $\mathrm{U}^{r'+1}(\mathscr{A}'_{\mathrm{E}})$ stabilizes $\mathrm{U}^{s}(\mathscr{A}_{i})$ for all i and s. Our next step in the proof of Lemma 3.17 is

LEMMA 3.18. — *With notation and assumptions as above*, π *contains* $\theta\cdot\psi_{\alpha}$ *upon restriction to* $\mathrm{U}^{r'+1}(\mathscr{A}'_{\mathrm{E}})\mathrm{U}^{m_{i}}(\mathscr{A}_{i})$ *for* $0\leqq i\leqq l$.

Proof. — We proceed by induction on i. Since the lemma holds for $i=0$ by assumption, assume it holds for i. To prove it holds for $i+1$ we consider two cases. First consider the case e_{i} odd. Then

$$(3.18.1)\qquad \begin{aligned}\mathscr{P}_{i}^{m_{i}}&=\mathrm{P}_{\mathrm{E}}^{(k-1)/2}\mathscr{P}_{i}^{(e_{i}+1)/2}\\&=\mathrm{P}_{\mathrm{E}}^{(k-1)/2}\mathscr{P}_{i}^{e_{i}+1/2}.\end{aligned}$$

Now one verifies by direct calculation that since $e_{i+1}=e_{i}+1$ we have that

$$(3.18.2)\qquad \mathscr{P}_{i}^{e_{i}+1/2}\supseteqq\mathscr{P}_{i+1}^{(e_{i+1}+2)/2}.$$

Thus (3.18.1) implies that

$$\mathscr{P}_i^{m_i} \supseteq \mathscr{P}_{i+1}^{m_{i+1}} \tag{3.18.3}$$

so that the lemma holds for $i+1$ by restriction.

Now assume e_i is even. Then

$$\begin{aligned}\mathscr{P}_i^{m_i} &= P_E^{(k-1)/2}\, \mathscr{P}_i^{(e_i+2)/2}\\ &= P_E^{(k-1)/2}\, \mathscr{P}_i^{(e_{i+1}+1)/2}\\ &\subseteq P_E^{(k-1)/2}\, \mathscr{P}_{i+1}^{(e_{i+1}+1)/2}\\ &= \mathscr{P}_{i+1}^{m_{i+1}}.\end{aligned} \tag{3.18.4}$$

Now one verifies as in (3.18.2) that

$$\mathscr{P}_{i+1}^{(e_{i+1}+1)/2} \subseteq \mathscr{P}_i^{e_i/2}. \tag{3.18.5}$$

Thus (3.18.4) implies that

$$\mathscr{P}_i^{m_i} \subseteq \mathscr{P}_{i+1}^{m_{i+1}} \subseteq \mathscr{P}_i^{m_i-1}. \tag{3.18.6}$$

Let $\mathscr{M} = \mathscr{P}_i^{m_i}$ and $\mathscr{N} = \mathscr{P}_{i+1}^{m_{i+1}}$. Note that $\mathscr{M}$ and $\mathscr{N}$ are sums of coordinates in $\mathscr{P}_i^{m_i-1}$. Note also that

$$\begin{aligned}\alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}_i^{m_i})^* &= (\mathscr{P}')^{r'+1}\, \mathscr{P}_i^{m_i-1}\\ &\subseteq \mathscr{P}\, \mathscr{P}_i^{m_i-1}\\ &\subseteq \mathscr{P}_i^{m_i}.\end{aligned} \tag{3.18.7}$$

Moreover, since $r' < e'$ and $k > 1$ (this is our first use of this assuption) we have that

$$\begin{aligned}(\mathscr{P}')^{r'+1} &\supseteq (\mathscr{P}')^{e'}\\ &\supseteq P_E^{(k-1)/2}\, \mathscr{A}'\\ &\supseteq P_E^{(k-1)/2}\, \mathscr{P}_{i+1}\\ &\supseteq P_E^{(k-1)/2}\, \mathscr{P}_{i+1}^{(e_{i+1}+1)/2}\\ &= \mathscr{P}_{i+1}^{m_{i+1}}.\end{aligned} \tag{3.18.8}$$

We also claim that

$$\mathscr{M}^* \cap A_{ii} = \mathscr{N}^* \cap A_{ii} \quad \text{for all } i. \tag{3.18.9}$$

Since $\mathscr{M}^* = \mathscr{P}_i^{1-m_i}$ and $\mathscr{N}^* = \mathscr{P}_{i+1}^{1-m_{i+1}} \supseteq \mathscr{P}_i^{-m_i}$, to prove (3.18.9) it suffices to show that e_i does not divide $1-m_i$. This can be verified directly. Now the lemma follows from Lemma 3.11.

By Lemma 3.18, π contains $\theta \cdot \psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}_E)\, U^{m_l}(\mathscr{A}_l)$. To proceed further with Lemma 3.17, let $L^l, L^{l+1}, L^{l+2}, \ldots, L^\lambda$ be a sequence of $\mathscr{O}_E$-lattice chains such that L^i thins L^{i-1} for $i>l$, $e_E(L^i)=e_E(L^{i-1})-1$ for $i>l$ and $L^\lambda=L'$. Define $\mathscr{A}_i$, $\mathscr{P}_i$, n_i, e_i and m_i as before. Since $i\geqq l$, we have that $\mathscr{A}_i\supseteqq\mathscr{A}_l\supseteqq\mathscr{P}'$ so that $U^{r'+1}(\mathscr{A}'_E)$ stabilizes $U^s(\mathscr{A}_{i,\,F})$ for all i and s. Now, to complete the proof of Lemma 3.17, it suffices to prove

LEMMA 3.19. — *With notation and assumptions as above,* π *contains* $\theta\cdot\psi_\alpha$ *upon restriction to* $U^{r'+1}(\mathscr{A}_E)\,U^{m_i}(\mathscr{A}_i)$ *for* $l\leqq i\leqq\lambda$.

Proof. — We proceed by induction. Since the lemma holds for $i=l$, assume it holds for i. To prove it holds for $i+1$ we consider two cases. First assume e_i is odd. Then, since L_{i+1} thins L_i and $e_{i+1}=e_i-1$

$$(3.19.1)\qquad \begin{aligned}\mathscr{P}_i^{m_i}&=P_E^{(k-1)/2}\,\mathscr{P}^{(e_i+1)/2}\\&=P_E^{(k-1)/2}\,\mathscr{P}_i^{(e_{i+1}+2)/2}\\&\supseteqq\mathscr{P}_{i+1}^{m_{i+1}}\end{aligned}$$

so that the lemma follows for $i+1$ by restriction.

Now suppose e_i is even. Then a computation similar to (3.19.1) yields that

$$(3.19.2)\qquad \mathscr{P}_i^{m_i}\supseteqq\mathscr{P}_{i+1}^{m_{i+1}+1}.$$

Thus π contains $\theta\cdot\psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}')\,M(\mathscr{M})$ where $\mathscr{M}=\mathscr{P}_i^{m_i}\cap\mathscr{P}_{i+1}^{m_{i+1}}$ is a sum of coordinates in $\mathscr{P}_i^{m_i+1}$ (with respect to an $\mathscr{O}_E$-basis for L^l). Let $\mathscr{N}=\mathscr{P}_{i+1}^{m_{i+1}}$. Since $k>1$ (this is only our second use of this hypothesis) and $r'<e'$,

$$(3.19.3)\qquad \begin{aligned}(\mathscr{P}')^{r'+1}&\supseteqq(\mathscr{P}')^{e'}\\&\supseteqq P_E^{(k-1)/2}\,\mathscr{A}'\\&\supseteqq P_E^{(k-1)/2}\,\mathscr{A}_{i+1}\\&\supseteqq\mathscr{P}_{i+1}^{m_{i+1}}\\&=\mathscr{N}.\end{aligned}$$

Now one can check that the lemma will follow from Lemma 3.11 once we prove that the following hold:

$$(3.19.4)\qquad \mathscr{M}^*\cap A_{jj}=\mathscr{N}^*\cap A_{jj}\quad\text{for all } j.$$

$$(3.19.5)\qquad \alpha^{-1}(\mathscr{P}')^{r'+1}\,\mathscr{M}^*\subseteqq\mathscr{M}.$$

We first consider (3.19.4). Since $\mathscr{P}_{i+1}^{-m_{i+1}}\supseteqq\mathscr{P}_{i+1}^{1-m_{i+1}}\cap\mathscr{P}_{i+1}^{1-m_{i+1}}\supseteqq\mathscr{P}_{i+1}^{1-m_{i+1}}$, to prove (3.19.4) it suffices to show that e_{i+1} does not divide m_{i+1}. But $m_{i+1}=(k-1)\,e_{i+1}/2+(e_{i+1}-1)/2$ so that (3.19.4) holds except the possibly in the case $e_{i+1}=1$. In this case it suffices to show that

$$(3.19.6)\qquad (\mathscr{P}_{i+1}^{1-m_{i+1}})_{jj}=(\mathscr{P}_i^{1-m_i})_{jj}\quad\text{for all } j.$$

But in this case $\mathscr{P}_{i+1}^{1-m_{i+1}}=P_E^{-(k+1)/2}\mathscr{P}_{i+1}$ and $\mathscr{P}_i^{1-m_i}=P_E^{-(k+1)/2}\mathscr{P}_i$ and (3.19.4) follows. To prove (3.19.5), it suffices to prove that

$$(3.19.7)\qquad \alpha^{-1}(\mathscr{P}')^{r'+1}(\mathscr{P}_i^{1-m_i}+\mathscr{P}_{i+1}^{1-m_{i+1}})\subseteqq\mathscr{P}_i^{m_i}\cap\mathscr{P}_{i+1}^{m_{i+1}}.$$

Since α^{-1} is in both $\mathscr{P}_i^{n_i}$ and $\mathscr{P}_{i+1}^{n_{i+1}}$ and since $(\mathscr{P}')^{r'+1}$ is contained in $\mathscr{P}_i$ and $\mathscr{P}_{i+1}$ it suffices to check that

$$(3.19.8)\qquad \alpha^{-1}(\mathscr{P}_{i+1})\,\mathscr{P}_i^{1-m_i}\subseteqq\mathscr{P}_{i+1}^{m_{i+1}}\quad\text{and}\quad\alpha^{-1}(\mathscr{P}_{i+1})\,\mathscr{P}_{i+1}^{1-m_{i+1}}\subseteqq\mathscr{P}_i^{m_i}.$$

The second of these follows from (3.19.1). To show that the first holds it suffices to show that $\mathscr{P}_i^{1-m_i}\subseteqq\mathscr{P}_{i+1}^{-m_{i+1}}$ which is in fact the dual of (3.19.2). This completes the proof of Lemma 3.19 and thus also the proof of Lemma 3.17.

