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SPECIAL VALUES OF ZETA FUNCTIONS ATTACHED TO SIEGEL MODULAR FORMS

By MICHAEL HARRIS (*)

Introduction

Let f be a cusp form of even weight k for the full Siegel modular group of degree $n \in 4\mathbf{Z}$; let ψ be a primitive Dirichlet character of conductor m; for simplicity we assume $\psi(-1)=1$. When f is an eigenform for all the associated Hecke operators, Andrianov and Kalinin have recently shown that, under certain restrictive conditions, a certain Dirichlet series with Euler product $L(f, s, \psi)$ attached to f and ψ can be extended to a meromorphic function on the entire complex s plane which (under still more restrictive hypotheses) satisfies a functional equation of the usual type (*cf.* [2]; their results are summarized in paragraph 5 below). We assume that the Fourier coefficients of f are algebraic numbers. The last Theorem of the present paper states that if $k>2n+1$, then, at each of the critical points (essentially in the sense of Deligne's paper [7]) of $L(f, s, \psi)$, at which $L(f, s, \psi)$ has no pole ([1]), the value of $L(f, s, \psi)$ is an *algebraic* multiple of $\pi^d \langle f, f \rangle_k$, where d is an integer depending only on k, s, and n, and $\langle\ ,\ \rangle_k$ is the Petersson inner product for modular forms of weight k. The proof gives an effective method for determining the field in which the algebraic number $L(f, s, \psi)/\pi^d \langle f, f\rangle_k$ lies.

The main object of this paper, however, is not to prove this Theorem, but rather to explain how the differential operators, originally introduced by Maaß, which have arisen in recent work of Shimura and Katz, as well as in the Andrianov-Kalinin paper, can be interpreted in terms both of representation theory and of algebraic geometry, and how these interpretations can be used to prove algebraicity Theorems of which the one mentioned in the last paragraph is a particular example. Other examples are the Theorems of Shimura on Rankin-Selberg type zeta functions for Hilbert modular forms ([34], [35], [37]) on which the arguments of this paper are loosely modeled (*cf.* [14]), and that of Sturm and Zagier ([46], [47]) on the symmetric square of the standard zeta function attached to a classical cusp form.

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([1]) When $k \gg n$ there are many such points.

What is common to all the results mentioned above is that, in each case, the special value of the zeta function is equal, up to a scalar in K_f multiplied by a power of π, to a Petersson inner product of the form $\langle f, h_1.\delta(h_2)\rangle_k$. Here h_1 and h_2 are holomorphic modular forms of weights l and λ respectively, and δ is a differential operator which takes modular forms of weight λ to (nonholomorphic) modular forms of weight $k-l$. Our basic argument consists in showing that:

$$\langle f, h_1.\delta(h_2)\rangle_k = \langle f, f_0\rangle_k,$$

where f_0 is *holomorphic* of weight k with Fourier coefficients in the field generated over $\mathbf{Q}$ by the Fourier coefficients of h_1 and h_2. This is done in three steps:

0.1. The forms f, h_1, and h_2 correspond respectively to functions φ, φ_1, and φ_2 on the group $G=\mathrm{Sp}(2n, \mathbf{R})$ which transform under the maximal compact subgroup:

$$K=\left\{\begin{pmatrix} A & -B \\ B & A\end{pmatrix}\in G \;\middle|\; A+B\,i \textit{ unitary}\right\},$$

by the formulas:

$$\varphi\left(g.\begin{pmatrix} A & -B \\ B & A\end{pmatrix}\right)=\det(A+B\,i)^{-k}\,\varphi(g),$$

$$\varphi_1\left(g.\begin{pmatrix} A & -B \\ B & A\end{pmatrix}\right)=\det(A+B\,i)^{-l}\,\varphi_1(g),$$

$$\varphi_2\left(g.\begin{pmatrix} A & -B \\ B & A\end{pmatrix}\right)=\det(A+B\,i)^{-\lambda}\,\varphi_2(g)\qquad g\in G,\ \begin{pmatrix} A & -B \\ B & A\end{pmatrix}\in K,$$

and, if $\Gamma=\mathrm{Sp}(2n, \mathbf{Z})\subset G$ and dg is the standard Haar measure on G:

$$\langle f, h_1.\delta(h_2)\rangle_k=\frac{1}{\mathrm{vol}(\Gamma\backslash G)}\int_{\Gamma\backslash G}\varphi.\overline{\varphi_1.D\varphi_2}\,dg \overset{def}{=} (\varphi, \varphi_1.D\varphi_2),$$

where D is a certain left-invariant operator on G.

0.2. We may write $\varphi_1.D\varphi_2$ as a finite sum of eigenfunctions $\varphi^{(i)}$ for $Z(\mathfrak{g}_{\mathbf{C}})$, the center of the universal enveloping algebra of the complexified Lie algebra of G. Now φ is an eigenfunction for $Z(\mathfrak{g}_{\mathbf{C}})$ with character χ_0; if the character χ_i of $Z(\mathfrak{g}_{\mathbf{C}})$ corresponds to $\varphi^{(i)}$, it follows from self-adjointness properties of $Z(\mathfrak{g}_{\mathbf{C}})$ that:

$$\langle f, h_1.\delta(h_2)\rangle_k=(\varphi, \varphi_1.D\varphi_2)=(\varphi, \varphi^{(0)}).$$

0.3. Finally, $\varphi^{(0)}$ corresponds (as in step 1) to a holomorphic modular form f_0 with the required rationality properties, and:

$$(\varphi, \varphi^{(0)})=\langle f, f_0\rangle_k.$$

Of these steps, the first is completely standard. The second depends on the decomposition of tensor products of "limits of holomorphic discrete series representations." as investigated, for example, by Jakobsen and Vergne in [16]. That δ, which is initially defined in terms of the Maaß operators, "comes from" the universal enveloping algebra follows directly from the transformation property of δ. The third step is based on an idea of Katz in [18]: We interpret the Maaß operators in terms of the *Gauss-Manin connection* on the relative algebraic de Rham cohomology of the universal family of abelian varieties over the Siegel upper half space. The rationality result is a consequence of this interpretation and the "q-expansion principle", Theorem 4.9. It should be mentioned that this part of the argument doesn't seem to work when either h_1 or h_2 is not of integral weight, as in the Sturm-Zagier example. However, in the one-dimensional case the differential operators are so explicit that the Fourier coefficients of f_0 can be shown directly to have the right rationality properties.

Several possible extensions of this theory should be noted:

0.4. Manin and Panchishkin have obtained results similar to those of Shimura ([34], [35]) by a different method, one which allows the estimation of how the p-adic absolute value of the "algebraic part" of the value of the Rankin-Selberg zeta function $D(s, f\chi, g)$ varies as $f\chi$ runs over the set of twists of f by Dirichlet characters χ of p-power conductor (*cf.* [24]). Since the enveloping algebra and the moduli space both have $\mathbf{Z}$-integral structures (as well as $\mathbf{Q}$-rational structures), it is possible that such results can be deduced by the methods of this paper.

0.5. The techniques used here are valid, in principle, for a very general class of modular forms; paragraph 7 contains a list of axioms which is probably sufficient to prove analogues of our Theorem 7.1. Of course, the zeta functions have yet to be defined in this degree of generality. The next candidates are the zeta functions attached by Shintani to holomorphic cusp forms for groups of type $U(2, 1)$ [39].

In the course of this paper, a number of artificially restrictive conditions are imposed upon our modular form. Some of these have no other motivation than the desire to avoid cluttering the final result with irrelevant notation. Others are required by the methods of Andrianov-Kalinin, in particular by the absence of detailed information on the analytic properties of Eisenstein series in the most general relevant case. It is enough to mention that most of the arguments in the body of the text depend entirely upon formal properties of the modular forms and differential operators in question, and that extensions of this method to higher level (for example) involve no new ideas.

The outline of this paper is as follows: paragraphs 1 and 2, which contain nothing original, review the theory of Siegel modular forms in the languages of [40] or [23], and [5] or [13], respectively. In paragraph 3 some scattered facts about tensor products of analytic continuations of discrete series representations are collected, and step 0.2 above is carried out. The principal novelty here is the use of a $\mathbf{Q}$-rational structure on the enveloping algebra. In paragraph 4 Siegel modular forms are investigated from the point of view of algebraic geometry; the Gauss-Manin connection is investigated, along the lines of [19], and a version of the "q-expansion principle" is proved. Section 5 is a résumé of the theory of Andrianov and Kalinin. The various differential operators introduced in the previous sections are compared in paragraph 6; paragraph 7 applies this work to derive the Theorem on special values alluded to above, and concludes with an axiomatic summary of what has gone before, with a view to future generalization. An appendix sketches two proofs of the most general q-expansion principle.

My acknowledgments are of two sorts. It goes without saying that this work depends on the previous efforts of many people, but a few should be mentioned specifically. The Theorem on special values should be regarded in the context of Deligne's conjecture [7] on the relations between special values of zeta functions and periods of integrals. Since I don't really understand what "motive" is attached to the zeta functions in question here, I can only say that Corollary 7.3 is not clearly inconsistent with Deligne's conjecture. More specifically, Theorem 7.1 and its proof are heavily indebted to the techniques introduced by Shimura in [34]. Shimura's work in other contexts plays an important role in paragraph 4, which is largely formulated, however, in a language based on that of Katz ([18], [19]). Of course, were it not for the work of Andrianov and Kalinin, the title of this paper would have been vacuous.

In the course of writing this paper, I have benefited from conversations with a number of mathematicians, whom I take this opportunity to thank. A remark of D. Kazhdan directed my attention initially to the significance of the canonical differential operators of paragraph 6. The derivation of Corollary 7.3 from Corollary 7.2 is based on a suggestion of G. Shimura. In preparing the final version of this paper, I was grateful for the suggestions of A. Mayer, J.-P. Serre, M. Vergne, and especially B. Mazur, who has encouraged me in this project since its inception. Most especially, I am grateful to H. P. Jakobsen, who patiently explained the theory of holomorphic representations to me; without his explanations, the theory of paragraph 3, and this paper, would never have come into being.

Notation

The symbols **Z**, **Q**, **R**, and **C** have their usual meaning, as does $\mathrm{GL}(n, \mathrm{R})$ for any ring R; $\mathrm{M}(n, \mathrm{R})$ is the algebra of $n \times n$ matrices with coefficients in R. The identity matrix is denoted I; its dimension will always be evident. The $2n \times 2n$ matrix J is given by:

$$\mathrm{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix};$$

then, for any ring R, $\mathrm{Sp}(2n, \mathrm{R})$ is the group of matrices $g \in \mathrm{M}(2n, \mathrm{R})$ such that $g\,\mathrm{J}\,g^t = \mathrm{J}$: Here g^t is the transpose of g. When such a g is written:

$$g = \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix},$$

A, B, C, and D will always denote $n \times n$ matrices. The determinant (resp. trace) of an $n \times n$ matrix g is written $\det g$ (resp. $\mathrm{Tr}\, g$).

The notation $e(z)$ denotes $e^{2\pi i z}$, for any complex number z. When G is a real matrix group and X is an element of its Lie algebra, then $\exp(X)$ is the corresponding element of G. Lie algebras are usually denoted by Gothic lower case letters.

If Y is a real symmetric square matrix, we write $Y>0$ (resp. $Y \geqq 0$) to indicate that Y is positive definite (resp. semidefinite).

If V is a vector space over a field L of characteristic zero, we write either $V^{\otimes n}$ or $\otimes_L^n V$ to denote the n-fold tensor product of V with itself; in the former case, the field is understood. The notation $\mathrm{Sym}^n V$ denotes the n-fold symmetric tensor product of V with itself; $\Lambda^n V$ the n-th exterior power of V. We let $V^{\otimes 0}=\mathrm{Sym}^0 V=\Lambda^0 V=L$. If X_1 and $X_2 \in V$, then define:

$$X_1 \circ X_2=\frac{1}{2}(X_1 \otimes X_2+X_2 \otimes X_1) \in \mathrm{Sym}^2 V ;$$

similarly for higher symmetric powers. The symmetric algebra on V is $S(V)=\bigotimes_{n=0}^{\infty} \mathrm{Sym}^n(V)$. In general, n can be a negative integer in the above notation; thus $V^{\otimes -n}$, for $n>0$, denotes $(V^*)^{\otimes n}$, where V^* is the dual space to V.

If X is a real manifold and V a complex vector space, $C^\infty(X, V)$ is the space of C^∞ V-valued functions on X. If X has a measure, then vol X is the volume of X with respect to that measure (usually implicit). If X is a complex manifold, or an algebraic variety, then $\mathcal{O}_X$ is its structure sheaf.

We denote by ζ_N a primitive N-th root of unity, for any integer $N>0$.

In paragraph 4, by a "section of the sheaf $\mathscr{S}$" we ordinarily mean a section of $\mathscr{S}$ over some open set, when the open set is not invoked explicitly.

If G is a group, a representation of G is denoted (ρ, V_ρ), where V_ρ is a vector space and ρ is a homomorphism from G to the group of automorphisms of V_ρ.

The symbol δ_{ij} is the Kronecker delta.

The field of all algebraic numbers, regarded as a subfield of $\mathbf{C}$, is denoted $\overline{\mathbf{Q}}$.

1. Review of Siegel modular forms

The basic references for this section are [23], [31], and [2].

1.0. We denote by $\mathfrak{S}_n$ the Siegel upper half-space of degree n:

$$\mathfrak{S}_n=\{Z=X+iY \in M(n, \mathbf{C}) \mid X, Y \in M(n, \mathbf{R}), X=X^t, Y=Y^t, Y>0\}.$$

The group $G=\mathrm{Sp}(2n, \mathbf{R})$ acts on $\mathfrak{S}_n$ in the usual way: If $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, $Z \in \mathfrak{S}_n$, then:

$$g(Z)=(AZ+B)(CZ+D)^{-1}.$$

1.0.1. The point iI belongs to $\mathfrak{S}_n$; the map $g \mapsto g(iI)$, $g \in G$, induces an isomorphism:

$$G/K \xrightarrow{\sim} \mathfrak{S}_n,$$

where K is the subgroup of G whose elements are matrices of the form $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. The group K is a maximal compact subgroup of G, and is canonically identified with the unitary group $U(n)$ by the map:

1.0.2.
$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto A + Bi.$$

1.0.3. For any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, $Z \in \mathfrak{S}_n$, we define the *canonical automorphy factor:*

$$J(g, Z) = CZ + D \in GL(n, \mathbf{C}).$$

If $\rho : GL(n, \mathbf{C}) \to GL(V_\rho)$ is a representation on some finite dimensional complex vector space V_ρ, we let $J_\rho(g, Z) = \rho(J(g, Z))$. For any ρ, J_ρ satisfies the cocycle condition:

1.0.4.
$$J_\rho(g_1 g_2, Z) = J_\rho(g_1, g_2 Z) \circ J_\rho(g_2, Z).$$

We note that, for $k \in K$, the map 1.0.2 identifies k with $J(k, iI)$.

1.0.5. Given a point $Z = X + iY \in \mathfrak{S}_n$, it is sometimes useful to know an explicit representative for Z in G under the isomorphism $G/K \simeq \mathfrak{S}_n$. We let $Y^{1/2}$ be the positive-definite symmetric matrix obtained by taking square roots of the eigenvalues of Y, satisfying $(Y^{1/2})^2 = Y$; then:

$$g_Z \overset{\text{def}}{=} \begin{pmatrix} Y^{1/2} & XY^{-1/2} \\ 0 & Y^{-1/2} \end{pmatrix}$$

is a representative for Z in G.

1.1. Let $\Gamma \subset Sp(2n, \mathbf{Z})$ be a subgroup of finite index. A *Siegel modular form of weight k for* Γ is a holomorphic function $f : \mathfrak{S}_n \to \mathbf{C}$ such that:

1.1.1.
$$f(\gamma Z) = (\det J(\gamma, Z))^k f(Z), \qquad \gamma \in \Gamma.$$

When $n = 1$ one adds the usual conditions of holomorphy at the cusps, which is automatic when $n > 1$.

More generally, if $\rho : GL(n, \mathbf{C}) \to GL(V_\rho)$ is a holomorphic representation, a Siegel modular form of type ρ for Γ is a holomorphic function $f : \mathfrak{S}_n \to V_\rho$ such that:

1.1.2.
$$f(\gamma Z) = J_\rho(\gamma, Z) f(Z), \qquad \gamma \in \Gamma.$$

If f is a C^∞-function satisfying 1.1.1 (resp. 1.1.2), we say f is a C^∞ modular form (2) of

(2) This is in contradiction with the usual terminology, which imposes a growth condition at infinity. In practice, this standard hypothesis will be satisfied, and this will be indicated. For *holomorphic* f, $n = 1$, this hypothesis is part of the definition.

weight k (resp. of type ρ) for Γ. The set of Siegel modular forms of weight k (resp. type ρ) for Γ is denoted $G_k(\Gamma)$ (resp. $G_\rho(\Gamma)$]; the C^∞ modular forms are denoted $G_k^\infty(\Gamma)$, $G_\rho^\infty(\Gamma)$.

We define, for any integer N, the subgroup $\Gamma(N) \subset \mathrm{Sp}(2n, \mathbf{Z})$ of matrices congruent to the identity modulo N; the *full modular group* is $\Gamma(1) = \mathrm{Sp}(2n, \mathbf{Z})$.

1.2. Let $M \in \mathbf{Z}$. We define:

$$\mathscr{S}_M = \{ N = N^t \in M(n, \mathbf{Q}) \mid N \geqq 0, \operatorname{Tr}(NN') \in \mathbf{Z}, \forall N' = N'^t \in M(n, \mathbf{Z}), N' \equiv 0 \pmod{M} \}.$$

Any $f \in G_\rho(\Gamma(M))$ has a Fourier expansion:

1.2.1. $$f = \sum_{N \in \mathscr{S}_M} a(N) e(\operatorname{Tr}(NZ)), \qquad a(N) \in V_\rho;$$

moreover $a(ANA^t) = a(A)$ for any $A \in SL(n, \mathbf{Z})$. If $a(N) \neq 0$ implies that $N > 0$, then f is a *cusp form;* the set of all cusp forms of weight k (resp. type ρ) for $\Gamma(M)$ is denoted $S_k(\Gamma(M))$ [resp. $S_\rho(\Gamma(M))$].

If $f_1 \in S_k(\Gamma(M))$, $f_2 \in G_k^\infty(\Gamma(M))$, we define the Petersson inner product of f_1 and f_2 to be:

1.2.2. $$\langle f_1, f_2 \rangle_k = \frac{1}{\operatorname{vol}(\mathscr{D})} \int_{\mathscr{D}} f_1(Z) \overline{f_2(Z)} (\det Y)^k \frac{dX\, dY}{(\det Y)^{n+1}},$$

whenever this integral converges absolutely; here $\mathscr{D}$ is a fundamental domain for $\Gamma(M)$ in $\mathfrak{S}_n$, X and Y are the standard coordinates on $\mathfrak{S}_n$, and $dX\, dY/(\det Y)^{n+1}$ is the G-invariant volume form on $\mathfrak{S}_n$.

1.4. We now introduce some specific Siegel modular forms which arise in Andrianov's theory. First, let $N \in \mathscr{S}_1$, $N > 0$ and let χ be a primitive Dirichlet character modulo some positive integer m. We define the theta-series attached to N and χ:

1.4.1. $$\Theta_{2N}(Z, \chi) = \sum_{M \in M(n, \mathbf{Z})} \chi(\det(M)) e(\operatorname{Tr}(MNM^t Z))$$

this is a modular form of weight $n/2$ for $\Gamma(2 \det(2N) m^2)$ (*cf.* [1], § 5).

Let m be as above; let:

$$\Gamma_0(m) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(1) \mid C \equiv 0 \pmod{m} \right\};$$

let:

$$\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma(1) \right\}.$$

For any $k \in \mathbf{Z}$, $s \in \mathbf{C}$, and any (not necessarily primitive) character χ modulo m, we define (formally) the Eisenstein series:

1.4.2. $$E_k(Z, s, \chi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(m)} \det(J(\gamma, Z))^{-k} \frac{\chi(d(\gamma))(\det Y)^s}{|\det J(\gamma, Z)|^{2s}},$$

where:

$$Z=X+iY, \qquad d(\gamma)=\det(D) \qquad \text{if} \quad \gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

This Eisenstein series converges absolutely when $k+\operatorname{Re}(2s)>n+1$ to an element of $G_k^{\chi}(\Gamma(m))$. When $s=0$ and $k>n+1$, k even, $E_k(Z,\chi)\overset{\text{def}}{=}E_k(Z,0,\chi)$ belongs to $G_k(\Gamma(m))$, and its Fourier coefficients are *rational* if $m=1$, *cyclotomic* in general ([40], [51]).

