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INDUCED AND AMENABLE ERGODIC ACTIONS OF LIE GROUPS

BY ROBERT J. ZIMMER

SUMMARY. — As with unitary representations, one can induce an ergodic action of a closed subgroup of a locally compact group G to obtain an ergodic action of G. We show that every amenable ergodic action of a real algebraic group or a connected semi-simple Lie group with finite center is induced from an action of an amenable subgroup (which is not true for amenable actions of general locally compact groups). The proof depends on the result, of independent interest, that the orbit of any probability measure on real projective space under the action of the general linear group is locally closed in the weak-$\star$-topology. Combined with recent results on the group-measure space construction of von Neumann algebras, this enables us to deduce that any free ergodic action of a real algebraic group or connected semi-simple Lie group with finite center determines a hyperfinite von Neumann algebra via this construction if and only if it is induced from a free ergodic action of an amenable subgroup. Another implication of this result is that a cocycle of an ergodic amenable group action with values in a real algebraic group or connected semi-simple Lie group with finite center is cohomologous to a cocycle taking values in an amenable subgroup.

I. — Introduction

One of the most important methods of constructing unitary representations of groups is that of inducing: to each unitary representation of a closed subgroup of a locally compact group G, there is a naturally associated "induced" unitary representation of G. As pointed out by G. W. Mackey ([12], [13]), one can define induced ergodic actions in an analogous manner. Thus, if G is a locally compact second countable group and H is a closed subgroup, then to each ergodic action of H there is a naturally associated "induced" ergodic action of G. As with unitary representations, the construction is of a very concrete and explicit nature, and one can thus hope to answer many questions about the induced action by examining the action of the smaller and hopefully simpler group H. For a given ergodic action, it is therefore of considerable interest to know when it can be expressed as an action induced from some (perhaps given) subgroup. This is, of course, parallel to a basic theme in the theory of unitary representations. This

point of view turns out to be quite relevant to the study of a broad class of ergodic actions recently introduced by the author in [20], namely amenable ergodic actions.

Amenable ergodic actions play a role in ergodic theory parallel in many respects to the role played by amenable groups in group theory and arise naturally in a variety of situations. For example, with each ergodic group action there is a naturally associated von Neumann algebra first introduced by Murray and von Neumann and subsequently generalized by a variety of authors, by the group-measure space construction. For free ergodic actions, this von Neumann algebra will be approximately finite dimensional (i. e., hyperfinite) if and only if the action is amenable. This was shown by the author for actions of countable discrete groups in [22] and for actions of general locally compact groups by J. Feldman, P. Hahn, and C. C. Moore in [6], Th. 8.10, using a reduction to the countable case. Other results concerning amenable ergodic actions with applications to problems in ergodic theory and its relation to probability and von Neumann algebras can be found in the author's papers [19]-[23], and the paper of P. Hahn [25].

It follows from the results of [20] that every ergodic action of a group that is induced from an action of an amenable subgroup is an amenable action. Not surprisingly, the converse assertion is false in general as we show by example in section 6 below. However, one of the main points of this paper is to prove the converse for ergodic actions of a suitable class of groups, thus for many purposes reducing the study of amenable actions of such groups to the study of the ergodic actions of amenable subgroups. Specifically, we have:

THEOREMS 5.7, 5.10. — *If* G *is a real algebraic group or a connected semi-simple Lie group with finite center, then every amenable ergodic action of* G *is induced from an ergodic action of an amenable subgroup.*

This theorem has immediate applications to von Neumann algebras and to the cohomology theory of ergodic actions. We also present theorems concerning the structure of amenable ergodic actions of more general Lie groups.

The outline of this paper is a follows. In section 2 we discuss the inducing process for ergodic actions and some of its general properties. In section 3 we recall the definition of amenability for ergodic actions and make some further observations of a general nature concerning these actions. Section 4 is devoted to an examination of the orbits of probability measures on a real projective space under the action of the general linear group. Specifically, we show that every orbit is locally closed in the weak-$\star$-topology. This result is an important step in proving Theorems 5.7 and 5.10, and seems to be of independent interest as well. The proof of this result in turn depends upon a technique of H. Furstenberg for examining the asymptotic behavior of measures on projective space under the general linear group action. Section 5 contains the remainder of the proof of the main theorems, applications of the theorem, and theorems concerning the amenable ergodic actions of more general groups. In section 6 we present an example of an amenable ergodic action that is not induced from an action of an amenable subgroup. Specifically, the action of a lattice subgroup of SL (2, **C**) on the projective space of a 2-dimensional complex vector space has this property.

II. — Induced ergodic actions

In this section we develop the material we shall need concerning induced ergodic actions. A good deal of this material is implicit in the discussions of Mackey ([12], [13]), but we shall here formulate the notion so as to emphasize the similarity with inducing for unitary representations.

Let G be a locally compact group. Throughout this paper, all locally compact groups will be assumed to be second countable. By a Borel G-space we mean a standard Borel space S together with a jointly Borel action $S \times G \to S$ of G on S. If μ is a σ-finite measure on S, quasi-invariant under the action of G, then μ is called ergodic if $A \subset S$ is a measurable set with $\mu(A\, g\, \Delta\, A) = 0$ for all $g \in G$ implies A is null or conull. We shall then call (S, μ), or sometimes just S, an ergodic G-space. If (S, μ) and (T, ν) are ergodic G-spaces, they are called equivalent, or isomorphic, if there is a G-isomorphism $B(T, \nu) \to B(S, \mu)$ of the associated Boolean σ-algebras of Borel sets modulo null sets. Equivalently [11], there is a conull G-invariant Borel set $S_0 \subset S$ and a measure class preserving G-map $(S_0, \mu) \to (T, \nu)$. In particular, changing the measure μ to a measure in the same measure class (i. e., same null sets) does not change the equivalence class of the action. We may thus assume the measures at hand to be probability measures if we wish. An ergodic G-space is called essentially transitive if there is a conull orbit, or equivalently, if it is isomorphic to the action on a homogeneous space G/H. (We write G/H to be cosets of the form Hg, and H\G to be cosets of the form gH). An ergodic G-space is called properly ergodic if every orbit is a null set. By ergodicity every ergodic action is either essentially transitive or properly ergodic.

Suppose now that $H \subset G$ is a closed subgroup and that (S, μ) is an ergodic H-space. We wish to construct in a natural fashion an associated ergodic G-space. We present two different constructions of this action, both originally described by Mackey. These are in fact analogues of two ways of constructing induced representations: the first as translations on a space of functions on G that transform according to the H-representation (Mackey's original description of induced representations [10]); the second as functions on the quotient G/H where the representation is then defined *via* a cocycle that corresponds to the representation of H (*See* [17] for a discussion of induced representations from the latter point of view.)

The first construction of the induced ergodic action is as follows. Let (S, μ) be an ergodic H-space. Then H acts on $S \times G$ by $(s, g)\, h = (sh, gh)$ and the product of μ with (a probability measure in the class of) Haar measure is quasi-invariant. Let (X, ν) be the space of H-orbits with the quotient Borel structure and quotient measure. As we shall see in a moment, X is a standard Borel space. There is also a G-action on $S \times G$ given by $(s, g).g_0 = (s, g_0^{-1} g)$ and this commutes with the H-action. There is thus an action of G induced on the orbit space X which clearly leaves ν quasi-invariant. Any G-invariant set in X corresponds to a set $S_0 \times G$ where $S_0 \subset S$ is H-invariant, and so the action of G on X is clearly ergodic.

DEFINITION 2.1. — The space (X, ν) constructed above is called the ergodic G-space induced from the ergodic H-space (S, μ).

To see that X is actually a standard Borel space, it suffices to show that there is a Borel subset of $S \times G$ that meets each H-orbit exactly once. Let $\theta : H\backslash G \to G$ be a Borel section of the natural projection and let $W = \theta(H\backslash G)$, which is a Borel set. Then it is straightforward that $S \times W$ meets each orbit exactly once.

The second construction we present depends on the notion of a cocycle, which the above definition does not require. However, this second approach is also geometrically appealing and often of considerable technical use. Suppose Y is an ergodic G-space and M is a standard Borel group. A Borel function $\alpha : Y \times G \to M$ is called a cocycle if for all $g_1, g_2 \in G$, $\alpha(y, g_1 g_2) = \alpha(y, g_1)\,\alpha(yg_1, g_2)$ for almost all $y \in Y$. The cocycle is called strict if this identity holds for all (y, g_1, g_2). Two cocycles $\alpha, \beta : Y \times G \to M$ are called equivalent or cohomologous if there is a Borel function $\varphi : Y \to M$ such that for each g, $\varphi(y)\,\alpha(y, g)\,\varphi(yg)^{-1} = \beta(y, g)$ for almost all y. If α and β are strict cocycles, they are called strictly equivalent if there exists φ so that this last identity holds for all (y, g). If G is transitive on Y, so that we can write $Y = G/G_0$ for some closed subgroup $G_0 \subset G$, then the strict equivalence classes of strict cocycles $G/G_0 \times G \to M$ correspond to the conjugacy classes of homomorphisms $G_0 \to M$. This correspondence is defined by taking a strict cocycle α and observing that the restriction to $\{[e]\} \times G_0$ is a homomorphism. Full details of the correspondence can be found in [17]. If $H \subset G$ is a closed subgroup, one has the identity homomorphism $H \to H$ and this will correspond to (an equivalence class of) a strict cocycle $\alpha : G/H \times G \to H$. This cocycle can be defined explicitly as follows: Choose a Borel section $\theta : G/H \to G$ of the natural projection with $\theta([e]) = e$, and define $\alpha(x, g) = \theta(x)\, g\, \theta(xg)^{-1}$.