Our proof of Proposition 3.15 and hence Theorem 2.21 will be complete once we prove

LEMMA 3.20. — *With notation and hypotheses as above, Proposition* 3.15 *holds if* $k=1$ *and* $r'<e'$.

Proof. – Define $L^0, \ldots, L^\lambda$ as in the proof of Lemma 3.17. Also define $\mathscr{A}_i$, $\mathscr{P}_i$, $\mathscr{A}_{i,\,E}$, $\mathscr{P}_{i,\,E}$, n_i, e_i and m_i for $0\leqq i\leqq\lambda$ as in that proof. Now, however, let

$$\mathscr{H}_i=\mathscr{P}_i^{m_i}\cap(\mathscr{P}')^{r'+1}.$$

Note that $U^{r'+1}(\mathscr{A}'_E)$ normalizes $M(\mathscr{H}_i)$ for $i\leqq 0\leqq\lambda$. Note also that $U^{r'+1}(\mathscr{A}'_E)\,M(\mathscr{H}_0)$ is contained in $U^{r'+1}(\mathscr{A}'_E)\,U^{m_0}(\mathscr{A}_0)$ so that π contains $\theta\cdot\psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}'_E)\,M(\mathscr{H}_0)$. We claim that for $0\leqq i\leqq\lambda$ π contains $\theta\cdot\psi_\alpha$ upon restriction to $U^{r'+1}(\mathscr{A}_E)\,M(\mathscr{H}_i)$. Since $\mathscr{H}_\lambda=(\mathscr{P}')^{m'}$, this claim implies the lemma. To prove the claim we proceed as in Lemma 3.18 and 3.19. The intersection with $(\mathscr{P}')^{r'+1}$ is necessary since, in the case $k=1$, $(\mathscr{P}')^{r'+1}$ does not necessarily contain $\mathscr{P}_i^{m_i}$ [*see* (3.18.8) and (3.19.3)]. The only difficulty in arguing as in Lemma 3.18 and Lemma 3.19 is that $\mathscr{H}_i$ and $\mathscr{H}_i\cap\mathscr{P}_{i+1}^{m_{i+1}}$ are not necessarily sums of coordinates so that Lemma 3.11 is not immediately applicable [verification of the other hypotheses of Lemma 3.11, in particular (ii), is tedious but straighforward]. However, one can check that although we do not have sums of coordinates the maps S_α and A_α do give exact pairs sufficient to generalize Lemma 3.11 to this context and then the lemma follows as did Lemma 3.17.

4. The supercuspidal case

This section has two goals. First we consider ways in which Theorem 2.21 may be strengthened in case the representation is supercuspidal. Our main result here is

THEOREM 4.1. — *Let* π *be an irreductible supercuspidal representation of* G *with contains a proper alfalfa stratum. Suppose that* $l_r(\pi)>0$. *Then* π *contains a second order alfalfa stratum.*

Second we consider the implications of Theorem 4.1 in case N is the product of two primes. Here our result is

THEOREM 4.2. — *Let π be an irreducible supercuspidal representation of* $G=GL_N(F)$ *with* N *the product of two (not necessarily distinct) primes. Suppose that π contains a proper alfalfa stratum and that $l_r(\pi)>0$. Then Theorem 0 holds for π.*

In order to prove Theorem 4.1, we first call some results from [K 3]. We fix E/F, V and so on as in Section 2.

DEFINITION 4.3 ([K 3], 2.1). — Let L be an $\mathcal{O}_E$-lattice chain in V and suppose (V^1, V^2) is a pair of non-zero E-subspaces of V for which $V=V^1 \oplus V^2$. Set $L_n^i = L_n \cap V^i$, $i=1,2$ and denote the sequence lattices L_n^i by L^i. Then we say that (V^1, V^2) is a *splitting for* L *over* E (or just a splitting) if the following conditions are satisfied:

(i) $L_n = L_n^1 \oplus L_n^2$ for all n.

(ii) L^1 is a uniform $\mathcal{O}_E$-lattice chain in V^1. (In particular, $L_n^1 \supsetneqq L_{n+1}^1$ for all n.)

Suppose that (V^1, V^2) is a splitting for L over E. Set $\mathscr{V}=\mathscr{V}_F=\mathrm{Hom}_F(V^2, V^1)$ and $\mathscr{V}_E=\mathrm{Hom}_E(V^2, V^1)$. For an integer n, denote by $\mathscr{L}_n=\mathscr{L}_{n,\,F}$ the $\mathcal{O}_F$-lattice of elements g in $\mathscr{V}_F$ such that $g L_k^2 \subseteq L_{k+n}^1$ for all k and set $\mathscr{L}_{n,\,E}=\mathscr{L}_{n,\,E} \cap \mathscr{V}_E$. Then we have the following lemma which is proved in a manner of analogous to the proof of Lemma 2.2 in [K 3].

LEMMA 4.4. — *With notation as above, $\mathscr{L}_F=(\mathscr{L}_{n,\,F}\}_{n\in\mathbb{Z}}$ is a uniform $\mathcal{O}_F$-lattice chain in $\mathscr{V}_F$ of period $e_F(L)$ and $\mathscr{L}_E=\{\mathscr{L}_{n,\,E}\}_{n\in\mathbb{Z}}$ is a uniform $\mathcal{O}_E$-lattice chain in $\mathscr{V}_E$ of period* $e_E(L)$.

We now fix an $\mathcal{O}_E$-lattice chain L which has a splitting (V^1, V^2) over E and continue with notation as above. Let b be an element A_E and suppose that $b(V^i)\subseteq V^i$, $i=1,2$. Set $b^i=b|_{V^i}$ and denote by $\varphi=\varphi_b$ the E-endomorphism of $\mathscr{V}_E$ given by $\varphi_b(x)=b^1 x - x b^2$. We say that b^1 (respectively φ_b) is nondegenerate of level k for L^1 (respectively $\mathscr{L}_E$) if $b^1 L_n^1 = L_{n+k}^1$ (respectively $\varphi_b(\mathscr{L}_{n,\,E})=\mathscr{L}_{n+k,\,E}$) for all n.

DEFINITION 4.5 ([K 3], 2.3). — With notation as above, we say that (V^1, V^2) *splits* b *over* E (or just splits b) if b_1 and φ_b are nondegenerate of level $\nu_{\mathscr{A}(L)}(b)$ for L^1 and $\mathscr{L}_E$ respectively. We say that b is *split over* E *by* L if L is an $\mathcal{O}_E$-lattice chain which has a splitting over E which splits b.

Recall that an irreducible admissible representation τ of a compact subgroup K of G is called (G, K)-*principal* (or just principal) [K 2] if no supercuspidal representation of G can contain τ upon restriction to K.

A key step in the proof of Theorem 4.1 will be the following proposition.

PROPOSITION 4.6. — *Let $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ be a fundamental relative alfalfa stratum of positive relative level; set $E=F[\alpha]$ and suppose that $\mathscr{A}=\mathscr{A}(L)$. Suppose further that $(r, e(\mathscr{A}_E))=1$ and that $\eta=\psi_{\alpha,\,b}$ where b is split over* E *by* L. *Then the representation $(\eta\otimes\theta)\cdot\psi_\alpha$ of $U^r(\mathscr{A}_E)\,U^m(\mathscr{A}_F)$ is principal.*

Proof. — This proposition will follow from a suitable modification of the agruments and to prove Proposition 2.4 of [K 3]. As in [K 3], let (V^1, V^2) be a splitting of L over

E which splits b. For each nonnegative integer k, let $\mathscr{Q}_F^k$ be the set of g in A such that for all j the following hold:

(i) $g\,L_j^1 \subseteqq L_{j+k+1}$

(ii) $g\,L_j^2 \subseteqq L_{j+k}^1 + L_{j+k+1}^2$.

We define $\mathscr{Q}_E^k$ to be $\mathscr{Q}_F^k \cap A_E$. Set $Q_F^k = M(\mathscr{Q}_F^k)$ and let $Q_F = Q_F^0$; similarly define Q_E^k and Q_E. Note that $\mathscr{Q}_F^k$ is a sum of coordinates in $\mathscr{P}_F^k$ with respect an appropriate $\mathcal{O}_E$-basis.

Our next step in proving Proposition 4.6 is the following lemma which can be proved by direct calculation as in [K 3], 2.7.

