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ON SPREADING MODELS IN $L^1(E)$

PAR MICHEL TALAGRAND (*)

ABSTRACT. — We construct a Banach space E which has the Schur property (hence l^1 is its only spreading model) but such for each family $(a_{n,k})$, with $a_{n,k} \geq 1$, $\lim_n a_{n,k} = +\infty$, there is a sequence (f_n) in $L^1(E)$ for which $\| \sum_{k \leq l \leq n} \pm f_l \| \leq a_{n,k}$. In particular, $L^1(E)$ has a spreading model isomorphic to $c_0(\mathbb{N})$.

1. Introduction

Let E be a separable Banach space and (Ω, Σ, μ) a (standard) measure space. We denote by $L^1(E)$ the space of integrable functions $\Omega \rightarrow E$. It is known that if E does not contain $c_0 = c_0(\mathbb{N})$, then $L^1(E)$ does not contain c_0 [2]. The purpose of this work is to show in an opposite direction that even when E is by no way close to c_0 , $L^1(E)$ can contain sequences which somehow behave like the unit basis of c_0 . Recall that a Banach space has the Schur property if weak null sequences go to zero in norm.

We shall show the following.

THEOREM A. — *There exists a separable Banach space E which has the Schur property, such that for each family $a_{n,k}$ of real $a_{n,k} \geq 1$, such that:*

$$(1) \quad \forall k, \quad \lim_n a_{n,k} = +\infty,$$

there exists a sequence $f_n \in L^1(E)$, such that:

$$(2) \quad \forall \omega, \quad \|f_n(\omega)\| = 1,$$

(3) *\forall finite set I , with $\text{card } I = n$ and $\inf I \geq k$ one has, for $(b_i) \in \mathbb{R}^I$:*

$$\inf |b_i| \leq \left\| \sum_{i \in I} b_i f_i \right\| \leq a_{n,k} \sup |b_i|.$$

(*) This paper was written while the author was visiting the Ohio State University.

Since E has the Schur property it follows from Rosenthal's theorem [3] that each sequence (X_n) of E which does not converge in norm has a subsequence equivalent to the basis of l^1 . However the sequence (f_n) of $L^1(E)$ has a behavior which is close to the basis of c_0 . Since it is possible to choose $(a_{n,k})$ such that for each n $\lim_k (a_{n,k}) = 1$, in the language of spreading models, $L^1(E)$ has c_0 as a spreading model, while E has l^1 as unique spreading model.

The whole difficulty of the construction is that in E there should be "very few" sequences equivalent to the basis of l^1 .

2. Setting of the construction

Let us set $T_n = \{0, 1\}^n$, $T = \bigcup T_n$. For $s \in T$ let $|s|$ be the unique n for which $s \in T_n$. For $s, t \in T$, $|s| = n$, $|t| = m$, $n \leq m$, $s = (s_1, \dots, s_n)$, $t = (t_1, \dots, t_m)$, we write $s < t$ if $\forall i \leq n$, $s_i = t_i$. With this order, T is the usual dyadic tree. For $t \in T$, $n \leq |t|$, we write $t|n$ the unique $s \in T_n$ for which $s < t$.

Let us denote by $(e_t)_{t \in T}$ the canonical basis of \mathbb{R}^T . In the next paragraph, we shall construct a family H of \mathbb{R}^T , and we shall define for $x \in \mathbb{R}^T$:

$$(4) \quad \|x\| = \sup \{ |\langle g, x \rangle|, g \in H \}.$$

Let E be the completion of $(\mathbb{R}^T, \|\cdot\|)$. It will be true that $\|e_t\| = 1$. We denote by e_t^* the element of E^* given by $e_t^*(e_{t'}) = 1$ if $t = t'$ and zero otherwise.

Let $\Omega = \prod_n T_n$, and let μ be the canonical measure on Ω (i.e. the product measure when each T_n is given the measure which puts weight 2^{-n} at each point).

