

ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4^e série, tome 14, n° 1 (1981), p. 41-59

http://www.numdam.org/item?id=ASENS_1981_4_14_1_41_0

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ON THE CHOW GROUPS OF CERTAIN RATIONAL SURFACES

BY SPENCER BLOCH ⁽¹⁾

Let X be a smooth projective surface defined over a field k . X is said to be *rational* if X_k is birational with \mathbb{P}_k^2 for some k'/k . I will say X is *split* over k' if X_k can be obtained by blowing up a finite number of k' -points starting from \mathbb{P}_k^2 and then blowing down a finite number of exceptional curves of the first kind defined and absolutely irreducible over k' . Any rational surface over k admits a splitting field k' finite over k . The following notation will be used throughout: X =rational surface/ k , k' =splitting field for X , and:

$X' = X_{k'}$; N =Néron-Severi group of X' .

F =quotient field of X ; $F' = F k'$.

$CH_0(X)$ =Chow group of zero cycles on X modulo rational equivalence.

$A_0(X) = \text{Ker}(CH_0(X) \xrightarrow{\text{deg}} \mathbb{Z}) = \text{Ker}(CH_0(X) \rightarrow CH_0(X'))$.

X^i =points of codim i on X ; $K_2(\cdot) = K_2$ group Milnor [11].

$\text{Br}(\cdot)$ =Brauer group; $k_*(\cdot) = \text{Mod } 2$ Milnor ring [10] [e. g. $k_1(k) = k^*/k^{*2}$].

Finally, to avoid technical problems associated with characteristic p , we will work only with rational surfaces X which are *separably split*, i. e. such that the splitting field k' can be taken to be separable over k .

The objective of this paper is to study a certain map Φ associating to a cycle of degree 0 in the Chow group $A_0(X)$ of a rational surface X a *torseur* for the Néron-Severi torus:

$$\Phi : A_0(X) \rightarrow H^1(k'/k, N \otimes k^*). \quad (2)$$

We show Image Φ is finite when k is local or global, and that Image $\Phi=0$ if k is non-archimedean local and X has good reduction over the closed fibre. When X is a *conic bundle surface*, i. e., when there exists $\pi : X \rightarrow \mathbb{P}_k^1$ a rational map with generic fibre a conic curve we

(1) Partially supported by National Science Foundation under NSF MCS77-01931.

(2) Colliot-Thélène has pointed out to me that the existence of such a map Φ is immediate from [5] and [15], where results like (0.2) and an even stronger version of the finiteness of $X(k)/B$ are also announced. The reader of the present paper should perhaps focus on the role of K -theory in evaluating $\text{Ker } \Phi$ and $\text{coker } \Phi$ as in (0.1.1), (0.3) and (0.4).

also show $\text{Ker } \Phi$ is finite when k is local or global and indeed $\text{Ker } \Phi = 0$ if $k \notin \mathbb{R}$. Interesting examples of such surfaces are the *Châtelet surfaces* $X : T_0^2 - aT_1^2 = x(x - a_1)(x - a_2)$. It is possible to give examples of such surfaces with $A_0(X) \neq (0)$ [4]. In an appendix, we verify a compatibility between Φ and the notion of *Brauer equivalence* defined in [9]. Using this and the above results, we show that the set $X(k)/B$ of k -points of X modulo Brauer equivalence is finite for k global and X any rational surface.

Here, in more precise form are the main results.

THEOREM (0.1). — *Let X be a smooth projective surface over a field k . Assume X is rational, split by a galois extension k'/k . Then there is an exact sequence:*

$$(0.1.1) \quad \Gamma(k'/k, N \otimes k'^*) \rightarrow H^1(k'/k, K_2(F')/K_2(k')) \xrightarrow{\mu} A_0(X) \\ \xrightarrow{\Phi} H^1(k'/k, N \otimes k'^*) \xrightarrow{\theta} H^2(k'/k, K_2(F')/K_2(k')).$$

If the g.c.d. of the degrees of all closed points on X is 1, then there is a map:

$$H^2(k'/k, K_2(F')/K_2(k')) \rightarrow \prod_{x \in X^1} \text{Br}(k(x)),$$

extending the above exact sequence one place to the right.

THEOREM (0.2). — *With notation as above, if k a local field and k'/k is an arbitrary (not necessarily finite) galois extension, then $H^1(k'/k, N \otimes k'^*)$ is finite. If, moreover, k is non-archimedean and there exists a lifting of X to \tilde{X} smooth and projective over the ring of integers of k , then $\text{Im } \Phi = 0$. Finally, if k is global and k'/k is finite galois, then $\text{Im } \Phi$ is finite.*

THEOREM (0.3). — *Let k be a field of characteristic $\neq 2$, which is either local, global, or C_i for $i \leq 3$, and let \bar{k} be the separable closure of k . Let X be a separably split conic bundle surface over k . Then $H^1(\bar{k}/k, K_2(F\bar{k})/K_2(\bar{k}))$ is subquotient of $k_3(k)$. In particular, for k non-archimedean local, C_1, C_2 , or totally imaginary global this group is zero. For k arbitrary local or global the group is finite.*

THEOREM (0.4). — *Let X be a conic bundle surface defined over a field k of characteristic $\neq 2$:*

- (i) *If k is a C_1 field then $A_0(X) = (0)$.*
- (ii) *If k is a local or global field, then $A_0(X)$ is finite.*
- (iii) *If k is non-archimedean local and X has good reduction in the sense that there exists \tilde{X} as in (0.2), then $A_0(X) = (0)$.*

The conic bundle surfaces given birationally by:

$$X : T_1^2 - a T_2^2 = \prod_{i=1}^3 (x - a_i) \cdot T_0^2 \quad \text{in } \mathbb{P}_k^2 \times \mathbb{A}_k^1, \quad a, a_i \in k^*$$

with homogeneous coordinates T_0, T_1, T_2 and affine coordinate x were studied by Châtelet [3]. Finiteness of $A_0(X)$ for such surfaces with k a number field was proven by Colliot-Thélène and Coray ([4], Remarques 6.7) using results of Châtelet [3] which can be found in Manin [9].

Manin ([8], [9]) constructed a pairing $A_0(X) \times H^1(k'/k, N) \rightarrow \text{Br}(k)$ which he used to show (among other things) non-triviality of $A_0(X)$. From our point of view, this pairing arises via:

$$A_0(X) \otimes H^1(k'/k, N) \rightarrow H^1(k'/k, N \otimes k'^*) \otimes H^1(k'/k, N) \xrightarrow[\text{intersection product}]{} H^2(k'/k, k'^*).$$

In considering cycles on rational varieties of dimension > 2 one wants to define the Néron-Severi torus to be the torus with character group N , i. e.:

$$\text{Néron-Severi torus}(k) = \text{Hom}_{\mathbb{Z}}(N, k'^*)^{\text{Gal}(k'/k)}.$$

This definition coincides with the one used in this paper because for X rational of dimension 2, N is self-dual. A very beautiful paper of Colliot-Thélène and Sansuc [5] contains among other results a proof that for X a smooth compactification of a torus T of arbitrary dimension, the group of torseurs for the Néron-Severi torus is isomorphic to the group $T(k)/R$. Here points on T are said to be R -equivalent if they can be connected by a chain of rational curves.