LEMMA 4.7. — *For k a nonnegative integer, let $\bar{\mathscr{Q}}_F^k$ be the set of elements x in $\mathscr{Q}_F^k$ such that $x\,V^2 \subseteqq V^1$ and $x\,V^1 = \{0\}$ and let $\underline{\mathscr{Q}}_F^k$ be the set of x in $\mathscr{Q}_F^k$ such that $x\,V^2 \subseteqq V^2$; similarly define $\bar{\mathscr{Q}}_E^k$ and $\underline{\mathscr{Q}}_E^k$. Set $\bar{Q}_F^k = M(\bar{\mathscr{Q}}_F^k)$, $\underline{Q}_F^k = M(\underline{\mathscr{Q}}_F^k)$, $\bar{Q}_E^k = M(\bar{\mathscr{Q}}_E^k)$ and $\underline{Q}_E^k = M(\underline{\mathscr{Q}}_E^k)$. Then the following hold:*

(i) *The map $g \to (g-1)|_{V^2}$ is an isomorphism of abelian groups of $\bar{Q}_F^k$ into $\mathscr{V}_F$ with image $\mathscr{L}_{k,\,F}$.*

(ii) $\bar{\mathscr{Q}}_F^k \cap \underline{\mathscr{Q}}_F^k = \{0\}$, $\bar{\mathscr{Q}}_F^k + \underline{\mathscr{Q}}_F^k = \mathscr{Q}_F^k$ *and* $\mathscr{P}_F^k = \bar{\mathscr{Q}}_F^k + \underline{\mathscr{Q}}_F^{k-1}$.

(iii) $\bar{Q}_F^k \cap \underline{Q}_F^k = \{1\}$, $Q_F^k = \bar{Q}_F^k\,\underline{Q}_F^k$ *and* $U^{k+1}(\mathscr{A}_F) = \bar{Q}_F^{k+1}\,\underline{Q}_F^k$.

(iv) *The above statements hold with* F *replaced by* E.

Remark 4.8. — In our proof of Proposition 4.6 we now distinguish as a special case the situation $r<2m-n$. Since $r \geqq 2m-n-1$, this possibility can only occur when either $m=n$ and $r=m-1$ or n is even, $m=(n+2)/2$ and $r=1$. We will exclude the case $r<2m-n$ at first and return to it later.

Thus we continue the proof of Proposition 4.6 with

LEMMA 4.9. — *Set* $J = U^r(\mathscr{A}_E)\,U^m(\mathscr{A}_F)$, $H = Q_E\,Q_F^{n-m}$ *and* $K = U^1(\mathscr{A}_E)\,U^{n-m+1}(\mathscr{A}_F)$. *Suppose that $r \geqq 2m-n$. Then* J *is a normal subgroup of* H *and the stability subgroup in* H *of* $(\eta \otimes \theta) \cdot \psi_\alpha$ *as a representation of* J *is* K.

Proof. — We note first that Q_E normalizes Q_F^{n-m} so that H makes sense as a subgroup of G. It is clear that J is a subgroup of H and that Q_E normalizes J and Q_F^m. Now, since $\mathscr{P}^{n-m}\mathscr{P}^r \subseteqq \mathscr{P}^{n-m+r} \subseteqq \mathscr{P}^m$ by assumption and $Q_F^{n-m} \subseteqq \mathscr{P}_F^{n-m}$, one can check that if g is an element of Q^{n-m}, then $g\,U^r(\mathscr{A}_E)\,g^{-1} \subseteqq U^m(\mathscr{A}_F)$ so that J is normal in H.

Now one can check, by an argument similar to that at the beginning of Lemma 3.11, that K stabilizes $(\eta \otimes \theta) \cdot \psi_\alpha$ since $(n-m+1)+r \geqq m$. Thus, by Lemma 4.7, to show that the stabilizer of $(\eta \otimes \theta) \cdot \psi_\alpha$ is K it suffices to show that if an element g of $\bar{Q}_E^0\,\bar{Q}_F^{n-m}$ stabilizes $(\eta \otimes \theta) \cdot \psi_\alpha$ then the element is in K. Write $g = h_1 h_2$ with h_1 in $\bar{Q}_E^0$ and h_2 in $\bar{Q}_F^{n-m}$. Then, since $\bar{Q}_E^0$ stabilizes ψ_α, h_2 must also stabilize ψ_α. Write $h_2 = 1 + x$. Then a computation similar to (3.6.1) implies that $A_\alpha(x)$ is in $\mathscr{P}_F^{n-m+1}$ and thus that h_2 is in K. Thus we may assume g is in $\bar{Q}_E^0$. Now the proposition follows as in [K 3], 2.6 (here one uses that φ_b is nondegenerate of level $-r$).

Remark 4.10. — Once we prove the following lemma, Proposition 4.6 will follow in the case $r \geqq 2m-n$ from Lemma 4.9 above and Proposition 1.9 of [K 2].

LEMMA 4.11. — *With notation and assumptions as aboven write* $1=rs+e(\mathscr{A}_E)t$ *where* s *and* t *are integers and then define* z *in* G_E *to be the automorphism of* V *which satisfies* $zv^1=\varpi_E^{-t}(b^1)^s v^1$ *for* v^1 *in* V^1 *and* $zv^2=v^2$ *for* v^2 *in* V^2. *Then the following conditions hold:*

(i) z *intertwines* $(\eta\otimes\theta)\cdot\psi_\alpha$.

(ii) *The double cosets* $Z(G)Hz^mH$ *are pairwise disjoint and* $Hz^mHzH=Hz^{m+1}H$ *for* m *a nonnegative integer where* $Z(G)$ *is the center of* G.

(iii) $J\cap zJz^{-1}$ *is normal in* J *with abelian quotient.*

(iv) zKz^{-1} *is contained in* H; *furthermore* $[\langle K, zKz^{-1}\rangle : K]=[J:J\cap zJz^{-1}]$ *where* $\langle K, zKz^{-1}\rangle$ *is the subgroup of* G *generated by* K *and* zKz^{-1}.

Proof. — (i), (ii) and (iii) follow from straightforward modifications of the arguments used to prove the analogous statements in Lemma 2.8 of [K 3]. We turn now to (iv).

Here again, just as in the proof of Lemma 2.8 (iv) of [K 3], one can check that $\langle K, zKz^{-1}\rangle=H$ since b^1 is nondegenerate of level $-r$. Now

$$\begin{aligned}(4.11.1)\quad [H:K] &= [Q_E Q_F^{n-m} : U^1(\mathscr{A}_E)U^{n-m+1}(\mathscr{A}_F)]\\ &= [\bar{Q}_E\bar{Q}_F^{n-m}U^1(\mathscr{A}_E)U^{n-m+1}(\mathscr{A}_F) : U^1(\mathscr{A}_E)U^{n+m-1}(\mathscr{A}_F)]\\ &= [\bar{Q}_E\bar{Q}_F^{n-m} : \bar{Q}_E^1\bar{Q}_F^{n-m+1}]\\ &= [\mathscr{L}_{0,E}+\mathscr{L}_{n-m,F} : \mathscr{L}_{1,E}+\mathscr{L}_{n-m+1,F}]\\ &= [\mathscr{L}_{0,E}:\mathscr{L}_{1,E}][\mathscr{L}_{n-m,F}:\mathscr{L}_{n-m+1,F}][\mathscr{L}_{n-m,E}:\mathscr{L}_{n-m+1,E}]^{-1}\\ &= [\mathscr{L}_{n-m,F}:\mathscr{L}_{n-m+1,F}]\end{aligned}$$

by Lemma 4.4 and Lemma 4.7.

One the other hand consider $[J:J\cap zJz^{-1}]$. By (ii),

$$\begin{aligned}(4.11.2)\quad [J:J\cap zJz^{-1}] &= [\mathscr{P}_E^r+\mathscr{P}_F^m : (\mathscr{P}_E^r+\mathscr{P}_F^m)\cap z(\mathscr{P}_E^r+\mathscr{P}_F^m)z^{-1}]\\ &= [\mathscr{P}_E^r+\mathscr{P}_F^m+z(\mathscr{P}_E^r+\mathscr{P}_F^m)z^{-1} : \mathscr{P}_E^r+\mathscr{P}_F^m].\end{aligned}$$

Now one check that since b^1 is nondegenerate of level $-r$ for L^1, $z\underline{\mathscr{Q}}_F^m z^{-1}\subseteqq\underline{\mathscr{Q}}_F^m$, $z\bar{\mathscr{Q}}_F^m z^{-1}=\bar{\mathscr{Q}}_F^{m-1}$, $z\underline{\mathscr{Q}}_E^r z^{-1}\subseteqq\underline{\mathscr{Q}}_E^r$ and $z\underline{\mathscr{Q}}_E^r z^{-1}=\bar{\mathscr{Q}}_E^{r-1}$. Thus (4.11.2), Lemma 4.4 and Lemma 4.7 imply that

$$\begin{aligned}(4.11.3)\quad [J:J\cap zJz^{-1}] &= [\mathscr{P}_E^r+\mathscr{P}_F^m+\bar{\mathscr{Q}}_E^{r-1}+\bar{\mathscr{Q}}_F^{m-1} : \mathscr{P}_E^r+\mathscr{P}_F^m]\\ &= [\bar{\mathscr{Q}}_E^{r-1}+\bar{\mathscr{Q}}_F^{m-1} : (\bar{\mathscr{Q}}_E^{r-1}+\bar{\mathscr{Q}}_F^{m-1})\cap(\mathscr{P}_E^r+\mathscr{P}_F^m)]\\ &= [\bar{\mathscr{Q}}_E^{r-1}+\bar{\mathscr{Q}}_F^{m-1} : \bar{\mathscr{Q}}_E^r+\bar{\mathscr{Q}}_F^m]\\ &= [\mathscr{L}_{r-1,E}+\mathscr{L}_{m-1,F} : \mathscr{L}_{r,E}+\mathscr{L}_{m,F}]\\ &= [\mathscr{L}_{r-1,E}:\mathscr{L}_{r,E}][\mathscr{L}_{m-1,F}:\mathscr{L}_{m,F}][\mathscr{L}_{m-1,E}:\mathscr{L}_{m,E}]^{-1}\\ &= [\mathscr{L}_{m-1,F}:\mathscr{L}_{m,F}]\end{aligned}$$

whence (iv).

We now indicate how to modify the above arguments in case $r<2m-n$; as was mentioned in Remark 4.8, in fact, $r=2m-n-1$.