If $\alpha : Y \times G \to M$ is a strict cocycle and Z is a Borel M-space, then one can define an action of G on $Y \times Z$ by $(y, z).g = (yg, z\,\alpha(y, g))$. It is exactly the cocycle identity of α that implies that this in fact defines an action. We shall sometimes denote this G-space by $Y \times_\alpha Z$. If α and β are strictly equivalent strict cocycles, then the corresponding actions are easily seen to be equivalent. Applying this procedure to the cocycle defined at the end of the preceding paragraph, we obtain our second description of an induced ergodic action. More precisely, suppose S is an ergodic H-space, where H is a closed subgroup of G. Let $\alpha : G/H \times G \to H$ be a strict cocycle corresponding to the identity homomorphism. Form the G-action $G/H \times_\alpha S$, i. e., $([g_1], s)\, g = ([g_1]\, g, s\,\alpha([g_1], g))$, which preserves the product measure class. One can easily check that this action is ergodic, but this also follows from the following.

PROPOSITION 2.2. — $G/H \times_\alpha S$ *is equivalent to the ergodic action of* G *induced from the ergodic* H*-space* S.

Proof. — Let $\theta : G/H \to G$ be a Borel section of the natural projection and define $\Phi : G/H \times S \to X$ by $\Phi(y, s) = p(s, \theta(y)^{-1})$ where $p : S \times G \to (S \times G)/H = X$ is the natural map. One can readily check that Φ is a measure class preserving Borel isomorphism. To see that Φ is a G-map, it suffices to see that

$$\Phi(yg, s\,\theta(y)\,g\,\theta(yg)^{-1}) = p(s, g^{-1}\theta(y)^{-1}),$$

i. e. that $(s\,\theta(y)\,g\,\theta(yg)^{-1},\ \theta(yg)^{-1})$ and $(s,\ g^{-1}\,\theta(y)^{-1})$ are in the same H-orbit. But acting upon the latter by $\theta(y)\,g\,\theta(yg)^{-1} \in H$ we obtain the former.

There is of course a great similarity between the construction of $G/H \times_\alpha S$ and the construction via cocycles of the unitary representations of G induced from S (*See* [17]). We also remark that the first construction we have given of induced ergodic actions is a special case of Mackey's "range-closure" (or Poincaré flow [5]) construction for arbitrary cocycles into locally compact groups, which generalizes the flow built under a function [12]. The inducing process is exactly the Poincaré flow construction applied to the cocycle $\alpha : S \times H \to G$ defined by $\alpha(s, h) = h$. We shall on occasion make further mention of the range-closure construction and some of its properties, and we refer the reader to [12] and [15] as general references for this material.

Example 2.3. — (*a*) If S is the H-space H/K where K is a closed subgroup, then the induced G-space is G/K. One can see this immediately from Proposition 2.2, since G will clearly be transitive on $G/H \times_\alpha H/K$ and K is a stability group. In particular, the action induced from the trivial action of H (on a point) is just the action of G on G/H, and the action induced from translation of H on H is translation of G on G. We note that this is analogous to facts in representation theory concerning the induced representation of a trivial or regular representation.

(*b*) If $G = \mathbf{R}$ and $H = Z$, then for a Z-space S the induced $\mathbf{R}$-action is just the flow built under the constant function 1 ([1], [12]). If $H = Zc$ for some fixed $c \in \mathbf{R}$, then the induced action is just the flow built under the constant function *c*.

We now present two useful facts concerning induced actions that are direct parallels of results in the theory of unitary representations, namely "inducing in stages" and a parallel of the imprimitivity theorem.

PROPOSITION 2.4. — *Suppose* $K \subset H \subset G$ *are closed subgroups of* G, *and that* S *is ae ergodic* K-*space. Let* T *be the ergodic* H-*space induced from* S. *Then the* G-*spaces obtained by inducing the* H-*action on* T *to* G *and inducing the* K-*action on* S *to* G *aru isomorphic.*

Proof. — Let $\alpha : G/H \times G \to H$ and $\beta : H/K \times H \to K$ be strict cocycles corresponding to the identity homomorphism. Then the action of G induced from T is $G/H \times_\alpha T = G/H \times_\alpha (H/K \times_\beta S)$. The G-action on $G/H \times H/K \times S$ is given by $(x, y, s)\,g = (xg,\ y\,\alpha(x, g),\ s\,\beta(y,\ \alpha(x, g)))$. We can identify $G/H \times_\alpha H/K$ with G/K in such a way that $([e], [e])$ corresponds to $[e]$. Then we can consider

$$\gamma : \quad (G/H \times_\alpha H/K) \times G \to K$$

defined by $\gamma((x, y), g) = \beta(y, \alpha(x, g))$ to be a strict cocycle $G/K \times G \to K$, and this will correspond to the identity homomorphism $K \to K$. It follows that $G/H \times_\alpha T$ is equivalent to $G/K \times_\gamma S$, with proves the proposition.

To describe the analogue of the imprimitivity theorem, we must first recall the notions of extensions and factors of ergodic actions. If (X, μ) and (Y, ν) are ergodic G-spaces, then X is called an extension of Y, and Y a factor of X, if there is a conull G-invariant Borel set $X_0 \subset X$ and a measure-class preserving G-map $X_0 \to Y$. Equivalently

([18], Prop. 2.1), there is a G-embedding of Boolean σ-algebras $B(Y, \nu) \to B(X, \mu)$. The following analogue of the imprimitivity theorem provides a criterion for determining when a given action is induced.

THEOREM 2.5. — *If* X *is an ergodic* G*-space and* $H \subset G$ *is a closed subgroup, then* X *is induced from an ergodic action of* H *if and only if* G/H *is a factor of* X.

Proof. — We provide an indication of the proof, leaving some measure theoretic details to the reader. One of the implications in the theorem is taken care of by Proposition 2.2, so we assume G/H is a factor of X. Passing to a G-invariant conull Borel set if necessary, we have a measure class preserving G-map $\varphi : X \to G/H$. Let μ be the given probability measure on X and ν the probability measure on G/H. By the transitivity of G on G/H, we can assume $X = G/H \times I$, $\mu = \nu \times m$, where (I, m) is the unit interval with some probability measure. For each g and almost all x, the map $\{x\} \times I \to \{xg\} \times I$ defined by the G-action will be measure class preserving and letting $\alpha(x, g)$ be the induced transformation on the Boolean algebra, $\alpha(x, g) : B(\{xg\} \times I, m) \to B(\{x\} \times I, m)$, one readily verifies that α is a cocycle on $G/H \times G$ with values in $\mathrm{Aut}(B(I, m))$, the group of automorphisms of the Boolean σ-algebra $B(I, m)$. It is not difficult to see that $\mathrm{Aut}(B(I, m))$ is a standard Borel group (in fact, it is a weakly closed subgroup of the unitary group on $L^2(I, m)$) and that α is Borel. It follows from the discussion of cocycles on transitive G-spaces in [17] that α is equivalent to a strict cocycle into $\mathrm{Aut}(B(I, m))$ which is in turn equivalent to a strict cocycle β with

$$\beta(G/H \times G) = \beta([e] \times H).$$

Each cocycle $G/H \times G \to \mathrm{Aut}(B(I, m))$ defines a Boolean action of G on $B(G/H \times I)$ and hence an action of G that is equivalent to the action on X since α and β are cohomologous. But using Proposition 2.2, this action defined by β is equivalent to the action induced from the H-action defined by the Boolean H-action on $B(I, m)$ given by $\beta \mid [e] \times H$.

Next we present another useful criterion that an action be induced.

COROLLARY 2.6. — *If* X *is an ergodic* G*-space, let* $\alpha(x, g) = g$ *so that* $\alpha : X \times G \to G$ *is a cocycle. Then* X *is induced from an ergodic* H*-space if and only if* α *is equivalent to a cocycle taking values in* H.

Proof. — Since the range of the cocycle α is the G-space X, the corollary follows from Theorem 2.5 and the fact that G/H is a factor of the range-closure of α if and only if α is equivalent to a cocycle into H ([18], Th. 3.5).

We now present some other results of a general nature concerning induced actions. We suppose throughout the remainder of this section that $H \subset G$ is a closed subgroup.