Let $p_n : \Omega \rightarrow T_n$ be the projection of rank n . Let $h_n(\omega) = e_{p_n(\omega)}$. The reader has noticed that the setting of this construction is very similar to the setting of the construction [4] of a space E with the Dunford-Pettis property such that $\mathcal{C}([0, 1], E)$ fails the Dunford-Pettis property. However the idea of the construction of the norm is rather different.

3. Construction of the norming functionals

We start with $H_0 = \{e_t^*; t \in T\}$. We shall construct inductively subsets H_n of \mathbb{R}^T .

Let X^n be the set of subsets $A = \{t_1, \dots, t_p\}$ of T with the following property:

$$(5) \quad \forall 1 \leq i \leq p, \quad |t_i| \geq n.$$

$$(6) \quad \exists s \in T_n, \quad s < t_i, \quad \forall i \leq p.$$

$$(7) \quad \text{If } |t_i| = c_i, \text{ for } 1 \leq i < j \leq p \text{ one has } t_j|c_i = t_{i+1}|c_i.$$

The element s will be called the *stem* of A and be denoted by $s(A)$. Let $H_1^n = \{1/4 \sum_{t \in A} e_t^*; A \in X^n\}$. For $g \in H_1^n$, we call $s(A)$ the stem of g , also denoted by $s(g)$. We set $H_1 = \bigcup_n H_1^n$.

For $g \in \mathbb{R}^{(T)}$, let $V(g) = \sup\{|t|; \langle g, e_t \rangle \neq 0\}$. Let $n > 0$. Consider a sequence $k(1) = n < k(2) < \dots < k(p)$ and a sequence $g(i) \in H_1^{k(i)}$ such that:

- (8) $\forall 1 \leq i \leq p, \quad V(g(i)) < k(i+1).$
- (9) $\exists s \in T_n, \quad \forall i, \quad s < s(g(i)).$
- (10) $\forall i < j \leq p, \quad s(g(i+1)) \mid V(g(i)) = s(g(j)) \mid V(g(i)).$

[The reader should make a picture of the supports of the $g(i)$.] We define H_2^n as the set of sums $1/4 \sum_{i \leq p} g(i)$ of the above type, and H_2 as $\bigcup_{n \geq 1} H_2^n$.

The construction continues in the same way. Notice that each $g \in H_n$ is of the type $4^{-n} \sum_{t \in A} e_t^*$. Moreover, as is seen by induction, if $B \subset A$, $g' = 4^{-n} \sum_{t \in B} e_t^*$ still belongs to H_n .

Let H' be the set of finite sums $\sum_{i \geq 2} g_i$, where $g_i \in H_i$. Let $H = H_0 \cup H_1 \cup H'$.

4. E has the Schur property

By standard arguments of approximation, it is enough to show that if a sequence $(f_n) \in E$ such that $f_n = \sum_{t \in A_n} x_t^n e_t$ for A_n disjoint sets, $\|f_n\| = 1$ it cannot go to zero weakly.

1st case. — The following holds:

$$(11) \quad \forall m, \quad \limsup_n \{|\langle g, f_n \rangle|; g \in H_m\} = 0.$$

For each n , there is $g_n \in H$ with $|\langle g_n, f_n \rangle| \geq 1/2$. From (11) it follows that $g_n \in H'$ for n large enough. Then we can write $g_n = \sum_{2 \leq i \leq k(n)} g_n^i$ where $g_n^i \in H_i$. By taking a subsequence one can assume from (11) that:

$$\left| \sum_{i \leq k(n-1)} g_n^i(f_n) \right| \leq 1/4.$$

If:

$$g'_n = \sum_{k(n-1) < i \leq k(n)} g_n^i$$

one has $|g'_n(f_n)| \geq 1/4$. Let \bar{g}_n^i obtained from g_n^i by restricting its support to A_n . Then $\bar{g}_n^i \in H_n$. Let:

$$g''_n = \sum_{k(n-1) < i \leq k(n)} \bar{g}_n^i.$$

Then $|g''_n(f_n)| \geq 1/4$. Moreover, $g''_n(f_p) = 0$ for $p \neq n$. Let $h_n = \sum_{p \leq n} g''_p$. Then $h_n \in H'$. Indeed, $h_n = \sum_{i < k(n)} h^i$ where $h^i = \bar{g}_p^i$ for the unique p such that $k(p-1) < i \leq k(p)$. We have $|h_n(f_p)| > 1/4$ for $p < n$. Hence if h is a weak* cluster point of (h_n) , we have $|h(f_p)| \geq 1/4 \forall p$, which finishes the proof in this case.

2nd case. — There is $m, \alpha > 0$ and a sequence k_n such that $\sup\{|\langle g, f_{k_n} \rangle|; g \in H_m\} > \alpha \forall n$. One can suppose $k_n = n$. One can also suppose that m is the smallest integer for which the above is true, i. e.:

$$(12) \quad \limsup_{n \rightarrow \infty} \{|\langle g, f_n \rangle|; g \in H_{m-1}\} = 0.$$

For convenience of notation suppose now on that $m \geq 1$. (The same argument works for $m = 0$.)

Let $g_n \in H_m$ with $|\langle g_n, f_n \rangle| > \alpha$. One can suppose that g_n is supported by A_n . It follows from the definition of H_m that for each k one can write $g_n = g_n^1 + \dots + g_n^k + g'_n$ where $g_n^i \in H_{m-1}$ for $i \leq k$, and $g'_n \in H_m^k$. It follows, by taking a subsequence, that one can assume $g_n \in H_m^n$ and $|\langle g_n, f_n \rangle| \geq \alpha/2$. Another extraction of subsequence will give $g_n \in H_m^{k(n)}$ where $k(n) > V(g_{n-1})$. Let $s_n = s(g_n) \in T_{k(n)}$. By taking a subsequence, one can assume that for each p , the sequence $s_n|_p$ is eventually constant. A further subsequence will satisfy $s_n|_V(g_p) = s_{p+1}|_V(g_p)$ for $n \geq p+1$.

It follows from the definition of H_{m+1} that for each n , $h_n = 1/4 \sum_{p \leq n} g_p \in H_{m+1}$. Moreover, for $p \leq n$ we have $|h_n(f_p)| > \alpha/8$. Let h be a weak* cluster point of (h_n) . Then $|h(f_p)| \geq \alpha/8$ for each p , which finishes the proof.

5. Construction of (f_n)

In fact, (f_n) will be a subsequence of h_n .

LEMMA. — Let (u_i) be a sequence of independent random variables uniformly distributed in $\{1, \dots, g\}$. Let $P(q, n) = \text{Prob}(\exists i, j \leq n, u_i = u_j)$. Then $\lim_{q \rightarrow \infty} P(q, n) = 0$.

Moreover, $P(q, n)$ is increasing in n and decreasing in q .

Proof. — $P(q, n) \leq q^{-2} (n(n-1))/2$.

Let $(a_{n,k})$ be the sequence of theorem A. One can suppose that $a_{n,k} \leq a_{n+1,k}$ and $a_{n,k} \geq a_{n,k+1}$ for each n, k . Let $n(k)$ be the smallest integer such that $a_{n(k),k} \geq k+1$. From the lemma, there exists an increasing sequence $q(k)$ such that for each $k \geq 1$ one has the following conditions :

$$(13) \quad n(k) P(2^{-q(k)}, n(k)) \leq \frac{1}{2}.$$

$$(14) \quad \text{For each integer } n \text{ such that } a_{n,k} \leq 2, n P(2^{-q(k)}, n) \leq a_{n,k} - 1.$$

We shall prove that the sequence $f_n = h_{q(n)}$ satisfies the theorem. Let I be a finite set of integers, with $k = \text{Inf } I$ and $\text{card } I = n$. Let l the greatest integer such that $l+1 \leq a_{n,k}$. (It is possible that $l=0$.) Let $m = k + l + 1$.