LEMMA 4.12. — *In the above case, an admissible representation of* G *which contains* $(\eta\otimes\theta)\cdot\psi_\alpha$ *upon restriction* $U^r(\mathscr{A}_E)\,U^m(\mathscr{A}_F)$ *also contains* $(\eta\otimes\theta)\cdot\psi_\alpha$ *upon restriction to* $U^r(\mathscr{A}_E)\,Q^{m-1}(\mathscr{A}_F)$.

Proof. — This lemma follows from Lemma 4.13 below. The proof of Lemma 4.13 is similar to the proof of Lemma 3.11 and thus we leave it to the reader.

LEMMA 4.13. — *With notation and hypotheses as in Lemma* 3.11, *suppose that* η *is an irreducible representation of* $U^{r'}(\mathscr{A}'_E)/U^{r'+1}(\mathscr{A}'_E)$. *Suppose further that, in fact,* $U^r(\mathscr{A}'_E)$ *normalizes* $M(\mathscr{M})$ *and* $M(\mathscr{N})$ *and that* $\alpha^{-1}(\mathscr{P}')^{r'}\mathscr{M}^*\subseteqq\mathscr{M}$. *Then an admissible representation of* π *which contains the representation* $(\eta\otimes\theta)\cdot\psi_\alpha$ *upon restriction to* $U^r(\mathscr{A}'_E)\,M(\mathscr{M})$ also contains it upon restriction to $U^r(\mathscr{A}'_E)\,M(\mathscr{N})$.

Now, for k a nonnegative integer, let $\tilde{\mathscr{Q}}^k_F$ be the set of g in $\mathscr{P}^k_F$ for which $g\,L^1_j\subseteqq L^1_{j+k}+L^2_{j+k+1}$ and $\tilde{\mathscr{Q}}^k_E=\tilde{\mathscr{Q}}^k_F\cap A_E$. For k positive, let $\tilde{Q}^k_F=M(\tilde{Q}^k_F)$ and set $\tilde{Q}^k_E=\tilde{Q}^k_F\cap A_E$. Then, Proposition 4.6 will follow in the case $r=2m-n-1$ and thus in general from

LEMMA 4.14. — *The restriction of* $(\eta\otimes\theta)\cdot\psi_\alpha$ *to* $\tilde{Q}^r_E\,Q^{m-1}_F$ *is principal.*

Proof. — Take $H=Q_E\,Q^{n-m}_F$ and $J=\tilde{Q}^r_E\,Q^{m-1}_F$ and then argue as in Lemma 4.9 and Lemma 4.11.

Proof of Theorem 4.1. — By Corollary 2.22, π contains a fundamental relative alfalfa stratum $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ of minimal relative level which also has the property that $(r, e(\mathscr{A}_E))=1$ where $E=F[\alpha]$. Write $\eta=\psi_{\alpha,b}$ and $\mathscr{A}=\mathscr{A}(L)$. By Proposition 4.6, b is not split over E. Then it follows as in the proof of Theorem 3.2 of [K2] that L is uniform and, morevover, there exists an integer $c\geqq 1$ such that if $\mathscr{A}'$ is the hereditary order with radical $\mathscr{P}'$ attached to a uniform $\mathscr{O}_E$-lattice chain L' which is a uniform refinement of L by a factor of c, then η is trivial on $U^{r'+1}(\mathscr{A}'_E)$ and the coset $b+(\mathscr{P}')^{1-r'}$ contains an element α' such that Definition 2.19.2 is satisfied for $\mathscr{A}'$ where $r'=re$. Note that $\mathscr{P}^r\supseteqq(\mathscr{P}')^{r'}\supseteqq(\mathscr{P}')^{r'+1}\supseteqq\mathscr{P}^{r+1}$ and let η' be the restriction of η to $U^{r'}(\mathscr{A}'_E)$. Also let $k'=-v_E(\alpha)$, $e=e(\mathscr{A}_E)$, $e'=e(\mathscr{A}'_E)=ec$ and $n'=ke'$. Finally let $m'=[(n'+2)/2]$ if $m=[(n+2)/2]$ and let $m'=r'+1$ otherwise. Note that π contains $(\eta'\otimes\theta)\,\psi_\alpha$ upon restriction to $U^r(\mathscr{A}'_E)\,U^m(\mathscr{A}_F)$. We claim that π also contains $(\eta'\otimes\theta)\cdot\psi_\alpha$ upon restriction to $U^{r'}(\mathscr{A}'_E)\,U^{m'}(\mathscr{A}'_F)$. Since $r'/e'=r/e$ and $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ is of minimal relative level in π, the claim, Lemma 3.6, Remark 2.17 and Lemma 3.9 imply that η' does not factor through the determinant whence $(\mathscr{A}', m', n', \alpha, r', \eta', \theta)$ is a second order alfalfa stratum. Hence, to prove Theorem 4.1, it suffices to prove the claim.

To the above end, we first consider the case $m'<[(n'+2)/2]$. By Lemma 3.1,

$$\begin{aligned}\mathscr{P}^{m-1}&=\mathscr{P}^r\\&\supseteqq(\mathscr{P}')^{r'+1}\\&=(\mathscr{P}')^{m'}\\&\supseteqq\mathscr{P}^m.\end{aligned}$$

Now one can check that Lemma 4.13 implies the claim.

If $m'=[(n'+2)/2]$ we consider three cases. First, suppose that k is even. Since $\mathscr{A} \supseteqq \mathscr{P}' \supseteqq \mathscr{P}$, $P_E^{k/2}\mathscr{A} \supseteqq P_E^{k/2} \supseteqq P_E^{k/2}\mathscr{P}$. This may be rewritten as

$$\mathscr{P}^{m-1} \supseteqq (\mathscr{P}')^m \supseteqq \mathscr{P}^m.$$

Using this one can check that Lemma 4.13 implies the claim in this case also.

Next suppose that k is odd and e is even. Then, by Lemma 3.1,

$$\begin{aligned}\mathscr{P}^{m-1} &= P_E^{(k-1)/2}\,\mathscr{P}^{e/2}\\ &\supseteqq P_E^{(k-1)/2}\,(\mathscr{P}')^{ce/2}\\ &=(\mathscr{P}')^{m'-1}\\ &\supseteqq (\mathscr{P}')^{m'}.\end{aligned}$$

Moreover, also by Lemma 3.1,

$$\begin{aligned}(\mathscr{P}')^{m'} &= P_E^{(k-1)/2}\,(\mathscr{P}')^{(ce+2)/2}\\ &\supseteqq P_E^{(k-1)/2}\,\mathscr{P}^{(e+2)/2}\\ &=\mathscr{P}_F^m\end{aligned}$$

Now one can check that Lemma 4.13 implies the claim in this case.

Finally, suppose k and e are odd, then

$$\begin{aligned}\mathscr{P}^{m} &= P_E^{(k-1)/2}\,\mathscr{P}^{(e+1)/2}\\ &\supseteqq P_E^{(k-1)/2}\,(\mathscr{P}')^{c\,(e+1)/2}\\ &=(\mathscr{P}')^{m''}\end{aligned}$$

where $m''=c(ek+1)/2$. Now, once again, Lemma 4.13 implies the claim and the proof is complete.

PROPOSITION 4.15. — *Let* $(\mathscr{A}, m, n, \alpha, r, \eta, \theta)$ *be a second order alfalfa stratum of degree* N *and let* σ *be an irreducible constituent of* $\mathrm{Ind}(\mathrm{K}(\mathscr{A}_F), \mathrm{U}^r(\mathscr{A}_E)\mathrm{U}^m(\mathscr{A}_f); (\eta\otimes\theta)\cdot\psi_\alpha)$ *where* $\mathrm{K}(\mathscr{A})$ *is the normalizer of* $\mathscr{A}_F^\times$ *in* $A_F^\times$. *Then* $\mathrm{Ind}(\mathrm{G}, \mathrm{K}(\mathscr{A}_F); \sigma)$ *is irreducible and supercuspidal.*

Proof. — We first show that any element of G which intertwines $(\eta\otimes\theta)\cdot\psi_\alpha$ must lie in $\mathrm{K}(\mathscr{A}_F)$. If z in G intertwines $(\eta\otimes\theta)\cdot\psi_\alpha$ then it interwines ψ_α. Thus, by Theorem 2.4 of [KM2], z must lie in $\mathrm{U}^{n+1-m}(\mathscr{A}_F)\,\mathrm{G}_E\,\mathrm{U}^{n+1-m}(\mathscr{A}_F)$. But then as in the start of the proof of Lemma 3.11 one can check that $\mathrm{U}^{n+1-m}(\mathscr{A}_F)$ stabilizes $(\eta\otimes\theta)\cdot\psi_\alpha$ so that we may take z in G_E. Now, since z commutes with α it follows that z intertwines η. Then, as in [Ca], the assumption on the degree implies that z is in $\mathrm{K}(\mathscr{A}_E)=\mathrm{K}(\mathscr{A}_F)\cap A_E$ as desired. Proposition 4.15 now follows from

LEMMA 4.16. — *Let* H *and* K *be any open compact subgroups of* G *such that* $\mathrm{H}<\mathrm{K}$ *and let* ρ *be an irreducible respresentation of* H. *Then the map* $\mathrm{F}\to\Phi_\mathrm{F}$ *from the Hecke* *algebra* $\mathcal{H}(G, \rho)$ *to the Hecke algebra* $\mathcal{H}(G, \operatorname{Ind}(K, H; \rho))$ *given by*

$$(\Phi_F(g)f)(x) = \frac{1}{[K:H]} \sum_{y \in J} F(xgy^{-1}) f(y)$$

where f *is in the space of* ρ *and* J *is a set of coset representatives for* K/H *as a set is an algebra isomorphism. Moreover,* $\operatorname{supp}(\Phi_F) = K \operatorname{supp}(F) K$ *where* supp *denotes support.*

Proof. — Straightforward.