PROPOSITION 2.7. — *If* S *is an ergodic* H*-space, then:*

(i) S *is properly ergodic if and only if the induced action of* G *is properly ergodic;*

(ii) S *is essentially free (i. e. almost all stability groups are trivial) if and only if the induced action of* G *is essentially free.*

Proof. — Straightforward.

PROPOSITION 2.8. — *Suppose* S *and* T *are ergodic* H*-spaces and that* X *and* Y *are the corresponding induced* G*-actions. If* S *is an extension of* T, *then* X *is an extension of* Y.

Proof. — Let $\alpha : G/H \times G \to H$ correspond to the identity homomorphism $H \to H$. If $\varphi : S \to T$ is a measure class preserving H-map, then $\psi : G/H \times_\alpha S \to G/H \times_\alpha T$ defined by $\psi(x, s) = (x, \varphi(s))$ is a measure class preserving G-map and the result follows by Proposition 2.2.

It follows from Theorem 2.5 that any extension of an ergodic G-space which is induced from an H-action is also induced from an H-action. The following statement is somewhat sharper. However, as we shall make no use of it below, we omit the proof.

PROPOSITION 2.9. — *Suppose* Y *is an ergodic* G*-space induced from the* H*-space* T. *If* X *is an extension of* Y, *then* X *is induced from an extension of* T.

PROPOSITION 2.10. — *Suppose that* X *is an ergodic* G*-space, and let* X_H *be the* H*-space which has* X *as the underlying set and the restriction of the* G*-action to* H *as the* H*-action. Suppose* H *is ergodic on* X_H. *Then the action of* G *induced from the* H*-action on* X_H *is the product* G*-space* $G/H \times X$.

Proof. — Let $\beta : G/H \times G \to G$ be $\beta(y, g) = g$. Then the product G-space $G/H \times X$ is just the action defined by β, i. e., $G/H \times_\beta X$. But β is strictly equivalent to a strict cocycle α with $\alpha(G/H \times G) = H$ corresponding to the identity homomorphism $H \to H$, and so $G/H \times_\beta X \cong G/H \times_\alpha X$. But the latter is just the action induced from X_H.

We conclude this section with a remark on the ergodic equivalence relation of induced actions. We refer the reader to [5], [6], [16] for the notion of an approximately finite (i. e., hyperfinite) ergodic equivalence relation.

PROPOSITION 2.11. — *If* (X, G) *is induced from* (S, H), *then the ergodic equivalence relation on* X *defined by* G *is approximately finite if and only if the ergodic equivalence relation on* S *defined by* H *is approximately finite.*

Proof. — Writing $X = G/H \times_\alpha S$, the result is clear once we observe that for $x_1 = (y_1, s_1)$, $x_2 = (y_2, s_2)$, we have $x_1 \sim x_2$ if and only if $s_1 \sim s_2$.

III. — Amenable actions

In this section we recall the definition and some properties of amenable ergodic actions and present some further results we shall subsequently require. We refer the reader to [20] for a more detailed and motivated account of amenable actions.

Let E be a separable Banach space, E* the dual Banach space, and E_1^* the unit ball in E*, which is a compact convex set with the $\sigma(E^*, E)$ topology. The group of isometric isomorphisms of E, which we denote by Iso (E), is a separable metrizable group in the strong operator topology, and the associated Borel structure is standard ([20], Lemma 1.1]). If S is a Borel space, by a Borel field of compact convex sets in E_1^* we mean an assignment $s \to A_s$, where $A_s \subset E_1^*$ is compact and convex such that $\{(s, A_s)\} \subset S \times E_1^*$ is Borel.

If S is an ergodic G-space and $\alpha : S \times G \to \mathrm{Iso}\ (E)$ is a cocycle, A_s is called α-invariant if for all g, $\alpha^*(s, g) A_{sg} = A_s$ for almost all s, where α^* is the adjoint cocycle $\alpha^*(s, g) = [\alpha(s, g)^{-1}]^*$. The following "fixed point property" then defines amenability. We call S an amenable ergodic G-space if for all such $(E, \alpha, \{A_s\})$, there is a Borel function $\varphi : S \to E_1^*$ such that $\varphi(s) \in A_s$ a. e. and $\alpha^*(s, g)\, \varphi(sg) = \varphi(s)$ a. e. for each $g \in G$. Then φ is called an α-invariant section. The reader should find drawing a sketch in $S \times E_1^*$ helpful in understanding the definition. We record the following for later reference.

PROPOSITION 3.1. — (1) *Any ergodic action of an amenable group is amenable* ([20], Th. 2.1);

(2) *if* S *is an amenable* G*-space and there is a* G*-invariant mean on* $L^\infty(S)$, *then* G *is amenable* ([20], Prop. 4.3, 4.4);

(3) *an extension of an amenable action is amenable* ([20], Th. 2.4);

(4) *a transitive action is amenable if and only if the stability groups are amenable* ([20], Th. 1.9).

The following relates amenability to inducing.

PROPOSITION 3.2. — *If* (X, G) *is induced from* (S, H), *then* X *is an amenable* G*-space if and only if* S *is an amenable* H*-space.*

Proof. — Since X is the range of a cocycle $S \times H \to G$, ([20], Th. 3.3) shows that S amenable implies X amenable. Suppose conversely that $X = G/H \times_\alpha S$ is an amenable G-space where $\alpha : G/H \times G \to H$ corresponds to the identity $H \to H$. Suppose $\gamma : S \times H \to \mathrm{Iso}\ (E)$ is a cocycle and A_s is a γ-invariant field. As in the proof of [20], Th. 2.1, we can define a representation $T : H \to \mathrm{Iso}\ (L^1(S, E))$ by

$$[T(h) f](s) = r(s, g) \gamma(s, g) f(sg)$$

where r is the Radon-Nikodym cocycle of the action, and

$$B = \{ f \in L^\infty(S, E^*) \mid f(s) \in A_s \text{ a. e.} \}$$

will be a compact convex subset of the unit ball $L_1^\infty(S, E^*)$ that is invariant under the adjoint representation T*. It suffices to show that there is a fixed point in B under T*. Let $F = L^1(S, E)$. With α as above, we can define a strict cocycle $\beta : G/H \times G \to \mathrm{Iso}\ (F)$ by $\beta(y, g) = T(\alpha(y, g))$. By the argument in part (ii) of the proof of ([20], Th. 1.9), it suffices to show that there is a β-invariant section $G/H \to B \subset F_1^*$. There is a natural isomorphism $L^\infty(G/H \times S, E_1^*) \to L^\infty(G/H, L^\infty(S, E^*))$. Suppose $\varphi : G/H \to L^\infty(S, E^*)$ and $\psi : G/H \times S \to E_1^*$ correspond under the isomorphism. Then it is straightforward that φ is a β-invariant section if and only if ψ is a δ-invariant section, where $\delta : G/H \times_\alpha S \to \mathrm{Iso}\ (E)$ is the cocycle $\delta((y, s), g) = \gamma(s, \alpha(y, g))$. Furthermore $\varphi(y) \in B$ a. e. if and only if $\psi(y, s) \in A_{y,s} = A_s$ a. e. Amenability of $G/H \times_\alpha S$ ensures the existence of such a function ψ which completes the proof.

Suppose S is an ergodic G-space and $H \subset G$ is a closed subgroup such that H is ergodic on S. We wish to examine the relation between amenability of the H-action and amenability of the G-action. We begin with the following.

LEMMA 3.3. — *S is an amenable* H*-space if and only if the product action of* G *on* $S \times G/H$ *is amenable.*

Proof. — This follows from Propositions 3.2 and 2.10.

PROPOSITION 3.4. — *Suppose* S *is an ergodic* G*-space and* $H \subset G$ *is a closed subgroup such that* H *is also ergodic on* S :

(i) *if* S *is an amenable* G*-space, it is also an amenable* H*-space;*

(ii) *if* S *is an amenable* H*-space and* G/H *has a* G*-invariant probability measure, then* S *is an amenable* G*-space.*

Proof. — (i) follows from Lemma 3.3 and Proposition 3.1 (3). On the other hand, if $S \times G/H$ is an amenable G-space and G/H has a finite invariant measure, (ii) follows from the proof of [20], Prop. 2.6.

We now demonstrate the existence of "minimal" amenable ergodic actions.

DEFINITIONS 3.5. — An amenable ergodic action is called minimal amenable if every factor action is non-amenable (other than considering the action as a factor of itself by the identity map).

A transitive G-space G/G_0 has only transitive factors and so will be minimal amenable if and only if G_0 is a maximal amenable subgroup. Every subgroup of G is contained in a maximal amenable subgroup ([9], Th. IV.1) and the next result is a generalization of this fact.