For any k, χ, $E_k(Z, s, \chi)$ can be continued to a meromorphic function in s which satisfies a functional equation if $m=1$ ([2]; Prop. 3.2). The poles of $E_k(Z, s, \chi)$ have yet to be completely determined (but *cf*. Prop. 3.2 of [2]); in any event, we will mainly be concerned with those s for which the defining series converges.

1.5. The $E_k(Z, s, \chi)$ for different (k, s), are connected with one another by a differential operator which was first introduced by Maaß in [22], and which will reappear in various guises throughout this paper.

Let d/dZ be the matrix $(((1+\delta_{ij})/2).(\partial/\partial Z_{ij}))$, where the subscript ij refers to the matrix entry in $\mathfrak{S}_n$. For any $\alpha\in\mathbf{R}$, $\alpha\geqq 0$, we define a differential operator on $\mathfrak{S}_n$, following Maaβ, [23]:

1.5.1. $M_\alpha=M_\alpha(Z)=\det(Z-\overline{Z})^{((n+1)/2)-\alpha}\det\left(\dfrac{d}{dZ}\right)\det(Z-\overline{Z})^{\alpha-((n-1)/2)}.$

We now define:

$$\delta_\alpha=\delta_\alpha(Z)=\left(-\frac{1}{4\pi}\right)^n(\det Y)^{-1}M_\alpha.$$

It follows immediately from the results of Maaß on M_α in [23], § 19, that if:

$$\varepsilon_n(\alpha)=\alpha\left(\alpha-\frac{1}{2}\right)\dots\left(\alpha-\frac{n-1}{2}\right),$$

1.5.2. $$\delta_\alpha E_\alpha(Z, s, \chi)=\frac{\varepsilon_n(\alpha)}{(-4\pi)^n}E_{\alpha+2}(Z, s-1, \chi)$$

and, if we write:

$$(f\,|\,g_\alpha)(Z)=\det(J(g, Z))^{-\alpha}f(gZ), \qquad g\in G,\ f\in C^\infty(\mathfrak{S}_n, \mathbf{C}),$$

that:

1.5.3. $$\delta_\alpha(f\,|\,g_\alpha)=(\delta_\alpha f)\,|\,g_{\alpha+2}, \qquad f\in C^\infty(\mathfrak{S}_n, \mathbf{C}), \quad g\in G.$$

In particular, for $k\in\mathbf{Z}$, δ_k induces a map:

$$\delta_k:\ G_k^\infty(\Gamma)\to G_{k+2}^\infty(\Gamma)$$

for any subgroup $\Gamma\subset\Gamma(1)$ of finite index.

1.6. For any holomorphic representation (ρ, V_ρ) of $GL(n, \mathbf{C})$, we define the G-homogeneous vector bundle $\mathscr{E}_\rho$ over $\mathfrak{S}_n$:

1.6.1. $\mathscr{E}_\rho = \mathfrak{S}_n \times V_\rho$, with G action defined by:

$$g(Z, v) = (g(Z), J_\rho(g, Z)v); \qquad g \in G, \quad Z \in \mathfrak{S}_n, \quad v \in V_\rho.$$

That this is indeed a G-action follows from the cocycle condition 1.0.4. For any subgroup $\Gamma \subset G$, $\mathscr{E}_\rho$ descends to a bundle denoted $\mathscr{E}_\rho(\Gamma)$, over $\Gamma \backslash \mathfrak{S}_n$. Then there are isomorphisms, assuming Γ acts without fixed points on $\mathfrak{S}_n$:

1.6.2. $$H^0_\infty(\Gamma \backslash \mathfrak{S}_n, \mathscr{E}_\rho(\Gamma)) \overset{\text{def}}{=} \left\{ \begin{matrix} \text{Global } C^\infty \text{ sections} \\ \text{of } \mathscr{E}_\rho(\Gamma) \text{ over } \Gamma \backslash \mathfrak{S}_n \end{matrix} \right\} \simeq G^\infty_\rho(\Gamma).$$

1.6.3. $$H^0(\Gamma \backslash \mathfrak{S}_n, \mathscr{E}_\rho(\Gamma)) \overset{\text{def}}{=} \left\{ \begin{matrix} \text{Global holomorphic sections} \\ \text{of } \mathscr{E}_\rho(\Gamma) \text{ over } \Gamma \backslash \mathfrak{S}_n \end{matrix} \right\} \simeq G_\rho(\Gamma).$$

Here it is assumed that Γ is of finite index in $\Gamma(1)$; when $n=1$ we impose the usual cuspidal condition on the left-hand side in 1.6.3.

When $\rho = \det^k$, we write $\mathscr{E}_k$, $\mathscr{E}_k(\Gamma)$ instead of $\mathscr{E}_\rho$, $\mathscr{E}_\rho(\Gamma)$.

1.6.4. By the very definition of $\mathscr{E}_\rho$, we see that each vector $v \in V_\rho$ gives rise to a global "constant" section $\tilde{v}$ of $\mathscr{E}_\rho$ over $\mathfrak{S}_n$: At every point $Z \in \mathfrak{S}_n$, $\tilde{v}(Z) = (Z, v)$ in the trivialization 1.6.1.

2. Lifting to the group

2.0. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{k}$ that of K. We write the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset k$. Let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$; define $\mathfrak{k}_{\mathbf{C}}$, $\mathfrak{p}_{\mathbf{C}}$, etc., correspondingly. The adjoint action of K on $\mathfrak{g}_{\mathbf{C}}$ induces a decomposition:

2.0.1. $$\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

as follows: We identify K with $U(n)$ as in 1.0.2. The inclusion $U(n) \to GL(n, \mathbf{C})$ is called the *standard representation* of K, and is denoted. St: it extends to the identity representation $K_{\mathbf{C}} \overset{\text{def}}{=} GL(n, \mathbf{C}) \to GL(n, \mathbf{C})$, also denoted $\underline{\mathrm{St}}$. The dual of $\underline{\mathrm{St}}$, denoted $\underline{\mathrm{St}}^*$, takes the matrix $k = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K$ to the matrix $A - Bi \in GL(n, \mathbf{C})$. Now 2.0.1 can be defined by requiring that, under the adjoint representation of K:

$$\mathfrak{p}^+ \simeq \mathrm{Sym}^2(\underline{\mathrm{St}}^*), \qquad \mathfrak{p}^- \simeq \mathrm{Sym}^2(\underline{\mathrm{St}}).$$

We may represent $\mathfrak{p}^+ \subset \mathfrak{g}_{\mathbf{C}} \subset M(2n, \mathbf{C})$ [resp. $\mathfrak{p}^- \subset M(2n, \mathbf{C})$] as the set of matrices of the form:

2.0.2. $$\left\{ \begin{array}{l} p_+(\alpha) = \dfrac{1}{2}\begin{pmatrix} \alpha & i\alpha \\ i\alpha & -\alpha \end{pmatrix}, \qquad \alpha = \alpha^t \in M(n, \mathbf{C}), \\ \left[\text{resp. } p_-(\alpha) = \dfrac{1}{2}\begin{pmatrix} \alpha & -i\alpha \\ -i\alpha & -\alpha \end{pmatrix}, \ \alpha = \alpha^t \in M(n, \mathbf{C}) \right]. \end{array} \right.$$

If k is as above, then:

$$\mathrm{Ad}\,k\,(p_+(\alpha))=p_+((\mathrm{A}-\mathrm{B}\,i)\,\alpha\,(\mathrm{A}^t-\mathrm{B}^t\,i))$$
$$\mathrm{Ad}\,k\,(p_-(\alpha))=p_-((\mathrm{A}+\mathrm{B}\,i)\,\alpha\,(\mathrm{A}^t+\mathrm{B}^t\,i)).$$

We note that $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are *Abelian Lie subalgebras* of $\mathfrak{g}_{\mathbf{C}}$.

2.1.0. We identify $\mathfrak{k}_{\mathbf{C}}$ with the set of matrices;

2.1.1. $$\left\{\begin{array}{l} p_0(\alpha,\beta)=\begin{pmatrix}\alpha & -i\beta\\ i\beta & \alpha\end{pmatrix},\\ \alpha^t=-\alpha,\qquad \beta^t=\beta,\qquad \alpha,\beta\in\mathrm{M}(n,\mathbf{C}).\end{array}\right.$$

Let $\mathfrak{k}_{\mathbf{Q}}$ be the set of $p_0(\alpha,\beta)$ with $\alpha,\beta\in\mathrm{M}(n,\mathbf{Q})$, as in 2.1.1; Let:

$$\tilde{\mathfrak{p}}_{\mathbf{Q}}^+\ (\text{resp. } \tilde{\mathfrak{p}}_{\mathbf{Q}}^-)=\{p_+(\alpha)\,|\,\alpha\in\mathrm{M}(n,\mathbf{Q}),\ \alpha=\alpha^t\}\ (\text{resp. } p_-(\alpha)\,|\,\alpha\in\mathrm{M}(n,\mathbf{Q}),\ \alpha=\alpha^t].$$

Then $\tilde{\mathfrak{g}}_{\mathbf{Q}}=\mathfrak{k}_{\mathbf{Q}}\oplus\tilde{\mathfrak{p}}_{\mathbf{Q}}^+\oplus\tilde{\mathfrak{p}}_{\mathbf{Q}}^-$ is a $\mathbf{Q}$-Lie algebra such that $\tilde{\mathfrak{g}}_{\mathbf{Q}}\otimes_{\mathbf{Q}}\mathbf{C}=\mathfrak{g}_{\mathbf{C}}$. Under the linear transformation $\mathrm{X}\to\mathrm{CXC}^{-1}$, where:

$$\mathrm{C}=\begin{pmatrix}-i\mathrm{I} & \mathrm{I}\\ i\mathrm{I} & i\mathrm{I}\end{pmatrix},$$

$\tilde{\mathfrak{g}}_{\mathbf{Q}}$ is taken to the Lie algebra $\mathfrak{sp}(2n,\mathbf{Q})$ of $\mathrm{Sp}(2n,\mathbf{Q})$, embedded in the canonical way in $\mathfrak{sp}(2n,\mathbf{C})$.

2.2. Let (ρ,V_ρ) be any holomorphic representation of $\mathrm{K}_{\mathbf{C}}=\mathrm{GL}(n,\mathbf{C})$; $\rho:\mathrm{K}_{\mathbf{C}}\to\mathrm{GL}(\mathrm{V}_\rho)$. If $\varphi\in\mathrm{C}^\infty(\mathrm{G},\mathrm{V}_\rho)$ satisfies:

2.2.1. $$\varphi(gk)=\rho(k)^{-1}\varphi(g),\qquad k\in\mathrm{K},\quad g\in\mathrm{G},$$

we say φ is of type ρ with respect to K. The space of all such φ is denoted $\mathrm{C}^\infty(\mathrm{G},\mathrm{V}_\rho)_\rho$. If $\rho=\det^\lambda$ for some integer λ, we say φ is of type λ, and write V_λ, $\mathrm{C}^\infty(\mathrm{G},\mathrm{V}_\lambda)_\lambda$, etc., in place of V_ρ, etc.

If $f\in\mathrm{C}^\infty(\mathfrak{S}_n,\mathrm{V}_\rho)$, define:

2.2.2. $$\varphi_f(g)=\varphi_{f,\rho}(g)=\mathrm{J}_\rho(g,i\mathrm{I})^{-1}f(g(i\mathrm{I})),\qquad g\in\mathrm{G};$$

then $\varphi_f\in\mathrm{C}^\infty(\mathrm{G},\mathrm{V}_\rho)_\rho$. Conversely, given any $\varphi\in\mathrm{C}^\infty(\mathrm{G},\mathrm{V}_\rho)_\rho$, we may define.

2.2.3. $$f_\varphi(\mathrm{Z})=f_{\varphi,\rho}(\mathrm{Z})=\mathrm{J}_\rho(g,i\,\mathrm{I})\,\varphi(g),\qquad \mathrm{Z}\in\mathfrak{S}_n,$$

where $g\in\mathrm{G}$ is any element such that $g(i\mathrm{I})=\mathrm{Z}$; evidently f_φ is well defined. This correspondence, denoted simply $f\leftrightarrow\varphi$, is a one-to-one correspondence:

$$\mathrm{C}^\infty(\mathfrak{S}_n,\mathrm{V}_\rho)\leftrightarrow\mathrm{C}^\infty(\mathrm{G},\mathrm{V}_\rho)_\rho.$$

We make the following well-known observation:

2.2.4. Let $\Gamma \subset \Gamma(1)$ be a discrete subgroup of finite index. Then, if $f \leftrightarrow \varphi$:

$$f \in G_\rho^\infty(\Gamma) \Leftrightarrow \varphi \in C^\infty(\Gamma \backslash G, V_\rho)_\rho,$$

where the latter condition signifies, in addition to 2.2.1, that $\varphi(\gamma g)=\varphi(g)$ for all $\gamma \in \Gamma$, $g \in G$.

If $\varphi \in C^\infty(G, V)$ for any complex vector space V, and if $X \in \mathfrak{g}$, we define:

2.2.5. $$X \star \varphi = \frac{d}{dt} f(g \exp(tX))\,|_{t=0}.$$

This action extends linearly to $\mathfrak{g}_{\mathbf{C}}$, and to its enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$.

2.3. Proposition. — *The function f is holomorphic on $\mathfrak{S}_n$ if and only if $X \star \varphi = 0$ for all $X \in \mathfrak{p}^-$ (we than say φ is of* holomorphic *type*).

Proof. — We first observe that this is essentially Lemma 5.7 of [4]; we sketch the proof briefly.

By the product rule:

$$X \star \varphi(g) = (X \star (J_\rho(g, i\mathrm{I}))^{-1}) f(g(i\mathrm{I})) + J_\rho(g, i\mathrm{I})^{-1} X \star f(g(i\mathrm{I})).$$

Now one can compute directly, using the methods of 6.2 below, that $X \star J_\rho(g, i\mathrm{I})^{-1} = 0$ for all $X \in \mathfrak{p}^-$: One checks this first for $\rho = \underline{\mathrm{St}}$, then uses the chain rule for general ρ. It thus suffices to check that:

2.3.1. f is holomorphic $\Leftrightarrow X \star f(g(i\mathrm{I}))=0$, for all $X \in \mathfrak{p}^-$ we may as well assume $V_\rho = \mathbf{C}$.

Let $X_{kj} = \mathfrak{p}_-(\alpha_{kj})$, where $\alpha_{kj} \in M(n, \mathbf{C})$ has zeroes in all except the kj and jk places, and where the kj and jk entries are $(1+\delta_{kj})/2$:

2.3.2. $$\alpha_{kj} \left\{ \begin{array}{l} = \begin{pmatrix} & \frac{1}{2} \\ \frac{1}{2} & \end{pmatrix} k \neq j \\ = (\ 1\ )\ k = j. \end{array} \right\} k, j = 1, \ldots, n.$$

Let $\overline{D} f$ be the symmetric $n \times n$ matrix whose kj entry is $(1+\delta_{kj})/2\,(\partial f / \partial \overline{Z}_{kj})$, where Z_{kj} is the coordinate in $\mathfrak{S}_n$.

Since an analogous computation will be carried out in paragraph 6 for $\mathfrak{p}^+$, the following computation will be merely sketched. First, to verify the right hand assertion of 2.3.1, it is enough to check the case $g = g_Z$ for $Z = X + iY$ (*cf.* 1.0.5). One then computes that:

2.3.3. $$\frac{1}{2i} X_{kj} \star f(g_Z(i\mathrm{I})) = \{ kj \text{ coordinate of the matrix } Y^{1/2}\, \overline{D} f\, Y^{1/2} \}.$$

Since $Y^{1/2}$ is an invertible matrix, 2.3.1, and consequently the proposition, follow from 2.3.3.

2.4. We mentioned in 1.6 that the elements $f \in G_\rho^\infty(\Gamma)$ can be identified with global C^∞ sections $\hat{f}$ of the holomorphic vector bundle $\mathscr{E}_\rho(\Gamma)$ over $\Gamma\backslash\mathfrak{S}_n$. We may construct $\mathscr{E}_\rho$ in another way: Let K act on the product $G \times V_\rho$ by the formula:

$$2.4.1. \qquad (g, v)k = (gk, \rho(k)^{-1}v); \qquad g \in G, \quad k \in K, \quad v \in V_\rho.$$

The quotient $(G \times V_\rho)/K$, with G acting on the first factor, is then easily seen to be equivariantly equivalent to the homogeneous vector bundle $\mathscr{E}_\rho$ on $\mathfrak{S}_n$; $\Gamma\backslash G \times V_\rho/K$ is equivalent to $\mathscr{E}_\rho(\Gamma)$. The formula 2.4.1 makes manifest the correspondence of 1.6.2 between $\varphi \in C^\infty(\Gamma\backslash G, V_\rho)_\rho$ and global sections $\hat{f} \in H_\infty^0(\Gamma\backslash\mathfrak{S}_n, \mathscr{E}_\rho(\Gamma))$. Of course φ is of holomorphic type if and only if $\hat{f}$ is a holomorphic section (by Prop. 2.3). We summarize our three one-to-one correspondences:

2.5. Proposition. — *Let Γ be a discrete subgroup of* G, *of finite index in* $\Gamma(1)$. *There is a one-to-one correspondence between:*

(*a*) *functions* $f \in G_\rho^\infty(\Gamma)$ [resp., $f \in C^\infty(G, V_\rho)$];

(*b*) *functions* $\varphi \in C^\infty(\Gamma\backslash G, V_\rho)_\rho$ [resp., $\varphi \in C^\infty(G, V_\rho)_\rho$];

(*c*) *sections* $\hat{f} \in H_\infty^0(\Gamma\backslash\mathfrak{S}_n, \mathscr{E}_\rho(\Gamma))$ (*resp.*, C^∞ *sections* $\hat{f}$ of $\mathscr{E}_\rho$ *over* $\mathfrak{S}_n$). *This correspondence is compatible with tensor products, direct sums, and duality. Finally* f *is holomorphic* $\Leftrightarrow$ φ *is of holomorphic type* $\Leftrightarrow$ $\hat{f}$ *is holomorphic.*

The correspondences are symbolized $f \leftrightarrow \varphi$, $f \leftrightarrow \hat{f}$, $\varphi \leftrightarrow f$, $\hat{f} \leftrightarrow \varphi$, etc.

2.6. We now suppose $\Gamma = \Gamma(M)$, for some integer $M > 0$. Let dg be the left-invariant measure on G such that, for all $f \in L^1(G/K)$:

$$\int_G f\, dg = \int_{\mathfrak{S}_n} f \frac{dX\, dY}{(\det Y)^{n+1}}.$$

Let $f_1 \in S_k(\Gamma)$, $f_2 \in G_k^\infty(\Gamma)$, $f_i \leftrightarrow \varphi_i$. Then we have the formula, whenever the integrals involved converge absolutely:

$$2.6.1. \quad \langle f_1, f_2 \rangle_k = \frac{1}{\operatorname{vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \varphi_1(g)\overline{\varphi_2(g)}\, dg \overset{\text{def}}{=} (\varphi_1, \varphi_2) \ (L^2 \text{ inner product}).$$

This interpretation has the following advantage: The center $Z(\mathfrak{g}_{\mathbf{C}})$ of $U(\mathfrak{g}_{\mathbf{C}})$ is spanned over $\mathbf{C}$ by elements ζ which are *self-adjoint* with respect to (,) (*cf.* [43], pp. 268-269): that is,

$$2.6.2. \qquad (\zeta \star \varphi_1, \varphi_2) = (\varphi_1, \zeta \star \varphi_2), \qquad \forall\, \varphi_1, \varphi_2 \in L^2(\Gamma\backslash G).$$

2.6.3. Lemma. — *Suppose* f_1 *is a (holomorphic) cusp form for* Γ *of weight* k, $f_1 \leftrightarrow \varphi_1'$. *Let* $f_2 \in G_k^\infty(\Gamma)$, $f_2 \leftrightarrow \varphi_2'$ *such that* φ_2' *is an automorphic form in the sense of Harish-Chandra* [5]: *i.e.,* φ_2' *is* $Z(\mathfrak{g}_{\mathbf{C}})$*-finite and slowly increasing at infinity. Suppose* $\zeta \in Z(\mathfrak{g}_{\mathbf{C}})$ *satisfies* 2.6.2. *Then:*

$$(\zeta \star \varphi_1', \varphi_2') = (\varphi_1', \zeta \star \varphi_2').$$

Proof. — It follows from Chapter I of [13] (*cf.* especially Lemma 14 and the proof of Lemma 15) that both $F_1=(\zeta\star\varphi'_1).\varphi'_2$ and $F_2=\varphi'_1.(\zeta\star\varphi'_2)$ are *bounded* on $\Gamma\backslash G$. Let $U_1\subset U_2\subset\ldots\subset U_n\subset\ldots\subset\Gamma\backslash G$ be a sequence of relatively compact open subsets of $\Gamma\backslash G$ such that $\bigcup_n U_n=\Gamma\backslash G$. Since $\Gamma\backslash G$ has finite volume, it follows that:

$$(\zeta\star\varphi'_1,\varphi'_2)=\lim_{n\to\infty}\int_{U_n}F_1,$$

$$(\varphi'_1,\zeta\star\varphi'_2)=\lim_{n\to\infty}\int_{U_n}F_2.$$

But since the function obtained by cutting off φ'_2 outside U_n is in $L^2(\Gamma\backslash G)$, it follows from 2.6.2 that, for each n, $\int_{U_n}F_1=\int_{U_n}F_2$. The Lemma follows immediately.