5 Proof of Theorem 0: the zero relative level case

In this section we will prove

PROPOSITION 5.1. — *Suppose* π *is an irreducible supercuspidal representation of* $G = GL_N(F)$ *where* N *is the product of two (not necessarily distinct) primes and that* π *contains a proper alfalfa stratum. Assume further that* $l_r(\pi) = 0$. *Then there exists an open compact-modulo-center subgroup* K *of* G *and an irreducible representation* ρ *of* K *such that* $\pi = \operatorname{Ind}(G, K; \rho)$.

Remark 5.2. — In light of the results of the previous section, the proof of Theorem 0 will be complete once Proposition 5.1 is proved. In proving Proposition 5.1, we will proceed in a manner similar to how we previously proved [KM1], 4.9 a special case of the proposition.

Proof of Proposition 5.1. — Suppose that π contains the relative alfalfa stratum $(\mathcal{A}, m, n, \alpha, r, \eta, \theta)$. Let $E = F[\alpha]$, $[E:F] = S$, $R = N/S$ and $k = -\nu_E(\alpha)$. Then by Corollary 2.22 we may assume $e(\mathcal{A}_E) = 1$. Then, since $U(\mathcal{A}_E)/U^1(\mathcal{A}_E) \cong GL_R(k_E)$, we may view η as a representation of $U(\mathcal{A}_E)$ or as a representation of $GL_R(k_E)$ as appropriate.

For the first part of this section we assume that k is odd.

LEMMA 5.3. — *With notation as above, if* η *is cuspidal as a representation of* $GL_R(k_E)$, *then Proposition* 5.1 *holds for* π.

Proof. — Let $H = U(\mathcal{A}_E) U^m(\mathcal{A}_F)$ and set $\sigma = (\eta \otimes \theta) \cdot \psi_\alpha$ as a representation of H. Then, by Lemma 4.16, it suffices to show that $\operatorname{supp} \mathcal{H}(G, \sigma)$ is contained in $Z(G) U(\mathcal{A}_F)$. Suppose g is an element of $\operatorname{supp} \mathcal{H}(G, \sigma)$. Then g is also an element of $\operatorname{supp} \mathcal{H}(G, \psi_\alpha)$ where ψ_α is viewed as a representation of $U^m(\mathcal{A}_F)$. Thus, by [KM2], 2.4, g is in $U^m(\mathcal{A}_F) G_E U^m(\mathcal{A}_F)$ since $2m = n+1$. Therefore we may assume g is in G_E. But then g is in $\mathcal{H}(G_E, \eta \otimes \theta)$ so that the lemma follows from well known results (*see*, e. g., [Ca]).

We may now assume that π contains $(\eta \otimes \theta) \cdot \psi_\alpha$ with η noncuspidal. Before proceeding further, we establish some notation. Choosing an appropriate $\mathcal{O}_E$-basis $\mathcal{B} = \{v_i\}_{i=1}^R$ for L, we may write $V = \bigoplus_{i=1}^{R} E v_i$ and $L_0 = \bigoplus_{i=1}^{R} \mathcal{O}_E v_i$. Then we may identify A_E with

$M_R(E)$ and A_F with $R\times R$-block matrices with each block being an element of $A_F(E)$. To be precise, if B is in A_F, we write $B=(b_{ij})_{i\leqq i,\,j\leqq R}$ where b_{ij} is an element of $A_F(E)$ for each (i, j) and we view A_E as the subgroup of A_F consisting of $B=(b_{ij})$ for which b_{ij} is an element E for each (i, j).

For an integer r such that $1\leqq r<R$, we define a period 2 $\mathcal{O}_E$-refinement L^r of L by setting $L_0^r=L_0$,

$$L_1^r=(\bigoplus_{i=1}^{r}\mathcal{O}_E V_i)\oplus(\bigoplus_{i=r+1}^{R}P_E V_i)$$

and extending in the usual manner to L_i^r. Similarly, we define a length R $\mathcal{O}_E$-refinement L^0 of L by setting $L_0^0=L_0$ and

$$L_i^0=(\bigoplus_{j=1}^{R-i}\mathcal{O}_E v_j)\oplus(\bigoplus_{j=R-i+1}^{R}P_E V_j)$$

for $1\leqq i<R$. For $0\leqq r<R$ we set $\mathcal{A}_{r,F}=\mathcal{A}(L^r)$ and $\mathcal{A}_{r,E}=\mathcal{A}_{r,E}\cap A_E$. We retain $\mathcal{A}_E$ and $\mathcal{A}_F$ as before. For $0\leqq r<R$ we let $P(r;k_E)$ be the parabolic subgroup of $GL_R(k_E)$ corresponding to the flag of subspaces $\{L_i^r/P_E L_0^r\}_{i=0}^{e_E(L^r)}$ of $L_0^r/P_E L_0^r$. We write $P(r;k_E)=M(r;k_E)N(r;k_E)$ for the associated Levi decomposition of $P(r;k_E)$ with $M(r;k_E)$ the reductive portion. We also let $P(r;E)$ be the obvious parabolic subgroup of G_E with the property that $P(r;E)\cap U^0(\mathcal{A}_F)/P(r;E)\cap U^1(\mathcal{A}_F)\cong P(r;k_E)$ and we let $P(r;F)$ be the obvious parabolic subgroup of G_F with the property that $P(r;F)\cap G_E=P(r;E)$.

Since η is not cuspidal and since R is prime, it follows from [Sp] that either there exists an integer $0<r<R$ and an irreducible representation τ of $P(r;k_E)$ trivial on $N(r;k_E)$ such that $\eta=\mathrm{Ind}(GL_R(k_E), P(r;k_E);\tau)$ or there exists a character τ of $k_E^\times$ such that η is contained in $\mathrm{Ind}(GL_R(k_E), P(0;k_E);\bigotimes_{i=1}^{R}\tau)$ where $\bigotimes_{i=1}^{R}\tau$ is the representation of $P(0;k_E)$ trivial on $N(0;k_E)$ and defined componentwise on $M(0;k_E)\cong\prod_{i=1}^{R}k_E^\times$.

LEMMA 5.4. — *With notation as above, if* $r>0$ *then the representation* $(\eta\otimes\theta)\cdot\psi_\alpha$ *of* $U(\mathcal{A}_E)U^m(\mathcal{A}_F)$ *is principal.*

Proof. — Let

$$x=\begin{bmatrix}\bar{\omega}_E I_r & 0\\ 0 & I_{R-r}\end{bmatrix}$$

in $GL_R(E)$, $H=U(\mathcal{A}_E)U^m(\mathcal{A}_F)$, $J=U(\mathcal{A}_F)$ and $K=U(\mathcal{A}_{r,E})U^m(\mathcal{A}_F)$. Also let $\sigma=\tau\otimes\theta$ as a representation of $U(\mathcal{A}_{r,E})$. Then it suffices to check that the hypotheses of [K2], 1.7 hold. First note that J contains K and K^x with finite index and x interwines $\sigma\cdot\psi_\alpha$. Now let $U(\mathcal{A}_{r,E})=\bar{U}\underline{U}$ be the Iwahori factorization of $U(\mathcal{A}_{r,E})$ with respect to $P(r;E)$. To be precise, let $\bar{U}$ be the subgroup of $U(\mathcal{A}_{r,E})$ consisting of elements of the form

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where A is $r \times r$ and D is $(R-r) \times (R-r)$ and let $\underline{U}$ be the subgroup of $U(\mathscr{A}_{r,E})$ consisting of elements of the form

$$\begin{bmatrix} I_r & 0 \\ C & I_{R-r} \end{bmatrix}.$$

Then, for j a positive integer,

$$\begin{aligned} \text{(B.4.1)} \qquad K x^{j-1} K x K &= K x^{j-1} U^m(\mathscr{A}_F) \bar{U} \underline{U} x K \\ &= K x^{j-1} U^m(\mathscr{A}_F) \bar{U} x K \\ &= K x^{j-1} \bar{U} U^m(\mathscr{A}_F) x K \\ &= K x^{j-1} U^m(\mathscr{A}_F) x K. \end{aligned}$$

Similarly, using the Iwahori factorization of $U^m(\mathscr{A}_F)$ with respect to $P(r; F)$, one can check that $K x^{j-1} U^m(\mathscr{A}_F) x K = K x^j K$ whence $K x^{j-1} K x K = K x^j K$. One can also verify by direct computation that the double cosets $Z(G) K x^j K j = 1, 2, \ldots$ are distinct.

We now claim that $(\ker \sigma \cdot \psi_\alpha)(K \cap x^{-1} K x) = K$. Since

$$\text{(5.4.2)} \qquad \begin{bmatrix} \bar{\omega}_E^{-1} I_r & 0 \\ 0 & I_{R-r} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{\omega}_E I_r & 0 \\ 0 & I_{R-r} \end{bmatrix} = \begin{bmatrix} \bar{\omega}_E^{-1} A \bar{\omega}_E & \bar{\omega}_E^{-1} B \\ C \bar{\omega}_E & D \end{bmatrix}$$

where A is an $r \times r$ matrix with entries in $A_F(E)$, D is an $(R-r) \times (R-r)$ matrix with entries in $A_F(E)$ and similarly for B and C, to prove the claim it suffices to show that matrices of the form

$$\begin{bmatrix} I_r & B \\ 0 & I_{R-r} \end{bmatrix}$$

in $U(\mathscr{A}_{r,E})$ and of the same form in $U^m(\mathscr{A}_F)$ are in $\ker(\sigma \cdot \psi_\alpha)$. This, however, is clear. Thus, to show that [K2], 1.7 is applicable, it suffices to show that $\operatorname{Ind}(J, K; \sigma \otimes \psi_\alpha)$ is irreducible. By assumption, $\sigma' = \operatorname{Ind}(H, K; \sigma \otimes \psi_\alpha)$ is irreducible. Then one checks that, as a $U^m(\mathscr{A}_F)$-space, σ' decomposes as a sum of copies of ψ_α. Then, since the stabilizer of ψ_α in $U(\mathscr{A}_F)$ is $U(\mathscr{A}_E) U^m(\mathscr{A}_F)$ by [KM2], 2.4, it follows that $\operatorname{Ind}(J, H; \sigma')$ and thus $\operatorname{Ind}(J, K; \sigma \otimes \psi_\alpha)$ are irreducible. The proof of Lemma 5.4 is now complete.