PROPOSITION 3.6. — *Suppose* X *is an amenable ergodic* G*-space. Then* X *has a minimal amenable factor.*

Proof. — We use Zorn's lemma. Suppose X_ξ is a totally ordered set of factors of X with corresponding G-invariant Boolean σ-algebras $B(X_\xi) \subset B(X)$. Then $\bigcap B(X_\xi) = B$ is a Boolean G-space and hence corresponds to an ergodic G-space Y which is a factor of each X_ξ. It suffices to show that Y is an amenable G-space. Let $\gamma : Y \times G \to \mathrm{Iso}(E)$. Let $p : X \to Y$ be the factor map and define $\alpha(x, g) = \gamma(p(x), g)$. Suppose A_y is a γ-invariant field in E_1^*, so that $A_x = A_{p(x)}$ is an α-invariant field. There is an induced representation T* of G on $L^\infty(X, E^*)$ defined by $(T^*(g)f)(x) = \alpha^*(x, g)f(xg)$, and $\tilde{A} = \{f \in L^\infty(X, E^*) \mid f(x) \in A_x \text{ a. e.}\}$ is a compact convex G-invariant subset. For each ξ, we can identify $L^\infty(X_\xi, E^*)$ as a subspace of $L^\infty(X, E^*)$ and similarly we can so identify $L^\infty(Y, E^*)$. For each ξ, let $A_\xi = \{f \in L^\infty(X_\xi, E^*) \cap \tilde{A} \mid T^*(g)f = f \text{ for all } g\}$. Then A_ξ is a decreasing sequence of compact convex sets, and amenability of X_ξ ensures that A_ξ is non-empty. Choose $\varphi \in \bigcap A_\xi$. Then φ is measurable with respect to each $B(X_\xi)$ and hence is measurable with respect to B. Hence φ is the required γ-invariant section.

In section 5 we will explicitly identify all minimal amenable ergodic actions of real algebraic groups and connected semi-simple Lie groups with finite center.

IV. — Orbits of measures on projective space

Let $\mathbf{P}_n$ be the real projective space of lines in $\mathbf{R}^{n+1}$. Let $M(\mathbf{P}^n)$ be the space of probability measures on $\mathbf{P}^n$ with the weak-$\star$-topology. The general linear group $GL(n+1, \mathbf{R})$ (we shall suppress the $\mathbf{R}$ hereafter in this section) acts continuously in a natural fashion on $\mathbf{P}^n$ and hence on $M(\mathbf{P}^n)$. The aim of this section is to prove the following theorem.

THEOREM 4.1. — *For every* $\mu \in M(\mathbf{P}^n)$ *the orbit of* μ *under* $GL(n+1)$ *is locally closed.*

The approach we take in proving this theorem is to employ a technique of H. Furstenberg. If $\mu \in M(\mathbf{P}^n)$ and ν is a limit point of the orbit of μ not contained in the orbit, then Furstenberg has shown in [7], Lemma 1.5, that ν is supported on a union of two proper projective subspaces. Since the space of probability measures supported on a union of two proper projective subspaces is closed, one can deduce immediately that the orbit of any μ which is not so supported must be locally closed. The idea of the proof of Theorem 4.1 is to expand upon these remarks to obtain the theorem for an arbitrary μ.

We begin with notation and some basic facts. $\mathbf{P}^n$ is a compact metrizable space and hence the set $\mathscr{C}$ of closed subsets of $\mathbf{P}^n$ is a compact metric space with the Hausdorf metric. If $A \in \mathscr{C}$, let $M(A)$ denote the set of probability measures on $\mathbf{P}^n$ supported in A. Then $M(A) \subset M(\mathbf{P}^n)$ is a closed subset. If $\mathscr{A} \subset \mathscr{C}$, let $M_{\mathscr{A}} = \bigcup_{A \in \mathscr{A}} M(A)$.

LEMMA 4.2. — *If* $A_n \in \mathscr{C}$, $A_n \to A$, *and* $\mu_n \in M(A_n)$ *with* $\mu_n \to \mu \in M(\mathbf{P}^n)$, *then* $\mu \in M(A)$.

Proof. — Let f be a continuous function on $\mathbf{P}^n$ with $\operatorname{supp}(f) \cap A = \emptyset$. Since $A_n \to A$, for sufficiently large n, $\operatorname{supp}(f) \cap A_n = \emptyset$. Thus $\int f \, d\mu_n = 0$ for sufficiently large n, so $\int f \, d\mu = 0$.

COROLLARY 4.3. — *If* $\mathscr{A} \subset \mathscr{C}$ *is closed, then* $M_{\mathscr{A}}$ *is closed.*

There is a natural map $\mathbf{R}^{n+1} - \{0\} \to \mathbf{P}^n$ and if $V \subset \mathbf{R}^{n+1}$ is a non-trivial subspace we will denote the image of V in $\mathbf{P}^n$ by $[V]$. We call $[V]$ a projective subspace of $\mathbf{P}^n$. Define $\mathscr{A}$ to be the subset of $\mathscr{C}$ consisting of elements $A \in \mathscr{C}$ of the form $A = \bigcup_{i=1}^{k} [V_i]$ where $[V_i]$ are (non-empty) projective subspaces such that $V_i \not\subset V_j$ for $i \neq j$, and $\sum \dim V_i \leqq n+1$. Then define $n(A) = k$, $d(A) = \sum \dim V_i$, and $D(A) = \dim \sum V_i$. These numbers are uniquely determined by the set A. We note that $1 \leqq n(A), d(A), D(A) \leqq n+1$.

The proofs of the following facts are straightforward.

LEMMA 4.4. — (*a*) *Let* $\mathscr{A}_k = \{ A \in \mathscr{A} \mid n(A) \leqq k \}$, $\mathscr{A}^d = \{ A \in \mathscr{A} \mid d(A) \leqq d \}$ *and* $\mathscr{A}(D) = \{ A \in \mathscr{A} \mid D(A) \leqq D \}$. *Then these sets are all closed subsets of* $\mathscr{C}$.

(*b*) *Let* $\mathscr{B}_k^d = \{ A \in \mathscr{A} \mid n(A) = k \text{ and } d(A) = d \}$. *If* $A_i \in \mathscr{B}_k^d$ *and* $A_i \to A \in \mathscr{C}$, *then* $A \in \mathscr{B}_k^d \cup \mathscr{A}^{d-1}$. *Hence* $\mathscr{B}_k^d \cup \mathscr{A}^{d-1}$ *is closed in* $\mathscr{C}$.

Now let $\mu \in M(\mathbf{P}^n)$. Define $d(\mu) = \min \{ d(A) \mid A \in \mathscr{A}$ and $\mu \in M(A) \}$ and $n(\mu) = \max \{ n(A) \mid A \in \mathscr{A}$, $\mu \in M(A)$, and $d(A) = d(\mu) \}$. Fix an element $A \in \mathscr{A}$ with $\mu \in M(A)$ and $d(A) = d(\mu)$, $n(A) = n(\mu)$, and let $D(\mu) = D(A)$. We note that we actually have $d(A) = D(A)$ for $D(A) < d(A)$ would contradict the definition of $d(\mu)$ since we have $\mu \in M([\sum V_i])$. Let $\mathscr{K}(\mu) \subset \mathscr{C}$ be the closed (by Lemma 4.4) set

$$\mathscr{K}(\mu) = \mathscr{B}^{d(\mu)}_{n(\mu)+1} \cup \mathscr{A}^{d(\mu)-1} \cup \mathscr{A}(D(\mu)-1).$$

Let $\mathscr{U}(\mu)$ be the complement in $M(\mathbf{P}^n)$ of $M_{\mathscr{K}(\mu)}$. Then by the choice of $d(\mu)$, $n(\mu)$, $D(\mu)$, $\mathscr{U}(\mu)$ is an open neighborhood of μ in $M(\mathbf{P}^n)$. Furthermore, the orbit of μ under $GL(n+1)$ is also contained in $\mathscr{U}(\mu)$. Therefore the proof of Theorem 4.1 is reduced to the following.

LEMMA 4.5. — *If* $g_i \in GL(n+1)$ *and* $\mu.g_i \to \nu$ *where* $\nu \in \mathscr{U}(\mu)$, *then* ν *is in the orbit of* μ.

The proof of Lemma 4.5 depends in turn upon the following formulation of Furstenberg's technique. Let $V \subset \mathbf{R}^{n+1}$ be a subspace and suppose $g_i : V \to \mathbf{R}^{n+1}$ are linear maps of determinant 1. Suppose $[V g_i] \to [W]$ where W is a subspace (necessarily of the same dimension as V). Let μ be the probability measure on $[V] \subset \mathbf{P}^n$ and suppose that $\mu.g_n \to \nu$, so that by Lemma 4.2, ν is a measure on $[W]$.

LEMMA 4.6. — *Either* $\{ g_i \}$ *is bounded, or* ν *is supported on a union* $[Y] \cup [Z]$ *where* $[Y]$, $[Z]$ *are projective subspaces with* $\dim Y, \dim Z \geqq 1$ *and* $\dim Y + \dim Z = \dim W$.