2.6.4. COROLLARY. — *Under the above hypotheses, let φ'_1 be an eigenfunction for $Z(\hat{\mathfrak{g}}_Q)$, the center of $U(\hat{\mathfrak{g}}_Q)$, with character χ_1. Let $\varphi'_2=\sum_{i=1}^{n}\varphi_{2,i}$, where $\varphi_{2,i}$ is an eigenfunction with character χ_i for $Z(\hat{\mathfrak{g}}_Q)$, such that $\chi_i\neq\chi_j$ for $i\neq j$. Then:*

$$(\varphi'_1,\varphi'_2)=(\varphi'_1,\varphi_{2,1}).$$

Proof. — We have only to note that, since $Z(\tilde{\mathfrak{g}}_Q)$ and the space of self-adjoint elements in $Z(\mathfrak{g}_C)$ both span $Z(\mathfrak{g}_C)$ over $\mathbf{C}$, two characters which agree on one of these subalgebras necessarily agree on the other.

2.7. Let $U=U(\mathfrak{g}_C)$. We have defined an action of U on $C^\infty(G, V)$ for any complex vector space V. Now let f be a holomorphic automorphic form of type ρ for Γ, and let $f\leftrightarrow\varphi\in C^\infty(G,V_\rho)_\rho$. The cyclic *representation generated by* f is the representation $(\pi_f,\mathscr{V}_f)$ of U on the space of functions:

$$\mathscr{V}_f=\{\Delta\star\varphi\mid\Delta\in U\}\subset C^\infty(G,V_\rho).$$

In the language of [42 *a*], $(\pi_f,\mathscr{V}_f)$ is an admissible $(\mathfrak{g}, K)$-module, which is to say that K acts on $\mathscr{V}_f$ (by right translation) in a manner compatible with the action of $\mathfrak{g}$, and that the representation is K-finite: Every vector $v\in\mathscr{V}_f$ is contained in a finite dimensional K-invariant subspace.

Assume ρ is an irreducible representation of K_C. Let V_0 be the smallest K-invariant subspace of $\mathscr{V}_f$ containing φ. One checks, using Proposition 2.3, that:

2.7.1. $$X\star v_0=0,\qquad X\in\mathfrak{p}^-,\quad v_0\in V_0.$$

2.7.2. The representation of K on V_0 is equivalent to the representation (ρ^*, V^*_ρ) contragredient to (ρ, V_ρ).

We say that V_0 is the *highest K-type* subspace of $\mathscr{V}_f$ (*sic*).

It follows from 2.7.1 and 2.7.2 that:

2.8. LEMMA. — *As a* $U(\mathfrak{g}_C)$*-module,* $\mathcal{V}_f$ *is isomorphic to a quotient of:*

$$D_\rho = U(\mathfrak{g}_C) \otimes_{U(\mathfrak{k}_C \oplus \mathfrak{p}^-)} V_\rho^*;$$

here $\mathfrak{k}_C$ *operates on* V_ρ^* *by the (differential of the) representation* ρ^*, *and* $\mathfrak{p}^-$ *acts trivially on* V_ρ^*.

The properties of the representations D_ρ are discussed in the next section.

3. Properties of antiholomorphic representations

3.0. When ρ is a representation of K_C, we frequently denote by ρ the corresponding representation of $\mathfrak{k}_C$ as well.

We have identified $\mathfrak{k}_C$ with $\mathfrak{gl}(n, \mathbf{C})$, in paragraph 2. Let $\mathfrak{h}$ be the diagonal Cartan subalgebra of $\mathfrak{k}_C$; $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}_C$. We identify characters of $\mathfrak{h}$ in the usual way with n-tuples of complex numbers. We may thus index the finite dimensional irreducible representations ρ of K_C by n-tuples of integers $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n$, where $(\alpha_1, \ldots, \alpha_n)$, as a character of $\mathfrak{h}$, is the highest weight of ρ relative to the standard ordering of the roots of $\mathfrak{h}$ in $\mathfrak{k}_C$. With this indexing, ρ^* corresponds to $-\alpha_n \geqq -\alpha_{n-1} \geqq \ldots \geqq -\alpha_1$. For example, the adjoint action of K_C on $\mathfrak{p}^-$ (resp. $\mathfrak{p}^+$) corresponds to the n-tuple $2 \geqq 0 \geqq \ldots \geqq 0$ (resp. $0 \geqq \ldots \geqq 0 \geqq -2$).

Let $\mathfrak{b}_0 \supset \mathfrak{h}$ be the Borel subalgebra of $\mathfrak{k}_C$ corresponding to the above ordering of the roots; then $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{p}^-$ is a Borel subalgebra of $\mathfrak{g}_C$. The character δ = half the sum of the positive roots of $\mathfrak{g}_C$ (with respect to $\mathfrak{h}$ and $\mathfrak{b}$) corresponds to the n-tuple $(n, n-1, \ldots, 2, 1)$.

We shall make frequent use of the following proposition, which is a special case of a result of Schmid:

3.1. PROPOSITION [30]. — *The adjoint representation of* K_C *on* $S(\mathfrak{p}^+) = U(\mathfrak{p}^+)$ *decomposes into the direct sum of the finite-dimensional representations corresponding to n-tuples* $\alpha_1 \geqq \ldots \geqq \alpha_n$ *where* $\alpha_1 \leqq 0$ *and each* α_i *is an even integer. Each such representation occurs with multiplicity one. Finally, the representation corresponding to* $\alpha_1 \geqq \ldots \geqq \alpha_n$ *occurs in* $\mathrm{Sym}^m(\mathfrak{p}^+)$ *where:*

$$m = -\frac{1}{2}(\alpha_1 + \ldots + \alpha_n).$$

3.2. In 2.7 we defined the representation D_ρ, for any finite-dimensional holomorphic representation (ρ, V_ρ) of K_C. We call D_ρ the *antiholomorphic representation* of $U = U(\mathfrak{g}_C)$ with highest K-type ρ^*. As a representation of K, $D_\rho = S(\mathfrak{p}^+) \otimes_C V_{\rho^*}$. When $\rho = \det^k$, we write D_k instead of D_ρ. It follows from Proposition 3.1 that.

3.2.1. The representations of K_C which occur in D_k are those of the form $-k+\alpha_1 \geqq \ldots \geqq -k+\alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are as in 3.1; furthermore, each such representation occurs with multiplicity one.

Since every holomorphic representation ρ^* of K_C corresponds to a rational representation $(\rho^*(\mathbf{Q}), V_{\rho^*(\mathbf{Q})})$ of $\mathfrak{k}_\mathbf{Q}$, unique up to isomorphism, we may define in the obvious way:

$$\tilde{D}_{\rho, \mathbf{Q}} = U(\tilde{\mathfrak{g}}_\mathbf{Q}) \otimes_{U(\mathfrak{k}_\mathbf{Q} \oplus \tilde{\mathfrak{p}}^-_\mathbf{Q})} V_{\rho^*}(\mathbf{Q});$$

then $D_\rho \simeq \tilde{D}_{\rho, \mathbf{Q}} \otimes_\mathbf{Q} \mathbf{C}$.

More generally, if β is a nonzero complex number, we let $\mathfrak{g}_{\mathbf{Q},\beta}=\mathfrak{k}_{\mathbf{Q}}\oplus\beta\,\tilde{\mathfrak{p}}_{\mathbf{Q}}^{+}\oplus\beta^{-1}\,\tilde{\mathfrak{p}}_{\mathbf{Q}}^{-}$. Then $\mathfrak{g}_{\mathbf{Q},\beta}$ is a $\mathbf{Q}$-form of $\mathfrak{g}_{\mathbf{C}}$. We define $\mathrm{D}_{\rho,\mathbf{Q},\beta}$ in the obvious way, and let $\mathrm{U}_{\beta}=\mathrm{U}(\mathfrak{g}_{\mathbf{Q},\beta})$, $\mathrm{Z}_{\beta}=$ the center of U_{β}.

3.3. Lemma. — *As a subalgebra of* $\mathrm{U}(\mathfrak{g}_{\mathbf{C}})$, Z_{β} *is independent of the choice of* β.

Proof. — Choose $\mathbf{Q}$-bases $\{\mathrm{X}_1, \ldots, \mathrm{X}_d\}$, $\{\mathrm{Y}_n, \ldots, \mathrm{Y}_d\}$ for $\tilde{\mathfrak{p}}_{\mathbf{Q}}^{+}$ and $\tilde{\mathfrak{p}}_{\mathbf{Q}}^{-}$, respectively. Then $\{\beta\mathrm{X}_1, \ldots, \beta\mathrm{X}_d\}$ (resp. $\{\beta^{-1}\mathrm{Y}_1, \ldots, \beta^{-1}\mathrm{Y}_d\}$) is a $\mathbf{Q}$-basis for $\beta\,\tilde{\mathfrak{p}}_{\mathbf{Q}}^{+}$ (resp. $\beta^{-1}\,\tilde{\mathfrak{p}}_{\mathbf{Q}}^{-}$). It is well known (*cf.* [31], pp. 10-05, 10-06) that any element $\zeta\in\mathrm{Z}_{\beta}$ can be written:

3.3.1. $$\zeta=\mathrm{K}_{0,\,0}+\sum_{m,\,n}(\beta\mathrm{X})^{m}\,\mathrm{K}_{m,\,n}(\beta^{-1}\mathrm{Y})^{n},\qquad \mathrm{K}_{i,\,j}\in\mathrm{U}(\mathfrak{k}_{\mathbf{Q}});$$

here m, n run over the set of d-tuples of nonnegative integers, not all of whose entries are zero, and $(\beta\mathrm{X})^{m}$ is multi-index notation. We know further that:

3.3.2. $\mathrm{K}_{0,\,0}$ is in the center of $\mathrm{U}(\mathfrak{k}_{\mathbf{Q}})$;

3.3.3. $$\mathrm{K}_{m,\,n}\neq 0 \Rightarrow |m|=|n|,$$

where $|m|$ is the sum of the entries of m. The Lemma follows from 3.3.3.

We write $\mathrm{Z}=\mathrm{Z}_{\beta}$ for any β.

3.4. Proposition. — *Assume* ρ *is irreducible. The algebra* Z *acts on* D_{ρ} *through a character* $\chi_{\rho}:\mathrm{Z}\rightarrow\mathbf{Q}$.

Proof. — Since D_{ρ} is generated over U by V_0, it is enough to show that Z acts as required on V_0. We write $\zeta\in\mathrm{Z}$ as in 3.3.1. If $v\in\mathrm{V}_0$, then, by the definition of D_{ρ}:

$$\zeta(v)=\mathrm{K}_{0,\,0}(v).$$

The Proposition is reduced, by 3.3.2, to the statement that the center of $\mathrm{U}(\mathfrak{k}_{\mathbf{Q}})$ acts through a rational character on the irreducible $\mathrm{K}_{\mathbf{C}}$-space V_{ρ^*}; this is well known.

We call χ_{ρ}, or its linear extension to $\mathrm{Z}(\mathfrak{g}_{\mathbf{C}})$, the *infinitesimal character* associated to D_{ρ}. As indicated in the proof of 2.6.4, χ_{ρ} is determined by its restriction to the $\mathbf{R}$-subalgebra of selfadjoint elements of $\mathrm{Z}(\mathfrak{g}_{\mathbf{C}})$.

3.5. In the notation of 3.0, let $\lambda:\mathfrak{h}\rightarrow\mathbf{C}$ be a linear form, and define the *Verma module*:

$$\mathrm{M}(\lambda)=\mathrm{U}\otimes_{\mathrm{U}(\mathfrak{b})}\mathbf{C},$$

where $\mathfrak{b}$ acts on $\mathbf{C}$ through the character λ of $\mathfrak{h}$. If λ is the highest weight (relative to $\mathfrak{b}_0$) of the representation ρ^* of $\mathfrak{k}_{\mathbf{C}}$, then D_{ρ} is naturally isomorphic to a U-quotient of $\mathrm{M}(\lambda)$; furthermore, $\mathrm{Z}(\mathfrak{g}_{\mathbf{C}})$ acts on $\mathrm{M}(\lambda)$ through the character χ_{ρ} (*cf.* [8], Chap. 7).

3.5.1. Proposition. — *Suppose* ρ_1, ρ_2 *are representations of* $\mathrm{K}_{\mathbf{C}}$ *corresponding to n-tuples* $\alpha_1^1\geqq\ldots\geqq\alpha_n^1$, $\alpha_1^2\geqq\ldots\geqq\alpha_n^2$, *respectively, such that* $\alpha_n^j\geqq n$, $j=1, 2$. *Then* $\chi_{\rho_1}=\chi_{\rho_2}$ *if and only if* $\rho_1=\rho_2$.

Proof. — Let $\lambda_i=(-\alpha_n^i, \ldots, -\alpha_1^i)$ be the highest weight of ρ_i^*, $i=1, 2$. By the preceding remarks, we need only check that the infinitesimal characters of $\mathrm{M}(\lambda_i)$, $i=1, 2$, coincide if and only if $\lambda_1=\lambda_2$. Now for any characters λ_1 and λ_2 of $\mathfrak{h}$, we know by 7.4.7 of [8] that the infinitesimal characters of $M(\lambda_i)$ coincide if and only if, for some w in the Weyl group W of $\mathfrak{g}_{\mathbf{C}}$ (relative to $\mathfrak{h}$), we have:

3.5.2. $$w(\lambda_1+\delta)=\lambda_2+\delta$$

with $\delta=(n, n-1, \ldots, 1)$ as in 3.0).

In our case W is the semidirect product of the symmetric group on n letters, acting on the set of n-tuples, with the group generated by the reflections ($\alpha_i \mapsto -\alpha_i$; $\alpha_j \mapsto \alpha_j$, $j \neq i$) $i=1, \ldots, n$. We have to check that, if 3.5.2 holds for some $w \in W$, under the stated hypotheses on the α_i^j, then $\lambda_1=\lambda_2$. This simple combinatorial exercise is left to the reader.

3.5.3. *Remark.* — If we assume that $\alpha_n > n$, then D_ρ is infinitesimally equivalent to the space of differentiable vectors in a square-integrable representation, the complex conjugate of one of the discrete series representations constructed in [12].

3.6. The representations D_k have been investigated in some detail. As remarked in paragraph 1 of [15], we have:

3.6.1. If $k \geqq n/2$, then D_k is the infinitesimal representation arising from a *unitary* representation of the group G. (To see this, take $\alpha \geqq -1/2$ in 1.2.7 of [15]. Actually, Jakobsen's paper deals with the complex conjugate "holomorphic type" representations, but the results are equivalent.)

Furthermore, it follows from the methods of [28], or from the remarks in paragraph 2 of [16], that:

3.6.2. D_k is irreducible for $k \geqq n/2$.

Combining 3.6.1 and 3.6.2 with 2.6 of [16] and Proposition 3.1, we have:

3.7. Proposition. — *If k_1 and k_2 are integers $\geqq n/2$, then, as* U-*modules:*

3.7.1. $$D_{k_1} \otimes_{\mathbf{C}} D_{k_2} \simeq \bigoplus_{\rho} D_\rho,$$

where ρ runs through the components of $\det^{k_1+k_2} \otimes \mathrm{Sym}^r(\mathfrak{p}^-)$, for all $r \geqq 0$. Each D_ρ occurs with multiplicity one, and each ρ which occurs satisfies the hypotheses of Proposition 3.5.1.

3.8. Corollary. — *Under the hypotheses of* 3.7, *we have:*

3.8.1. $$D_{k_1, \mathbf{Q}, \beta} \otimes_{\mathbf{Q}} D_{k_2, \mathbf{Q}, \beta} \simeq \bigoplus_{\rho} D_{\rho, \mathbf{Q}, \beta}$$

as U_β-modules for any complex number β, with ρ as in 3.7.

Proof. — It follows from 3.5.1 that the decomposition of 3.7.1 is the eigenfunction decomposition of $D_{k_1} \otimes_{\mathbf{C}} D_{k_2}$ with respect to $Z=Z_\beta$. By Proposition 3.4, the eigenvalues of Z are all rational, which implies the Corollary.

3.9. Corollary. — *Under the above hypotheses, suppose $\varphi \in D_{k_1} \otimes_{\mathbf{C}} D_{k_2}$ satisfies:*

(*a*) *$\zeta(\varphi)=\chi_\rho(\zeta)\,\varphi$ for all $\zeta \in Z(\mathfrak{g}_{\mathbf{C}})$, where $\rho=\det^k$ for some integer k;*

(*b*) *φ is of type ρ for* K.

Then φ is of holomorphic type.

Proof. — By (*a*), we see that $\varphi \in D_k \subset D_{k_1} \otimes_{\mathbf{C}} D_{k_2}$; by (*b*), φ is a highest K-type vector in D_k. Our conclusion follows immediately.

3.10. Given any complex number β and any holomorphic modular form f of type ρ for some subgroup $\Gamma \subset \Gamma(1)$ of finite index, we may define $\mathscr{V}_{f, \mathbf{Q}, \beta}$ to be the cyclic U_β-module generated by φ, where $\varphi \leftrightarrow f$. If f_i is of weight k_i for Γ, $i=1, 2$, such that $k_i \geqq n/2$, then it follows from 3.8 that, as U_β-modules:

3.10.1. $$\mathscr{V}_{f_1, \mathbf{Q}, \beta} \otimes_{\mathbf{Q}} \mathscr{V}_{f_2, \mathbf{Q}, \beta} \simeq \bigoplus_\rho D_{\rho, \mathbf{Q}, \beta},$$

where ρ runs through the same set as in 3.7.

We denote by $\underline{R}$ the restriction of functions on $\Gamma \backslash G \times_\Gamma \backslash G$ to its diagonal $\Gamma \backslash G$. Elements of $\mathscr{V}_{f_1} \otimes_{\mathbf{C}} \mathscr{V}_{f_2}$ may be regarded in the usual way as functions on $\Gamma \backslash G \times \Gamma \backslash G$; we may thus apply $\underline{R}$ to such elements. The resulting functions are automorphic forms in the sense of [5], by [13], Lemma 14.

3.11. Lemma. — *Let f_i be a holomorphic modular form of weight $k_i \geqq n/2$, $i=1, 2$, for the arithmetic subgroup $\Gamma = \Gamma(N)$, for some integer $N > 0$. Let $\varphi' \in C^\infty(\Gamma \backslash G, V_k)_k$ be $\underline{R}$ of an element of $\mathscr{V}_{f_1, \mathbf{Q}, \beta} \otimes_{\mathbf{Q}} \mathscr{V}_{f_2, \mathbf{Q}, \beta}$ for some complex number β and some integer k; let $\varphi' \leftrightarrow f' \in G_k^\infty(\Gamma)$. Then there exists $\varphi_0 \in \mathscr{V}_{f_1, \mathbf{Q}, \beta} \otimes_{\mathbf{Q}} \mathscr{V}_{f_2, \mathbf{Q}, \beta}$ such that $\varphi_0 \leftrightarrow f_0 \in G_k(\Gamma)$ (f_0 is* holomorphic) *and such that, for any holomorphic cusp form $f \in G_k(\Gamma)$*:

3.11.1. $$\langle f, f' \rangle_k = \langle f, f_0 \rangle_k.$$

Proof. — Let $f \leftrightarrow \varphi \in C^\infty(\Gamma \backslash G, V_k)_k$. Write $\varphi' = \sum_\rho \underline{R}(\varphi'_\rho)$, with $\varphi'_\rho \in D_{\rho, \mathbf{Q}, \beta}$, as in 3.10.1. Since 3.10.1 is the eigenfunction decomposition with respect to Z (by 3.5.1), each $\underline{R}(\varphi'_\rho)$ belongs to $C^\infty(\Gamma \backslash G, V_k)_k$. Let $\varphi_0 = \underline{R}(\varphi'_\rho)$ with $\rho = \det^k$. By 3.5.1 and 2.6.4, we have:

3.11.2. $$(\varphi, \varphi') = (\varphi, \varphi_0),$$

which is equivalent (by 2.6.1) to 3.11.1. It remains to show (by 2.3) that φ_0 is of holomorphic type; but this follows from 3.9.