LEMMA 5.5. — *With notation as above, if* η *is contained in*

$$\operatorname{Ind}(GL_R(k_E), P(0; k_E); \bigotimes_{i=1}^{R} \tau),$$

then the representation $(\eta \otimes \theta) \cdot \psi_\alpha$ of $U(\mathscr{A}_E) U^m(\mathscr{A}_F)$ *is principal.*

Proof. — Let $\sigma = ((\bigotimes_{i=1}^{R} \tau) \otimes \theta) \cdot \psi_\alpha$ as a one-dimensional representation of $K = U(\mathscr{A}_{0,E}) U^m(\mathscr{A}_F)$. It suffices to show that σ is principal. If y in G intertwines σ, define a map $F_y : G \to \mathbb{C}$ by

$$F_y(g) = \begin{cases} 0 & \text{if } g \notin K y K \\ \sigma(k_1)\sigma(k_2) & \text{if } g = k_1 y k_2, \quad k_i \in K. \end{cases}$$

Then the F_y are in $\mathscr{H}(G, \sigma)$ and $\{F_y\}$ is a basis for the subspace of $(G, 0)$ consisting of functions supported on $K y K$. Note that if g is in G_E, then g intertwines σ. Let w_1 and w_2 be elements in $GL_R(E)$ defined as follows

$$w_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & & 1 \\ 1 & 0 & 0 & & & 0 \\ 0 & 1 & 0 & \ddots & & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & & & \ddots & 0 & 0 \\ 0 & & & & 1 & 0 \end{bmatrix}$$

and

$$w_2 = \begin{bmatrix} 0 & 1 & & & & 0 \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 0 & 1 \\ \bar{\omega}_E & & & & & 0 \end{bmatrix}$$

Then, w_1, w_2 and

$$x = w_1 w_2 = \begin{bmatrix} \bar{\omega}_E & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

intertwine σ.

Now let M be a right $\mathscr{H}(G, \sigma)$-module which is finite dimensional as a vector space over $\mathbb{C}$. We claim that F_{w_1} and F_{w_2} act as isomorphisms on M. Suppose this claim were true. Then, since w_2 normalizes $U(\mathscr{A}_{0,E})$ and w_2 normalizes $U^m(\mathscr{A}_F)$ we would have that $K w_1 K w_2 K = K x K$ and thus it would follow that F_x must act as an isomorphism on M. Thus, letting Φ_j be the j-fold convolution of F_x with itself, it would follow that Φ_j acts as an isomorphism on M. Now it also follows from appropriate Iwahori factorizations that $K x^{j-1} K x K = K x^j K$ and thus, by induction, Φ_j is supported on $K x^j K$. Then, since, as one can check, the double cosets $Z(G) K x^j K$ are distinct the lemma would follow from [K2], 1.5. Thus it suffices to show that F_{w_1} and F_{w_2} act as isomorphisms.

First consider F_{w_1}. Let W^a denote the affine Weyl group of $GL_R(E)$. For $i=1, \ldots, R-1$, let s_i be the element of W^a associated to the transposition $(i, i+1)$ in the permutation group on R letters. Note that the s_i intertwine σ. Now $w_1 = s_1 s_2 \ldots s_{R-1}$ is an expression of minimal length for w_1 in W^a, and thus $U(\mathscr{A}_{0,E}) s_1 U(\mathscr{A}_{0,E}) s_2 \ldots s_{R-1} U(\mathscr{A}_{0,E}) = U(\mathscr{A}_{0,E}) w_1 U(\mathscr{A}_{0,E})$. Therefore, since the s_i normalize $U^m(\mathscr{A}_F)$,

$$\begin{aligned} K s_1 K s_2 \ldots s_{R-1} K &= K s_1 U(\mathscr{A}_{0,E}) U^m(\mathscr{A}_F) \ldots s_{R-1} K \\ &= K s_1 U(\mathscr{A}_{0,E}) \ldots s_{R-1} K \\ &= K w_1 K. \end{aligned} \tag{5.5.1}$$

Thus, $F_{s_1} * F_{s_2} * \ldots * F_{s_{R-1}}$ is supported on $K w_1 K$. Therefore, to show that F_{w_1} acts as an isomorphism, it suffices to show that the F_{s_i} act as isomorphisms. To this end, note that

$$\begin{aligned} K s_i K s_i K &= K s_i U(\mathscr{A}_{0,E}) s_i K \\ &\subseteq K \cup K s_i K. \end{aligned}$$

Thus,

$$F_{s_i} * F_{s_i} = a_{1,i} F_1 + a_{2,i} F_{s_i} \tag{5.5.2}$$

for some constants $a_{1,i}$ and $a_{2,i}$. Evaluating (5.5.2) at 1 we see that $a_{1,i}$ is nonzero. Therefore, since F_1 is a multiple of the identity, it follows that F_{s_i} is invertible and thus acts as isomorphism.

Now consider F_{w_2}. We will write F_2 in place of F_{w_2}. We will also write $v \sim w$ for vectors v and w in some complex vector space if there exists a nonzero constant c such that $v = cw$. Let Ψ_j be the j-fold convolution of F_2 with itself. Then we claim that $F_{w_2^j} \sim \Psi_j$ for $j=1, \ldots R$. We will prove this claim by induction on j. The case $j=1$ being trivial, we assume the claim holds for $j-1$. Then

$$\begin{aligned} \Psi_j &= \Psi_{j-1} * F_2 \\ &\sim F_{w_2^{j-1}} * F_2. \end{aligned} \tag{5.5.3}$$

Now

$$\text{(5.5.4)}\quad \mathrm{K}\,w_2^{j-1}\,\mathrm{K}\,w_2\,\mathrm{K}=\mathrm{K}\,w_2^{j-1}\,\mathrm{U}^m(\mathscr{A}_\mathrm{F})\,w_2\,\mathrm{K}$$

$$=\bigcup_{a_{\mathrm{R}i}\in\mathscr{P}_\mathrm{F}^m(\mathrm{E})}\mathrm{K}\,w_2^{j-1}\begin{bmatrix}1 & & & & 0\\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ a_{\mathrm{R}1} & a_{\mathrm{R}2} & \dots & a_{\mathrm{RR}-1} & 1\end{bmatrix}w_2\,\mathrm{K}$$

$$=\bigcup_{a_{1k}\in\mathrm{Q}}\mathrm{K}\,w_2^{j}\begin{bmatrix}1 & a_{12} & \dots & a_{1\mathrm{R}}\\ & \ddots & 0 & \\ 0 & & \ddots & \\ & & & 1\end{bmatrix}\mathrm{K}$$

$$=\bigcup_{a_{1k}\in\mathrm{Q}}\mathrm{K}\,w_2^{j}\begin{bmatrix}1 & a_{12} & \dots & a_{1j} & 0 & \dots & 0\\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & 0 & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 1\end{bmatrix}$$

where Q is a set of coset representatives for $\mathscr{P}_\mathrm{F}^{m-1}(\mathrm{E})/(\mathscr{P}_\mathrm{F}^{m}(\mathrm{E})+\mathrm{P}_\mathrm{E}^{m-1})$. Let

$$\mathrm{A}=\begin{bmatrix}1 & a_{12} & \dots & a_{1j} & 0 & \dots & 0\\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 1\end{bmatrix}$$

where the a_{1k} are in Q. Then one can compute that the $\mathrm{R}-j+1$ row of the matrix $\alpha^{w_2^j \mathrm{A}}$ is

$$(0\quad 0\quad \dots\quad \alpha\ [a_{12},\alpha]\quad \dots\quad [a_{1j},\alpha]).$$

Computing $(w_2^j\,\mathrm{A})\,\mathrm{U}^m(\mathscr{A}_\mathrm{F})\,(w_2^j\,\mathrm{A})^{-1}\cap \mathrm{U}^m(\mathscr{A}_\mathrm{F})$, it follows that if $w_2^j\,\mathrm{A}$ intertwines ψ_α then $[a_{1k},\alpha]$ is in $\mathscr{P}_\mathrm{F}^{1-m}(\mathrm{E})$ for all k. But then, by [Ca], 3.5, it follows that a_{1k} is in $\mathrm{P}_\mathrm{E}^{m-1}+\mathscr{P}_\mathrm{F}^{m}(\mathrm{E})$. Thus, (5.5.3) and (5.5.4) imply that ψ_j is supported on $\mathrm{K}\,w_2^j\,\mathrm{K}$.

Therefore, $\Psi_j = a_j F_{w_2^j}$ for some number a_j. Thus, to prove the claim, it suffices to show that $\Psi_j(w_2^j) \neq 0$.