Proof. — The proof is essentially just that of Furstenberg ([7], Lemma 1.5), but we include this minor variation for the reader's convenience. If g_i is not bounded, let $h_i = g_i / \| g_i \|$. Then, perhaps by passing to a subsequence, we have $h_i \to h$ for some linear function $h : V \to \mathbf{R}^{n+1}$ with $\| h \| = 1$ and $\det(h) = 0$. Let $N = \ker(h) \subset V$, $Z = \text{range } (h) \subset W$, so $\dim N + \dim Z = \dim V (= \dim W)$. Again by passing to a subsequence we can assume $[N g_i] \to [Y] \subset [W]$. We claim $\nu = \lim \mu.g_i$ is supported on $[Y] \cup [Z]$. Write $\mu = \mu_1 + \mu_2$ where μ_1 is supported on $[N]$ and μ_2 is supported on $[V] - [N]$. Passing to a subsequence, we have $\nu = \lim(\mu_1.g_i) + \lim(\mu_2.g_i)$. Clearly $\lim(\mu_1.g_i)$ is supported on $[Y]$ so it suffices to see that $\nu_2 = \lim(\mu_2.g_i)$ is supported on $[Z]$. Each g_i can be extended to a linear map $\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ and then for f continuous on $\mathbf{P}^n$ with $f = 0$ on $[Z]$ we have

$$\int_{\mathbf{P}^n} f \, d\nu_2 = \lim \int_{\mathbf{P}^n} f \, d(\mu_2 g_i)$$

$$= \lim \int_{\mathbf{P}^n} f(x g_i) \, d\mu_2(x) = \lim \int_{[V]-[N]} f(x g_i) \, d\mu_2(x).$$

But $f(x g_i) \to 0$ pointwise for $x \in [V] - [N]$, so $\int f \, d\nu_2 = 0$ by the dominated convergence theorem, completing the proof of Lemma 4.6.

We now return to the proof of Lemma 4.5 and hence Theorem 4.1.

Proof (of Lemma 4.5). – Let $A = \bigcup [V_i]$ be chosen as above. As we remarked previously, since μ is supported on $[\sum V_i]$, we have $\dim \sum V_i = \sum \dim V_i$. In other words, the subspaces $V_1, \ldots, V_{n(A)}$ are independent. By passing to a subsequence, we can assume $[V_j]\, g_i \to [W_j]$ for each j, where $\dim W_j = \dim V_j$, and since $\nu \in M(\bigcup [W_j])$ and $\nu \in \mathscr{U}(\mu)$, $W_1, \ldots, W_{n(A)}$ will also be independent subspaces. For each j let $\mu_j = \mu \,|\, [V_j]$, $\nu_j = \nu \,|\, [W_j]$ and $h_{ij} = (g_i \,|\, V_j)\, [\det (g_i \,|\, V_j)]^{-1/\dim V_j}$. Thus $h_{ij} : V_j \to \mathbf{R}^n$, $\det h_{ij} = 1$, $h_{ij} \,|\, [V_j] = g_i \,|\, [V_j]$, and $\lim_{i \to \infty} \mu_j h_{ij} = \nu_j$. We claim that for each j, the sequence h_{ij} is bounded as $i \to \infty$. This is clear if $\dim V_j = 1$. If $\dim V_j \geqq 2$ and h_{ij} is not bounded, then Lemma 4.6 implies that μ_j is supported on $[Y_j] \cup [Z_j]$ where Y_j, $Z_j \neq 0$ and $\dim Y_j + \dim Z_j = \dim W_j$. If $Y_j \cap Z_j \neq 0$, this would imply that ν is supported on an element of $\mathscr{A}(D(\mu) - 1)$ which contradicts the fact that $\nu \in \mathscr{U}(\mu)$. On the other hand, if $Y_j \cap Z_j = 0$, then ν is supported on an element of $\mathscr{B}^{d(\mu)}_{n(\mu)+1}$ which again contradicts $\nu \in \mathscr{U}(\mu)$. Thus h_{ij} is bounded for each j and it follows that by passing to a subsequence, as $i \to \infty$ h_{ij} converges to an isomorphism $h_j : V_j \to W_j$ such that $\mu_j h_j = \nu_j$. Finally, let $h : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be an invertible linear opertor which agrees with h_j on V_j. Then $h \in GL(n+1)$ and $\mu h = \nu$. This completes the proof.

V. — Actions of algebraic and semi-simple Lie groups

In this section we prove that amenable ergodic actions of real algebraic groups and connected semi-simple Lie groups with finite center are induced from actions of amenable subgroups, as well as other less precise results for more general groups, and applications. We begin with an analysis of the amenable ergodic actions of $GL(n)$. The essential step involving Theorem 4.1 is Lemma 5.3.

If S is any ergodic G-space, where G is a locally compact group, by a finite ergodic extension of S we mean an ergodic extension $p : T \to S$ such that T is isomorphic as an extension of S to a space of the form $(S \times F, \mu \times m)$ where F is a finite set and m a measure on F. (By ergodicity one can see that this is in fact equivalent to only requiring that under a direct integral decomposition of the measure on T over (S, μ) that almost all fiber measures have finite support). Clearly, a finite extension of a finite extension is again finite. If T is any ergodic extension of S and $\alpha(s, g)$ is a cocycle defined on $S \times G$, then the restriction of α to $T \times G$ is the cocycle $\tilde{\alpha}(t, g) = \alpha(p(t), g)$.

LEMMA 5.1. – *Let* M *be a locally compact group and* $\alpha : S \times G \to M$ *a cocycle. Suppose* $M_0 \subset M$ *is a normal subgroup of finite index. Then* S *has a finite ergodic extension* T *such that the restriction of* α *to* T *is equivalent to a cocycle into* M_0.

Proof. – Let $q : M \to M/M_0$ be the natural projection. Then $q \circ \alpha$ is a cocycle into the finite group M/M_0. It follows that there is a subgroup $F \subset M/M_0$ such that $q\alpha$ is equivalent to a cocycle β with $\beta(S \times G) \subset F$ and $S \times_\beta F = T$ is ergodic ([18], section 3). Suppose $\varphi : S \to M/M_0$ such that $\varphi(s)\, q(\alpha(s, g))\, \varphi(sg)^{-1} = \beta(s, g)$ for each g, almost all s. Let $\theta : M/M_0 \to M$ be a section of q and define $\psi : S \times_\beta F \to M$ by $\psi(s, x) = \theta(x\varphi(s))$. To prove the lemma it suffices to show that for each g,

$\psi(t)\,\tilde{\alpha}(t, g)\,\psi(tg)^{-1} \in M_0$ a. e. But

$$q(\psi(s, x)\alpha(s, g)\psi(sg, x\beta(s, g))^{-1}) = x\varphi(s)\,q(\alpha(s, g))\big[x\beta(s, g)\varphi(sg)\big]^{-1}$$
$$= x\beta(s, g)\beta(s, g)^{-1}x^{-1} = e$$

for each x, g and almost all s, and so the result follows.

LEMMA 5.2. — *Suppose* $\alpha : S \times G \to M$ *is a cocycle and* $q : M \to N$ *a surjective homomorphism. If* $q \circ \alpha$ *is equivalent to a cocycle into a closed subgroup* $N' \subset N$, *then* α *is equivalent to a cocycle into* $M' = q^{-1}(N')$.

Proof. — If $\varphi : S \to N$ with $\varphi(s)\,q(\alpha(s, g))\,\varphi(sg)^{-1} \in N'$, then $\theta(\varphi(s))\,\alpha(s, g)\,\theta(\varphi(sg))^{-1}$ is the required cocycle where θ is a section of q.

The basic application of Theorem 4.1 in proving the main theorems is the following.