3.12. *Remark*. — When $0 \leqq k \leqq (n-1)/2$, the representation D_k is no longer irreducible if $2k \in \mathbf{Z}$. (Here we may think of D_k purely as a Lie algebra module, or else work with the two-fold covering group of G.) The question naturally arises whether the cyclic representation generated by a modular form of weight k is or is not irreducible when $k \leqq (n-1)/2$. It is shown in [52] that the results of Freitag [9] and Resnikoff [27], to the effect that (holomorphic) modular forms of these "singular" weights are annihilated by a certain class of differential operators, are equivalent to the statement that the cyclic representations generated by these forms are irreducible. It should be possible to prove such an irreducibility result independently, and thus provide an alternate proof of the Theorem of Freitag and Resnikoff.

Assuming that $(\pi_{f_1}, \mathscr{V}_{f_1})$ and $(\pi_{f_2}, \mathscr{V}_{f_2})$ are irreducible, when f_i is a holomorphic modular form of weight $k_i, i=1,2$, with (say) $k_1 \leqq (n-1)/2$, is it still true that $\mathscr{V}_{f_1} \otimes_{\mathbf{C}} \mathscr{V}_{f_2}$ is a direct sum of irreducible representations of U?

4. Review of the algebraic theory

4.0. Any point $\tau \in \mathfrak{S}_n$ corresponds to a symmetric $n \times n$ matrix (τ_{ij}) whose columns are denoted $\tau_1, \ldots, \tau_n$; the τ_i are vectors in $\mathbf{C}^n$. If $e_1, \ldots, e_n$ are the columns of the $n \times n$ identity matrix I (read from left to right), we let $\mathrm{L}_\tau \subset \mathbf{C}^n$ be the $\mathbf{Z}$-lattice generated by $\{e_1, \ldots, e_n, \tau_1, \ldots, \tau_n\}$. The complex torus $\mathscr{A}_\tau = \mathbf{C}^n/\mathrm{L}_\tau$ is an abelian variety, and there is an analytic family of abelian varieties:

$$\begin{array}{c} \mathscr{A} \\ {\scriptstyle\pi}\big\downarrow \\ \mathfrak{S}_n \end{array}$$

whose fiber over the point τ is $\mathscr{A}_\tau$.

If $\mathrm{N} \geqq 3$ is an integer, $\Gamma(\mathrm{N})$ acts without fixed points on $\mathscr{A}$; the quotient is a smooth algebraic family $\mathscr{A}_\mathrm{N}$ of abelian varieties with level N structure over the quasi-projective variety $\mathscr{M}_\mathrm{N} = \Gamma(\mathrm{N}) \backslash \mathfrak{S}_n$. The family $\mathscr{A}_\mathrm{N}/\mathscr{M}_\mathrm{N}$, with its canonical polarization (*cf.* 4.4 below) is (a connected component of) the universal family of principally polarized abelian varieties of dimension n with level N structure; as such it is defined over $\mathbf{Q}(\zeta_\mathrm{N})$. For these facts, and for much of what follows, the reader is referred to [3], [6], [25], [26], and the exposes of Shimura in [31].

4.1. The fiber varieties $\mathscr{A}$ and $\mathscr{A}_\mathrm{N}$ give rise to a series of vector bundles over $\mathfrak{S}_n$ and $\mathscr{M}_\mathrm{N}$. For example, we have the relative algebraic De Rham cohomology bundles (or sheaves) $\mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n)$ and $\mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}_\mathrm{N}/\mathscr{M}_\mathrm{N})$ of dimension $2n$ over $\mathfrak{S}_n$ and $\mathscr{M}_\mathrm{N}$ respectively. The latter is an algebraic vector bundle, defined over $\mathbf{Q}(\zeta_\mathrm{N})$. For any integer N and any field $\mathrm{L} \supset \mathbf{Q}(\zeta_\mathrm{N})$, let $\mathscr{A}_{\mathrm{N,L}}$ (resp. $\mathscr{M}_{\mathrm{N,L}}$) be $\mathscr{A}_\mathrm{N}$ (resp. $\mathscr{M}_\mathrm{N}$), thought of as a variety over L. We then define $\mathscr{H}^1(\mathrm{N}, \mathrm{L})$ to be the bundle (or sheaf) of L-rational sections of $\mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}_{\mathrm{N,L}}/\mathscr{M}_{\mathrm{N,L}})$. (For algebraic de Rham cohomology, *cf.* [11].)

The C^∞ vector bundle associated to $\mathscr{H}^1_{\mathrm{DR}}$ (over $\mathfrak{S}_n$ or $\mathscr{M}_\mathrm{N}$) is denoted $\mathscr{H}^1_\infty$. It splits as a direct sum:

$$(4.1.1) \qquad \mathscr{H}^1_\infty \simeq \mathscr{H}^{1,0}_\infty \oplus \mathscr{H}^{0,1}_\infty;$$

this splitting induces the Hodge decomposition on the de Rham cohomology of each fiber. The summand $\mathscr{H}^{1,0}_\infty$ is itself the C^∞ bundle associated to a holomorphic (or algebraic) subbundle $\boldsymbol{\omega} \subset \mathscr{H}^1_{\mathrm{DR}}$; $\boldsymbol{\omega}$ is the bundle of relative 1-forms for either $\mathscr{A}/\mathfrak{S}_n$ or $\mathscr{A}_\mathrm{N}/\mathscr{M}_\mathrm{N}$.

The bundle of holomorphic 1-forms on $\mathfrak{S}_n$, or on $\mathscr{M}_\mathrm{N}$, is denoted Ω; the same symbol is used to denote its associated C^∞ bundle.

4.2. We now define some global sections of $\boldsymbol{\omega}$, Ω, and $\mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n)$ over $\mathfrak{S}_n$. We denote the standard coordinates of $\mathbf{C}^n$ by $u_1, \ldots, u_n$, then $\{du_1, \ldots, du_n\}$ forms a basis for the fiber $\boldsymbol{\omega}_\tau$ of $\boldsymbol{\omega}$ over every point $\tau \in \mathfrak{S}_n$.

The fiber of $\mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n)$ at the point τ is naturally interpreted (*via* singular cohomology) as the complex vector space $\mathrm{Hom}_{\mathbf{C}}(\mathrm{L}_\tau \otimes_{\mathbf{Z}} \mathbf{C}, \mathbf{C})$. Consider the sections $\alpha_i, \beta_i \in \mathrm{H}^0(\mathfrak{S}_n, \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n))$, $i=1, \ldots, n$:

$$(4.2.1)\qquad \left\{\begin{array}{l} \alpha_i\left(\sum_{j=1}^n a_j e_j + \sum_{j=1}^n b_j \tau_j\right) = a_i \\ \beta_i\left(\sum_{j=1}^n a_j e_j + \sum_{j=1}^n b_j \tau_j\right) = b_i \end{array}\right\} \text{over } \tau \in \mathfrak{S}_n.$$

The sections α_i and β_i represent global relative 1-forms with *constant periods* along the fibers of $\mathscr{A}/\mathfrak{S}_n$; the sheaf $\mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n)$ is free over $\mathcal{O}_{\mathfrak{S}_n}$ with basis $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$. We have the obvious equalities:

$$(4.2.2.1)\qquad du_i = \alpha_i + \sum_{j=1}^n Z_{ij}\beta_j,$$

$$(4.2.2.2)\qquad d\bar{u}_i = \alpha_i + \sum_{j=1}^n \bar{Z}_{ij}\beta_j, \qquad i=1, \ldots, n.$$

4.2.3. As mentioned in 4.0, $\Gamma(\mathrm{N})$ acts on $\mathscr{A}/\mathfrak{S}_n$ and thus on $\mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n)$. The latter action extends to a right action of G on $\mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n)$ covering the standard action of G on $\mathfrak{S}_n$. Under this action, $\boldsymbol{\omega}$ is preserved, and the formula for the action of G on $\boldsymbol{\omega}$ is given by:

$$(4.2.3.2)\qquad g\left(Z, \begin{pmatrix} du_1 \\ \vdots \\ du_n \end{pmatrix}\right) = \left(g(Z), (J(g, Z)^t)^{-1}\begin{pmatrix} du_1 \\ \vdots \\ du_n \end{pmatrix}\right), \qquad g \in G, \quad Z \in \mathfrak{S}_n$$

(*cf.* [3], p. 346; [36]); here $\begin{pmatrix} du_1 \\ \vdots \\ du_n \end{pmatrix}$ is regarded as a column vector of n global sections of $\boldsymbol{\omega}$. This action evidently gives rise, upon taking quotients by $\Gamma(\mathrm{N})$, to the structure of $\boldsymbol{\omega}$ as a holomorphic vector bundle over $\mathscr{M}_{\mathrm{N}}$. In particular, we see that $\boldsymbol{\omega}$ is holomorphically equivalent to the bundle denoted $\mathscr{E}_{\underline{\mathrm{St}}}$ in paragraphs 1 and 2. Moreover, the $2n$-dimensional complex subspace $\mathrm{H}^1 \subset \mathrm{H}^0(\mathfrak{S}_n, \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n))$, spanned by $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$, is also preserved by G. One computes that, in terms of the basis $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$, the element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ has the matrix representation $\begin{pmatrix} A & -B \\ -C & D \end{pmatrix}^t$ on H^1. We note that $\begin{pmatrix} A & -B \\ -C & D \end{pmatrix}^t$ is again a symplectic matrix, and that every element of G arises in this way.

Finally, the coordinates Z_{ij} on $\mathfrak{S}_n$ give rise to a trivialization of Ω with respect to the set of everywhere linearly independent sections $\{dZ_{ij}\}$. Of course G acts on $\mathfrak{S}_n$, and therefore on Ω by functoriality; the formula for this action is given on p. 3-05 of [31]:

$$(4.2.3.2)\qquad g(Z, (dZ)) = (g(Z), (J(g, Z)^t)^{-1}(dZ)\, J(g, Z)^{-1});$$

here (dZ) is the $n \times n$ matrix (dZ_{ij}), $g \in G$, $Z \in \mathfrak{S}_n$.

Thus Ω is holomorphically equivalent to $\mathscr{E}_{\mathrm{Sym}^2(\underline{\mathrm{St}})}$. Over $\mathscr{M}_{\mathrm{N}}$, $\boldsymbol{\omega}$ and Ω are respectively equivalent to $\mathscr{E}_{\underline{\mathrm{St}}}(\Gamma(\mathrm{N}))$ and $\mathscr{E}_{\mathrm{Sym}^2(\underline{\mathrm{St}})}(\Gamma(\mathrm{N}))$. In particular, Ω is holomorphically equivalent, as a homogeneous vector bundle over $\mathscr{S}_n$, to $\mathrm{Sym}^2\,\boldsymbol{\omega}$; we define an *algebraic* isomorphism in 4.4.

4.3. The extent to which the global sections du_i of $\boldsymbol{\omega}$ fail to have constant periods is computed by the Gauss-Manin connection [20]:

$$\nabla:\quad \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n)\to \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n)\otimes_{\mathscr{O}_{\mathfrak{S}_n}}\Omega.$$

In terms of the isomorphism $\mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}/\mathfrak{S}_n)\simeq \mathrm{H}^1\otimes_{\mathbf{C}}\mathscr{O}_{\mathfrak{S}_n}$, ∇ is just $1\otimes d$, where d is exterior differentiation and 1 is the identity map on H^1. In particular, by 4.2.2.1:

$$\nabla(du_i)=\sum_{j=1}^{n}\beta_j\otimes d\mathrm{Z}_{ij}. \tag{4.3.1}$$

The important fact for our purposes is that ∇ descends to a differential operator on algebraic vector bundles over $\mathscr{M}_{\mathrm{N}}$, for any integer $\mathrm{N}\geqq 3$:

$$\nabla:\quad \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}_{\mathrm{N}}/\mathscr{M}_{\mathrm{N}})\to \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}_{\mathrm{N}}/\mathscr{M}_{\mathrm{N}})\otimes_{\mathscr{O}_{\mathscr{M}_{\mathrm{N}}}}\Omega.$$

In fact, ∇ may be defined in the algebraic category, as in [20]; it may then be verified (*cf.* [17], 4.1.2) that ∇ lifts to the required connection over $\mathfrak{S}_n$.

The C^{∞} vector bundles $\mathscr{H}^{1,0}_{\infty}$ and $\mathscr{H}^{0,1}_{\infty}$ are generated over every point of $\mathfrak{S}_n$ by the sections $\{du_1,\ldots,du_n\}$ and $\{d\bar{u}_1,\ldots,d\bar{u}_n\}$, respectively. We denote by $\underline{\mathrm{Split}}$ the projection $\mathscr{H}^1_{\infty}\to\mathscr{H}^{1,0}_{\infty}$ of 4.1.1, and define similarly, for each integer $m\geqq 0$:

$$\underline{\mathrm{Split}}:\quad (\mathscr{H}^1_{\infty})^{\otimes m}\to(\mathscr{H}^{1,0}_{\infty})^{\otimes m}.$$

Following Katz, [19], we define the differential operator:

$$\vartheta:\quad \mathscr{H}^{1,0}_{\infty}\to\mathscr{H}^{1,0}_{\infty}\otimes_{\mathrm{C}^{\infty}(\mathfrak{S}_n,\,\mathbf{C})}\Omega$$

to be the composite:

$$\begin{tikzcd}
\mathscr{H}^{1,0}_{\infty} \arrow[r] \arrow[ddr, dashed, "\vartheta"'] & \mathscr{H}^{1}_{\infty} \arrow[d, "\nabla"] \\
 & \mathscr{H}^{1}_{\infty}\otimes_{\mathrm{C}^{\infty}(\mathfrak{S}_n,\mathbf{C})}\Omega \arrow[d, "\underline{\mathrm{Split}}\otimes 1"] \\
 & \mathscr{H}^{1,0}_{\infty}\otimes_{\mathrm{C}^{\infty}(\mathfrak{S}_n,\mathbf{C})}\Omega
\end{tikzcd}$$

We are here making use of the C^{∞} versions of ∇ and Ω. Using (4.2.2.1) and (4.3.1), one computes easily that:

$$\vartheta(du_k)=\sum_{j=1}^{n}((\mathrm{Z}-\overline{\mathrm{Z}})^{-1}du)_j\otimes d\mathrm{Z}_{kj},\qquad k=1,\ldots,n; \tag{4.3.3}$$

here $((\mathrm{Z}-\overline{\mathrm{Z}})^{-1}du)_j$ is the j-th entry of the column vector $(\mathrm{Z}-\overline{\mathrm{Z}})^{-1}\begin{pmatrix}du_1\\ \vdots\\ du_n\end{pmatrix}$.

Of course, the Gauss-Manin connection may be extended by the product rule to an algebraic connection:

$$\nabla:\quad (\mathscr{H}^1_{\mathrm{DR}})^{\otimes m} \to (\mathscr{H}^1_{\mathrm{DR}})^{\otimes m} \otimes_{\mathcal{O}_{\mathscr{M}_{\mathrm{N}}}} \Omega;$$

similarly for $\mathrm{Sym}^m \mathscr{H}^1_{\mathrm{DR}}$, $\Lambda^m \mathscr{H}^1_{\mathrm{DR}}$, etc. Composing with Split, we may define the series of differential operators:

$$\vartheta:\quad (\mathscr{H}^{1,0}_{\infty})^{\otimes m} \to (\mathscr{H}^{1,0}_{\infty})^{\otimes m} \otimes_{\mathrm{C}^{\infty}(\mathfrak{S}_n,\, \mathrm{C})} \Omega$$

and their analogues for $\Lambda^m \mathscr{H}^{1,0}_{\infty}$, $\mathrm{Sym}^m \mathscr{H}^{1,0}_{\infty}$, etc. With this definition, we see that ϑ satisfies the product rule:

$$\vartheta(a \otimes b) = \vartheta(a) \otimes b + a \otimes \vartheta(b),$$

4.4. The canonical principal polarization on the family $\mathscr{A}_{\mathrm{N}}/\mathscr{M}_{\mathrm{N}}$ gives rise to an everywhere non-degenerate alternating form:

$$\langle\ ,\ \rangle_{\mathrm{DR}}:\quad \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}_{\mathrm{N}}/\mathscr{M}_{\mathrm{N}}) \otimes_{\mathcal{O}_{\mathscr{M}_{\mathrm{N}}}} \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}_{\mathrm{N}}/\mathscr{M}_{\mathrm{N}}) \to \mathcal{O}_{\mathscr{M}_{\mathrm{N}}}.$$

This can be computed over $\mathfrak{S}_n$ in terms of the global sections α_i, β_i:

$$(4.4.1)\qquad \begin{cases} \langle \beta_j, \alpha_k \rangle_{\mathrm{DR}} = \dfrac{1}{2\pi i}\delta_{jk} = -\langle \alpha_k, \beta_j \rangle_{\mathrm{DR}}, & j, k = 1, \ldots, n, \\ \langle \beta_j, \beta_k \rangle_{\mathrm{DR}} = \langle \alpha_j, \alpha_k \rangle_{\mathrm{DR}} = 0, & j, k = 1, \ldots, n. \end{cases}$$

(The divergence by $1/2\pi i$ from the usual formula for the Chern class of the canonical theta divisor on $\mathscr{A}_{\mathrm{N}}/\mathscr{M}_{\mathrm{N}}$; *cf.* [26], [44], is the divergence between cup products in algebraic de Rham and singular cohomology.) We note that $\langle\ ,\ \rangle_{\mathrm{DR}}$, restricted to the space H^1, is invariant under the action of G defined in 4.2.3.

We can now define a canonical isomorphism:

$$\mathrm{Sym}^2\, \boldsymbol{\omega} \simeq \Omega$$

as follows: For any abelian variety A let Lie (A) denote the Lie algebra of A, and let A′ denote the abelian variety dual to A. For any $\tau \in \mathfrak{S}_n$, let $\psi_\tau : \mathrm{H}^1_{\mathrm{DR}}(\mathrm{A}_\tau) \to \mathrm{Lie}(\mathrm{A}_\tau)$ be the composite of the isomorphism, given by the polarization, of $\mathrm{H}^1_{\mathrm{DR}}(\mathrm{A}_\tau)$ with $\mathrm{H}^1_{\mathrm{DR}}(\mathrm{A}'_\tau)$, followed by the canonical surjection $\mathrm{H}^1_{\mathrm{DR}}(\mathrm{A}'_\tau) \to \mathrm{H}^1(\mathrm{A}'_\tau, \mathcal{O}_{\mathrm{A}'_\tau}) \simeq \mathrm{Lie}(\mathrm{A}_\tau)$. The set of maps ψ_τ gives rise to a map of algebraic vector bundles $\psi : \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}_{\mathrm{N}}/\mathscr{M}_{\mathrm{N}}) \to \boldsymbol{\omega}^*$, for any $\mathrm{N} \geqq 3$. For any section $\mathrm{X} \in \mathrm{H}^0(\mathrm{U}, \Omega^*)$, for some open set $\mathrm{U} \subset \mathscr{M}_{\mathrm{N}}$, consider the map:

$$\psi_{\mathrm{X}}:\quad \mathrm{H}^0(\mathrm{U}, \boldsymbol{\omega}) \xrightarrow{\nabla(\mathrm{X})} \mathrm{H}^0(\mathrm{U}, \mathscr{H}^1_{\mathrm{DR}}(\mathscr{A}_{\mathrm{N}}/\mathscr{M}_{\mathrm{N}})) \xrightarrow{\psi} \mathrm{H}^0(\mathrm{U}, \boldsymbol{\omega}^*);$$

here $\nabla(\mathrm{X})$ is differentiation with respect to X and the connection ∇. The map:

$$\mathrm{X} \mapsto \psi_{\mathrm{X}} \in \mathrm{Hom}_{\mathcal{O}_{\mathrm{U}}}(\mathrm{H}^0(\mathrm{U}, \boldsymbol{\omega}), \mathrm{H}^0(\mathrm{U}, \boldsymbol{\omega}^*))$$

gives rise by duality to the *Kodaira-Spencer isomorphism* of vector bundles over $\mathscr{M}_{\mathrm{N}}$:

$$(4.4.2)\qquad \mathrm{Sym}^2\, \boldsymbol{\omega} \simeq \Omega.$$

4.4.3. Using (4.4.1) and (4.2.2.1), we see that the isomorphism (4.4.2) identifies $2\pi i\, du_j \circ du_k \in \mathrm{H}^0(\mathfrak{S}_n, \mathrm{Sym}^2\, \boldsymbol{\omega})$ with $d\mathrm{Z}_{ij} \in \mathrm{H}^0(\mathfrak{S}_n, \Omega)$.

