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FILTRATIONS ON VERMA MODULES

By Dan BARBASCH ⁽¹⁾

1. Introduction

The purpose of this note is to make several remarks on the Jantzen filtration that follow from the paper of O. Gabber and A. Joseph [G-J]. The first is to show that their techniques show that in fact the filtration has to coincide with the Socle filtration for Verma modules. The second is to apply their result to calculate Ext^1 between two irreducible Verma modules.

Since the publication of [G-J], the conjecture that the Jantzen filtration is hereditary has been proved by A. Beilinson and J. Bernstein (unpublished).

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2. Notation and preliminary results

We adopt the notation and conventions of [G-J] particularly section 4. We assume that the Verma modules all have fixed regular infinitesimal character ρ (to keep the notation to a minimum). We write $M(w)$ for $M(-w\rho)$ if $w \in W$, the Weyl group. Let α be a simple root, $s = s_\alpha$ the corresponding simple reflection and assume $ws > w$ in the Bruhat ordering. Fix an inclusion $M(w) \rightarrow M(ws)$. If $\{M^j\}$ is the Jantzen filtration, then the Jantzen conjecture states that

$$(2.1) \quad M(w)^j = M(ws)^{j+1} \cap M(w).$$

Set $M_j = M^j/M^{j+1}$. Let $\theta_\alpha = \varphi_\alpha \psi_\alpha$ be the coherent continuation functor. Write $X = M(ws)$, $Z = M(w)$ and $Y = \theta_\alpha X = \theta_\alpha Z$.

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Then there is an exact sequence

$$(2.2) \quad 0 \rightarrow X \xrightarrow{i} Y \xrightarrow{\pi} Z \rightarrow 0.$$

Y also has a canonical filtration satisfying the following properties ([G-J], section 4)

$$(2.3) \quad \begin{cases} Y^j = \theta_x Z^j = \theta_x X^{j+1}, \\ Y_j = \theta_x Z_j = \theta_x X_{j+1}, \end{cases}$$

$$(2.4) \quad \pi(Y^j) \subset Z^j,$$

$$(2.5) \quad \begin{aligned} Y^{j+1} \cap X &\subset X^{j+1} \subset Y^j, \\ Z^{j+1} &\subset \pi(Y^j). \end{aligned}$$

Then (cf. [G-J], section 4) one can set up the following exact sequences. Let

$$X_{j+1}^a = X^{j+1} / (Y^{j+1} \cap X),$$

$$X_{j+1}^b = (Y^{j+1} \cap X) / X^{j+2}.$$

Then

$$(2.7) \quad 0 \rightarrow X_{j+1}^b \rightarrow X_{j+1} \rightarrow X_{j+1}^a \rightarrow 0.$$

The sequence splits, X_{j+1} is completely reducible and X_{j+1}^b is the largest submodule so that $\theta_x(X_{j+1}^b) = 0$.

Let

$$Z_j^a = \pi(Y^j) / Z^{j+1}$$

$$Z_{j+1}^b = Z^{j+1} / \pi(Y^{j+1}).$$

Then

$$(2.8) \quad 0 \rightarrow Z_{j+1}^a \rightarrow Z_{j+1} \rightarrow Z_{j+1}^b \rightarrow 0.$$

Z_{j+1} is completely reducible and Z_{j+1}^a is the smallest submodule so that $\theta_x(Z_{j+1} / Z_{j+1}^a) = 0$.

Let

$$Y_j^x = (Y^j \cap X) / (Y^{j+1} \cap X),$$

$$Y_j^z = \pi(Y^j) / \pi(Y^{j+1}).$$

Then

$$(2.9.1) \quad 0 \rightarrow Y_j^x \rightarrow Y_j \rightarrow Y_j^z \rightarrow 0,$$

$$(2.9.2) \quad 0 \rightarrow X_{j+1}^a \rightarrow Y_j^x \rightarrow X_j^b \rightarrow 0,$$

$$(2.9.3) \quad 0 \rightarrow Z_{j+1}^b \rightarrow Y_j^z \rightarrow Z_j^a \rightarrow 0,$$

$$(2.10) \quad X_{j+1}^a \simeq Z_j^a.$$

Let now $v \in W$ be such that $vs > v$ and $L = L(v)$, the corresponding irreducible quotient of $M(v)$. Then there are exact sequences

$$(2.10.1) \quad 0 \rightarrow L \rightarrow \theta_\alpha L \rightarrow Q_\alpha \rightarrow 0,$$

$$(2.10.2) \quad 0 \rightarrow U_\alpha L \rightarrow Q_\alpha \rightarrow L \rightarrow 0,$$

which do not split.