Among the open questions remaining in this area let me single out the question of finiteness of $A_0(X)$ for any smooth rational surface over a local or global field, as well as the question of injectivity of $A_0(X) \rightarrow H^1(k'/k, N \otimes k'^*)$. I know of no counter-example to these assertions over any ground field. The relation between the proof of (0.3) and the Eichler norm Theorem suggests one should study the injectivity question when $k = \mathbb{R}$.

I should like to acknowledge many helpful conversations with J.-L. Colliot-Thélène, D. Coray and J.-J. Sansuc ⁽³⁾.

1. The fundamental exact sequence

The following complex is defined for any variety X [12]:

$$(1.1) \quad K_2(F) \xrightarrow{\text{tame}} \prod_{x \in X^1} k(x)^* \rightarrow \prod_{x \in X^2} \mathbb{Z}.$$

(Here "tame" denotes the tame symbol [11].) Suppose for simplicity X is a smooth surface over k and let $\pi : Y \rightarrow X$ be obtained by blowing up a k -point on X . Let $E \cong \mathbb{P}_k^1 \subset Y$ be the

⁽³⁾ *Added in proof:* in a recent paper, On the Chow groups of certain rational surfaces: complements to a paper of S. Bloch, Colliot-Thélène and Sansuc have strengthened a number of results found here. In particular they calculate the H^1 group in (3.4) for conic bundle surfaces and give examples where it is $\neq 0$.

exceptional fibre, and let $e \in E$ be the generic point. The top row and middle and right-hand columns of the following diagram are exact:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & k(e)^* & \rightarrow & \prod_{E^1}^0 \mathbb{Z} & \rightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 0 & \rightarrow & k^* & \rightarrow & k(e)^* & \rightarrow & \prod_{E^1}^0 \mathbb{Z} \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \prod_{y \in Y^1} k(y)^* & \rightarrow & \prod_{Y^2} \mathbb{Z} \\
 & & & & \downarrow & & \downarrow \\
 & & & & \prod_{x \in X^1} k(x)^* & \rightarrow & \prod_{X^2} \mathbb{Z} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0 \\
 \end{array}$$

(1.2)

[Here $\prod_{\sum}^0 \mathbb{Z} = \text{Ker} (\prod \mathbb{Z} \xrightarrow{\text{sum}} \mathbb{Z})$.]

PROPOSITION (1.3). – (i) *The complex (1.1) computes the cohomology of the Zariski sheaf \mathcal{K}_2 on X , i. e., it has cohomology groups reading from left to right $\Gamma(X, \mathcal{K}_2)$, $H^1(X, \mathcal{K}_2)$, and $H^2(X, \mathcal{K}_2) \simeq \text{CH}_0(X)$.*

(ii) *With Y as above, $\Gamma(Y, \mathcal{K}_2) \simeq \Gamma(X, \mathcal{K}_2)$ and $H^1(Y, \mathcal{K}_2) \simeq H^1(X, \mathcal{K}_2) \oplus k^*$.*

Proof. – (i) is a standard result in algebraic K-theory [12]. For (ii), using the contravariant functoriality of $\Gamma(\cdot, \mathcal{K}_2)$ and the covariant functoriality of (1.1) for proper maps ⁽⁴⁾, one gets:

$$\begin{array}{ccc}
 \Gamma(Y, \mathcal{K}_2) & & \\
 \pi_* \updownarrow \pi^* & \searrow & \\
 \Gamma(X, \mathcal{K}_2) & & K_2(F)
 \end{array}$$

where $\Gamma(X, \mathcal{K}_2) \simeq \Gamma(Y, \mathcal{K}_2)$. The isomorphism $H^1(X, \mathcal{K}_2) \oplus k^* \simeq H^1(Y, \mathcal{K}_2)$ now follows from a diagram chase using (1.2).

Q.E.D.

COROLLARY (1.4). – *Assume X is a smooth projective rational surface split by k . Then $\Gamma(X, \mathcal{K}_2) \simeq K_2(k)$ and $H^1(X, \mathcal{K}_2) \simeq N \otimes k^*$, where $N = \text{Néron-Severi group of } X$.*

Proof. – Follows from (1.3) and the corresponding result for $X = \mathbb{P}_k^2$ [14].

Remark (1.5). – With X as above, we have $N \cong H^1(X, \mathcal{O}_X^*)$ and the isomorphism $H^1(X, \mathcal{O}_X^*) \otimes k^* \rightarrow H^1(X, K_2)$ is given by the symbol map.

⁽⁴⁾ This functionality results from covariant functoriality of the localization sequence. The reader who wishes a detailed discussion in an analogous (cohomological) context is referred to S. BLOCH and A. OGUS, *Gersten's Conjecture and the Homology of Schemas* (Ann. scient. Éc. Norm. Sup., T. 7, fasc. 2, 1974).

THEOREM (1.6). — *Let X be a smooth projective rational surface over a field k , and let k' be galois over k splitting X . Then there is an exact sequence:*

$$\Gamma(k'/k, \underset{\mathbb{Z}}{\mathbb{N} \otimes k'^*}) \rightarrow H^1(k'/k, K_2(F')/K_2(k')) \\ \xrightarrow{\mu} A_0(X) \xrightarrow{\Phi} H^1(k'/k, \mathbb{N} \otimes k'^*) \xrightarrow{\Theta} H^2(k'/k, K_2(F')/K_2(k')).$$

Proof. — One knows that $\text{CH}_0(X') \cong \mathbb{Z}$, so there are exact sequences (defining \mathbb{Z}):

$$(1.7) \quad \begin{cases} 0 \rightarrow \mathbb{Z} \rightarrow \coprod_{x' \in X^1} k'(x')^* \rightarrow \coprod_{X^2}^0 \mathbb{Z} \rightarrow 0, \\ 0 \rightarrow K_2(F')/K_2(k') \rightarrow \mathbb{Z} \rightarrow \mathbb{N} \otimes k'^* \rightarrow 0. \end{cases}$$

One has:

$$H^*(k'/k, \coprod_{X^1} k'(x')^*) \cong \coprod_{x \in X^1} H^*(k'/k_x, k'(x')^*),$$

where $k_x = k' \cap k(x)$, and on the right $x' \in X^1$ is chosen lying above $x \in X$. Also:

$$\Gamma\left(k'/k, \coprod_{X^2}^0 \mathbb{Z}\right) \cong \coprod_{X^2}^0 \mathbb{Z}.$$

It follows from the first sequence in (1.7) that $A_0(X) \cong H^1(k'/k, \mathbb{Z})$, and substituting this value in the cohomology sequence from the second exact sequence yields the Theorem.

Remark (1.7). — If the g. c. d. of the degrees of all closed points of X is 1 [e. g., if $X(k) \neq \emptyset$], then $H^1\left(k'/k, \coprod_{X^2}^0 \mathbb{Z}\right) = 0$ and the above sequence can be extended one more term to the right via a map:

$$H^2(k'/k, K_2(F')/K_2(k')) \rightarrow \coprod_{x \in X^1} \text{Br}(k(x)).$$

2. Finiteness of $\text{Im } \Phi$

THEOREM (2.1). — *Let k be a local field, k'/k a (not necessarily finite) galois extension. Let \mathbb{N} be a free f.g. \mathbb{Z} -module with a continuous $\text{Gal}(k'/k)$ -action. Let $\mathbb{N}^v = \text{Hom}_{\mathbb{Z}}(\mathbb{N}, \mathbb{Z})$. Then $H^1(k'/k, \underset{\mathbb{Z}}{\mathbb{N} \otimes k'^*})$ is a finite group, dual to $H^1(k'/k, \mathbb{N}^v)$.*

Proof. — This is a standard result in local duality [13].