4.4.4. We reinterpret the map ϑ as a C^∞ differential operator:

$$\vartheta: \quad \mathscr{H}_\infty^{1,0} \to \mathscr{H}_\infty^{1,0} \otimes_{C^\infty(\mathfrak{S}_n, C)} \mathrm{Sym}^2(\mathscr{H}_\infty^{1,0})$$

using the isomorphism (4.4.2). In this way we may use the product rule to define higher order iterates $\vartheta^{(m)}$ of ϑ, as well as maps such as:

$$\mathrm{Sym}^m \vartheta: \quad \mathscr{H}_\infty^{1,0} \to \mathscr{H}_\infty^{1,0} \otimes_{C^\infty(\mathfrak{S}_n, C)} \mathrm{Sym}^m(\mathrm{Sym}^2(\mathscr{H}_\infty^{1,0})),$$

etc.

4.5. For any field $L \supset \mathbf{Q}(\zeta_N)$, let

$$\mathscr{T}_m(N, L) = (\mathscr{H}(N, L))^{\otimes m}, \quad \mathscr{T}(N, L) = \bigoplus_{m=0}^{\infty} \mathscr{T}_m(N, L).$$

The Gauss-Manin connection is a derivation of the sheaf of algebras $\mathscr{T}(N, L)$ [thanks to (4.4.2)], for all N and L. Let $\mathscr{T}_\infty(N) = \bigoplus_{m=0}^{\infty} (\mathscr{H}_\infty^1)^{\otimes m}$ (we are thinking of $\mathscr{H}_\infty^1$ as a sheaf over $\mathscr{M}_N$), and let $\mathscr{I}_\infty(N)$ be the sheaf of two-sided ideals in $\mathscr{T}_\infty(N)$ generated by $\mathscr{H}_\infty^{0,1}$. We have a natural inclusion (of sections) $\mathscr{T}(N, \mathbf{C}) \subset \mathscr{T}_\infty(N)$; let $\mathscr{I}(N) = \mathscr{T}(N, \mathbf{C}) \cap \mathscr{I}_\infty(N)$.

The map Split may be interpreted as an injection:

$$\text{(4.5.1)} \qquad \underline{\text{Split}}: \quad \mathscr{T}(N, \mathbf{C})/\mathscr{I}(N) \hookrightarrow \bigoplus_{m=0}^{\infty} (\mathscr{H}_\infty^{1,0})^{\otimes m}.$$

The image of $\mathscr{T}(N, L)$ under Split is denoted $\mathscr{R}(N, L)$, the sheaf of algebras of *pseudo-arithmetic modular forms* over L of level N. The map Split is compatible with the graded structure on both sides of (4.5.1), and we write $\mathscr{R}(N, L) = \bigoplus_{m=0}^{\infty} \mathscr{R}_m(N, L)$, in the obvious way. Since $\mathscr{I}(N)$ is horizontal with respect to ∇, the operator ϑ is a derivation of the sheaf of $\mathcal{O}_{\mathscr{M}_{N, L}}$ algebras $\mathscr{R}(N, L)$.

Of course, there is an inclusion (of sections) $\bigoplus_{m=0}^{\infty} \boldsymbol{\omega}^{\otimes m} \subset \mathscr{R}(N, \mathbf{C})$, arising from the inclusion of $\boldsymbol{\omega}$ in $\mathscr{H}_{DR}^1$ and the projection Split, and which takes the L-rational sections of $\boldsymbol{\omega}^{\otimes m}$ to $\mathscr{R}_m(N, L)$. A section of $\mathscr{R}(N, \mathbf{C})$ which lies in the image of $\bigoplus_{m=0}^{\infty} \boldsymbol{\omega}^{\otimes m}$ is called simply *holomorphic*, and the algebra of holomorphic elements of $\mathscr{R}(N, \mathbf{C})$ may be identified with the algebra $\bigoplus_{m=0}^{\infty} \boldsymbol{\omega}^{\otimes m}$. A section of $\bigoplus_{m=0}^{\infty} \boldsymbol{\omega}^{\otimes m}$ which is rational over L is called *arithmetic* over L. We must check the following fundamental compatibility:

4.6 THEOREM. — *If a section $\tilde{f}$ of $\mathscr{R}(N, L)$, for some $L \supset \mathbf{Q}(\zeta_N)$ is holomorphic, then $\tilde{f}$ is* arithmetic *over* L.

Proof. — We intend to find a dense subset Σ of $\overline{\mathbf{Q}}$-algebraic points of $\mathcal{M}_N$ with the following properties:

(4.6.1) Σ is invariant under $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$,

(4.6.2) the restriction $\tilde{f}_\Sigma$ of $\tilde{f}$ to Σ is rational over L.

This will prove the Theorem.

For Σ we take the set of points $x \in \mathcal{M}_N$ such that the fiber A_x of $\mathcal{A}_N$ at x is an abelian variety (of dimension n) with complex multiplication by a totally imaginary number field k_x of degree $2n$ over $\mathbf{Q}$. Obviously Σ satisfies 4.6.1. Let $\mathcal{H}^1_\Sigma$ (resp. ω_Σ) be the restriction of $\mathcal{H}^1_{DR}(\mathcal{A}_N/\mathcal{M}_N)$ (resp. ω) to Σ; let $H^1_{\Sigma,x}$ (resp. $\omega_{\Sigma,x}$) be its fiber at x. When $L' \subset \mathbf{C}$ contains the field of definition k'_x of x, we let $H^1_{\Sigma,x}(L')$ and $\omega_{\Sigma,x}(L')$ denote the L'-rational subspaces of $H^1_{\Sigma,x}$ and $\omega_{\Sigma,x}$, respectively. Then $H^1_{\Sigma,x}(\overline{\mathbf{Q}})$ is, under the natural action of $k_x \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ induced by the complex multiplication, a free rank one $k_x \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-module, and $\omega_{\Sigma,x}(\overline{\mathbf{Q}})$ is the sum of n distinct $k_x \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-eigenspaces of $H^1_{\Sigma,x}(\overline{\mathbf{Q}})$. Let $\omega_{\Sigma,x}(\overline{\mathbf{Q}})'$ denote the direct sum of the remaining n $k_x \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-eigenspaces of $H^1_{\Sigma,x}(\overline{\mathbf{Q}})$.

Now $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$ acts on $\mathcal{H}^1_\Sigma$, and the element τ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$ clearly takes the $k_x \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-eigenspaces of $H^1_{\Sigma,x}(\overline{\mathbf{Q}})$ to the $k_{\tau x} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-eigenspaces of $H^1_{\Sigma,\tau x}(\overline{\mathbf{Q}})$. But τ also takes $\omega_{\Sigma,x}(\overline{\mathbf{Q}})$ to $\omega_{\Sigma,\tau x}(\overline{\mathbf{Q}})$. This implies that the subbundle ω'_Σ of H^1_Σ, the $\overline{\mathbf{Q}}$-rational points of whose fiber at x is $\omega_{\Sigma,x}(\overline{\mathbf{Q}})'$, is actually defined over $\mathbf{Q}(\zeta_N)$, and the bundle H^1_Σ splits $\mathbf{Q}(\zeta_N)$-rationally as a direct sum $\mathcal{H}^1_\Sigma = \omega_\Sigma \oplus \omega'_\Sigma$. But, for any $x \in \Sigma$, $\omega_{\Sigma,x}(\overline{\mathbf{Q}})' \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$ is the fiber at x of $\mathcal{H}^{0,1}_\infty$ (*cf.* [19], 5.1.27). Thus the projection $\underline{\mathrm{Split}}$, restricted to $\bigoplus_{m=0}^{\infty} (\mathcal{H}^1_\Sigma)^{\otimes m}$, is just the projection modulo the sheaf of two-sided ideals generated by ω'_Σ. In particular:

(4.6.3) if F is any section of $\mathcal{T}(\mathrm{N}, \mathrm{L})$, then the restriction of $\underline{\mathrm{Split}}(\mathrm{F})$ to Σ is an algebraic L-rational section of $\bigoplus_{m=0}^{\infty} \omega^{\otimes m}$.

This implies 4.6.2, which implies the Theorem.

4.6.4. *Remark.* — We note that the statement 4.6.3, as interpreted by Katz in [19], implies some of the rationality Theorems in Shimura's paper [32] for special values of certain non-holomorphic automorphic forms.

A more general version of Theorem 4.6 will be proved in a forthcoming paper.

4.6.5. Let V be a finite dimensional vector space over a field L of characteristic zero; let $L_0 \subset L$ be a subfield. We say a subspace $W \subset \otimes^m_L V$, $m \geqq 0$, is *defined by* L_0-*rational symmetry conditions* if there is an element π_W in the group ring $L_0[P_m]$ of the permutation group P_m such that, under the natural action of $L_0[P_m]$ on $\otimes^m_L V$, π_W is a projection of $\otimes^m_L V$ onto W. A classical Theorem of Weyl ([45], esp. Chapters III and IV) asserts that every

$GL_L(V)$-invariant subspace of $\otimes_L^m V$ is defined by L-rational symmetry conditions. The same terminology may be applied to subbundles of $\boldsymbol{\omega}^{\otimes m}$:

4.6.6 COROLLARY. — *Let* $\mathscr{W} \subset \boldsymbol{\omega}^{\otimes m}$ *be defined by* L-*rational symmetry conditions, for some* $L \supset \mathbf{Q}(\zeta_N)$; *let* $\pi_{\mathscr{W}} : \boldsymbol{\omega}^{\otimes m} \to \mathscr{W}$ *be the corresponding projection operator in* $L[P_m]$. *Let* $\hat{f}$ *be a section of* $\mathscr{R}(N, L)$ *such that* $\pi_{\mathscr{W}}(\hat{f})$ *is holomorphic. Then* $\pi_{\mathscr{W}}(\hat{f})$ *is arithmetic over* L.

Proof. — If $\hat{f} = \underline{\text{Split}}(F)$, for some section F of $\mathscr{T}_m(N, L)$, then $\pi_{\mathscr{W}}(\hat{f}) = \underline{\text{Split}}(\pi_{\mathscr{W}}(F))$, where $\pi_{\mathscr{W}}$ acts on $\mathscr{T}_m(N, L)$ in the obvious way. The Corollary now follows immediately from Theorem 4.6.

4.7. Let V be an n-dimensional vector space over a field E of characteristic zero. We may regard $\mathrm{Sym}^n(\mathrm{Sym}^2(V))$ as the space of degree n polynomial functions in the values of symmetric bilinear forms B on V^*. One such function is

$$D(B): (v_1, \ldots, v_n, w_1, \ldots, w_n) = \det(B(v_i, w_j)), \qquad v_i, w_j \in V^*, \quad i, j = 1, \ldots, n.$$

As B varies, the set D(B) runs through a one-dimensional linear subspace $L \subset \mathrm{Sym}^n(\mathrm{Sym}^2(V))$. The standard action of GL(V) on V gives rise to a natural action of GL(V) on $\mathrm{Sym}^n(\mathrm{Sym}^2(V))$; L is invariant with respect to this action, and the representation of GL(V) on L is equivalent to $\det^2$. Furthermore, one checks (or else deduces from Schmid's Theorem, Proposition 3.1) that L is the unique subspace of $\mathrm{Sym}^n(\mathrm{Sym}^2(V))$ with this property, and is even the only one-dimensional invariant subspace of $\mathrm{Sym}^n(\mathrm{Sym}^2(V))$. It is thus canonically a direct factor of $\mathrm{Sym}^n(\mathrm{Sym}^2(V))$.

4.7.1. Using the identification $\Omega \simeq \mathrm{Sym}^2 \boldsymbol{\omega}$ of 4.4.2, we may thus define a sub-line bundle $\mathscr{L} \subset \mathrm{Sym}^n \Omega$ by the above procedure; then $\mathscr{L}$ is an algebraic direct factor of $\mathrm{Sym}^n \Omega$ (over any $\mathscr{M}_N$), and as a homogeneous line bundle is isomorphic to $\mathscr{E}_2$. It is globally trivialized over $\mathfrak{S}_n$ by the section det(DZ), where DZ is the matrix (dZ_{ij}), and where the multiplication in the determinant is symmetric in the dZ_{ij}.

4.7.2. Let ρ be a holomorphic representation of $GL(n, \mathbf{C})$. We have described a correspondence $f \leftrightarrow \hat{f}$ between $G_\rho^\infty(\Gamma)$ and $H_\infty^0(\Gamma \backslash \mathfrak{S}_n, \mathscr{E}_\rho(\Gamma))$ in 1.6; this correspondence is determined up to a constant factor, as long as ρ is irreducible. Now when $\Gamma = \Gamma(N)$, each $\mathscr{E}_\rho(\Gamma)$ can be given the structure of algebraic vector bundle, denoted $\mathscr{E}_\rho$ over $\mathscr{M}_N$: indeed, if ρ is realized as a canonical direct factor of $\underline{\mathrm{St}}^{\otimes m}$, for some $m \in \mathbf{Z}$ (which is always possible, by the Theorem of the highest weight) then $\mathscr{E}_\rho$ is the corresponding algebraic direct factor of $\boldsymbol{\omega}^{\otimes m}$. Furthermore, the holomorphic sections of $\mathscr{E}_\rho(\Gamma(N))$ are in one-to-one correspondence with *algebraic* regular sections in $H^0(\mathscr{M}_{N,C}, \mathscr{E}_\rho)$, by [4], Theorem 10.14 (with the usual cuspidal condition when $n=1$). We would like to know how to determine the field of rationality of $\hat{f}$ from properties of f (for this, of course, the correspondence must be normalized; i. e., the indeterminate constant must be specified.) In the remainder of this section we carry this out when $\rho = \det^k$ for an *even* integer k; in the appendix we sketch a general criterion.

4.7.3. We write $\rho \otimes k$ instead of $\rho \otimes \det^k$. A *meromorphic* section of $\mathscr{E}_\rho(\Gamma)$ is defined to be a rational section $\hat{f} = \tilde{g}/\tilde{h}$, where $\tilde{g}$ is a holomorphic section of $\mathscr{E}_{\rho \otimes k}(\Gamma)$ and $\tilde{h}$ is a holomorphic section of $\mathscr{E}_k(\Gamma)$, for some integer k. The notion of meromorphic modular form of type ρ for Γ is defined analogously; we write $A_\rho(\Gamma)$ for the space of all such forms.

Now let $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$ be a meromorphic modular form of type $\underline{\mathrm{St}}$ for $\Gamma(N)$. We define the meromorphic section $\hat{f}$ of $\boldsymbol{\omega}/\mathscr{M}_N$ to be:

$$\hat{f} = \sum_{j=1}^{n} (2\pi i) f_j \, du_j. \tag{4.7.4}$$

With this normalization we may define, for any $f \in A_\rho(\Gamma(N))$, a meromorphic section $\hat{f}$ of $\mathscr{E}_\rho/\mathscr{M}_N$, whenever $\mathscr{E}_\rho$ is represented as a direct factor of $\boldsymbol{\omega}^{\otimes m}$ for some m. The true theorem is the following (*cf.* Appendix):

4.8. THEOREM (q-expansion principle). — *Let* L *be a field containing* $\mathbf{Q}(\zeta_N)$. *Let* $f \in G_\rho(\Gamma(N))$ *correspond to the (algebraic) section* $\hat{f} \in H^0(\mathscr{M}_{N,\mathbf{C}}, \mathscr{E}_\rho)$ *by the above procedure, for some direct factor* $\mathscr{E}_\rho$ *of* $\boldsymbol{\omega}^{\otimes m}$. *Then* $\hat{f}$ *is rational over* L *if and only if* f *has Fourier coefficients in* L.

We denote by $G_\rho(\Gamma(N), L)$ the set of f in $G_\rho(\Gamma(N))$ with L-rational Fourier coefficients. We shall be content to prove the following:

4.9. THEOREM. — *Let k be a positive even integer, and let* $f \in G_k(\Gamma(N))$. *In the notation of* 4.7, *let* $\hat{f}$ *be the corresponding section of* $\mathscr{L}^{\otimes k/2}$. *Then the conclusion of Theorem* 4.8 *holds.*

Proof. — Under the procedure of 4.7, and the identification $\mathrm{Sym}^2 \boldsymbol{\omega} \simeq \Omega$ of 4.4, we see that:

$$\hat{f} = (2\pi i)^{nk/2} f \, (\det DZ)^{\otimes k \times 2}. \tag{4.9.1}$$

Our proof uses the methods of Shimura in [32]. Let g be any rational function on $\mathscr{M}_N$, arithmetic over L. Then dg is a meromorphic section of Ω, and as in 4.7 we may define $D(dg)$ as a section of $\mathscr{L}$ (*cf.* [33], p. 265): we find that:

$$D(dg) = \det\left(\frac{1+\delta_{ij}}{2}\,\frac{\partial g}{\partial Z_{ij}}\right). \det DZ = (\det dg).(\det DZ); \tag{4.9.2}$$

the latter equality is the *definition* of det dg.

As in [33], Theorem 3, we may write $g = g_1/g_2$, where $g_i \in G_\lambda(\Gamma(N), L)$, $i = 1, 2$, for an integer $\lambda \gg 0$. Here we are using the fact that Shimura's canonical model for $\mathscr{M}_N$ is the modular variety of [25]; *cf.* [6]. Let X be the divisor of $D(dg)$; it is rational over L. For any positive integer μ, denote by $\mathbf{L}(\mu X, L)$ the linear system (over L) associated to μX. Just as in the proof of Theorem 6 of [32], we see that:

$$(\det dg)^\mu \, \mathbf{L}(\mu X, L) = G_{2\mu}(\Gamma(N), L). \tag{4.9.3}$$

Since, as noted above, every function in $\mathbf{L}(\mu X, L)$ can be written as a quotient of elements of $G_\lambda(\Gamma(N), L)$, $\lambda \gg 0$, our theorem follows from (4.9.3), (4.9.1), and the Lemma:

4.9.4. LEMMA. — *Suppose* $f \in G_k(\Gamma(N))$ *can be written* $f = g_1/g_2$ *with* $g_1 \in G_{k+\lambda}(\Gamma(N, L)$, $g_2 \in G_\lambda(\Gamma(N), L)$, *for some integer* $\lambda > 0$. *Then* $f \in G_k(\Gamma(N, L)$.

Proof. — This is Lemma 1 of Baily's paper [3].

4.10. COROLLARY. — *If* $\hat{f} \in H^0(\mathcal{M}_{N, \mathbf{C}}, \mathcal{L}^{\otimes k/2})$ *is* pseudo-arithmetic *over* L, *then* $\hat{f}$ *corresponds to an element* $f \in G_k(\Gamma(N), L)$ (*by the procedure of* 4.7).

Proof. — We have only to remark that $\mathcal{L}$, regarded as a subbundle of $\boldsymbol{\omega}^{\otimes 2n}$ *via* the canonical embedding $\mathrm{Sym}^n(\mathrm{Sym}^2 \boldsymbol{\omega}) \subset \boldsymbol{\omega}^{\otimes 2n}$, is defined by $\mathbf{Q}$-rational symmetry conditions (*cf.* 4.6.5 and the reference there to Weyl). The Corollary now follows from 4.6.6 and 4.9.

4.11. In terms of the "good" basis $\{2\pi i\, du_j\}$, $j = 1, \ldots, n$, for $\boldsymbol{\omega}$, we may rewrite the formula (4.3.3) for ϑ:

$$\vartheta(2\pi i\, du_k) = \frac{1}{2\pi i} \sum_{j=1}^{n} ((Z - \overline{Z})^{-1}(2\pi i\, du))_j \otimes (2\pi i\, du_k) \circ (2\pi i\, du_j);$$

we have used the identification of 4.4.3.

5. Andrianov's zeta functions

5.0. In this section we state some of the Theorems obtained by Andrianov and Kalinin in their investigation of zeta functions of Rankin-Selberg type attached to holomorphic cusp forms for the Siegel modular group ([2], [49], [50]).

We do not state the most general Theorems, but rather those which can be expressed with a minimum of notation.