(5.5.5)
$$\begin{aligned}\Psi_j(w_2^j) &\sim \int_G F_{w_2^{j-1}}(w_2^j y^{-1}) F_w(y)\, dy \\ &\sim \int_K F_{w_2^{j-1}}(w_2^j h^{-1} w_2^{-1})\, \sigma(h)\, dh \\ &\sim \int_K F_{w_2^{j-1}}(w_2^j h w_2^{-1})\, \sigma(h^{-1})\, dh \\ &\sim \int_{U^m(\mathscr{A}_F)} F_{w_2^{j-1}}(w_2^j h w_2^{-1})\, \sigma(h^{-1})\, dh.\end{aligned}$$

Now if $j = R$, then $w_2^R = \bar{\omega}_E$ normalizes $U^m(\mathscr{A}_F)$ and stabilizes σ so that (5.5.5) implies $\Psi_j(w_2^j) \neq 0$. Thus suppose $j < R$. Then, considering $w_2 h w_2^{-1}$ and $w_2^j h w_2^{j-1}$, one checks that (5.5.5) implies that

(5.5.6)
$$\Psi_j(w_2^j) \sim \sum_{h \in H} F_{w_2^{j-1}}(w_2^j h w_2^{-1})\, \sigma(h)$$

where H is a set of coset representatives for $J/J \cap U^{m+1}(\mathscr{A}_F)$ where J is the set of matrices in $GL_N(F)$ of the form

(5.5.7)
$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{I_j} & 0 \\ \hline \begin{matrix} a_{1j+1} \\ \vdots \\ a_{1R} \end{matrix} & 0 & I_{R-j} \end{array}\right]$$

where each a_{1i} is in $\mathscr{P}_F^m(E)$. But then one checks that if h in H has the form given in (5.4.7), then $w_2^j h w_2^{-1}$ is not in $K w_2^{j-1} K$ unless a_{1i} is in $\mathscr{P}_F^{m+1}(E) + P_E^m$ for all i. Thus elements of H may be chosen of the form in (5.5.7) with a_{1i} in P_E^m for all i and then (5.5.6) implies $\psi_j(w_2^j) \neq 0$ whence the claim. As a consequence of the claim, $\Psi_R \sim F_{\bar{\omega}_E}$.

But now since $\bar{\omega}_E$ normalizes K it follows that $F_{\bar{\omega}_E}$ and thus F_{w_2} act as isomorphisms.

In proving Proposition 5.1, we may now assume that $k = -\nu_E(\alpha)$ is even. In considering this case, we will use the theory of the Heisenberg group and the oscillator (Weil) representation. For further details, *see*, in general, [W2] and, for the particular constructions we use here, *see* [H2] (p odd), [G], [Wa] (all p) and [KM1] (N = 4).

With notation as above, for i and j nonnegative integers set $H^{i,j} = U^i(\mathscr{A}_E)\, U^j(\mathscr{A}_F)$. Recall that $n = k e_E(L) + 1 = k + 1$ and $m = k/2 + 1$. Then, if we set $\Gamma = H^{1, m-1}/\ker(\theta \cdot \psi_\alpha)$ with $\theta \cdot \psi_\alpha$ regarded as a representation of $H^{1,m}$, Γ is a "Heisenberg group" with center $Z(\Gamma) = H^{1,m}/\ker(\theta \cdot \psi_\alpha)$ and thus has a unique irreducible representation χ with central character $\theta \cdot \psi_\alpha$.

There are other Heisenberg groups which will be of interest to us. Recall that for $0 \leq r < R$ we defined at the beginning of this section an $\mathcal{O}_E$-refinement L^r of L of period 2 if $r > 0$ and period R if $r = 0$; we also attached orders $\mathscr{A}_{r,E}$ and $\mathscr{A}_{r,F}$ to these refinements. Now for i and j nonnegative integers set $J_r^{i,j}=U^i(\mathscr{A}_{r,E})U^j(\mathscr{A}_{r,F})$. Also set $e_r=e_E(L^r)$, $n_r=ke_r+1$ and $m_r=ke_r/2+1$. Then if we $\Gamma^r=J_r^{1,m-1}/\ker(\theta\cdot\psi_\alpha)$ with $\theta\cdot\psi_\alpha$ regarded as a representation of J_r^{1,m_r}, Γ^r is a product of Heisenberg groups and has center $Z(\Gamma^r)=J^{1,m_r}/\ker(\theta.\psi_\alpha)$. We now describe these products. For $0<r<R$, let $V_r^1=\bigoplus_{i=1}^{R} E v_i$ and $V_r^2=\bigoplus_{i=r+1}^{R} E v_i$ so that $V=V_r^1\oplus V_r^2$ and for $i=1,\ldots,R$ let $V_0^i=Ev_i$ so that $V=\bigoplus_{i=1}^{R} V_0^i$. Let $A_{r,E}^i=A_E(V_r^i)$ and $A_{r,F}^i=A_F(V_r)$. Also let $L^{r,i}$ be the period one $\mathcal{O}_E$-lattice chain V_r^i defined by $L\cap V_r^i$ and let $\mathscr{A}_{r,E}^i$ and $\mathscr{A}_{r,F}^i$ be the the associated maximal orders in $A_{r,E}^i$ and $A_{r,F}^i$ respectively. Then we may view θ also as a character of $A_{r,E}^i$ *via* the determinant and ψ_α also as a character of $U^m(\mathscr{A}_{r,F}^i)/U^{n+1}(\mathscr{A}_{r,F}^i)$ where m and n are as before. In what follows we will either state the group that θ or ψ_α is being considered a character of or it will be clear from context. Now, if we set $\Gamma_r^i=U^1(\mathscr{A}_{r,E}^i)U^{m-1}(\mathscr{A}_{r,F}^i)/\ker(\theta\cdot\psi_\alpha)$ with $(\theta\cdot\psi_\alpha)$ regarded as a character of $U^1(\mathscr{A}_{r,E}^i)U^m(\mathscr{A}_{r,F}^i)$, then Γ_r^i is a Heisenberg group with center $Z(\Gamma_r^i)=U^1(\mathscr{A}_{r,E}^i)U^m(\mathscr{A}_{r,F}^i)/\ker(\theta\cdot\psi_\alpha)$ and thus has a unique irreducible representation χ_r^i with central character $\theta\cdot\psi_\alpha$. Then one checks that $\Gamma_r\cong\times_{i=1}^{e_r}\Gamma_r^i$ and thus, if we let $\chi_r=\bigotimes_{i=1}^{e_r}\chi_r^i$, χ_r is the unique irreducible representation of Γ_r with central character $\theta\cdot\psi_\alpha$. We will also view χ_r as a representation of J_r^{1,m_r-1}. Since $H^{1,m-1}$ stabilizes $\theta\cdot\psi_\alpha$ as a character of $H^{1,m}$ it follows from the uniqueness of χ that $\mathrm{Ind}(H^{1,m-1},H^{1,m};\theta\cdot\psi_\alpha)$ decomposes as a sum of copies of χ. Similarly, $\mathrm{Ind}(J_r^{1,m_r-1},J_r^{1,m_r};\theta\cdot\psi_\alpha)$ decomposes as a sum of copies of χ_r.

Now $H^{0,m-1}$ acts on $H^{1,m-1}$ by conjugation and this action stabilizes $\theta\cdot\psi_\alpha$ as a representation of $H^{1,m}$. Thus, by uniqueness of χ, there exists an extension Λ of χ to $H^{0,m-1}$. Similarly, there exists an extension Λ_r of χ_r to J_r^{0,m_r-1}. Moroever, these extensions may be chosen so that $\mathrm{Ind}(U(\mathscr{A}_{r,E})U^{m-1}(\mathscr{A}_F),J_r^{0,m_r};\Lambda_r)$ is the restriction of Λ to $U(\mathscr{A}_{r,E})U^{m-1}(\mathscr{A}_F)$.

We now return to the proof of Proposition 5.1. We may assume that, upon restriction $H^{1,m-1}$, π contains χ. Then, since Λ extends χ,

$$\mathrm{Ind}(H^{0,m-1},H^{1,m-1};\chi)\cong\bigoplus_{\tau\in(H^{0,m-1}/H^{1,m-1})^\wedge}(\dim\tau)\Lambda\otimes\tau.$$

Thus we may assume that upon restriction to $H^{0,m-1}\pi$ contains some representation $\Lambda\otimes\tau$ where τ is a representation of $H^{0,m-1}$ that may be viewed as a representation of $GL_R(k_E)$.

LEMMA 5.6. — *With notation as above, if* π *contains* $\Lambda\otimes\tau$ *with* τ *cuspidal, then Proposition* 5.1 *holds for* π.

Proof. — By Lemma 4.16, it suffices to check that $\mathrm{supp}\,\mathscr{H}(G,\Lambda\otimes\tau)$ is contained in $Z(G)U(\mathscr{A}_F)$. Suppose g is an element of $\mathrm{supp}\,\mathscr{H}(G,\Lambda\otimes\tau)$. Now $\Lambda\otimes\tau$ restricted to

$U^m(\mathscr{A}_F)$ decomposes as a sum of copies of ψ_α. Thus, g must lie in $\mathscr{H}(G, \psi_\alpha)$. Thus, by [KM2], 2.4, g is an element of $U^{m-1}(\mathscr{A}_F)\, G_E\, U^{m-1}(\mathscr{A}_F)$. Therefore, we may assume g is in G_E. Now g also intertwines $\Lambda \otimes \tau$ restricted to $H^{0, m}$. Since the action of $U(\mathscr{A}_E)$ by conjugation on G is trivial upon restriction to $Z(\Gamma)$ it follows form the construction of Λ that $\Lambda \otimes \tau$ restricted to $H^{0, m}$ decomposes as a sum of copies of $(\theta \otimes \tau) \cdot \psi_\alpha$. Thus g is in $\mathscr{H}(G_E, \tau)$ and the lemma follows.

In proving Proposition 5.1 we may now assume π contains $\Lambda \otimes \tau$ with τ noncuspidal. Now, by [Sp] as before, either there exists an interger $0 < r < R$ and an irreducible representation σ of $P(r; k_E)$ trivial on $N(r; k_E)$ such that $\tau = \mathrm{Ind}(GL_R(k_E), P(r; k_E); \sigma)$ or there exists a character ρ of $k_E^\times$ such that τ is contained in $\mathrm{Ind}(GL_R(k_E), P(0, k_E); \sigma)$ where σ is the representation of $P(0, k_E)$ trivial on $N(0; k_E)$ and defined componentwise on $M(0; k_E) = \times_{i=1}^{R} k_E$ by ρ. Then, since

$$\mathrm{Ind}(U(\mathscr{A}_{r, E})\, U^{m-1}(\mathscr{A}_F), J_r^{0, m_r}; \Lambda_r)$$

is the restriction of Λ to $U(\mathscr{A}_{r, E})\, U^{m-1}(\mathscr{A}_F)$ by construction, it suffices to show that $\Lambda_r \otimes \sigma$ is the principal where σ is now viewed as a representation of $J_r^{0, m_r - 1}$ trivial on $J_r^{1, m_r - 1}$.