THEOREM (2.2). — *Let k be a global field, k'/k a finite galois extension. Let X be a smooth projective rational surface defined over k split over k' . Then the kernel of the map:*

$$\theta : H^1(k'/k, \mathbb{N} \otimes k'^*) \rightarrow H^2(k'/k, K_2(F')/K_2(k')),$$

defined in (1.6) is finite.

Proof. — The geometric situation, viz. X together with an isomorphism $X_k \cong (\mathbb{P}_k^2$ with a finite number of k' points blown up and down), can be spread out over the ring of S -integers $\mathcal{O}_{k,S}$ for some finite set S of primes of k . Thus for each $\mathfrak{p} \notin S$ we may assume the reliction $X(\mathfrak{p})$ is a smooth projective rational surface over $k(\mathfrak{p})$, and that we are given $X(\mathfrak{p}) \times_{k(\mathfrak{p})} k'(\mathfrak{p}') \simeq [\mathbb{P}_{k(\mathfrak{p})}^2$ with finite number of $k'(\mathfrak{p}')$ -points blown up and down]. Enlarging S , we may assume it contains all primes which ramify in k' , and that the class group of $\mathcal{O}_{k',S'}$ is trivial, where $S' = \{\text{primes lying over } S\}$.

Tensoring the exact sequence:

$$0 \rightarrow \mathcal{O}_{k',S'}^* \rightarrow k'^* \rightarrow \prod_{\mathfrak{p}' \notin S'} \mathbb{Z} \rightarrow 0,$$

with N , we obtain a map $\psi : H^1(k'/k, N \otimes k'^*) \rightarrow \prod_{\mathfrak{p} \notin S} H^1(D_{\mathfrak{p}}, N)$, where $D_{\mathfrak{p}} \subset \text{Gal}(k'/k)$ is a decomposition group at \mathfrak{p} . Note $\text{Ker } \psi$ is a quotient of $H^1(k'/k, N \otimes \mathcal{O}_{k',S'}^*)$ which is finite since $\mathcal{O}_{k',S'}^*$ is *f.g.* Thus it will suffice to show $\text{Ker } \theta \subset \text{Ker } \psi$.

We may think now of X' as smooth and projective over $\text{Sp } \mathcal{O}_{k',S'}$, so $\mathfrak{p}' \notin S'$ defines a discrete valuation on F' , and hence a tame symbol $K_2(F') \rightarrow F'(\mathfrak{p}')^*$, where $F'(\mathfrak{p}') =$ function field of $X'(\mathfrak{p}')$. Our objective is to define injective maps:

$$\tau_{\mathfrak{p}} : H^1(D_{\mathfrak{p}}, N) \hookrightarrow H^2(D_{\mathfrak{p}}, F'(\mathfrak{p}')^*/k'(\mathfrak{p}')^*)$$

(where \mathfrak{p}' denotes a fixed prime above \mathfrak{p}) fitting into a commutative square:

$$(2.2.1) \quad \begin{array}{ccc} H^1(k'/k, N \otimes k'^*) & \xrightarrow{0} & H^2(k'/k, K_2(F')/K_2(k')) \\ \psi \downarrow & & \downarrow \text{tame} \\ \prod_{\mathfrak{p} \notin S} H^1(D_{\mathfrak{p}}, N) & \xrightarrow{\prod \tau_{\mathfrak{p}}} & \prod_{\mathfrak{p} \notin S} H^2(D_{\mathfrak{p}}, F'(\mathfrak{p}')^*/k'(\mathfrak{p}')^*) \end{array}$$

The existence of such a commutative square (2.2.1) will prove $\text{Ker } \theta \subset \text{Ker } \psi$ as claimed.

LEMMA (2.2.2). — Define $\tau_{\mathfrak{p}}$ to be the boundary map associated to the exact sequence:

$$0 \rightarrow F'(\mathfrak{p})^*/k'(\mathfrak{p}')^* \rightarrow \text{Div}(X'(\mathfrak{p}')) \rightarrow N \rightarrow 0.$$

Then $\tau_{\mathfrak{p}}$ is injective and the diagram (2.2.1) commutes.

Proof. — Notice first that

$$N = \text{Néron-Severi}(X') \cong \text{Néron-Severi}(X'(\mathfrak{p}'))$$

because both surfaces are split over their fields of definition, so the above exact sequence is defined. Moreover, if $C' \subset X'$ is an irreducible curve it can be “closed-up” and normalized to a normal family of curves \tilde{C} flat over $\mathcal{O}_{k',S'}$, so \mathfrak{p}' defines a divisor $C'(\mathfrak{p}')$ on \tilde{C} , hence a map:

$$k'(C')^* \xrightarrow{\alpha_{\mathfrak{p}'}} \prod \mathbb{Z},$$

where the sum runs over irreducible components of $C'(\mathfrak{p}')$.

Now let C' run through all irreducible curves on X' and identify components of $C'(p')$ with divisors on $X'(p')$ via cycle-theoretic direct image to get a square:

$$\begin{array}{ccc} K_2(F') & \xrightarrow{\text{tame}} & \prod_{x' \in X'} k'(x')^* \\ \text{tame} \downarrow & & \downarrow \alpha \\ F'(p')^* & \longrightarrow & \text{Div}(X'(p')) \end{array}$$

which commutes up to sign ⁽⁵⁾. We can now consider a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_2(F')/K_2(k') & \longrightarrow & Z & \longrightarrow & N \otimes k'^* \longrightarrow 0 \\ & & \text{tame} \downarrow & & \downarrow & & \downarrow V_{p'} & (6) \\ 0 & \longrightarrow & F'(p')^*/k'(p')^* & \longrightarrow & \text{Div}(X'(p')) & \longrightarrow & N \longrightarrow 0 \end{array}$$

where the middle vertical arrow is defined via α in the square above. This suffices to show that (2.2.1) commutes. Finally, note $\text{Div}(X'(p))$ is a direct limit of permutation modules so $H^1(D_p, \text{Div}(X'(p)))=0$ and τ_p is injective.

Q.E.D.

On reflection, it seems that the same argument should yield something in the local case as well:

THEOREM (2.3). — *Let X be a smooth, projective rational surface defined over a non-archimedean local field k with residue field \mathbb{F}_q and ring of integers \mathcal{O} . Assume X has good reduction in the sense that there exists an \tilde{X} smooth and projective over $\text{Sp } \mathcal{O}$ with generic fibre X . Then there exists k'/k finite unramified splitting X , and:*

$$\text{Im } \Phi = \text{Ker}(\theta : H^1(k'/k, N \otimes k'^*) \rightarrow H^2(k'/k, K_2(F')/K_2(k'))).$$

is zero.