5.1. Assume that f is a cusp form of weight $k \in 2\mathbf{Z}$ for the full Siegel modular group of genus n, and that f is an eigenfunction for the full Hecke algebra. We shall not go into the details here, but Satake's theory provides, for each prime p, a (one-to-many) correspondence between the set of characters λ_p of the local Hecke algebra H_p and the set of n-tuples $(\alpha_{1,p} \ldots \alpha_{n,p})$ of nonzero complex numbers. This correspondence is normalized in such a way that two n-tuples give rise to the same character if and only if they differ by an element of the Weyl group of $\mathrm{Sp}(2n)$, which operates on the set of $(\alpha_{1,p}, \ldots, \alpha_{n,p})$ in the obvious way (*cf.* the proof of Proposition 3.5.1). For more details, *cf.* [29], [2]. If $(\alpha_{1,p}, \ldots, \alpha_{n,p})$ is one of the n-tuples attached to the Hecke eigenform f for the prime p and ψ is a primitive Dirichlet character of conductor m (say), we define:

$$L_p(f, s, \psi) = \left[\left(1 - \frac{\psi(p)}{p^s}\right) \prod_{i=1}^{n} \left(1 - \frac{\psi(p)\alpha_{i,p}}{p^s}\right)\left(1 - \frac{\psi(p)\alpha_{i,p}^{-1}}{p^s}\right)\right]^{-1}$$

(this is evidently invariant under the Weyl group), and let:

$$L(f, s, \psi) = \prod_p L_p(f, s, \psi). \tag{5.1.1}$$

This is the Euler product attached in Langlands' monograph [21] to f and the standard representation of the L-group $SO(2n+1)$ of G; in particular, it converges absolutely in some right half-plane.

In the notation of 1.2, we write $\mathcal{S}$ for $\mathcal{S}_1$, and we assume f has the Fourier expansion 1.2.1. Choose $N \in \mathcal{S}$, $N > 0$, and define:

$$D_N(f, s, \psi) = \sum_{M \in SL(n, \mathbf{Z}) \backslash M^+(n, \mathbf{Z})} \psi(\det(M))\, a(MNM')(\det(M))^{-s-k+1},$$

where $M^+(n, \mathbf{Z})$ is the subset of $M(n, \mathbf{Z})$ of elements with positive determinant. Let:

$$Z_N(s, \psi) = L\left(s + \frac{n}{2}, \chi_N \psi\right) \prod_{i=0}^{(n/2)-1} L(2s+2i, \psi^2)$$

(a product of Dirichlet L-functions), where χ_N is the quadratic Dirichlet character defined in [1] and [2]:

$$(5.1.2)\quad \begin{cases} \chi_N(d) = (\operatorname{sign} d)^{n/2}\left(\dfrac{(-1)^{n/2}\det 2N}{|d|}\right), & d \text{ odd} \quad (d, \det 2N) = 1, \\ \chi_N(2) = \left(\dfrac{2}{(-1)^{n/2}\det 2N}\right) & (2, \det 2N) = 1, \\ \chi_N(p) = 0, & p \mid \det 2N. \end{cases}$$

Here $(-)$ is the Legendre symbol. As noted in [1], the conductor of χ_N divides $2\det 2N$. Of course $Z_N(s, \psi)$ has an Euler product $\prod_p Z_{N,p}(s, \psi)$:

$$Z_{N,p}(s, \psi) = \left[\left(1 - \frac{\chi_N(p)\psi(p)}{p^{s+(n/2)}}\right) \prod_{i=0}^{(n/2)-1}\left(1 - \frac{\psi^2(p)}{p^{2s+2i}}\right)\right]^{-1}.$$

The results of Andrianov and Kalinin may be summarized as follows:

5.2. THEOREM. – *Let* $q = \det 2N$. *Let* f *and* ψ *be as in* 5.1:

1. *for* Re s *sufficiently large*, $D_N(f, s, \psi)$ *converges absolutely to a holomorphic function of* s;

2. $D_N(f, s, \psi)$ *has an Euler product:*

$$D_N(f, s, \psi) = \prod_{\substack{p \\ (p, q)=1}} D_{N,p}(f, s, \psi) . D_q^1(f, s, \psi),$$

where:

3. *if* $(p, q) = 1$, *then:*

$$L_p(f, s, \psi)^{-1} D_{N,p}(f, s, \psi) = Z_{N,p}(s, \psi)^{-1},$$

and

4. $(\prod_{p \mid q} L_p(f, s, \psi)^{-1}).D_q^1(f, s, \psi)$ *is a finite Dirichlet series* $B_N(f, s, \psi) = \sum_{d \mid q^n} b_d d^{-s}$. *Let* N *be such that, for any* $N' \in \mathcal{S}$ *with* $\det N' < \det N$, *we have* $a(N') = 0$. *Such an* N *is called* minimal for f.) *Then*

5.2.4.1. $$B_N(f, s, \psi) = a(N) \prod_{p \mid q} B_{N,p}(f, s, \psi),$$

where $B_{N,p}(f, s, \psi)$ *is a polynomial in* p^{-s} *which divides (as a polynomial) the polynomial* $Z_{N,p}(s, \psi)^{-1}$ *in* p^{-s};

5. *the infinite product for* $L(f, s, \psi)$ *converges absolutely to a holomorphic function for* $\operatorname{Re} s > n+1$, *and extends meromorphically to the entire complex plane.*

This result is not stated as such in Andrianov's papers, but it follows immediately from Proposition 8.3 of [49], formulas 5.40 and 5.41 of [50], Theorem 1 of [2], and formulas 5.1.3 and 5.1.4 of [54].

5.3. As far as this author knows, a functional equation for $L(f, s, \psi)$ has only been proved when $\psi = \psi_0$ is trivial and when, for some $N_0 \in \mathscr{S}$, $a(N_0) \neq 0$ and $\det(2N_0) = 1$. In that case, the functional equation is:

5.3.1. $$\Psi(f, s) \overset{\text{def}}{=} \pi^{-((2n+1)/2)s}\,\Gamma\left(\frac{s}{2}\right)\prod_{i=1}^{n}\Gamma\left(\frac{s+k-i}{2}\right)\Gamma\left(\frac{s+k-i+1}{2}\right)L(f, s, \psi_0)$$
$$= \Psi(f, 1-s).$$

The *critical points* for $L(f, s, \psi_0)$, in Deligne's sense, are the integers s at which the Γ-factors for $\Psi(f, s)$ and $\Psi(f, 1-s)$ have no pole, and at which $L(f, s)$ has no pole. (That is, *if* there were a motive whose Euler product happened to be 5.1.1, and whose functional equation happened to be 5.3.1, *then* the critical points of that motive, in the sense of [7], would be determined by the above considerations.) Now for $\operatorname{Re} s > (n+2)/2$ the integral representation given in 5.4 below for $D_N(f, s, \psi)$ is holomorphic, so if there is an $N_0 \in \mathscr{S}$ with $a(N_0) \neq 0$ and $\det(2N_0) = 1$, then (*cf.* 5.3.3 below):

$$L(f, s, \psi_0) = \frac{1}{a(N_0)} Z_{N_0}(f, s, \psi_0) . D_{N_0}(f, s, \psi_0)$$

is holomorphic for $\operatorname{Re} s > (n+2)/2$. With this in mind, we define the *critical set* for f (actually, half of the critical set) to be:

5.3.2. $$s \in \mathbf{Z}, \quad s \text{ even}, \quad \frac{n+2}{2} < s < k-n+1.$$

Even if f is not known to have a functional equation, we may define, for any primitive Dirichlet character ψ such that $\psi(-1) = 1$, the *critical set* for (f, ψ) to be the set 5.3.2. It will be seen in 5.4, below, that $D_N(f, s, \psi)$ is holomorphic at such points for any $N \in \mathscr{S}$. It then follows easily from Theorem 5.2 that $L(f, s, \psi)$ is also holomorphic at these points: one has only to choose N *minimal* for f, such that $a(N) \neq 0$.

Our methods allow us to study the values of $L(f, s, \psi)$ at such points.

5.4. We now describe the integral representation for $D_N(f, s, \psi)$. Fix N; let $q = 2(\det 2N)m^2$, where m is the conductor of the (primitive) Dirichlet character ψ. Let $\mathscr{D}$

be a fundamental domain for $\Gamma(q)$ in $\mathscr{S}_n$. Now the formulas in paragraph 2, 4 of [2], and the duplication formula for the gamma function yield the representation:

5.4.1. $$D_N(f, s, \psi) = 2^{ns}(\det 2N)^{s/2} \frac{\varepsilon_0 \pi^{(n/2)(s+k-(n/2)-1)}}{\prod_{i=1}^{n/2} \Gamma(s+k-2i)}$$
$$\times \int_{\mathscr{D}} f(Z)\, \overline{\Theta_{2N}(Z, \overline{\psi}) E_{k-(n/2)}\left(Z, \frac{1}{2}(n-k+s), \overline{\psi}\chi_N\right)}\, (\det Y)^k \frac{dX\,dY}{(\det Y)^{n+1}},$$

where $\varepsilon_0 \in \mathbf{Q}^\times$, $\overline{\psi}$ is the complex conjugate of ψ, and $\Theta_{2N}(Z, \overline{\psi})$ and $E_{k-(n/2)}(Z, (1/2)(n-k+s), \overline{\psi}\chi_N)$, the functions defined in 1.4, are C^∞ modular forms of weights $n/2$ and $k-(n/2)$, respectively, with respect to $\Gamma(q)$. If s is in the critical set, then the Eisenstein series is absolutely convergent and $\prod_{i=1}^{n/2} \Gamma(s+k-2i) \in \mathbf{Q}^\times$, so, as in paragraph 2,5 of [2], $D_N(f, s, \psi)$ is holomorphic at s. Moreover, at such s, it follows from 5.4.1 and 1.5 that, for some $\varepsilon_1 \in \mathbf{Q}^\times$, we have:

$$D_N(f, s, \psi) = \varepsilon_1 \pi^{(n/2)(-(3n/2)+2k-1)}$$
$$\times \int_{\mathscr{D}} f(Z)\, \overline{\Theta_{2N}(Z, \overline{\psi})\, \delta_l^{r/2}(E_l(Z, \chi_N . \overline{\psi}))}\, (\det Y)^k \frac{dX\,dY}{(\det Y)^{n+1}}$$
$$= \varepsilon_1 \operatorname{vol}(\mathscr{D}) \pi^{(n/2)(-(3n/2)+2k-1)} \langle f, \Theta_{2N}(Z, \overline{\psi}) . \delta_l^{(r/2)}(E_l(Z, \chi_N . \overline{\psi})) \rangle_k$$

in the notation of paragraph 1, where we have set $r = k-n-s$, $l = k-(n/2)-r$, and where, for any positive integers a and b, we write $\delta_a^{(b)} = \delta_{a+2b-2} \ldots \delta_{a+2} . \delta_a$.

By a Theorem of Siegel ([41], p. 57):

$$\operatorname{vol}(\mathscr{D}) = \varepsilon_2 \pi^{n(n+1)/2}, \qquad \varepsilon_2 \in \mathbf{Q}^\times.$$

It follows that, when s is in the critical set, we have, for some $\varepsilon \in \mathbf{Q}^\times$,

5.4.2. $$D_N(f, s, \psi) = \varepsilon \pi^{(n/2)(2k-(n/2))} \langle f, \Theta_{2N}(Z, \overline{\psi}) . \delta_l^{r/2}(E_l(Z, \chi_N . \overline{\psi})) \rangle_k.$$

5.5.1. We now make a few remarks that will be used in paragraph 7. First, if:

$$G_{\mathbf{Q}^+} = \{ g \in M(2n, \mathbf{Q}) \mid g J g' = v(g) J, \text{ some } v(g) \in \mathbf{Q},\ v(g) > 0 \},$$

the Hecke operators can be defined on $G_k(\Gamma(1))$ by the action of double cosets $\Gamma(1) g \Gamma(1)$ with $g \in G_{\mathbf{Q}^+}$, as in [50]. Since $G_{\mathbf{Q}^+}$ preserves $G_k(\Gamma(1), \overline{\mathbf{Q}})$ (*cf.* [33], thm. 4), it follows that the eigenvalues of the Hecke operators are *algebraic* numbers.

5.5.2. Similarly, let k, $N \in \mathbf{Z}$, and let $G_k(N) \subset G_k(\Gamma(N))$ be the orthogonal complement to $S_k(\Gamma(1))$ with respect to $\langle\ ,\ \rangle_k$. The projection map:

$$\pi : \quad G_k(\Gamma(N)) \to G_k(\Gamma(1)),$$
$$\pi(f)(Z) = [\Gamma(1) : \Gamma(N)]^{-1} \sum_{\sigma \in \Gamma(1)/\Gamma(N)} \det(J(\sigma, Z))^{-k} f(\sigma(Z))$$

takes $G_k(\Gamma(N), \overline{\mathbf{Q}})$ to $G_k(\Gamma(1), \overline{\mathbf{Q}})$ ([33], Thm. 4), and its kernel is contained in $G_k(N)$, by the invariance properties of $\langle\ ,\ \rangle_k$. Thus, if $g \in G_k(\Gamma(N), \overline{\mathbf{Q}})$, there is an $h \in G_k(\Gamma(1), \overline{\mathbf{Q}})$, such that $\langle f, g\rangle_k = \langle f, h\rangle_k$ for all cusp forms $f \in S_k(\Gamma(1))$.

These remarks have obvious translations into the language of paragraph 4 above.

6. Canonical differential operators

6.0. All the differential operators introduced so far arise from the canonical differential operators defined by the action of the universal enveloping algebra $U(\mathfrak{p}^+)$. Thus, let $W \subset U(\mathfrak{p}^+) \simeq S(\mathfrak{p}^+)$ be a finite-dimensional subspace invariant under the adjoint action of K; let (ρ, V_ρ) be a finite-dimensional representation of $K_{\mathbf{C}}$. Then there is a differential operator:

6.0.1. $$D_W : \quad C^\infty(G, V_\rho)_\rho \to C^\infty(G, V_\rho \otimes W^*)_{\rho \otimes W^*};$$

here $\rho \otimes W^*$ is the representation of K on $V_\rho \otimes W^*$. If we regard $V_\rho \otimes W^*$ K-equivariantly as $\mathrm{Hom}(W, V_\rho)$, then D_W is defined by:

6.0.2 $$D_W(f)(X) = X \star f \in V_\rho, \qquad \forall f \in C^\infty(G, V_\rho), \quad X \in W.$$

Such an operator will be called a *transition operator;* it commutes with direct sums $\rho \simeq \rho_1 \oplus \rho_2$. The basic transition operator, denoted D, corresponds to the case $W = \mathfrak{p}^+$; all the others are derived from symmetric powers $D^n = \mathrm{Sym}^n D$ for positive integers n. The operator D acts *via* the product rule on tensor products $\rho \simeq \rho_1 \otimes \rho_2$.

Of course, since every irreducible ρ is equivalent to a direct factor of $\underline{\mathrm{St}}^{\otimes m}$ for some (possibly negative) integer m, all the maps 6.0.1 are determined by the product rule from the single map:

6.0.3. $$D_{\mathrm{can}} \overset{\mathrm{def}}{=} D : \quad C^\infty(G, V_{\underline{\mathrm{St}}})_{\underline{\mathrm{St}}} \to C^\infty(G, V_{\mathrm{St}} \otimes (\mathfrak{p}^+)^*)_{\mathrm{St} \otimes \mathrm{Sym}^2(\mathrm{St})}.$$

6.1. We recall that Proposition 3.1 completely describes the representations $\alpha_1 \geqq \ldots \geqq \alpha_n$ that occur in $U(\mathfrak{p}^+)$. The case $\alpha_1 = \ldots = \alpha_n = -2k$, for some positive integer k, corresponds to the representation $\det^{-2k}$ of $K_{\mathbf{C}}$, and occurs in $\mathrm{Sym}^{nk}(\mathfrak{p}^+)$ with multiplicity one. When $k=1$, the corresponding subspace of $\mathrm{Sym}^n(\mathfrak{p}^+)$ is just the one-dimensional subspace denoted L in 4.7, where we take the space V of 4.7 to be V_{St^*}. In general, we may regard $L^{\otimes k}$, $k = 0, 1, 2, \ldots$, as a subspace of $\mathrm{Sym}^{nk}(\mathfrak{p}^+)$, on which $K_{\mathbf{C}}$ acts through the representation $\det^{-2k}$. It follows from Proposition 3.1 that $D_{I\otimes k}$ is the unique transition operator from $C^{\prime}(G, V_{\rho})_{\rho}$ to $C^{\prime}(G, V_{\rho\otimes 2k})_{\rho\otimes 2k}$, in the notation of 4.7.3.

6.2. We want to compute the map D_{can}. For this, suppose $\varphi \in C^{\prime}(G, V_{St})_{St}$; we represent φ, as in 2.2, as a product $\varphi(g)=J(g, i\mathrm{I})^{-1} f(g(i\mathrm{I}))$, for some $f \in C^{\infty}(\mathfrak{S}_n, V_{St})$. If $g(i\mathrm{I})=Z \in \mathfrak{S}_n$, we write g uniquely as a product,

6.2.1. $g=g_Z.k$ for g_Z as in 1.0.5 and $k \in K$.

Let $p_+(\alpha)$ be as in 2.0.2. To compute $D_{\text{can}}(\varphi)$, it is enough to know $p_+(\alpha) \star \varphi$ for all symmetric $\alpha \in M(n, \mathbf{C})$; we may even assume $\alpha \in M(n, \mathbf{R})$. We write $p_+(\alpha)=X_{\alpha}^0+iX_{\alpha}^1$, where:

$$X_{\alpha}^0=\frac{1}{2}\begin{pmatrix}\alpha & 0\\ 0 & -\alpha\end{pmatrix}, \qquad X_{\alpha}^1=\frac{1}{2}\begin{pmatrix}0 & \alpha\\ \alpha & 0\end{pmatrix}.$$

Then:

6.2.2. $$(p_+(\alpha)) \star \varphi(g)=\frac{d}{dt}[\varphi(g\exp(tX_{\alpha}^0))+i\varphi(g\exp(tX_{\alpha}^1))]|_{t=0}.$$

Using the product rule, we may as well work out 6.2.2 for $f(g(i\mathrm{I}))$ and $J(g, i\mathrm{I})^{-1}$ separately.

Now $J(g\exp(tp_+(\alpha)), i\mathrm{I})^{-1}$ makes sense as a complex matrix, at least for t sufficiently small; we need not split $p_+(\alpha)$ into real and imaginary parts. By 6.2.1, we have:

$$J(g.\exp(tp_+(\alpha)), i\mathrm{I})^{-1}=J(g_Z(\exp(tkp_+(\alpha)k^{-1}))k, i\mathrm{I})^{-1}$$

$$=J(k, i\mathrm{I})^{-1}J(g_Z.(\exp(tkp_+(\alpha)k^{-1})), i\mathrm{I})^{-1}$$

(by 1.0.4)
$$=J(k, i\mathrm{I})^{-1}.J(g_Z.\exp(tp_+(k(\alpha))), i\mathrm{I})^{-1}$$

(by 2.0.3), where $k(\alpha)\overset{\text{def}}{=}(k^t)^{-1}\alpha k^{-1}$ is still symmetric. Using 1.0.4 again, we get:

$$J(g_Z.\exp(tp_+(k(\alpha))), i\mathrm{I})^{-1}$$

$$=J(\exp(tp_+(k(\alpha)), i\mathrm{I})^{-1}.J(g_Z, \exp(tp_+(k(\alpha))(i\mathrm{I}))^{-1}$$

$$=J(\exp(tp_+(k(\alpha)), i\mathrm{I})^{-1}.Y^{1/2} \quad (cf.\ 1.0.5).$$

Therefore, if $g=g_Zk$ as above, we have:

6.2.3. $p_+(\alpha) \star J(g, i\mathrm{I})^{-1}$

$$=J(k, i\mathrm{I})^{-1}\left(\frac{d}{dt}J(\exp(tp_+(k(\alpha)), i\mathrm{I}))^{-1}Y^{1/2}\right)=k^{-1}(k(\alpha))Y^{1/2},$$

as follows from a brief computation; we have identified k with $J(k, i\mathrm{I})$ as in 1.0.3.

Now we compute $p_+(\alpha) \star f(g(i\mathrm{I}))$. We have:

6.2.4.1. $f(g\exp(tX_{\alpha}^0)(i\mathrm{I}))=f(g_Z\exp(tX_{k(\alpha)}^0)(i\mathrm{I}))$

$$=f(g_Z(i\exp(tk(\alpha)))=f(X+iY^{1/2}(\exp(tk(\alpha)))Y^{1/2})$$

$$=f(X+i(Y+tY^{1/2}k(\alpha)Y^{1/2})+O(t^2))$$

whereas:

6.2.4.2. $f(g\exp(tX^1_\alpha)(iI))=f(g_Z\exp(tX^1_{k(\alpha)})(iI))$
$$=f(X+tY^{1/2}k(\alpha)Y^{1/2}+iY+O(t^2)).$$

It is convenient to compute $p_+(\alpha)\star f(g_Z(iI))$ when $\alpha=\alpha_{lj}$ as defined in 2.3.2. Let df/dZ be the symmetric $n\times n$ matrix whose lj entry is $((1+\delta_{lj})/2)(\partial f/\partial Z_{lj})$ (which is itself a column vector of length n). It follows from 6.2.4.1 and 6.2.4.2 that:

6.2.5. $p_+(\alpha_{lj})\star f(g_Z(iI))$ is the lj entry of the matrix $2iY^{1/2}(df/dZ(Z))Y^{1/2}$.