LEMMA 5.3. — *Suppose* $\alpha : S \times G \to GL(n)$ *and that* S *is an amenable* G*-space. Then* S *has a finite ergodic extension* T *such that the restriction of* α *to* T *is equivalent to a cocycle into a subgroup* $M \subset GL(n)$ *that either projects to a compact group in* $SL(n)$ *or leaves a proper subspace invariant.*

Proof. — Let $E = C(\mathbf{P}^{n-1})$ be the Banach space of continuous complex-valued functions on $\mathbf{P}^{n-1}$ and let $\pi : GL(n) \to Iso(E)$ be the representation induced by the action of $GL(n)$ on $\mathbf{P}^{n-1}$. Let π^* denote the adjoint representation on E^*. Then $\pi \circ \alpha : S \times G \to Iso(E)$ is a cocycle and $M(\mathbf{P}^{n-1})$ is compact convex and $(\pi \circ \alpha)$-invariant. By amenability, there is a Borel function $\varphi : S \to M(\mathbf{P}^{n-1})$ such that for each g and almost all s.

$$(\star) \qquad \pi^*(\alpha(s, g))\,\varphi(sg) = \varphi(s).$$

Let $\hat{M}(\mathbf{P}^{n-1}) = M(\mathbf{P}^{n-1})/GL(n)$ be the space of orbits in $M(\mathbf{P}^{n-1})$ under the general linear group. It is a consequence of Theorem 4.1 and [4] that $\hat{M}(\mathbf{P}^{n-1})$ is a standard Borel space with the quotient Borel structure. Equation $(\star)$ implies that for each g and almost all s, $\varphi(s) \equiv \varphi(sg)$ in $\hat{M}(\mathbf{P}^{n-1})$. Since $\hat{M}(\mathbf{P}^{n-1})$ is standard and G is ergodic on S, it follows that by changing φ on a null Borel set, $\varphi(S)$ will be contained in a single orbit in $M(\mathbf{P}^{n-1})$. Choose a point μ_0 in this orbit and let $M \subset GL(n)$ be the stabilizer of μ_0, so that the orbit can be identified with $GL(n)/M$. *Via* a Borel section $GL(n)/M \to GL(n)$ we can find a Borel map $\theta : \text{Orbit}(\mu_0) \to GL(n)$ such that $\pi^*(\theta(\mu))\,\mu_0 = \mu$ for all μ in the orbit. Define a cocycle $\beta \sim \alpha$ by

$$\beta(s, g) = \theta(\varphi(s))^{-1}\,\alpha(s, g)\,\theta(\varphi(sg)).$$

We claim that for each g, $\beta(s, g) \in M$ a. e. It suffices to show that $\pi^*(\beta(s, g))\,\mu_0 = \mu_0$. We can write $\varphi(s) = \pi^*(\theta(\varphi(s)))\,\mu_0$ for each s, and hence equation $(\star)$ becomes $\pi^*(\alpha(s, g)\,\theta(\varphi(sg)))\,\mu_0 = \pi^*(\theta(\varphi(s)))\,\mu_0$ from which the required identity follows immediately. Since for each g, $\beta(s, g) \in M$ a. e., changing β on a null set we can assume β is a cocycle into M. Since M is the subgroup of $GL(n)$ leaving μ_0 fixed, it follows from the proof of [7], Lemma 1.5, that either M is compact when projected to $SL(n)$ or has

a normal subgroup M_0 of finite index such that M_0 leaves a proper subspace invariant. In the former case there is nothing further to show and in the latter case the lemma follows from Lemma 5.2.

We now use an inductive argument to obtain the following result.

LEMMA 5.4. — *Suppose* $\alpha : S \times G \to GL(n)$ *and that* S *is an amenable* G*-space. Then* S *has a finite ergodic extension* T *such that the restriction of* α *to* T *is equivalent to a cocycle taking values in a subgroup* $M \subset GL(n)$ *which can be described as follows. There is a sequence of subspaces* $0 = V_0 \subset V_1 \subset \ldots \subset V_k = \mathbf{R}^n$ *and inner products* B_i *on* V_i/V_{i-1}, $i = 1, \ldots, k$, *such that* $M = \{ g \in GL(n) \mid V_i g = V_i$ *for all* i, *and* g *induces a similarity on* V_i/V_{i-1} *with respect to* B_i; i. e., $B_i.g$ *is a scalar multiple of* $B_i \}$.

Proof. — The proof is by induction on n. If $n = 1$, the lemma is clear. For $n > 1$, suppose the theorem is true for all integers strictly less than n. If S has a finite ergodic extension T such that the restriction of α to T is equivalent to a cocycle with values in $\mathbf{R}K$, where $K \subset SL(n)$ is compact, we are clearly done. If not, then by Lemma 5.3 we can choose a finite ergodic extension T such that α restricted to T is equivalent to a cocycle β into the group M_1 of all invertible transformations leaving a proper subspace V_1 invariant, and so that V_1 is of minimal dimension among all such choices of T and M_1. There is a natural surjective homomorphism $p : M_1 \to GL(V_1)$. Since T is an amenable G-space, we can apply Lemmas 5.3 and 5.2 to $p \circ \beta$ and conclude from the minimality property of V_1 that β is equivalent to a cocycle, which we still denote by β, taking values in the group M_0 consisting of all invertible transformations leaving V_1 invariant and inducing a similarity on (V_1, B_1) for some inner product B_1 on V_1. We can now take the surjective homomorphism $q : M_0 \to GL(\mathbf{R}^n/V_1)$ and apply the inductive hypothesis to $q \circ \beta : T \times G \to GL(\mathbf{R}^n/V_1)$. An application of Lemma 5.2 then completes the proof.

Let us pause to summarize our situation. The group M in the statement of Lemma 5.4 is a compact extension of a solvable group and is therefore amenable. If we take $G = GL(n)$ and $\alpha : S \times GL(n) \to GL(n)$ to be projection on $GL(n)$, then Lemma 5.4 and Corollary 2.6 imply that any amenable ergodic $GL(n)$ space has a finite ergodic extension that is induced from an ergodic action of an amenable subgroup M. We now wish to dispose of the need for taking a finite extension (perhaps changing M to another amenable subgroup in the process, of course). In order to do this, we need to recall one important fact about algebraic transformation groups which will be of use to us in another situation below as well.

If a locally compact group G acts in a Borel fashion on a standard Borel space X, the action is called smooth if the orbit space is a standard Borel space with the quotient Borel structure. If X is metrizable by a complete metric and the G-action is continuous, this is equivalent to all orbits being locally closed [4]. (We have used this fact in Lemma 5.3.) The result we will need is that algebraic actions are smooth. More precisely, if G is a real algebraic group acting algebraically on a real algebraic variety, then the action is smooth. For a proof, see the remarks in [3], p. 183-184.

We are now in a position to prove the main theorem for ergodic actions of $GL(n)$.

THEOREM 5.5. — *Every amenable ergodic action of* GL (n) *is induced from an action of a closed amenable subgroup.*

Proof. — Let $\alpha : S \times GL(n) \to GL(n)$ be $\alpha(s, g) = g$, and construct the extension $T \to S$ and the sequence of subspaces $0 = V_0 \subset V_1 \subset \ldots \subset V^k = \mathbf{R}^n$ as in Lemma 5.4. Let the cardinality of the fibers of $T \to S$ be the integer p (a. e.). If Z is a finite-dimensional real vector space, let B (Z) be the space of inner products on Z with two inner products identified if they differ by a scalar multiple. Thus B (Z) is a subset of the projective space of the linear space of bilinear maps $Z \times Z \to \mathbf{R}$ and the stability group in GL (Z) of an element in B (Z) is just the group of similarities of the inner product.

Let F be the set of all $2k$-tuples, $F = \{ (W_1, \ldots, W_k, D_1, \ldots, D_k) \mid W_i \subset \mathbf{R}^n$ is a subspace with $\dim W_i = \dim V_i$, $W_{i-1} \subset W_i$, and $D_i \in B(W_i/W_{i-1}) \}$. Then F has a natural Borel structure, GL (n) acts naturally on F and it is not difficult to see that this action is transitive. Thus we can identify F with GL $(n)/M$ where M is the subgroup given in Lemma 5.4. Since M is an algebraic group and algebraic actions are smooth, it follows that GL (n) is smooth on the product space F^p. Let S_p be the symmetric group on p letters which acts naturally on F^p and commutes with the GL (n) action. It follows that GL (n) is smooth on the quotient F^p/S_p as well.

Let $\tilde{\alpha}$ be the restriction of α to T and $\beta : T \times GL(n) \to M$ the cocycle equivalent to $\tilde{\alpha}$ given by Lemma 5.4. We can write $T = S \times I$ where I is some finite set. Let $\varphi : T \to GL(n)$ implement the equivalence of $\tilde{\alpha}$ and β, i. e., for each g,

$$\varphi(s, y)\alpha(s, g)\varphi((s, y)g)^{-1} = \beta(s, y, g)$$

for almost all $(s, y) \in T$. Let $V \in F$ be the element $V = (V_1, \ldots, V_k, B_1, \ldots, B_k)$ given by Lemma 5.4, and $V(s, y) \in F$ be $V.\varphi(s, y)$. We then can define a map $\tilde{V} : S \to F^p/S_p$ by $\tilde{V}(s) = \{ V(s, y) \mid y \in I \}$, and it is easy to check that $\tilde{V}$ is Borel. Furthermore, since $V.\beta(t, g) = V$, we have for each g, y, and almost all s, $V(s, y).\alpha(s, g) = V(sg, y')$ for some y'. In other words, $\tilde{V}(s)\alpha(s, g) = \tilde{V}(sg)$. This says that $\tilde{V}(s)$ and $\tilde{V}(sg)$ are in the same GL (n) orbit in F^p/S_p, and since the action of GL (n) on F^p/S_p is smooth, ergodicity of GL (n) on S implies that there is a single orbit in F^p/S_p such that $\tilde{V}(s)$ is in that orbit for almost all s. Arguing exactly as in the proof of Lemma 5.3, we see that α is equivalent to a cocycle γ taking values in the group $G_0 = \{ g \in GL(n) \mid \tilde{V}_0 g = \tilde{V}_0 \}$ where $\tilde{V}_0 \in F^p/S_p$ is some point in the orbit singled out above. We can write $\tilde{V}_0$ as a set $\{ V^1, V^2, \ldots, V^p \}$ where $V^j \in F$. Let $G_1 = \{ g \in G_0 \mid V^j g = V^j$ for all $j \}$. Then G_1 is a normal subgroup of G_0 with finite index and G_1 is of course a closed subgroup of $\{ g \in GL(n) \mid V^1 g = V^1 \}$. The latter is a compact extension of a solvable group and hence amenable, and so G_1 and therefore G_0 is amenable. The theorem now follows from Corollary 2.6.