Proof. — Let $X_0 \rightarrow \text{Sp } \mathbb{F}_q$ be the closed fibre of \tilde{X} . Then X_0 is rational [the canonical bundle has negative degree and $q=0$, since both these properties specialize from X ⁽⁷⁾]. The fact that X has an unramified splitting field follows from:

LEMMA (2.3.1). — *Let R be a complete discrete valuation ring with residue field \mathbb{F} and quotient field k . Let $f: \tilde{X} \rightarrow \text{Sp } R$ be smooth, projective, geometrically irreducible, and*

⁽⁵⁾ The referee is dubious. Quite generally, for a noetherian scheme Y , there are complexes

$$\prod_{y^0} K_n(k(y)) \rightarrow \prod_{y^1} K_{n-1}(k(y)) \rightarrow \prod_{y^2} K_{n-2}(k(y)) \rightarrow \dots \quad [12].$$

Apply this with $n=2$ and Y = "spreading out" of X' over the local ring at p' on \mathcal{O}_k .

⁽⁶⁾ If the curve $C' \subset X'$ discussed above has good reduction at p' , and if $a \in k'^*$, then clearly $\alpha_0(a) = V_{p'}(a)$. Since N is generated by such curves, we get $V_{p'}$ here.

⁽⁷⁾ Let ω = canonical bundle of $\tilde{X}/\text{Sp } \mathcal{O}$, $\omega_k = \omega|_X$, $\omega_0 = \omega|_{X_0}$. Then $0 > (\omega_k \cdot L_k) = (\omega_0 \cdot L_0)$. The assertion about q follows because q is one half the first Betti number in etale cohomology, and the etale Betti numbers are constant in smooth families.

assume the closed fibre X_0 is rational and split over \mathbb{F} . Then the generic fibre X is rational and split over k .

Proof. — An \mathbb{F} -valued point of X_0 lifts by smoothness to an \mathbb{R} -valued point of \tilde{X} , so a morphism $X_0(1) \rightarrow X_0$ obtained by blowing up an \mathbb{F} -point lifts to a diagram:

$$\begin{array}{ccc} X_0(1) & \subset & \tilde{X}(1) \\ \downarrow & & \downarrow \text{blow up an } \mathbb{R}\text{-point.} \\ X_0 & \subset & \tilde{X} \end{array}$$

We may thus build a figure:

$$\begin{array}{ccc} & \tilde{Y} \subset Y_0 & \\ \text{blowings up} \swarrow & & \searrow \text{blowings down} \\ \tilde{X} \subset X_0 & & \mathbb{P}_{\mathbb{F}}^2 \subset \mathbb{P}_{\mathbb{R}}^2 \end{array}$$

Let $E_0 \subset Y_0$ be an exceptional curve of the first kind. There exists a unique family of exceptional curves of the first kind $\tilde{E} \subset \tilde{Y}$ lifting E_0 . Indeed, the normal bundle of $E_0 \cong \mathbb{P}^1$ in Y_0 is $\mathcal{O}(-1)$, so both the zeroth and first cohomology of the normal bundle vanish, implying existence and uniqueness of lifting⁽⁸⁾. We may then blow down \tilde{E} on \tilde{Y} in much the same way as E_0 on Y_0 ⁽⁹⁾. Iterating we get a diagram:

$$\begin{array}{ccc} Y_0 \subset \tilde{Y} & & \\ \downarrow & & \downarrow \\ \mathbb{P}_{\mathbb{F}}^2 \subset \tilde{Z} & & \end{array}$$

with \tilde{Z} smooth and projective over $\text{Spec } \mathbb{R}$ with closed fibre $\mathbb{P}_{\mathbb{F}}^2$. Since $\mathbb{P}_{\mathbb{F}}^2$ is rigid also, it follows that $\tilde{Z} \cong \mathbb{P}_{\mathbb{R}}^2$. Passing now to the generic fibres we get $X_k \leftarrow Y_k \rightarrow Z_k \cong \mathbb{P}_k^2$, both morphisms being obtained by successive blowings up of k -points.

Q.E.D.

Returning to the proof of (2.3), let k'/k be unramified with residue field splitting X_0 . The units \mathcal{O}'^* in the ring of integers $\mathcal{O}' \subset k'$ are cohomologically trivial so $N \otimes \mathcal{O}'^*$ is cohomologically trivial ([13], Thm. 9, p. 152) and:

$$H^1(k'/k, N \otimes k'^*) \cong H^1(k'/k, N).$$

(8) For this sort of argument, cf. A. GROTHENDIECK, *SGA*, fasc. I, exp. III, Paris, 1960-1961.

(9) Let $\eta \rightarrow \text{Sp } \mathbb{R}$ be the generic point. Let A_η be the ring of germs of regular functions on Zariski neighborhoods of $E_\eta \subset Y_\eta$. Since E_η is rational with self-intersection -1 it blows down so A_η is smooth of dim. 2 over $k(\eta)$. Let $A \subset A_\eta$ be the \mathbb{R} -algebra of functions along $\tilde{E} \subset \tilde{Y}$, $I \subset A$ the ideal of functions vanishing along \tilde{E} . Then the germ of $\tilde{E} \in \tilde{Y}$ comes by blowing up $I \subset A$.

Writing \mathbb{F}' and \mathbb{F}'_0 for the residue field of k' and the function field of $X_0 \times_{\mathbb{F}'} \mathbb{F}'$ respectively we obtain as in (2.2.1):

$$\begin{array}{ccc} H^1(k'/k, N \otimes k'^*) & \xrightarrow{\nu} & H^2(k'/k, K_2(\mathbb{F}')/K_2(k')) \\ \downarrow \text{tr} & & \downarrow \text{tame} \\ 0 \rightarrow H^1(k'/k, N) & \xrightarrow{\tau} & H^2(k'/k, \mathbb{F}'_0^*/\mathbb{F}'^*) \end{array}$$

It follows that $\text{Ker } \theta = 0$.

Q.E.D.

3. Applications to conic bundle surfaces

PROPOSITION (3.1). — *Let X be as in paragraph 2 and let $k' \subset k''$ be galois extensions of k splitting X . Then the triangle:*

$$\begin{array}{ccc} H^1(k'/k, K_2(\mathbb{F}')/K_2(k')) & \longrightarrow & H^1(k''/k, K_2(\mathbb{F}'')/K_2(k'')) \\ & \searrow \mu' & \swarrow \mu'' \\ & A_0(X) & \end{array}$$

commutes. In particular, $\text{Image } \mu'' \supset \text{Image } \mu'$.

The proof is immediate.

We assume henceforth when discussing conic bundle surfaces that $\text{char } k \neq 2$.

THEOREM (3.2). — *Let X be a conic bundle surface, i. e., suppose given a rational map $\pi : X \rightarrow \mathbb{P}_k^1$ with generic fibre a conic curve (Severi-Brauer variety of dimension 1). Assume k is a local field, a global field or a C_i field for $i \leq 3$. Then $H^1(\bar{k}/k, K_2(\bar{\mathbb{F}})/K_2(\bar{k}))$ is a sub-quotient of $k_3(k)$, the third graded piece of the Milnor ring of k . (Here \bar{k} is the separable closure of k , $\bar{\mathbb{F}} = \mathbb{F}\bar{k}$.)*

COROLLARY (3.3). — *With hypotheses as above, if k is a C_2 -field, a local field $\neq \mathbb{R}$, or a totally imaginary global field, then $H^1(\bar{k}/k, K_2(\bar{\mathbb{F}})/K_2(\bar{k})) = 0$. If k is any number field, or $k = \mathbb{R}$, then $H^1(\bar{k}/k, K_2(\bar{\mathbb{F}})/K_2(\bar{k})) \cong \bigoplus_s \mathbb{Z}/2\mathbb{Z}$, where $s \leq$ the number of real places of k .*