Putting all this together, we see that we may represent $D_{\mathrm{can}}\varphi$ as an $n\times n$ symmetric matrix, whose entries are column vectors of length n; if $g=g_Z k$, then:

6.2.6. $$D_{\mathrm{can}}\varphi(g)=J(g,iI)^{-1}.2iY^{1/2}(k^t)^{-1}\left(\frac{df}{dZ}(Z)\right)k^{-1}Y^{1/2}+\Lambda,$$

where $J(g,iI)^{-1}$ acts on the column vectors which are the entries of the matrix, and where the jl entry of Λ is the column vector $k^{-1}(k(\alpha_{jl}))Y^{1/2}f(Z)$. The equality in 6.2.6 is with reference to the chosen basis $\{p_+(\alpha_{lj})\}$ of $\mathfrak{p}^+$.

6.3. We have written $D_{\mathrm{can}}\varphi$ in terms of the dual basis to $\{p_+(\alpha_{lj})\}$, and in terms of the standard basis $\{e_1,\ldots,e_n\}$ of $V_{\underline{St}}$. But $D_{\mathrm{can}}\varphi$ is of type $\underline{St}\otimes\mathrm{Sym}^2(\underline{St})$ with respect to K; in other words, in the notation of paragraph 2, 2.2:

$$D_{\mathrm{can}}\varphi=\varphi_{\Delta_{\mathrm{can}}f,\,\underline{St}\otimes\mathrm{Sym}^2(\underline{St})}$$

for some function $\Delta_{\mathrm{can}}f\in C^\infty(\mathfrak{S}_n, V_{\underline{St}}\otimes(\mathfrak{p}^+)^*)$. We may compute $\Delta_{\mathrm{can}}f$ explicitly: By definition:

6.3.1. $$\Delta_{\mathrm{can}}f(Z)=J_{\underline{St}\otimes\mathrm{Sym}^2(\underline{St})}(g,iI)\,D_{\mathrm{can}}\varphi(g)$$

if $g=g_Z k$ for any $k\in K$. When we unwind the action of $J_{\underline{St}\otimes\mathrm{Sym}^2\underline{St}}(g,iI)$, we find that:

6.3.2. $$\Delta_{\mathrm{can}}f(Z)=2i\frac{df}{dZ}(Z)+\Lambda',$$

where the jl entry of Λ' is the column vector:

$$Y^{-1}(\alpha_{jl})f(Z);$$

i.e., if we think of Λ' as a function from $\mathfrak{S}_n$ to $\mathrm{Hom}(\mathfrak{p}^+, V_{\underline{St}})$, then $\Lambda'(Z)(p_+(\alpha_{jl}))=Y^{-1}(\alpha_{jl})f(Z)$. This computation is carried out most easily using 6.2.3, and recalling the convention that if $w\in\mathrm{Hom}(V_{\rho_1},V_{\rho_2})$, $k\in GL(n,\mathbf{C})$, $v\in V_{\rho_1}$, then:

$$k(w)(v)=\rho_2(k)w(\rho_1^{-1}(k)v).$$

6.4. As in 4.0, let $e_1,\ldots,e_n$ be the standard basis for $\mathbf{C}^n$. As in 1.6.4, we may define the global sections $\tilde{e}_i$ of the homogeneous vector bundle $\mathscr{E}_{\mathrm{St}}$ over $\mathfrak{S}_n$. If, as above, we choose an

$f \in C^\infty(\mathfrak{S}_n, V_{\underline{St}})$, we may identify the corresponding global section $\tilde{f}$ of $\mathscr{E}_{\underline{St}}$ (*cf.* 2.5) with the sum:

$$\tilde{f} = \sum_{i=1}^{n} f_i \tilde{e}_i,$$

where $f_i \in C^\infty(\mathfrak{S}_n, \mathbf{C})$ is the i-th component of f.

As explained in 4.7, the good trivialization of the vector bundle $\boldsymbol{\omega}$ is given by the global sections $2\pi i\, du_1, \ldots, 2\pi i\, du_n$. The map which sends $\tilde{e}_j$ to $2\pi i\, du_j$, $j=1, \ldots, n$, identifies $\tilde{f}$ as defined above with $\tilde{f}$ as defined in 4.7.4, and defines an isomorphism of homogeneous vector bundles:

$$\gamma_1 : \quad \mathscr{E}_{\underline{St}} \xrightarrow{\sim} \mathscr{H}^{1,0}_{\alpha}.$$

We have identified $\mathscr{E}_{\mathrm{Sym}^2(\underline{St})}$ with $\mathscr{E}_{(\mathfrak{p}^+)^*}$, the homogeneous vector bundle associated to the adjoint representation of K on $(\mathfrak{p}^+)^*$. The dual basis in $\mathrm{Sym}^2(\underline{St})$ to the basis $\{p_+(\alpha_{jk})\}$ of $\mathfrak{p}^+$ is denoted $\{Y_{jk}\}$. We have also identified $\mathscr{E}_{\mathrm{Sym}^2(\underline{St})}$ with Ω. The map (notation 1.6.4):

$$\hat{Y}_{jk} \mapsto \left(\frac{2}{1+\delta_{jk}}\right) 2\pi i\, dZ_{jk}$$

is easily seen to define an isomorphism of homogeneous vector bundles:

$$\gamma_2 : \quad \mathscr{E}_{\mathrm{Sym}^2(\underline{St})} \xrightarrow{\sim} \Omega.$$

Of course $\hat{Y}_{jk} = \hat{Y}_{kj}$. The two isomorphisms γ_1 and γ_2 are compatible with one another, in view of 4.4.3.

The formula 6.3.2 is a computation of $\widetilde{\Delta_{\mathrm{can}}(f(Z))}$ in terms of the global trivialization $\{\tilde{e}_i \otimes \hat{Y}_{jk}\}$, $i, j, k = 1, \ldots, n$, $j \leqq k$. On the other hand, we have computed the differential operator ϑ in 4.3; it follows from 4.11 that:

$$\vartheta\left(\sum_{j=1}^{n} f_j (2\pi i\, du_j)\right)$$

$$= \sum_{j=1}^{n} (2\pi i\, du_j \otimes df_j) + \sum_{j=1}^{n} f_j \left(\sum_{j=1}^{n} (Z-\overline{Z})^{-1} \frac{1}{2\pi i} du\right)_k \otimes dZ_{jk}$$

$$= \sum_{j=1}^{n} 2\pi i\, du_j \otimes \sum_{k,\, l=1}^{n} \left(\frac{1+\delta_{kl}}{2}\right) \frac{1}{2\pi i} \frac{\partial f_j}{\partial Z_{kl}} . 2\pi i\, dZ_{kl}$$

$$+ \sum_{j=1}^{n} \frac{f_j}{2\pi i} \left(\sum_{j=1}^{n} (Z-\overline{Z})^{-1}\, 2\pi i\, du\right)_k \otimes 2\pi i\, dZ_{jk}.$$

An elementary computation, using the symmetry of the matrix Z, now yields:

6.5. THEOREM. — *The following diagram is commutative:*

$$\begin{tikzcd}
\mathscr{E}_{\underline{St}} \arrow[r, "\gamma_1"] \arrow[d, "-\frac{1}{4\pi}\tilde{\Delta}_{\mathrm{can}}"'] & \mathscr{H}^{1,0} \arrow[d, "\vartheta"] \\
\mathscr{E}_{\underline{St}} \otimes_{C^\infty(\mathfrak{S}_n, C)} \mathscr{E}_{\mathrm{Sym}^2(\underline{St})} \arrow[r, "\gamma_1 \otimes \gamma_2"] & \mathscr{H}^{1,0} \otimes_{C^\infty(\mathfrak{S}_n, C)} \Omega
\end{tikzcd}$$

where $\tilde{\Delta}_{\mathrm{can}}$ *is the differential operator on vector bundles corresponding* (via 2.5) *to the differential operator* Δ_{can} *of* 6.3.

6.6. The above theorem suggests that we define the *arithmetic* **Q**-form of $\mathfrak{g}_{\mathbf{C}}$, to be, in the notation of 3.2, $\mathfrak{g}_{\mathbf{Q}}=\mathfrak{g}_{\mathbf{Q},\beta}$, with $\beta=-1/4\pi$. We let $\mathrm{U}_{\mathbf{Q}}=\mathrm{U}(\mathfrak{g}_{\mathbf{Q}})$, the enveloping algebra of $\mathfrak{g}_{\mathbf{Q}}$ over **Q**. We define the transition operators $\mathrm{D}_{\mathrm{W},\mathbf{Q}}$ analogously. Through the correspondence $\varphi\leftrightarrow\tilde{f}$ of 2.5 [with the conventions (4.4.2) and (4.7.4)], the operators $\mathrm{D}_{\mathrm{W},\mathbf{Q}}$ act on the sheaf of algebras $\bigoplus_{m=0}^{\infty}(\mathscr{H}^{1,0}_{\infty})^{\otimes m}$. Using the product rule, it follows from Theorem 6.5 that:

6.6.1. Corollary. — *Let* W *be as in* 6.0. *The action of the operator* $\mathrm{D}_{\mathrm{W},\mathbf{Q}}$ *on* $\bigoplus_{m=0}^{\infty}(\mathscr{H}^{1,0}_{\infty})^{\otimes m}$ *preserves* $\mathscr{R}(\mathrm{N},\mathrm{L})$ *for any integer* N *and any field* $\mathrm{L}\supset\mathbf{Q}(\zeta_{\mathrm{N}})$.

6.7. The formula 1.5.3 states that the Maaß operator δ_α corresponds to a *homogeneous differential operator* on the line bundles over $\mathfrak{S}_n$:

$$\tilde{\delta}_\alpha:\ \mathscr{E}_\alpha\to\mathscr{E}_{\alpha+2},\qquad \alpha\in\mathbf{Z}$$

commutes with the action of G. It follows from general nonsense (*cf.* [42], 5.4.11), that there is an element $\delta'_\alpha\in\mathrm{U}(\mathfrak{g}_{\mathbf{C}})$ such that, under the correspondence $\varphi\leftrightarrow\tilde{f}$ of 2.5:

(6.7.1) $\delta'_\alpha\star\varphi\in\mathrm{C}^\infty(\mathrm{G},\mathrm{V}_{\alpha+2})$ corresponds to $\tilde{\delta}_\alpha\tilde{f}\in\mathrm{H}^0(\mathfrak{S}_n,\mathscr{E}_{\alpha+2})$.

Now suppose f is a holomorphic modular form of weight α, corresponding to a function $\varphi\in\mathrm{C}^\infty(\mathrm{G},\mathrm{V}_\alpha)_\alpha$ of holomorphic type. Then, for each integer $r>0$, there is an element $\delta''^{(r)}_\alpha\in\mathrm{U}(\mathfrak{p}^+)$ such that:

$$\delta''^{(r)}_\alpha\star\varphi=(\delta'_\alpha)^r\star\varphi \tag{6.7.2}$$

simply because $\mathrm{U}(\mathfrak{k}_{\mathbf{C}}\oplus\mathfrak{p}^-)$ acts through a character on φ.

On the other hand, it is clear that, for $\alpha\geqq n/2$, $\delta''^{(r)}_\alpha$ is uniquely determined by 6.7.2. In fact, by 3.6.2, D_α is irreducible; thus $\mathscr{V}_f$ is free of rank one over $\mathrm{U}(\mathfrak{p}^+)$ (*cf.* 2.7). It follows that $\delta''^{(r)}_\alpha$ is a transition operator of type $\det^{2r}$ (with respect to the adjoint action of K). On the other hand, in terms of the bases of V_{St} and $(\mathfrak{p}^+)^*$ introduced in 6.2 and 6.3, we may express the operator $\mathrm{D}_{\mathrm{L}^{\otimes r}}$ of 6.1 explicitly as an operator:

$$\mathrm{L}_r:\ \mathrm{C}^\infty(\mathrm{G},\mathrm{V}_\alpha)_\alpha\to\mathrm{C}^\infty(\mathrm{G},\mathrm{V}_{\alpha+2r})_{\alpha+2r}.$$

It follows from the remarks in 6.1 that:

$$\delta''^{(r)}_\alpha=\frac{\beta}{(-4\pi)^{nr}}\mathrm{L}_r\qquad\text{for some }\beta\in\mathbf{C}^\times. \tag{6.7.3}$$

We want to check that $\delta''^{(r)}\in\mathrm{U}_{\mathbf{Q}}$ for all r. For this it suffices, by 6.7.3, to check that $\beta\in\mathbf{Q}^\times$. If X is any differential operator, let X(0) be its term of highest degree. We have to see that:

$$\delta''^{(r)}(0)=\frac{\beta}{(-4\pi)^{nr}}\mathrm{L}_r(0)\qquad\text{for some}\quad\beta\in\mathbf{Q}^\times.$$

Now we have computed that $\Delta_{\mathrm{can}}(0)$ is $B=2i\,d/dZ$. The same method shows that, if:

$$\Delta_{\rho,\,\mathrm{can}}:\quad C^{\infty}(\mathfrak{S}_n,\,V_\rho)\to C^{\infty}(\mathfrak{S}_n,\,V_\rho\otimes(\mathfrak{p}^+)^*)$$

is the operator corresponding (*via* 2.5) to the operator D of 6.0, then:

$$\Delta_{\rho,\,\mathrm{can}}(0)=B\quad\text{for all }\rho.$$

Consequently, we see that:

$$L_r(0)=\left((2i)^n\det\left(\frac{d}{dZ}\right)\right)^r. \tag{6.7.4}$$

On the other hand, one sees from the expression for the Maaß operator on p. 309 of [23] that:

$$\delta_\alpha''^{(r)}(0)=\left(\left(\frac{1}{2\pi i}\right)^n\det\left(\frac{d}{dZ}\right)\right)^r. \tag{6.7.5}$$

Combining (6.7.4) and (6.7.5), we see that $\beta=1$ in (6.7.3); we have proved:

6.8. THEOREM. — *Let f be a holomorphic modular form of weight* $\alpha\geqq n/2$ *for* $\Gamma(N)$, *for some integer* N. *Let r be a positive integer. Then there is an operator* $\mathfrak{M}_\alpha^r\in U(\mathfrak{p}_{\mathbf{Q}}^+)$ *such that, under the isomorphism:*

$$G_{\alpha+2r}^{\infty}(\Gamma(N))\rightrightarrows C^{\infty}(\Gamma(N)\backslash G,\,V_{\alpha+2r})_{\alpha+2r},$$

the function $\delta_\alpha^{(r)}f$ *corresponds to* $\mathfrak{M}_\alpha^r\star\varphi$, *where* $\varphi=\varphi_f\in C^{\infty}(\Gamma(N)\backslash G,\,V_\alpha)_\alpha$.

Of course, the subscript "α" of $\mathfrak{M}_\alpha^r$ is irrelevant.

6.9. COROLLARY. — *Under the above hypotheses, assume f has Fourier coefficients in the subfield* $L\subset\mathbf{C}$. *Then* $\widetilde{\delta_\alpha^{(r)}f}$ *is pseudo-arithmetic over* L.

Proof. — This follows from Theorems 4.9 and 6.8 and Corollary 6.6.1.

6.10. *Remarks.* — 1. Actually, it follows from a density argument, based upon the fact that ϑ descends to $\mathscr{H}_\infty^{1,0}/\mathscr{M}_N$, that ϑ is a *homogeneous* differential operator. Thus the computation of 6.2-6.4 need only have been carried out for the highest order term, as in 6.7.

2. It's worthwhile explaining what Corollary 6.9 really means. By the q-expansion principle, modular forms are rational over L if their Fourier coefficients lie in L. Now the C^∞ modular forms produced by the Gauss-Manin connection and the map Split, or by the transition operators, acting on holomorphic modular forms, have Fourier series whose coefficients $a(N)$ are rational functions of the coordinates of $Y=\mathrm{Im}\,Z$. Those which are pseudo-arithmetic over L are those for which, in a natural sense, the "constant term" of $a(N)$ lies in L for all N. The "meaning" of 6.6 and 6.9 is simply that, when you differentiate a Fourier series with respect to Z_{jk}, a rational multiple of $2\pi i$ comes out as a coefficient of each term in the series; this explains the division by $2\pi i$.

The point is that, when a polynomial in L-pseudo-arithmetic modular forms turns out to be holomorphic (as happens in the next section), it automatically has L-rational Fourier coefficients; this is the meaning of Theorem 4.6.

All the algebraic geometry introduced in paragraph 4 could have been dispensed with, and the Fourier coefficients used throughout. But this has two disadvantages: (1) The computations become long and complicated; (2) From some points of view, the algebraic geometry serves as an explanation of the phenomena encountered when studying Fourier coefficients.

7. Special values of zeta functions

7.0. NOTATION. — We let f be a non-zero cusp form of even weight k for the full Siegel modular group $\Gamma(1)$ acting on the Siegel upper half space $\mathfrak{S}_n$; we assume $4 \mid n$. Assume f is an eigenform for the Hecke algebra, and define the functions $L(f, s, \psi)$ and $D_N(f, s, \psi)$ as in paragraph 5, where $N \in \mathcal{S} \overset{\text{def}}{=} \{ A \in \mathcal{S}_1 \mid A > 0 \}$. When we choose a point s in the critical set 5.3.2 for (f, ψ), we let $r = k - n - s$ and $l = k - (n/2) - r$; then r and l are even integers. To $N \in \mathcal{S}$ we attach the Dirichlet character χ_N as in 5.1.2. We choose a primitive Dirichlet character ψ of conductor m, and assume that $\psi(-1) = 1$. For $N \in \mathcal{S}$, let $q = q_N = 2(\det 2N) m^2$. The functions $\Theta_{2N}(Z, \overline{\psi})$ and $E_l(Z, \overline{\psi}\chi_N)$, defined as in 1.4, are, respectively, modular forms of weight $n/2$ and l, for $\Gamma(q)$. Finally, we define the differential operator $\delta_l^{(r/2)}$ as in 5.4.

7.1. THEOREM. — *Let s be in the critical set for (f, ψ). There is a holomorphic modular form $f_{N, \psi}$, of weight k for $\Gamma(q_N)$, and depending on r, k, ψ, and N, such that, for every cusp form g of weight k for $\Gamma(a)$, $a \in \mathbf{Z}$:*

$$\langle g, \Theta_{2N}(Z, \overline{\psi}) . \delta_l^{(r/2)}(E_l(Z, \overline{\psi}\chi_N)) \rangle_k = \langle g, f_{N, \psi} \rangle_k$$

and such that $f_{N, \psi}$ has cyclotomic Fourier coefficients. If $q_N = 1$, then the Fourier coefficients of f_N are rational.

7.2. COROLLARY. — *Let s satisfy the condition of 5.3.2, and let f be as in 7.0. Then, for some $\varepsilon \in \mathbf{Q}^\times$:*

$$D_N(f, s, \psi) = \varepsilon \pi^{(n/2)(2k-(n/2))} \langle f, f_{N, \psi} \rangle_k .$$

Corollary 7.2 is an immediate consequence of Theorem 7.1 and the formula 5.4.2.

Let $V_f \subset G_k(\Gamma(1))$ be the subspace of Hecke eigenforms with the same eigenvalues as f. If we assume that $k > 2n+1$, then it follows from Theorem 2.5 of [55] that every element of V_f is a *cusp* form. Of course, for any $f' \in V_f$, $L(f', s, \psi) = L(f, s, \psi)$. The proof of the following corollary was suggested by Shimura.

7.3. COROLLARY. — *Assume* $k>2n+1$. *Let* f, ψ, *and* s *be as in* 7.1; *let* $d=(n+1)(s+k)-k$. *If* f *has algebraic Fourier coefficients, then:*

$$\frac{L(f, s, \psi)}{\pi^d \langle f, f\rangle_k} \in \overline{\mathbf{Q}}. \tag{7.3.1}$$

Proof of 7.3 (*assuming Theorem* 7.1). — For any $g \in V_f$, we write its Fourier expansion:

$$g(Z)=\sum_{N\in\mathscr{S}} a(g, N)\, e(\operatorname{Tr}(NZ));$$

this is possible because, by our hypothesis on k, V_f consists of cusp forms. By 5.5.1 we know that V_f is spanned by elements with algebraic Fourier coefficients.