Extending the definition of U_α to semisimple modules, we get an exact sequence

$$(2.11) \quad 0 \rightarrow X_j^b \rightarrow U_\alpha Z_j^a \rightarrow Z_{j+1}^b \rightarrow 0.$$

Since $Y_j = \theta_\alpha Z_j^a$ and θ_α is exact, we can write

$$(2.12) \quad Y_j = \bigoplus_{vs > v} [L(v) : Z_j^a] \theta_\alpha L(v).$$

LEMMA. — For $y \in W$ we have

$$\text{Hom}[L(y) : Y_j^a] = \begin{cases} \text{Hom}[L(y) : X_{j+1}^a] & \text{if } ys > y, \\ 0 & \text{if } ys < y. \end{cases}$$

Proof. — Suppose $ys > y$. Then $\text{Hom}[L(y) : X_j^b] = 0$. The long exact sequence

$$0 \rightarrow \text{Hom}[L(y) : X_{j+1}^a] \rightarrow \text{Hom}[L(y) : Y_j^a] \rightarrow \text{Hom}[L(y) : X_j^b] \rightarrow$$

gives the assertion.

Suppose $ys < y$ but $\text{Hom}[L(y) : Y_j^a] \neq 0$. Then $\text{Hom}[L(y) : Y_j] \neq 0$. By (2.12) there must be v such that $\text{Hom}[L(y) : \theta_\alpha L(v)] \neq 0$. But the only submodule of $\theta_\alpha L(v)$ is $L(v)$ itself and v must be such that $vs > v$, a contradiction. This proves the lemma.

We now proceed to define the Socle filtration. Given $M = M(w)$, it is well known that the largest semisimple submodule of M is $M(\text{id})$. We label $M^{(l(w))} = M(\text{id})$. Suppose we have defined $M^{[j+1]}$. Then we define $M^{[j]}$ as the largest submodule so that $M_{[j]} = M^{[j]} / M^{[j+1]}$ is semisimple. This is a well-defined construction and $\{M^{[j]}\}$ is called the Socle filtration of M .

3. The main theorem

THEOREM. — $M^{[j]} = M^j$.

Proof. — The proof goes by ascending induction in $l(w)$ and descending induction in j . It is enough to assume the statement to be true for Z and for X up to $j+3$ and to show it is true for $j+2$.

It is clear that $X^{j+2} \subseteq X^{[j+2]}$. Suppose $X^{j+2} \neq X^{[j+2]}$. Let $k \leq j+1$ be the largest integer such that

$$X^{k, j+2} = X^{[j+2]} \cap X^k / X^{[j+2]} \cap X^{k+1} \neq 0.$$

Let $y \in W$ be such that $[L(y): X^{k, j+2}] \neq 0$.

Then

$$(3.1) \quad 0 \rightarrow X^{[j+2]} \cap X^k / X^{[j+2]} \cap X^{k+2} \rightarrow X^k / X^{k+2}$$

and its image under the map π

$$(3.2) \quad 0 \rightarrow X^{k+1} / X^{k+2} \rightarrow X^k / X^{k+2} \xrightarrow{\pi} X^k / X^{k+1} \rightarrow 0$$

is nonzero.

Then

$$(3.3) \quad \text{Hom}[L(y): X^k / X^{k+2}] > \text{Hom}[L(y): X^{k+1} / X^{k+2}].$$

We show that this cannot be.

Case 1. — $y < y_s$.

$$\text{Hom}[L(y): X^k] = \text{Hom}[L(y): Z^{k-1}] \quad \text{since} \quad [L(y): X^k / Z^{k-1}] = 0.$$

By the induction hypothesis, the socle of Z / Z^{k+1} is Z^k / Z^{k+1} , which implies

$$(3.4) \quad \text{Hom}[L(y): Z^{k-1} / Z^{k+1}] = \text{Hom}[L(y): Z^k / Z^{k+1}].$$

This is a contradiction.

Case 2. — $y > y_s$.

Since $X^{k+2} \subset Y^{k+1} \subset Y^k$, we get $Y^k \cap X^{k+2} = X^{k+2}$.