Proof. — With the exception of the C_2 case, all assertions follow from (3.2) together with results about $k_3(k)$ proved in [10]. For k C_2 (or non-archimedean local) any quadratic form in five variables represents zero. Let

$$l : k^* \rightarrow k_1(k) = k^*/k^{*2}$$

and consider an element

$$l(a) l(b) l(c) \in k_3(k).$$

Let A be the quaternion algebra $T^2 = a, U^2 = b, TU = -UT$. The reduced norm $N : A \rightarrow k$ is a quadratic form in four variables, so it is surjective and there exists $\alpha \in A, N(\alpha) = c$. If $\alpha \in k$, then $c \in k^{*2}$ so $l(a) l(b) l(c) = 0$. If $\alpha \notin k$, then $k(\alpha) \subset A$ is a commutative subfield splitting A . If k [and hence also $k(\alpha)$] is C_2 or local, the map $k_2(k) \rightarrow_2 \text{Br}(k), l(a) l(b) \rightarrow [A]$

is known to be injective [6], so the symbol $l(a)l(b) \rightarrow 0$ in $k_2(k(\alpha))$. Since the reduced norm on A coincides with the field norm from $k(\alpha)$ to k , we may use the projection formula [2]:

$$l(a)l(b)l(c) = l(a)l(b)N(l(\alpha)) = N_{k_3(k(\alpha))/k_3(k)}(l(a)l(b)l(\alpha)) = 0.$$

Q.E.D.

Remark (3.4). — I don't know whether $H^1(\bar{k}/k, K_2(\bar{F})/K_2(\bar{k}))$ is ever non-zero.

COROLLARY (3.5). — *Let X be a conic bundle surface over a local or global field k . Then $A_0(X)$ is finite.*

Proof. — Combine (3.1), (3.3), the results in paragraph 2 on finiteness of $\text{Im } \Phi$, and (1.6).

COROLLARY (3.6). — *Let X be a conic bundle surface over a non-archimedean local field, and assume X has good reduction over the residue field in the sense of (2.3). Then $A_0(X) = (0)$.*

Proof. — Combine (2.3), (3.1), (3.3), and (1.6).

For the proof of (3.2), fix a conic bundle structure $\pi : X^* \rightarrow \mathbb{P}_k^1$ on X . Let $K = k(\mathbb{P}^1)$, $\bar{K} = \bar{k}(\mathbb{P}^1)$. Note that $X_{\bar{K}} \cong \mathbb{P}_{\bar{K}}^1$ so there exists an exact sequence of $\text{Gal}(\bar{k}/k)$ -modules:

$$(3.7) \quad 0 \rightarrow K_2(\bar{K}) \rightarrow K_2(\bar{F}) \rightarrow \prod_{\bar{x} \in X_{\bar{K}}} \bar{K}(\bar{x})^* \xrightarrow{N} \bar{K}^* \rightarrow 0.$$

Note also:

$$(3.8) \quad K_2(\bar{K}) \cong K_2(\bar{k}) \oplus \prod_{\bar{y} \in \mathbb{P}_k^1} \bar{k}(\bar{y})^*.$$

Let $\mathcal{N} \subset K^*$ be the image under the norm map of $\prod_{\bar{x} \in X_{\bar{K}}} \bar{K}(\bar{x})^*$. It follows from (3.7) and (3.8) that there is an exact sequence:

$$(3.9) \quad 0 \rightarrow H^1(\bar{k}/k, K_2(\bar{F})/K_2(\bar{k})) \rightarrow K^*/\mathcal{N} \xrightarrow{\psi} \prod_{y \in \mathbb{P}_k^1} \text{Br}(k(y)).$$

Fix $a, b \in K^*$ such that $X_{\bar{K}}$ is isomorphic to the conic curve defined by $T_0^2 - aT_1^2 - bT_2^2 = 0$. We will work with $l(a)l(b) \in k_2(K)$ as well as with the quaternion algebra A defined by $T^2 = a, U^2 = b, TU = -UT$.

LEMMA (3.10). — *Let $N : A^* \rightarrow K^*$ be the reduced norm. Then $NA^* \subset \mathcal{N}$.*

Proof. — If $\alpha \in A, \alpha \notin K$ then $K(\alpha)$ splits A so $X_{\bar{K}}$ has a $K(\alpha)$ -rational point. Since the norm from A coincides on $K(\alpha)$ with the field norm we get $N(\alpha) \in \mathcal{N}$. If $\alpha \in K$ then $N(\alpha) = \alpha^2 \in \mathcal{N}$.

Q.E.D.

Returning to the proof of (3.2) we now have:

$$(3.11) \quad 0 \rightarrow H^1(\bar{k}/k, K_2(\bar{F})/K_2(\bar{k})) \rightarrow K^*/\mathcal{N} \xrightarrow{\psi} \coprod_{y \in \mathbb{P}^1_k} \text{Br}(k(y))$$

$$\begin{array}{ccc} & \uparrow & \nearrow \psi' \\ & K^*/\mathcal{N} & \end{array}$$

and it will suffice for the proof to show ψ' injective.

LEMMA (3.12). — *Let k be as in (3.2). The diagram below is commutative:*

$$\begin{array}{ccc} K^*/\mathcal{N} & \xrightarrow{\psi'} & \coprod \text{Br}(k(y)) \\ \downarrow l(a).l(b) & & \uparrow \text{galois symbol} \\ k_3(\mathbb{K}) & \xrightarrow{\text{tame symbol}} & \coprod k_2(k(y)). \end{array}$$

Note that $\mathcal{N} \cdot l(a).l(b) = (0)$ by the sort of projection formula argument already used in the proof of (3.3). I will postpone the proof of (3.12) to go directly to the main point.

LEMMA (3.13). — *The maps in the above square have the following properties:*

- (i) “galois symbol” is injective.
- (ii) “tame symbol” has kernel $k_3(k) \subset k_3(\mathbb{K})$.
- (iii) $l(a).l(b)$ is injective.

Proof. — Injectivity of the galois symbol follows from results of [6] together with our assumptions on k . The fact that $\text{Ker}(\text{tame}) = k_3(k)$ is proved in [10]. To show injectivity of the left-hand arrow, we consider the map defined in *op. cit.* :

$$k_3(\mathbb{K}) \rightarrow I^3/I^4,$$

$$l(a)l(b)l(c) \rightarrow (\langle c \rangle + \langle -1 \rangle)(\langle b \rangle + \langle -1 \rangle)(\langle a \rangle + \langle -1 \rangle),$$

where I is the augmentation ideal in the Witt ring of \mathbb{K} . The above quadratic form is a *Pfister form* in the terminology of [7]. In particular, this form lies in I^4 if and only if it is *hyperbolic* (Theorem of Arason-Pfister, *op. cit.*, Cor. 3.4, p. 290). In particular, $l(a)l(b)l(c) = 0$ in $k_3(\mathbb{K})$ implies there exists $x_1, \dots, x_8 \in \mathbb{K}$ not all zero such that:

$$abcx_1^2 - abx_2^2 - acx_3^2 - bcx_4^2 + ax_5^2 + bx_6^2 + cx_7^2 - x_8^2 = 0.$$

Formally, then we may factor and write:

$$c = \frac{abx_2^2 - ax_5^2 - bx_6^2 + x_8^2}{abx_1^2 - ax_3^2 - bx_4^2 + x_7^2}.$$

Notice that both numerator and denominator are norms from A^* . We may assume $A \not\cong M_2(\mathbb{K})$ else $X_{\mathbb{K}} \cong \mathbb{P}^1_{\mathbb{K}}$ and the whole discussion is silly. Thus the denominator in the above equation vanishes if and only if $x_1 = x_3 = x_4 = x_7 = 0$. But vanishing of the

denominator implies vanishing of the numerator, and hence of all the x_i , a contradiction. Hence, neither numerator nor denominator vanishes and we have written c as a norm from A^* .