For the analysis of amenable actions of other groups, we will need the following lemma.

LEMMA 5.6. — *Suppose* X *is a standard Borel* G*-space and that* H $\subset$ G *is a subgroup of finite index. Then* G *acts smoothly on* X *if and only if* H *acts smoothly on* X.

Proof. — We first observe that it suffices to consider the case where H is normal, since $\bigcap_g g\,H\,g^{-1}$ will be normal in G and of finite index in both G and H. If H is normal and smooth on X, then G will be smooth since $X/G \cong (X/H)/G/H$ and finite group actions are smooth. Conversely, if G acts smoothly, then to see that H does as well it suffices to remark that if μ is a properly ergodic quasi-invariant measure under H, then $\sum \mu.g_i$ will be properly ergodic and quasi-invariant under G, where $\{g_i\}$ are a set of representatives of G/H.

We now extend Theorem 5.5 to algebraic groups.

THEOREM 5.7. — *Every amenable ergodic action of a real algebraic group is induced from an action of an amenable subgroup.*

Proof. — Let G be a real algebraic group and (S, μ) an amenable ergodic G-space. Let X be the induced ergodic $GL(n)$ action, and represent $X = GL(n)/G \times_\alpha S$ where $\alpha : GL(n)/G \times GL(n) \to G$ corresponds to the identity as in Proposition 2.2. By Proposition 3.2 and Theorem 5.5, there is a closed amenable subgroup $M \subset GL(n)$ such that $GL(n)/M$ is a factor $GL(n)$-space of X. We can clearly assume that M is a maximal closed amenable subgroup. One can readily check that a conull $GL(n)$-invariant Borel subset of X is of the form $GL(n)/G \times_\alpha S_0$ where $S_0 \subset S$ is conull, Borel, and G-invariant. Thus we can assume that we have a $GL(n)$-map

$$GL(n)/G \times_\alpha S \to GL(n)/M.$$

Restricting this map to $[e] \times S$ gives a (not necessarily measure-class preserving) G-map $\varphi : S \to GL(n)/M$. Let $\nu = \varphi^*(\mu)$, so that ν is quasi-invariant and ergodic under G. Suppose for the moment that ν is supported on a G-orbit. Then S has a G-space factor of the form $G/g\,M\,g^{-1} \cap G$ for some $g \in GL(n)$, and since $g\,M\,g^{-1} \cap G$ is a closed amenable subgroup of G, the result follows from Theorem 2.5. Thus to prove the theorem, it suffices to show that every G-ergodic measure on $GL(n)/M$ is supported on an orbit, i. e. that G is smooth on $GL(n)/M$. This will be the case if and only if M is smooth on $GL(n)/G$. By [9], Th. IV.2, M has a normal subgroup M_1 of finite index which is a real algebraic group. Since G is algebraic and algebraic actions are smooth, M_1 is smooth on $GL(n)/G$ and by Lemma 5.6, M is as well, completing the proof.

To obtain the corresponding result for connected semi-simple Lie groups with finite center we use the following two lemmas.

LEMMA 5.8. — *Suppose* $H \subset G$ *is a normal subgroup of finite index and that every ergodic amenable* G*-action is induced from an action of an amenable subgroup. Then the same is true for* H.

Proof. — If S is an amenable ergodic H-space, form the induced G-space $G/H \times_\alpha S$ and suppose that this has a factor G-space G/G_0 where $G_0 \subset G$ is amenable. Arguing as in the proof of Theorem 5.7, it suffices to show that H is smooth on G/G_0. But this is clear since G_0 is obviously smooth on G/H.

LEMMA 5.9. — *Suppose* $p : G \to H$ *is a surjective homomorphism with a compact kernel* K. *If every amenable ergodic* H*-action is induced from the action of an amenable subgroup, the same is true for* G.

Proof. — Let S be an amenable ergodic G-space. Let $\hat{S} = S/K$, i. e., the space of K-orbits in S. Since K is normal, there is a naturally defined ergodic action of G on $\hat{S}$ which is amenable by [20], Prop. 2.6. This action factors to an action of H which will clearly be amenable, and so there is an amenable subgroup $H_0 \subset H$ and a map of H-spaces $\hat{S} \to H/H_0$. Thus there is a map of G-spaces $S \to G/G_0$ where $G_0 = p^{-1}(H_0)$. Since G_0 satisfies the exact sequence $0 \to K \to G_0 \to H_0 \to 0$ with K, H_0 amenable, it follows that G_0 is amenable. The lemma now follows from Theorem 2.5.

THEOREM 5.10. — *Let* G *be a connected semi-simple Lie group with finite center. Then every amenable ergodic action of* G *is induced from an ergodic action of an amenable subgroup.*

Proof. — This follows from Theorem 5.7, Lemmas 5.8 and 5.9, and the fact that $G/Z(G) \cong Ad(G)$ is the connected component of a real algebraic group.

In the proof of Theorem 5.7, the only point at which the condition that G be algebraic was used was in deducing smoothness of the action of G on $GL(n)/M$. Thus for general closed subgroups of $GL(n)$ one obtains the following by the same proof.

THEOREM 5.11. — *Let* G *be a closed subgroup of* $GL(n)$ *and* S *an amenable ergodic action of* G. *Then there is an amenable subgroup* $M \subset GL(n)$ *and a probability measure* ν *on* $GL(n)/M$ *quasi-invariant and ergodic under* G *such that* $(GL(n)/M, \nu)$ *is a factor of* S.

One can also regard the theorems above as identifying the minimal amenable actions of Lie groups.

COROLLARY 5.12. — (i) *If* S *is a minimal amenable ergodic action of a closed subgroup* $G \subset GL(n)$, *then* S *is of the form* $(GL(n)/M, \nu)$ *where* M *is maximal amenable in* G *and* ν *is quasi-invariant and ergodic under* G.

(ii) *The minimal amenable ergodic actions of a real algebraic group or a connected semi-simple Lie group with finite center are of the form* G/M *where* M *is a maximal amenable subgroup of* G.

It would be interesting to obtain some sort of description and classification of those measures ν appearing in the above corollary, in general or in some specific cases. For example, a case of interest would be to take M to be the triangular matrices and G to be a lattice subgroup of $SL(n)$. Another general case of interest would be that of Poisson boundaries. In [20] we showed that the Poisson boundary of an étalée random walk on G is an amenable ergodic G-space. It follows that if G is a closed subgroup of $GL(n)$ the Poisson boundary has a factor of the form described by Corollary 5.12. It would be interesting to determine which M and ν can arise in this situation. Furstenberg, of course, has very precise results if G is a semi-simple Lie group ([8], Th. 13.4).

Every closed amenable subgroup of $GL(n)$ is a compact extension of a solvable normal subgroup ([9], Th. IV.3). From the above theorems, one can in a certain sense reduce

amenable actions to compact extensions and actions of solvable subgroups. If $X \to Y$ is an extension of ergodic G-spaces, we call X a compact group extension of Y if X is isomorphic as an extension of Y to $Y \times_\alpha K$ where K is a compact group and $\alpha : Y \times G \to K$ is a cocycle. These extensions have been studied in [18] for example.

COROLLARY 5.13. — *If* S *is amenable ergodic* G*-space where* G *is a real algebraic group or a connected semi-simple Lie group with finite center, then* S *has a compact group extension* T *which is induced from an ergodic action of a solvable subgroup of* G.

Proof. — Let $\alpha : S \times G \to G$ be projection on G. Then by Theorems 5.7, 5.10 and Corollary 2.6, α is equivalent to a cocycle into an amenable subgroup M. Let $M_0 \subset M$ be a normal solvable subgroup such that M/M_0 is compact ([9], Th. IV.3). Using the argument of Lemma 5.1, we see that there is a compact group extension $T \to S$ such that the restriction of α to T is equivalent to a cocycle into M_0. An application of Corollary 2.6 then completes the proof.

Using Theorems 5.7, 5.10, and the results of [22] and [6], Th. 8.10, we obtain the following corollary.

THEOREM 5.14. — *Suppose* G *is a real algebraic group or a connected semi-simple Lie group with finite center, and that* S *is a free ergodic* G*-space. Then the von Neumann algebra associated to the action by the group-measure space construction is approximately finite dimensional (i. e., hyperfinite) if and only if the action is induced from a (free) ergodic action of an amenable subgroup of* G.