Assume $L(f, s, \psi)\neq 0$ (this is true for $s>n+1$, by 5.2.5); otherwise we have nothing to prove. Since the Hecke operators are Hermitian with respect to $\langle\ ,\ \rangle_k$, it follows from 5.5.1 and 5.5.2 that $\forall N\in S$, $\exists g_{N,\psi}\in V_f\cap G_k(\Gamma(1), \overline{\mathbf{Q}})$ such that, in the notation of Theorem 7.1:

$$\langle g, f_{N,\psi}\rangle_k=\langle g, g_{N,\psi}\rangle_k, \qquad \forall g\in V_f.$$

Now let $N_0\in\mathscr{S}$ be minimal for every $g\in V_f$, in the sense of 5.2.4, with the additional requirement that $a(f, N_0)\neq 0$; in order that this be possible, we may have to replace f by another element of V_f. By Corollary 7.2 and Theorem 5.2, we then have:

$$a(g, N_0) \prod_{p\mid \det 2N_0} B_{N_0,p}(g, s, \psi)\, L(g, s, \psi)$$
$$=\varepsilon\pi^{(n/2)(2k-(n/2))}\Big(\prod_{p\mid\det 2N_0} Z_{N_0,p}(s, \psi)^{-1}\Big)\, Z_{N_0}(s, \psi)\,\langle g, g_{N_0,\psi}\rangle_k$$

for some $\varepsilon\in\mathbf{Q}^\times$, $\forall g\in V_f$.

Let $g_0=g_{N_0,\psi}$. At the s in question, $Z_{N_0,p}(s,\psi)$ and $B_{N_0,p}(s,\psi)$ are non-zero and algebraic for all p; since $\tilde{\chi}_{N_0}(-1)=\psi(-1)=1$ (*cf.* 5.1.2), the well-known formulas for the special values of Dirichlet L-functions imply that, for some $\varepsilon_{N_0,\psi}(g)\in\overline{\mathbf{Q}}^\times$:

$$a(g, N_0)\, L(g, s, \psi)=\varepsilon_{N_0,\psi}(g)\, \pi^d\langle g, g_0\rangle_k, \qquad \forall g\in V_f. \tag{7.3.2}$$

Since we have assumed $L(g, s, \psi)\neq 0$, it follows that the space:

$$g_0^\perp \overset{\text{def}}{=} \{g\in V_f \mid \langle g, g_0\rangle_k=0\}$$

is the space $\{g\in V_f \mid a(g, N_0)=0\}$. By hypothesis, $g_0^\perp\neq V_f$, so $g_0\neq 0$. Since V_f is spanned by elements with $\overline{\mathbf{Q}}$-rational Fourier coefficients, the above characterization of $g_0^\perp$ implies that $g_0^\perp$ is also spanned by elements with $\overline{\mathbf{Q}}$-rational Fourier coefficients. Furthermore (7.3.2) implies that:

$$\frac{L(f, s, \psi)}{\pi^d\langle g_0, g_0\rangle_k}=\frac{L(g_0, s, \psi)}{\pi^d\langle g_0, g_0\rangle_k}\in\overline{\mathbf{Q}}. \tag{7.3.3.0}$$

Now let $N_1 \in \mathscr{S}$ be minimal for all the elements in $g_0^{\perp}$, and such that $\exists f_1 \in g_0^{\perp}$ such that $a(f_1, N_1) \neq 0$. Define $g_1 = g_{N_1, \psi}$. Again, $g_1 \neq 0$. Repeating the above argument, we find that:

$$(7.3.3.1) \qquad \frac{L(f, s, \psi)}{\pi^d \langle g_1, g_1 \rangle_k} = \frac{L(g_1, s, \psi)}{\pi^d \langle g_1, g_1 \rangle_k} \in \overline{\mathbf{Q}}.$$

Continuing in this way, we find a basis $g_0, g_1, \ldots, g_\mu$ for V_f over $\mathbf{C}$, such that $g_i \in G_k(\Gamma(1), \overline{\mathbf{Q}})$ for $i=0, \ldots, \mu$, and such that:

$$(7.3.3.i) \qquad \frac{L(f, s, \psi)}{\pi^d \langle g_i, g_i \rangle_k} = \frac{L(g_i, s, \psi)}{\pi^d \langle g_i, g_i \rangle_k} \in \overline{\mathbf{Q}}, \qquad i=0, \ldots, \mu.$$

In particular:

$$(7.3.4) \qquad \frac{\langle g_i, g_i \rangle_k}{\langle g_j, g_j \rangle_k} \in \overline{\mathbf{Q}}, \qquad i, j=0, \ldots, \mu.$$

Thus, since f has algebraic Fourier coefficients:

$$(7.3.5) \qquad \frac{\langle g, g \rangle_k}{\langle f, f \rangle_k} \in \overline{\mathbf{Q}} \quad \text{for all } g \in V_f \cap G_k(\Gamma(1), \overline{\mathbf{Q}}).$$

The Corollary follows immediately from 7.3.5 and 7.3.3.i.

Remark. – The relation 7.3.5, pointed out to me by Shimura, may be of independent interest. Note that it depends on the hypothesis $k > 2n+1$, and the existence, under this hypothesis, of a special point s (namely, $s=n+2$), such that $L(f, s, \psi) \neq 0$.

7.4. *Proof of Theorem* 7.1. – For simplicity, we write Θ for $\Theta_{2N}(Z, \overline{\psi})$, and E for $E_l(Z, \chi_N \overline{\psi})$. We define the representations V_Θ and V_E as in 2.7. The weights of Θ and E are even, and both at least $n/2$ by hypothesis.

We let $\Theta \leftrightarrow \varphi_\Theta \overset{\text{def}}{=} \varphi_1$, $E \leftrightarrow \varphi_E \overset{\text{def}}{=} \varphi_2$, and let $V_{\Theta, Q}$ and $V_{E, Q}$ be the U_Q subrepresentations of V_Θ and V_E generated by φ_1 and φ_2 respectively. In the notation of 3.10:

$$V_{\Theta, Q} = V_{\Theta, Q, -1/4\pi}; \qquad V_{E, Q} = V_{E, Q, -1/4\pi}.$$

By 6.8 there is an element $\mathfrak{M}_\alpha^{(r/2)} \star \varphi_2 \in V_{E, Q}$ which corresponds to $\delta_{k-n/2-r}^{(r/2)}(E)$ (under the correspondence of 2.5). Everything except the assertions about Fourier coefficients now follows from Lemma 3.11, with $\Gamma = \Gamma(q_N)$.

Let K denote the extension of $\mathbf{Q}(\zeta_{q_N})$ generated by the Fourier coefficients of Θ and E. By the description of the Fourier coefficients of Θ and E given in 1.4, K is a cyclotomic field; thus we see that Theorem 7.1 is implied by the assertion that $f_{N, \psi}$ has Fourier coefficients in K, or even in $K(\zeta_a)$ for all integers $a \geqq 3$ ([3]). Now we know, by Theorem 4.9, that the sections $\hat{\Theta}$ and $\hat{E}$ of $\mathscr{L}^{\otimes n/4}/\mathscr{M}_{aq_N}$ and $\mathscr{L}^{\otimes k/2-(n/4)-(r/2)}/\mathscr{M}_{aq_N}$, respectively, are both rational over $K(\zeta_a)$ for any integer $a \geqq 3$.

([3]) The results of Karel [51] seem to imply that $K \subset \mathbf{Q}(\zeta_{q_N})$.

As in 6.4, we may identify $\mathscr{E}_{(p^+)^*}$ with Ω; we denote this identification:

$$\gamma'_2: \quad \mathscr{E}_{(p^+)^*} \xrightarrow{\sim} \Omega$$

The maps $\mathrm{Sym}^t(\gamma'_2)$ thus identify $\mathscr{E}_{(\mathrm{Sym}^t(p^+))^*}$ with $\mathrm{Sym}^t\Omega$, $t=0, 1, 2, \ldots$ Now we may define, for each integer $t \geqq 0$, sections:

$$\widetilde{\mathrm{D}^t\Theta} \overset{\text{def}}{=} \widetilde{\mathrm{D}_{\mathrm{Sym}^t(p^+), \mathbf{Q}}(\varphi_1)} \quad \text{and} \quad \widetilde{\mathrm{D}^t \mathrm{E}} \overset{\text{def}}{=} \widetilde{\mathrm{D}_{\mathrm{Sym}^t(p^+), \mathbf{Q}}(\varphi_2)}$$

of:

$$\mathscr{E}_{\mathrm{Sym}^t((p^+)^*)\otimes n/2} \quad \text{and} \quad \mathscr{E}_{\mathrm{Sym}^t((p^+)^*)\otimes(k-(n/2)-r)},$$

respectively, with the obvious notation, by combining the operators of 6.6 with the identification 2.5. Using the maps $\mathrm{Sym}^t(\gamma'_2)$, $\widetilde{\mathrm{D}^t\Theta}$ and $\widetilde{\mathrm{D}^t\mathrm{E}}$ may be regarded, respectively, as C^∞-sections of $\mathscr{L}^{\otimes n/4}\otimes\mathrm{Sym}^t\Omega$ and $\mathscr{L}^{\otimes k/2-(n/4)-(r/2)}\otimes\mathrm{Sym}^t\Omega$ over $\mathscr{M}_a$, $a\geqq 3$. These sections are, moreover, *pseudo-arithmetic* over $\mathrm{K}(\zeta_a)$, by 6.6.1. We regard $\mathscr{L}$ and Ω as subbundles of $\boldsymbol{\omega}^{\otimes 2n}$ and $\boldsymbol{\omega}^{\otimes 2}$, respectively, as in paragraph 4, and we consider the algebra $\mathscr{A}$ of global sections of $\mathscr{R}(q_{\mathrm{N}}\, a, \mathrm{K}(\zeta_a))$ (notation as in 4.5) generated by sections of the form $\widetilde{\mathrm{D}^t\Theta}$ and $\widetilde{\mathrm{D}^t\mathrm{E}}$, $t=0, 1, 2, \ldots$ It follows from Lemma 3.11 that the section $f_{\mathrm{N}, \psi}$ of $\mathscr{L}^{\otimes k/2}$ is the image of an element of $\mathscr{A}$ under the natural projection of $\boldsymbol{\omega}^{\otimes kn}$ onto its direct summand $\mathscr{L}^{\otimes k/2}$. Since $\mathscr{L}^{\otimes k/2}\subset\boldsymbol{\omega}^{\otimes kn}$ is defined by $\mathbf{Q}$-rational symmetry conditions, it follows from Corollary 4.6.6, that $\tilde{f}_{\mathrm{N}, \psi}$ is arithmetic over $\mathrm{K}(\zeta_a)$. The Theorem now follows from Theorem 4.9.

7.5. *Question.* – The functions $f_{\mathrm{N}, \psi}$ are effectively computable in terms of derivatives of Θ and E. Do their Fourier series have a simple expression? In this context, the work of Manin-Panchishkin [24] and Zagier [46] on the one-dimensional case may be relevant.

7.6. It is worthwhile reviewing the key ingredients of the proof of Theorem 7.1, in order to indicate how the arguments may be generalized.

7.6.1. First we need a theory of "arithmetic automorphic forms", as discussed by Shimura in [38]. Such a theory has to be consistent with Shimura's theory of arithmetic automorphic functions, and has to include holomorphic Eisenstein series and theta series as special cases (*cf.* § 7 of [38]).

7.6.2. The theory in 7.6.1 has to be connected with the tautological theory of arithmetic automorphic forms (such as in described in our case in 4.1, more or less) by a variant of the "q-expansion principle". Alternatively, one has to show that the automorphic forms which are arithmetic with respect to a given moduli problem have "arithmetic" Fourier-Jacobi series.

7.6.3. The Maaß operators should be defined in terms of the enveloping algebra directly. It should then be proved that they transform the simplest Eisenstein series (those modelled on $\mathrm{E}_k(\mathrm{Z}, s)$, but with values in representations ρ of dimension greater than one, in general) among one another, up to well-determined scalar factors. This cannot be completely trivial, since it seems likely that the scalar factors, which should depend on ρ, will contain information about the unitarizable degenerate representations beyond the analytic continuation of the discrete series.

7.6.4. The remaining steps—namely those discussed in paragraphs 3 and 6—should be reducible either to tautologies or to tautological consequences of possibly deep general facts about representations, homogeneous vector bundles, and so on.

Appendix

THE q-EXPANSION PRINCIPLE

We briefly sketch two proofs of Theorem 4.8. Our terminology is as follows: A section (regular or meromorphic) of $\mathscr{E}_\rho \subset \boldsymbol{\omega}^{\otimes m} \mathscr{M}_N$ is said to be Ar^1-arithmetic over L if it is arithmetic in Shimura's sense—i. e., if it is a quotient of two modular forms (of appropriate types) with Fourier coefficients in L; it is Ar^2-arithmetic over L if it is rational over L with respect to the L-rational structure on $\boldsymbol{\omega}/\mathscr{M}_N$ defined in 4.0. We also refer to Ar^1 and Ar^2 as "theories of arithmetic automorphic forms"; sub-theories include the theories of arithmetic automorphic functions, etc. Theorem 4.8 asserts that these theories conicide.

FIRST PROOF.

STEP I. — *The subtheories of Ar^1 and Ar^2 of arithmetic automorphic functions coincide.*

Proof. — This is Theorem 3 of [33].

STEP 2. — *Assume the space of meromorphic forms of type ρ which are Ar^1-arithmetic over* L *is non-trivial. Then:*

$$\dim_L G_\rho\ (\Gamma(N), L) = \dim_L H^0(\mathscr{M}_{N,L}, \mathscr{E}_\rho).$$

Proof. — Both subspaces of $G_\rho(\Gamma(N), \mathbf{C})$ generate $G_\rho(\Gamma(N), \mathbf{C})$ under our hypotheses: the former by Proposition 5.2 of [38], the latter by general facts about vector bundles.

STEP 3. — *The theories Ar^1 and Ar^2 coincide for the spaces of* meromorphic *sections of* $\boldsymbol{\omega}^{\otimes m}$ *for all m (i.e., they coincide over the generic point of $\mathscr{M}_N$).*

Proof. — It is enough to check this for $m=1$. Consider the matrix labeled P in Proposition 1.2 of Shimura's paper [36]; let $p_1, \ldots, p_n$ be the columns of P. Shimura proves that:

1. $p_1, \ldots, p_n$ are meromorphic modular forms of type $\underline{\mathrm{St}}$.
2. They are generically linearly independent.
3. They are Ar^1-arithmetic over $\mathbf{Q}$.

In view of Step 1, it is enough to check that $\tilde{p}_1, \ldots, \tilde{p}_n$ are Ar^2-arithmetic over $\mathbf{Q}(\zeta_N)$. But this fact is implicit in the original construction (e. g., in Shimura [31] or Baily [3]) of arithmetic models of $\mathcal{M}_N$ by means of theta-functions.

STEP 4. — *End of the sketch.*

In view of Step 2, it is enough to show that every global section $\tilde{f} \in H^0(\mathcal{M}_{N,1}, \mathcal{E}_\rho)$ comes from an $f \in G_\rho(\Gamma(N), L)$. We know that f is Ar^1-arithmetic as a meromorphic modular form by Step 3. The q-expansion principle now follows from Lemma 4.9.4.

SECOND PROOF.

We show that the Ar^2-arithmetic forms are Ar^1-arithmetic (for any L, any ρ). Since the L-dimension of the Ar^2-arithmetic forms (over L) is, for trivial reasons, at least as great as the L-dimension of the Ar^1-arithmetic forms, this is sufficient to prove the Theorem.

To prove that (in shorthand) $\mathrm{Ar}^2 \subset \mathrm{Ar}^1$, we interpret the Fourier expansion of an automorphic form, just as in the one-dimensional case (*cf.* [18]), as the *value* of the automorphic form at the generic fiber of a "degenerating family" of abelian varieties over a certain scheme S, constructed according to Mumford's article [48]; the value is taken relative to a trivialization of $\boldsymbol{\omega}$ over S.

Let:

$$K_0 = \mathbf{Q}[q_{ij}, q_{ij}^{-1}; i, j = 1, \ldots, n]/(q_{ij} - q_{ji}; i, j = 1, \ldots, n).$$

Let R_0 denote the (infinitely generated) $\mathbf{Q}$-subalgebra generated by products $\prod_{i,j=1}^{n} (q_{ij})^{a_{ij}}$, $a_{ij} \in \mathbf{Z}$, such that the matrix $(1/2(a_{ji} + a_{ij}))$ is positive semi-definite. Let $R_1 \supset R_0$ be a finitely generated $\mathbf{Q}$-subalgebra of K_0, with fraction field K_0, and such that R_1 is integrally closed in K_0. [Such R_1 correspond to polyhedral cones with rational sides and non-empty interior, contained (except for the origin) in the open cone in $\mathrm{Sym}^2(\mathbf{R}^{n*})$ of positive definite symmetric matrices; *cf.* Kempf *et al.*, *Toroidal Embeddings*, *Lecture Notes in Mathematics*, 339, 1973.] Let R be the completion of R_1 at the maximal ideal generated by the non-constant monomials in R_1. Let I be the unique maximal ideal of R, and let K be the fraction field of R; we have $K \subset \mathbf{Q}((q_{ij}; i, j = 1, \ldots, n))/(q_{ij} - q_{ji}; i, j = 1, \ldots, n)$.

Let $G = \mathbf{G}^n_{m/R} = \mathrm{Spec}\, R[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. Consider the subgroup $Y \subset G(K)$, the image of the morphism:

$$\alpha: \quad \mathbf{Z}^n = \mathbf{Z} e_1 \oplus \ldots \oplus \mathbf{Z} e_n \to G(K)$$

such that $t_j(\alpha(e_i)) = q_{ij}$. The theory of Mumford allows us to construct an abelian variety A/K, with a group theoretic isomorphism $\beta: G(K)/Y \to A(K)$. In fact, as is proved in [48], we need only check that Y has a polarization, i. e. a map φ from Y to the character group $X = \langle t_i, \ldots, t_n \rangle$ of G, such that:

(i) $\varphi(y)(z) = \varphi(z)(y)$ for $y, z \in Y$;

(ii) $\varphi(y)(y) \in I$ for all $y \in Y$, $y \neq 1$.

[In fact, A extends to a semi-abelian scheme over Spec (R), but we don't need to know this.]

For φ we take the map which sends $\alpha(e_i) \in Y$ to $t_i \in X$. Condition (i) follows from the equality $q_{ij} = q_{ji}$ in K. Condition (ii) follows from the evident fact that $\varphi(y)(y)$, for any $y \in Y$, is a monomial $\prod_{i,j=1}^{n} (q_{ij})^{a_{ij}}$ such that the matrix (a_{ij}) is symmetric positive semi-definite; this monomial is constant only for $y = 1$.

A K-basis for the space ω_A of invariant differential forms on A (K) is given by the forms dv_i such that $\beta^*(dv_i) = dt_i/t_i$. Given any element f of the space $\omega_A^{\otimes m} \otimes_{\mathbf{Q}} L$, where L is any field containing $\mathbf{Q}$, we may write f as a sum:

$$f = \sum p_{i_1 \ldots i_m} dv_{i_1} \ldots dv_{i_m}, \qquad i_1, \ldots, i_m = 1, \ldots, n,$$

where $p_{i_1 \ldots i_m} \in K \otimes_{\mathbf{Q}} L$. If f comes from an automorphic form for $\Gamma(1)$, it follows immediately by base change to $\mathbf{C}$ that the set $\{p_{i_1 \ldots i_m}\}$ *is* the Fourier series expansion of f, written as a vector via the identification of 4.7.4, and where we have identified q_{ij} with $e(z_{ij})$. (Of course, dv_j becomes $2\pi i\, du_j$, in our previous notation, after base change to $\mathbf{C}$.) The inclusion $\mathrm{Ar}^2 \subset \mathrm{Ar}^1$ is just the statement that each $p_{i_1 \ldots i_m}$ is a power series in $\{q_{ij}\}$ with L-rational coefficients. (That our abelian variety A/K is principally polarized, hence a suitable test object for our automorphic forms, follows from the fact that φ is an isomorphism.)

This completes the sketch of the proof of Theorem 4.8 in the case of level one. The abelian variety A obtains a level N structure over the field $K_N = K(q_{ij}^{1/N}, \zeta_N)$; the level N case of Theorem 4.8 is proved over K_N in the same way as the level one case.

It should be mentioned that, in an unpublished set of notes, M. Rapoport has generalized Mumford's construction to the case where G is a semi-abelian scheme (of constant torus rank), rather than simply a torus. Using this construction, one should be able to interpret the "arithmetic Fourier-Jacobi series" of Shimura's paper [38] in an algebraic way. A similar idea has been suggested by J.-L. Brylinski in [53].

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Note added in proof: The author has carried out the program of 7.6, in the general context of Shimura varieties parametrizing "families of abelian varieties of Hodge type". These results will appear in a forthcoming series of papers. It should be added that J. Sturm has recently obtained results which refine and generalize our Corollary 7.3; his methods differ substantially from ours.