Q.E.D.

The proof of (3.2) is now immediate. Indeed, $H^1(\bar{k}/k, K_2(\bar{F})/K_2(\bar{k}))$ is a quotient of $\text{Ker } \psi'$ by (3.11), and $\text{Ker } \psi' \subset k_3(k)$ by (3.12).

Q.E.D.

It remains to prove (3.12).

LEMMA (3.14). — Let X_K be a conic curve over $\text{Sp } K$ with \bar{K}/K galois splitting X_K . Let $[X_K] \in H^2(\bar{K}/K, \bar{K}^*)$ be the class of X_K as a Severi-Brauer variety, and assume $[X_K] \neq 0$. Let $\bar{F} = \text{quotient field of } X_{\bar{K}}$ and consider the exact sequence of $\text{Gal}(\bar{K}/K)$ -modules:

$$0 \rightarrow \bar{K}^* \rightarrow \bar{F}^* \rightarrow \coprod_{\bar{x} \in X_{\bar{K}}} \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0.$$

Then $[X_K] = \partial_2 \partial_1(1)$ where ∂_i are the boundary maps associated to this exact sequence.

Proof. — $(\coprod_{\bar{x} \in X_{\bar{K}}} \mathbb{Z})^{\text{Gal}(\bar{K}/K)} = \coprod_{x \in X_K} \mathbb{Z}$. Since any $x \in X_K$ has even degree over $\text{Sp } K$ [this follows from a norm argument using the fact that X_K splits over $K(x)$], we get an exact sequence:

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow H^2(\bar{K}/K, \bar{K}^*) \rightarrow H^2(\bar{F}/F, \bar{F}^*),$$

$$1 \mapsto \partial_2 \partial_1(1).$$

Since $[X_K] \rightarrow 0$ in $H^2(\bar{F}/F, \bar{F}^*)$, we are done.

Q.E.D.

Tensoring the sequence in (3.14) with K^* and using the symbol map, we obtain a commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & K^* \otimes K^* & \rightarrow & \bar{F}^* \otimes K^* & \rightarrow & \coprod_{\bar{x} \in X_{\bar{K}}} K^* \rightarrow K^* \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \downarrow \\ 0 & \rightarrow & K_2(\bar{K}) & \rightarrow & K_2(\bar{F}) & \rightarrow & \coprod \bar{K}(\bar{x})^* \rightarrow \bar{K}^* \rightarrow 0. \end{array}$$

Writing ∂'_i for the boundary maps on cohomology associated to the bottom row, we find a commutative triangle:

$$\begin{array}{ccc} K^* & \xrightarrow{[X_K] \otimes} & H^2(\bar{K}/K, \bar{K}^*) \otimes K^* \\ \partial'_0 \circ \partial'_1 \searrow & & \swarrow \text{symbol} \\ & & H^2(\bar{K}/K, K_2(\bar{K})) \end{array}$$

Now take $K = k(\mathbb{P}^1)$, $\bar{K} = \bar{k}(\mathbb{P}^1)$ as in (3.12) and compose with the tame symbol to get a commutative square:

$$\begin{array}{ccc} K^* & \xrightarrow{\psi} & \prod_{y \in \mathbb{P}^1} \text{Br}(k(y)) \\ \downarrow \text{[X}_K\text{]}\otimes & & \uparrow \text{tame} \\ H^2(\bar{k}/k, \bar{K}^*) \otimes K^* & \longrightarrow & H^2(\bar{k}/k, K_2(\bar{K})) \end{array}$$

The proof of (3.12) now follows from:

LEMMA (3.15). — Let k be a field of characteristic $\neq 2$, \bar{k} = separable closure of k . Let K be an extension field of transcendence degree 1 over k , and write $\bar{K} = K\bar{k}$. Let $a, b \in K^*$ and write $l(a)l(b) \in k_2(K)$, $(a, b) \in {}_2\text{Br}(K)$. Then for torsion prime to the characteristic:

$$\text{Br}(K) \cong H^2(\bar{k}/k, \bar{K}^*).$$

Moreover, if y is a place of K over k with residue field $k(y)$, the diagram:

$$(3.15.1) \quad \begin{array}{ccccc} & & H^2(\bar{k}/k, \bar{K}^*) \otimes K^* & & \\ & \nearrow^{(a,b)\otimes} & & \searrow & \\ K^* & & & & H^2(\bar{k}/k, K_2(\bar{K})) \\ & \searrow_{l(a):l(b)} & & \nearrow_{T_1} & \\ & & k_3(K) & \xrightarrow{\text{Gal} \circ T_2} & \text{Br}(k(y)) \end{array}$$

(where T_1 and T_2 are tame symbols) commutes.

Proof. — Replacing k by its perfect closure, we may assume $k(y)$ separable over k . Next replacing K by its completion at y we may suppose $k(y) \subset K$ and $\bar{K} = \prod \bar{K}_i$, one copy for each place lying over y . Now replacing the galois group by the decomposition group for one of the \bar{K}_i we may assume $k = k(y)$, $K = k((\pi))$.

We now have split exact sequences:

$$0 \rightarrow \mathcal{O}^* \rightarrow K^* \rightarrow \mathbb{Z} \rightarrow 0, \quad 0 \rightarrow \bar{\mathcal{O}}^* \rightarrow \bar{K}^* \rightarrow \mathbb{Z} \rightarrow 0.$$

$\pi \curvearrowright \searrow \swarrow 1$

Note that $1 + \pi\bar{\mathcal{O}}$ is cohomologically trivial, so:

$$H^2(\bar{k}/k, \bar{K}^*) \cong H^2(\bar{k}/k, \bar{k}^*) \oplus H^2(\bar{k}/k, \mathbb{Z}).$$

Case 1. — $a, b \in \mathcal{O}^*$. Let $a_0, b_0 \in k^*$ denote the mod π reductions of a, b . In this case $(a, b) = (a_0, b_0) \in H^2(\bar{k}/k, \bar{k}^*)$. It is easy enough to see that going either way around (3.15.1), $f \in K^*$ gets taken to $\text{ord}(f) \cdot (a_0, b_0) \in \text{Br}(k)$.

Using linearity, it remains only to consider.