LEMMA 5.7. — *With notation as above, if* $r > 0$, $\Lambda_r \otimes \sigma$ *is principal.*

Prof. – Our proof of this lemma will be similar to our proof of Lemma 5.4. As in that proof, let

$$x = \begin{bmatrix} \bar{\omega}_E I_r & 0 \\ 0 & I_{R-r} \end{bmatrix}$$

in $GL_R(E)$. Also, let $K = J_R^{0, m_r - 1}$, $J = H^{0, m-1}$ and $H = U(\mathscr{A}_{r, E})\, U^{m-1}(\mathscr{A}_F)$. Then by appropriate Iwhaori factorizations, $K x^{j-1} K x K = K x^j K$ for $j = 1, 2, \ldots$. Further one can check that the double cosets $Z(G) K x^j K j = 1, 2, \ldots$ are distinct and that J contains both K and K^x with finite index. Now $\Lambda_r = \Lambda_r^1 \otimes \Lambda_r^2$ where Λ_r^1 and Λ_r^2 are extensions of χ_r^1 and χ_r^2. Then, since conjugation by $\omega_E I_r$ preserves the central character of Λ_r^1, it follows that x intertwines $\Lambda_r \otimes \sigma$. Finally, one can check that

$$(\ker(\Lambda_r \otimes \sigma))(K \cap x^{-1} K x) = K.$$

Thus, by [K2], 1.7, it suffices to show that $\mathrm{Ind}(J, K; \Lambda_r \otimes \sigma)$ is irreducible. But one can check that $\mathrm{Ind}(J, K; \Lambda_r \otimes \sigma) = \mathrm{Ind}(J, H; \Phi_r \otimes \sigma)$ where Φ_r is the restriction of χ to H. Then, since Φ_r is irreducible and extends to J while $\mathrm{Ind}(GL_R(k_E), P(r; k_E); \sigma)$ is irreducible the lemma follows.

Now, to prove Proposition 5.1, it suffices to prove

LEMMA 5.8. — *If* ρ *is a character of* $k_E^\times$ *and* σ *is the representation of* $J_0^{0, m-1}$ *obtained from the representation of* $P(0; k_E)$ *which is trivial on* $N(0; k_E)$ *and defined componentwise on* $M(0; k_E)$ *by* ρ, *then* $\Lambda_0 \otimes \sigma$ *is principal.*

Proof. — We will use an argument similar to the one that was used in proving Lemma 5.5. As in that proof, let

$$w_1 = \begin{bmatrix} 0 & 0 & \cdots & & & 1 \\ 1 & 0 & 0 & & & \vdots \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & 0 \\ & & & & 1 & 0 \end{bmatrix},$$

$$w_2 = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ \bar{\omega}_E & & & & 0 \end{bmatrix},$$

$x = x_1 w_2$ and for $i = 1, \ldots R-1$ let s_i be the element of the affine Weyl group W^a of $GL_R(E)$ associated to the transposition $(i, i+1)$ in the permutation group on R letters. Finally, let $K = J^{0, m_r}$. Then one can check that $K x^{j-1} K x K = K x^j K$ for $j = 1, 2, \ldots$ and also that the double cosets $Z(G) K x^j K j = 1, 2, \ldots$ are distinct. Note also that $K w_1 K w_2 K = K x K$ since w_2 normalizes K.

Now $\Lambda_0 = \bigotimes_{i=1}^{s} \Lambda_0^i$ and since $\mathrm{Ind}(U(\mathscr{A}_{0,E}) U^{m-1}(\mathscr{A}_F), J_0^{0, m_0}, \Lambda_0)$ is the restriction of χ to $U(\mathscr{A}_{0,E}) U^{m-1}(\mathscr{A}_F)$ it follows that for $i = 1, \ldots R-1$ s_i intertwines Λ_0 (and thus $\Lambda_0 \otimes \sigma$). Thus $\Lambda_0^i \cong \Lambda_0^j$ for $1 < i, j < R$. It then follows, by an argument similar to that used in the previous lemma to show that the x of that lemma intertwines, that the x of this lemma intertwines $\Lambda_0 \otimes \sigma$ and that the associated intertwining operator, T_x say, is unique (up to scalar). Then

$$F_x(g) = \begin{cases} 0 & \text{if } g \notin K x^j K \\ (\Lambda_0 \otimes \sigma)(k_1) T_x (\Lambda_0 \otimes \sigma)(k_2) & \text{if } g = k_1 x K_2 k_i \in K \end{cases}$$

forms a basis for the subspace of $\mathscr{H}(G, \Lambda_0 \otimes \sigma)$ consisting of functions with support $K x K$. Similarly

$$F_{x^j}(g) = \begin{cases} 0 & \text{if } g \notin K x^j K \\ (\Lambda_0 \otimes \sigma)(k_1)(T_x)^j (\Lambda_0 \otimes \sigma)(k_2) & \text{if } g = k_1 x^j k_2, \quad k_1 \in K \end{cases}$$

for $j = 2, \ldots$ is the unique (up to scalars) element of $\mathscr{H}(G, \Lambda_0 \otimes \sigma)$ with support $K x^j K$. Let Ψ_j be the j-fold convolution product of F_x. We claim that $F_{x^j} \sim \Psi_j$. Since F_{x^j} and Ψ_j have the same support, to prove the claim, it suffices to show that F_x acts as an isomorphism on any right $\mathscr{H}(G, \Lambda_0 \otimes \sigma)$-module M which is finite dimensional as a vector space over $\mathbb{C}$. The lemma will then follow from [K2], 1.5.

Let T_i be the unique (up to scalar) intertwining operator associated to s_i and set

$$F_i(g)=\begin{cases} 0 & \text{if } g \notin K s_i K \\ (\Lambda_0 \otimes \sigma(k_1)) T_i (\Lambda_0 \otimes \sigma(k_2)) & \text{if } g=k_1 s_1 k_2, \quad k_i \in K. \end{cases}$$

Then $\{F_i\}$ is a basis for the subspace of $\mathscr{H}(G, \Lambda_0 \otimes \sigma)$ consisting of functions supported on $K s_i K$. Also let

$$F_I(g)=\begin{cases} 0 & g \notin K \\ (\Lambda_0 \otimes \sigma)(g), & g \in K. \end{cases}$$

Then $\{F_I\}$ is a basis for the subspace of $\mathscr{H}(G, \Lambda_0 \otimes \sigma)$ consisting of functions supported on K. Then one checks that $K s_i K s_i \subseteq K \cup K s_i K$ and thus $F_i * F_i = a_i F_I + b_i F_i$ for some numbers a_i and b_i. Evaluation at I yields that $a_i \neq 0$ and thus that F_i must act as an isomorphism on M. Then one checks that w_1 intertwines $\Lambda_0 \otimes \sigma$ with intertwining operator $T_0 = T_1 T_2 \ldots T_{R-1}$ and that this operator is unique up to scalar. One further checks that $K s_1 K s_2 \ldots K s_{R-1} K = K w_1 K$. Therefore $F_1 * F_2 * \ldots * F_{R-1} \sim F_{w_1}$ and thus F_{w_1} acts as an isomorphism on M where

$$F_{w_1}(g)=\begin{cases} 0, & \text{if } g \notin K w_1 K \\ (\Lambda_0 \otimes \sigma)(k_1) T_0 (\Lambda_0 \otimes \sigma)(k_2) & \text{if } g=k_1 w_1 k_2, \quad k_i \in K. \end{cases}$$

Now consider w_2. One checks that w_2 intertwines $\Lambda_0 \otimes \sigma$ with associated intertwining operator $T' = T_{R-1} \ldots T_1 T_x$. As before, T' is unique up to scalar multiplication. Set

$$F_{w_2}(g)=\begin{cases} 0 & \text{if } g \notin K w_2 K \\ (\Lambda_0 \otimes \sigma)(k_1) T' (\Lambda_0 \otimes \sigma)(k_2) & \text{if } g=k_1 w_2 k_2, \quad k_i \in K. \end{cases}$$

Then $F_{w_1} * F_{w_2} = a F_x$ for some number a. Thus, to complete the proof of Lemma 5.8, it suffices to show that F_{w_2} acts as an isomorphism on M.

Note first that w_2^j intertwines $\Lambda_0 \otimes \sigma$ and that the (unique up to a scalar) associated intertwining operator is $(T')^j$. Note also that $(T')^R$ is the unique (up to scalar intertwining operator associated to $\bar{\omega}_E$. Define $F_{w_2^j}$ in the usual manner and let Ψ_j be the j-fold convolution product of F_{w_2}. We claim that $\Psi_j \sim F_{w_2^j}$. The claim being trivial for $j=1$ assume it holds for $j-1$.

Then since $K w_2^{j-1} K w_2 K = K w_2^j K$ it suffices to show that $F_{w_2^{j-1}} * F_{w_2}(w_2^j) \neq 0$. But

$$\begin{aligned} F_{w_2^{j-1}} * F_2^j(w_2) &= \int_G F_{w_2}^{j-1}(w_2^j y^{-1}) F_{w_2}(y)\, dy \\ &\sim \int_K F_{w_2}^{j-1}(w_2^j k^{-1} w_2^{-1}) T' (\Lambda_0 \otimes \sigma)(k)\, dk \\ &= \int_K F_{w_2^{j-1}}(w_2^{j-1}(w_2 k^{-1} w_2^{-1}))(\Lambda_0 \otimes \sigma)(w_2 k w_2^{-1}) T'\, dk \\ &= \int_K (T')^j\, dk \\ &\neq 0. \end{aligned}$$

Thus, $\Psi_R \sim F_{\bar{\omega}_E}$. As in Lemma 5.5 this implies that F_{w_2} acts as an isomorphism as desired.

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