We have:

$$a_{n,m} \leq a_{n,k} < l+2 \leq m \leq a_{n(m),m} \quad \text{so} \quad n \leq n(m).$$

Hence:

$$(15) \quad n P(2^{-q(m)}, n) \leq \frac{1}{2}.$$

Let us define $a_i(\omega)$ by $f_i(\omega) = e_{a_i(\omega)}$. Let:

$$Z = \{ \omega \in \Omega; \exists i, j \in I, i, j \geq m, i \neq j, a_i(\omega) | q(m) = a_j(\omega) | q(m) \}.$$

Since the maps $\omega \rightarrow a_i(\omega)$ are independent and $a_i(\omega) | q(m)$ takes for value each element of $T_{q(m)}$ with equal probability, one has $\mu(Z) \leq P(2^{-q(m)}, n)$. For $\omega \in Z$, we have the trivial estimate $\| \sum_{i \in I} f_i(\omega) \| \leq n$.

We show by induction over p that for $\omega \notin Z$ and $g \in H_p$, we have:

$$(16) \quad | \langle g, \sum_{i \in I} f_i(\omega) \rangle | \leq 2^{-p}(l+1).$$

The result is obvious for $p=0$. Assume it has been proved for p . Let $g \in H_{p+1}$. Then we have a decomposition $g = 1/4 \sum_{i \leq r \leq n} g(r)$ which satisfy (8) to (10). Let j be the largest integer $j \leq n$ for which $V(g(j)) < m$.

Let $g' = 1/4 \sum_{i \leq r \leq j} g(r)$. Then $g' = 4^{-p-1} \sum_{t \in A} e_t^*$ where $\sup \{ |t|; t \in A \} < m$. Since there are at most l indexes i for which $|a_i(\omega)| < m$ we have $| \langle g', \sum_{i \in I} f_i(\omega) \rangle | \leq 4^{-p-1} l$.

If $j=p$, the proof is finished. Otherwise $| \langle g(j+1), \sum_{i \in I} f_i(\omega) \rangle | \leq 2^{-p}(l+1)$ by induction hypothesis. If $j+1=p$, the proof is finished. Otherwise let $g'' = \sum_{r > j+1} g(r)$. It follows from condition (10) that there is $s \in T_m$ such that for each $t \in T$ one has $s < t$. But since there is at most one $i \in I$ for which $s < a_i(\omega)$, we have $| \langle g'', \sum_{i \in I} f_i \rangle | \leq 4^{-p-1}$. Adding these three estimates gives (16). It follows that for $g \in H$ one has:

$$| \langle g, \sum_{i \in I} f_i(\omega) \rangle | \leq \sup \left(1, \frac{l+1}{2} \right)$$

and hence $\left\| \sum_{i \in I} f_i(\omega) \right\| \leq \sup(1, (l+1)/2)$. So we have:

$$\begin{aligned} \left\| \sum_{i \in I} f_i \right\|_1 &\leq \int_Z \left\| \sum_{i \in I} f_i(\omega) \right\| d\mu(\omega) + \int_{\Omega \setminus Z} \left\| \sum_{i \in I} f_i(\omega) \right\| d\mu(\omega) \\ &\leq n \mu(Z) + \sup\left(1, \frac{l+1}{2}\right), \\ &\leq n P(2^{-q(m)}, n) + \sup\left(1, \frac{l+1}{2}\right). \end{aligned}$$

If $l=0$, we have $a_{n,k} \leq 2$, so $n P(2^{-q(m)}, n) \leq a_{n,k} - 1$ from (14) and since $q(m) \geq q(k)$, so the right hand side is $\leq a_{n,k}$. If $l \geq 1$, we have $n P(2^{-q(m)}, n) \leq (1/2)$ from (14), so the right hand side is less than $l/2 + 1 \leq l + 1 \leq a_{n,k}$ which concludes the proof of the theorem.

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