Then we have an exact sequence

$$(3.5) \quad 0 \rightarrow (Y^k \cap X) / X^{k+2} \rightarrow X^k / X^{k+2} \rightarrow X^k / Y^k \cap X \rightarrow 0$$

But $X^k / Y^k \cap X = X_k^a$ and $[L(y): X_k^a] = 0$.

Thus

$$(3.6) \quad \text{Hom}[L(y): X^k / X^{k+2}] = \text{Hom}[L(y): (Y^k \cap X) / X^{k+2}].$$

Also

$$(3.7) \quad 0 \rightarrow (Y^{k+1} \cap X) / X^{k+2} \rightarrow (Y^k \cap X) / X^{k+2} \rightarrow (Y^k \cap X) / (Y^{k+1} \cap X) \rightarrow 0.$$

But $(Y^k \cap X) / (Y^{k+1} \cap X) = Y_k^x$ so by Lemma 2.13

$$(3.8) \quad \text{Hom}[L(y): (Y^k \cap X) / X^{k+2}] = \text{Hom}[L(y): (Y^{k+1} \cap X) / X^{k+2}] \leq \text{Hom}[L(y): X_{k+1}].$$

Thus, combining (3.6) and (3.8) we get

$$\text{Hom}[L(y): X^k / X^{k+2}] = \text{Hom}[L(y): X_{k+1}].$$

This contradicts (3.3) so

$$X^{[j+2]} = X^{j+2}.$$

The proof is now complete.

4. Computation of Ext^1

Let $y < w$. We compute $\text{Ext}^1[L(y), L(w)]$. Consider the exact sequences

$$(4.1) \quad 0 \rightarrow M(w)^1 \rightarrow M(w) \rightarrow L(w) \rightarrow 0,$$

$$(4.2) \quad 0 \rightarrow M(y)^1 \rightarrow M(y) \rightarrow L(y) \rightarrow 0$$

(4.1) gives rise to a long exact sequence

$$0 \rightarrow \text{Hom}[L(w), L(y)] \rightarrow \text{Hom}[M(w), L(y)] \rightarrow \text{Hom}[M^1(w), L(y)] \rightarrow \\ \rightarrow \text{Ext}^1[L(w), L(y)] \rightarrow \text{Ext}^1[M(w), L(y)] \rightarrow$$

But $\text{Hom}[M(w), L(y)] = 0$ because $L(w)$ is the unique irreducible quotient of $M(w)$ and $\text{Ext}^1[M(w), L(y)] = 0$ because it is equal to $H^1(n, L(y))^{w\rho - \rho}$ which is zero since $y < w$. Thus

$$(4.3) \quad \text{Ext}^1[L(w), L(y)] \simeq \text{Hom}[M^1(w), L(y)].$$

Next, consider the long exact sequence coming from (4.2)

$$0 \rightarrow \text{Hom}[L(y), L(w)] \rightarrow \text{Hom}[M(y), L(w)] \rightarrow \text{Hom}[M(y)^1, L(w)] \rightarrow \\ \rightarrow \text{Ext}^1[L(y), L(w)] \rightarrow \text{Ext}^1[M(y), L(w)] \rightarrow.$$

Since $\text{Hom}[M(y)^1, L(w)] = 0$

$$(4.4) \quad \dim \text{Ext}^1[L(y), L(w)] \leq \dim \text{Ext}^1[M(y), L(w)] = \mu(y, w)$$

(from the Kazhdan-Lusztig conjecture as phrased in [V], conjecture 3.4 and the fact that $\text{Ext}^1[M(y), L(w)] \simeq H^1[n, L(w)]^{y\rho - \rho}$).

Putting (4.3) and (4.4) together, we get

$$(4.5) \quad \dim \text{Hom}[M(w)^1, L(y)] \leq \dim \text{Ext}^1[L(w), L(y)] = \dim \text{Ext}^1[L(y), L(w)] \leq \mu(y, w).$$

By [G-J], Corollary 4.9, $\dim \text{Hom}[M^1(w), L(y)] \geq \mu(w w_0, y w_0)$.

By [K-L], Corollary 3.2, $\mu(y, w) = \mu(w w_0, y w_0)$. We summarize our result.

PROPOSITION. — *Let $y < w$. Then*

$$\dim \text{Ext}^1[L(y), L(w)] = \mu(y, w).$$

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