Case 2. — $b = \pi, a \in \mathcal{O}^*$. In this case let $G = \text{Gal}(\bar{k}/k)$ and let $\rho : G \times G \rightarrow \mathbb{Z}$ be a 2-cocycle representing the image of a under the composition:

$$(3.15.2) \quad \left\{ \begin{array}{l} \mathcal{O}^* \rightarrow H^1(\bar{K}/K, \mu_2) \xrightarrow{\delta} H^2(\bar{K}/K, \mathbb{Z}), \\ a \mapsto \chi_a. \end{array} \right.$$

Here δ is the coboundary from the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$. Then (a, b) is represented by the cocycle:

$$\mathbf{G} \times \mathbf{G} \rightarrow \overline{\mathbf{K}}^*, \quad (g_1, g_2) \mapsto \pi^{p(g_1, g_2)}.$$

Indeed, one knows (cf. [13], p. 214) that:

$$(a, \pi) = \pi \cdot \delta \chi_a. \quad (\text{cup product}).$$

With reference to (3.15.1) we must show:

$$\mathbf{T}_1 \{ \pi^{p(g_1, g_2)}, f \} = \mathbf{T}_2 (l(a) l(\pi) l(f)).$$

If $f = -\pi$ this is clear as both sides are trivial. If, on the other hand, f is a unit with residue class f_0 , we reduce to showing that $l(a_0) l(f_0) \in k_2(k)$ maps to the element in $\text{Br}(k)$ represented by the cocycle $f_0^{p(g_1, g_2)}$. This follows as above with \mathbf{K} in (3.15.2) replaced by k and $\overline{\mathbf{K}}$ replaced by \overline{k} . This completes the proof of (3.15) and (3.12).

Brauer equivalence. Appendix

Let X be a rational surface over k , and let $\prod_X^0 \mathbb{Z}$ be the group of 0-cycles of degree 0 on X . Following Manin ([8], [9]), we define a pairing:

$$(\quad): \prod_X^0 \mathbb{Z} \times (\text{Br}(X)/\text{Br}(k)) \rightarrow \text{Br}(k),$$

as follows. Given a cycle:

$$\sum n_i(x_i) \quad \text{and} \quad a \in \text{Br}(X), \quad \left(\sum n_i(x_i), a \right) = \prod_i \text{cor}_{k(x_i)/k} (a(x_i))^{n_i}.$$

Two k -points x_1, x_2 are said to be Brauer equivalent (written $x_1 \sim_B x_2$) if $(x_1 - x_2, a) = 1$ for all $a \in \text{Br}(X)$. Manin shows (*op. cit.*, Thm. 44.2) that $X(k)/B$ is finite for X a smooth cubic surface over a global field. In this appendix, I want to establish a compatibility between the Manin pairing and the map $\Phi: A_0(X) \rightarrow H^1(\overline{k}/k, N \otimes \overline{k}^*)$. It will follow that $X(k)/B$ is finite for any smooth rational surface over a global field, and that all points are Brauer equivalent at good reduction places. This explains and generalizes a number of Manin's calculations (*op. cit.*, 45.5, 45.6, 45.11, 45.12).

The key fact is:

PROPOSITION (A. 1). — *The following diagram commutes:*

$$\begin{array}{ccc}
 \prod_{p \in X}^0 \mathbb{Z} \times (\text{Br}(X)/\text{Br}(k)) & \xrightarrow{(\cdot)} & \text{Br}(k) = \text{H}^2(\bar{k}/k, \bar{k}^*) \\
 \downarrow & & \uparrow \text{Intersection pairing} \\
 A_0(X) \times \text{H}^1(\bar{k}/k, \mathbb{N}) & \xrightarrow{\Phi \times 1} & \text{H}^1(\bar{k}/k, \mathbb{N} \otimes \bar{k}^*) \times \text{H}^1(\bar{k}/k, \mathbb{N})
 \end{array}$$

[Here the map $\text{Br}(X)/\text{Br}(k) \rightarrow \text{H}^1(\bar{k}/k, \mathbb{N})$ arises from the spectral sequence $\text{H}^p(\bar{k}/k, \text{H}^q(\bar{X}, \mathbb{G}_m)) \Rightarrow \text{H}^{p+q}(X, \mathbb{G}_m)$.]

COROLLARY (A. 2). — *Let k be a local or global field. Then $X(k)/\mathbb{B}$ is finite.*

Proof. — Fix $x_0 \in X(k)$ [if $X(k) = \emptyset$, of course, there is not much to prove] and define $\Psi: X(k) \rightarrow \text{H}^1(\bar{k}/k, \mathbb{N} \otimes \bar{k}^*)$ by $\Psi(x) = \Phi((x) - (x_0))$. Of course $\text{Image } \Psi \subset \text{Image } \Phi$ is finite. On the other hand, if $x, y \in X(k)$ and $x \not\sim_B y$ then $\Psi(x) \neq \Psi(y)$. Hence $\#(X(k)/\mathbb{B}) \leq \# \text{Image } \Phi$. Q.E.D.

COROLLARY (A. 3). — *If k is non-archimedean local and X has good reduction over the residue field of k , then $\#(X(k)/\mathbb{B}) \leq 1$.*

Proof. — As above, using $\#(\text{Image } \Phi) = 1$. Q.E.D.

We turn now to the proof of (A.1). Let $S \subset X$ be a union of divisors, and let $\bar{S} = \bar{S}_k \subset \bar{X}$. Let $\text{Div}_{\bar{S}}(\bar{X}) \subset \text{Div}(\bar{X}) = \prod_{\bar{D} \subset \bar{X}} \mathbb{Z}$ be the group generated by divisors supported on \bar{S} , and assume S sufficiently large so $\text{Div}_{\bar{S}}(\bar{X}) \twoheadrightarrow \mathbb{N}$. There is then an exact sequence:

$$(A.4) \quad 0 \rightarrow \Gamma(\bar{X} - \bar{S}, \mathcal{O}_{\bar{X}}^*/k^*) \rightarrow \text{Div}_{\bar{S}}(\bar{X}) \rightarrow \mathbb{N} \rightarrow 0.$$

Given $\bar{D} \subset \bar{X}$ an irreducible divisor with $\bar{D} \not\subset \bar{S}$, let $\pi: \tilde{D} \rightarrow \bar{X}$ be the normalization, and let $\bar{k}(\bar{D})_{\bar{S}}^*$ denote those meromorphic functions on \tilde{D} which are invertible at points of $\pi^{-1}(\bar{S})$. Define $Z_{\bar{S}} \subset Z$ and $(\prod_{\bar{D} \not\subset \bar{S}} \mathbb{Z})_{\bar{S}} \subset \prod_{\bar{D} \subset \bar{X}} \mathbb{Z}$ so the sequence:

$$(A.5) \quad 0 \rightarrow Z_{\bar{S}} \rightarrow \prod_{\bar{D} \not\subset \bar{S}} \bar{k}(\bar{D})_{\bar{S}}^* \rightarrow \left(\prod_{\bar{D} \subset \bar{X}} \mathbb{Z} \right)_{\bar{S}} \rightarrow 0,$$

is exact. We define a pairing:

$$(A.6) \quad \langle \cdot \rangle: \prod_{\bar{D} \not\subset \bar{S}} \bar{k}(\bar{D})_{\bar{S}}^* \times \text{Div}_{\bar{S}}(\bar{X}) \rightarrow \bar{k}^*,$$

as follows. Let f be a function on \tilde{D} with divisor (f) supported off $\pi^{-1}(\bar{S})$. Let $\Delta \subset \bar{S}$ be a divisor and write:

$$\pi^{-1} \Delta = \sum n_i (d_i) \quad (\text{as a cycle}).$$

Define:

$$(A.7) \quad \langle f|_{\bar{D}}, \Delta \rangle = \prod f(d_i)^{n_i}.$$

LEMMA (A.8). — *The following diagram is a commutative diagram of pairings:*

$$\begin{array}{ccc} \prod_{\bar{D} \neq \bar{S}} \bar{k}(\bar{D})_{\bar{S}}^* \times \text{Div}_{\bar{S}}(\bar{X}) & \longrightarrow & \bar{k}^* \\ \downarrow & \uparrow \text{div} & \parallel \\ \left(\prod_{\bar{X}}^0 \mathbb{Z} \right)_{\bar{S}} \times \Gamma(\bar{X} - \bar{S}, \mathcal{O}_{\bar{X}}^*) / \bar{k}^* & \longrightarrow & \bar{k}^* \end{array}$$

Here the top pairing is defined in (A.6) and the bottom is evaluation. The left hand vertical arrow is from (A.5).

Proof. — Given $g \in \Gamma(\bar{X} - \bar{S}, \mathcal{O}_{\bar{X}}^*)$ and given

$$\beta = \sum n_j(b_j) \in \left(\prod_{\bar{S}}^0 \mathbb{Z} \right)_{\bar{S}},$$

there exists $\sum f_i|_{\bar{D}_i} \mapsto \beta$ with $f_i \in \bar{k}(\bar{D}_i)_{\bar{S}}^*$. We have:

$$\langle g, \beta \rangle = h(0)h(\infty)^{-1}, \quad h = \prod N_{g, \bar{D}_i}(f_i).$$

Note the equality of divisors $(h) = g_*(\beta) = \sum n_j g(b_j)$. Thus $h = C \cdot \prod (t - g(b_j))^{n_j}$, where t is the standard parameter on \mathbb{P}^1 and C is some constant. It follows that:

$$h(0)h(\infty)^{-1} = \prod g(b_j)^{n_j}.$$

Q.E.D.

The following abstract result in group cohomology comes from setting $C = C'$, $C'' = 0$ in ([16], Thm. 6, p. 112).

LEMMA (A.9). — *Let G be a group, A, A', A'', B, B', B'' , and C all G -modules. Suppose given exact sequences:*

$$\begin{array}{c} 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0, \\ 0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0 \end{array}$$

and a G -pairing $A \times B \xrightarrow{\langle \rangle} C$. Assume $\langle A', B' \rangle = 0$ so there are induced pairings:

$$A' \times B'' \rightarrow C, \quad A'' \times B' \rightarrow C.$$

Then given $a \in H^p(G, A'')$, $b \in H^q(G, B'')$ we have:

$$\langle \partial_A a, b \rangle = (-1)^{p+1} \langle a, \partial_B b \rangle \in H^{p+q+1}(G, C).$$

(Here ∂_A and ∂_B are boundary maps for the corresponding long exact sequence of cohomology.)

Applying this result to the sequences (A.4), (A.5) and the pairing into \bar{k}^* described above, we get a commutative diagram of pairings:

$$(A.10) \quad \begin{array}{ccc} H^0(\bar{k}/k, (\prod \mathbb{Z})_{\bar{S}}) \times H^2(\bar{k}/k, \Gamma(\bar{X} - \bar{S}, \mathcal{O}^*)/\bar{k}^*) & & \\ \downarrow \partial & \uparrow \partial & \searrow \text{evaluate} \\ H^1(\bar{k}/k, \mathbb{Z}_{\bar{S}}) \times H^1(\bar{k}/k, N) & \xrightarrow{\langle \rangle} & Br(k) = H^2(\bar{k}/k, \bar{k}^*) \end{array}$$

[Commutativity means $\langle \partial z, n \rangle = \text{evaluation}(z, \partial n)$.]

The next step is to show that the pairing:

$$\mathbb{Z}_{\bar{S}} \times N \rightarrow \bar{k}^*,$$

factors through the intersection pairing $N \otimes \bar{k}^* \otimes N \rightarrow \bar{k}^*$ by means of $\mathbb{Z}_{\bar{S}} \rightarrow \mathbb{Z} \rightarrow N \otimes \bar{k}^*$. This is essentially a consequence of Weil reciprocity as follows. Think of $N \otimes \bar{k}^* \simeq H^1(\bar{X}, \mathcal{K}_2)$. Given $\Delta \subset \bar{S}$ an irreducible divisor, let $\pi: \tilde{\Delta} \rightarrow \bar{X}$ be the normalization. There is a diagram:

$$\begin{array}{ccccccc} K_2(\bar{k}(\tilde{\Delta})) & \rightarrow & \prod_{x \in \tilde{\Delta}} \bar{k}(x)^* & \rightarrow & H^1(\tilde{\Delta}, \mathcal{K}_2) & \rightarrow & 0 \\ & \searrow 0 & \downarrow \text{norm} & & \swarrow & & \\ & & \bar{k}^* & & & & \end{array}$$

with exact top row. The diagonal arrow is zero by the reciprocity [cf. Lemma (A.8)], whence an induced map $H^1(\tilde{\Delta}, \mathcal{K}_2) \rightarrow \bar{k}^*$.

If we associate to Δ the composed map:

$$\varphi_{\Delta}: H^1(\bar{X}, \mathcal{K}_2) \rightarrow H^1(\tilde{\Delta}, \mathcal{K}_2) \rightarrow \bar{k}^*,$$

we obtain a pairing:

$$H^1(\bar{X}, \mathcal{K}_2) \times \text{Div}_{\bar{S}}(\bar{X}) \rightarrow \bar{k}^*.$$

One verifies quite easily ⁽¹⁰⁾ that the diagram of pairings:

$$(A.11) \quad \begin{array}{ccc} & \mathbb{N} \otimes \bar{k}^* \times \mathbb{N} & \\ & \uparrow & \text{intersection pairing} \\ & \mathbb{H}^1(\bar{X}, \mathcal{K}_2) \times \text{Div}_{\bar{S}}(\bar{X}) & \longrightarrow \bar{k}^* \\ \uparrow & \parallel & \nearrow \langle \cdot \rangle \\ \mathbb{Z}_{\bar{S}} \times \text{Div}_{\bar{S}}(\bar{X}) & & \end{array}$$

commutes.

Finally, noting that $\text{Pic}(\bar{X} - \bar{S}) = (0)$, one obtains a commutative diagram up to sign ([5], appendice):

$$(A.12) \quad \begin{array}{ccc} \text{Br}(X)/\text{Br}(k) & \longrightarrow & \text{Br}(X - S)/\text{Br}(k) \\ \downarrow & & \uparrow \\ \mathbb{H}^1(\bar{k}/k, \mathbb{N}) & \xrightarrow{\partial} & \mathbb{H}^2(\bar{k}/k, \Gamma(\bar{X} - \bar{S}, \mathcal{O}^*)/\bar{k}^*). \end{array}$$

Combining (A.10), (A.11), (A.12) one obtains the assertion of (A.1).

Q.E.D.

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⁽¹⁰⁾ Use functoriality of the map $\text{pic}(\bar{X}) \otimes \bar{k}^* \rightarrow \mathbb{H}^1(\bar{X}, \mathcal{K}_2)$ for the morphism $\tilde{\Delta} \rightarrow X$, the fact that $\text{Pic } \bar{X} \rightarrow \text{Pic } \tilde{\Delta} \xrightarrow{\text{deg}} \mathbb{Z}$ is the intersection number with $\tilde{\Delta}$, and the above diagram.

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(Manuscrit reçu le 15 mars 1980,
révisé le 15 juillet 1980.)