We note that the statement of Theorem 5.14 involves no concept that depends upon cohomology for its definition. The proof, however, depends from almost beginning to end upon cohomological considerations. One can, of course, formulate similar theorems based on other results above.

Another interesting consequence of Theorems 5.7 and 5.10 concerning cohomology is the following.

THEOREM 5.15. — *Suppose* S *is an amenable* G*-space and* $\alpha : S \times G \to H$ *where* H *is a real algebraic group or a connected semi-simple Lie group with finite center. Then* α *is equivalent to a cocycle into an amenable subgroup of* H.

Proof. — By [18], Th. 3.5, α is equivalent to a cocycle into an amenable subgroup $M \subset H$ if and only if H/M is a factor of the Poincare flow (range-closure) of α. But the Poincaré flow of α is amenable ([20], Th. 3.3) and so the result follows from Theorems 5.7, 5.10, 2.5.

VI. — A counterexample

In this section we present an amenable ergodic action of a discrete group that is not induced from an ergodic action of any amenable subgroup. In fact, we have the following result.

THEOREM 6.1. — *Let* $M \subset SL(2, \mathbf{C})$ *be the subgroup of all upper triangular matrices, and suppose* Γ *is a lattice subgroup of* $SL(2, \mathbf{C})$, *i. e.,* Γ *is discrete and* $SL(2, \mathbf{C})/\Gamma$ *has*

finite volume. Then $\mathrm{SL}(2, \mathbf{C})/\mathrm{M}$ *is an amenable ergodic* Γ*-space that is not induced from an action of any amenable subgroup of* Γ.

Proof. — We remark first that Γ is ergodic on G/M by Moore's theorem [14] and since M is amenable, the Γ action is amenable by Proposition 3.4. To begin the remainder of the proof, we need notation for some subgroups of M. Each element of M is of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

where $a, b \in \mathbf{C}$, $a \neq 0$. Let $\mathrm{N} \subset \mathrm{M}$ be the set of elements with $a = 1$, so in fact $\mathrm{N} = [\mathrm{M}, \mathrm{M}]$. Let A be the elements with $b = 0$, so that M is a semi-direct product AN. The group A is isomorphic to $\mathrm{K} \times \mathbf{R}$ where K is the unit circle and we shall identify K and $\mathbf{R}$ as the corresponding subgroups of M. For brevity, we denote $\mathrm{SL}(2, \mathbf{C})$ by G for the remainder of the proof.

Suppose that the Γ-action on G/M is induced from an action of an amenable subgroup $\Gamma_0 \subset \Gamma$. Then inducing both Γ-actions to G we *see* by Example 2.3, Theorem 2.5, and Proposition 2.8 that we have an extension of ergodic G-spaces $\mathrm{G}/\mathrm{M} \times \mathrm{G}/\Gamma \to \mathrm{G}/\Gamma_0$. By [9], Th. IV.3 and [2], Corol., p. 243, Γ_0 has a normal subgroup Γ_1 of finite index which is triangulable. Since $\mathrm{G}/\Gamma_0 \cong \mathrm{G}/g\,\Gamma_0\,g^{-1}$ as G-spaces for any $g \in \mathrm{G}$, we can assume that $\Gamma_1 \subset \mathrm{M}$. We can identify G/M with the complex projective space of $\mathbf{C}^2$. There are two M-orbits in G/M (i. e., two double cosets $\mathrm{M}\backslash\mathrm{G}/\mathrm{M}$), namely $\{[e]\}$ and its complement. We shall call the latter the non-trivial M-orbit, and we note that as an M-space this orbit is equivalent to M/A. In particular, N is also transitive on the non-trivial M-orbit.

We claim that $\left\{ a \in \mathrm{A} \mid \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \Gamma_1 \text{ for some } b \right\}$ is a discrete set in A. Suppose first that Γ_1 is abelian and that $\mathrm{B} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \Gamma_1$ with $a^2 \neq 1$. Suppose $\mathrm{B}_n = \begin{pmatrix} c_n & d_n \\ 0 & c_n^{-1} \end{pmatrix} \in \Gamma_1$ with c_n, c_n^{-1} bounded and c_n distinct. From multiplying out the equation $\mathrm{B}\mathrm{B}_n\,\mathrm{B}^{-1} = \mathrm{B}_n$, it follows that $d_n = ab\,(c_n - c_n^{-1})\,(a^2-1)^{-1}$. Therefore d_n is bounded which implies that B_n has a convergent subsequence, contradicting the discreteness of Γ_1. Thus, we need only consider the case in which Γ_1 is not abelian. Then $[\Gamma_1, \Gamma_1] \neq \{e\}$, so there is some $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \in \Gamma_1$ with $d \neq 0$. Conjugating this matrix by $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ we obtain $\begin{pmatrix} 1 & a^2 d \\ 0 & 1 \end{pmatrix}$. Thus if $\begin{pmatrix} a_n & b_n \\ 0 & a_n^{-1} \end{pmatrix} \in \Gamma_1$ with a_n bounded and distinct, then $\begin{pmatrix} 1 & a_n^2 d \\ 0 & 1 \end{pmatrix} \in \Gamma_1$ and since $d \neq 0$, this again contradicts the discreteness of Γ_1. This verifies our assertion.

We can now observe that $\Gamma_1 \mathrm{N} \subset \mathrm{M}$ is a closed subgroup. If

$$\begin{pmatrix} a_n & b_n \\ 0 & a_n^{-1} \end{pmatrix} \in \Gamma_1 \quad \text{and} \quad \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \in \mathrm{N},$$

then the product is

$$\begin{pmatrix} a_n & a_n x_n + b_n \\ 0 & a_n^{-1} \end{pmatrix}.$$

If this converges to $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ we can assume by the above remarks that $a_n = a$ for all n and so

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & b_1 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & (b-b_1)/a \\ 0 & 1 \end{pmatrix} \in \Gamma_1 N$$

and the latter is closed. Since $\Gamma_1 N$ is normal in M, the action of Γ_1 on $M/\Gamma_1 N$ is trivial. We claim that this implies that the action of Γ_1 on $G/\Gamma_1 N$ has infinitely many ergodic components. To see this, observe that we can consider $G/\Gamma_1 N$ as the G-action induced from the M-action on $M/\Gamma_1 N$. Thus we can write the G-space $G/\Gamma_1 N$ as $G/M \times_\alpha M/\Gamma_1 N$ where $\alpha : G/M \times G \to M$ is the strict cocycle $\alpha(x, g) = s(x)\, gs(xg)^{-1}$ where $s : G/M \to G$ is a Borel section with $s([e]) = e$. Recall that there is a point, say q, in the non-trivial M-orbit in G/M so that the stabilizer in M of q is A. One can readily check that $J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ is in the coset q. We can also choose the section s above so that $s(q) = J$. Since $\Gamma_1 N$ is transitive on the non-trivial M-orbit, $\Gamma_1 N$ will have infinitely many ergodic components on $G/M \times_\alpha M/\Gamma_1 N$ (which will certainly imply that Γ_1 has infinitely many as well) if and only if the action of the stabilizer of q in $\Gamma_1 N$ has infinitely many ergodic components on $\{q\} \times M/\Gamma_1 N$. To see this, one can argue as in [24], Th. 4.2, for example. The stabilizer of q in $\Gamma_1 N$ is $\Gamma_1 N \cap A$ since $\Gamma_1 N \subset M$. Furthermore, the action of $h \in \Gamma_1 N \cap A$ on $\{q\} \times M/\Gamma_1 N$ is translation of the conjugate $s(q)\, hs(q)^{-1}$ on $M/\Gamma_1 N$. This just follows from the definition of the action of G on $G/M \times_\alpha M/\Gamma_1 N$. But $s(q) = J$ and on A, conjugation by J is just the map $h \to h^{-1}$. Therefore $\Gamma_1 N \cap A$ acts trivially on $\{q\} \times M/\Gamma_1 N$, and so the verification that Γ_1 has infinitely many ergodic components on $G/\Gamma_1 N$ is complete. Since Γ_1 is of finite index in Γ_0, it is easy to *see* that this implies that Γ_0 cannot be ergodic on $G/\Gamma_1 N$. Equivalently ([24], Th. 4.2), the product action of G on $G/\Gamma_1 N \times G/\Gamma_0$ is not ergodic.

We now consider two cases. Suppose first that $\Gamma_1 \not\subset KN$. We have an extension $G/M \times G/\Gamma \to G/\Gamma_0$ and hence a measure-class preserving G-map

$$G/\Gamma_1 N \times G/M \times G/\Gamma \to G/\Gamma_1 N \times G/\Gamma_0.$$

We have shown that $G/\Gamma_1 N \times G/\Gamma_0$ is not ergodic, so to derive a contradiction, it suffices to show that $G/\Gamma_1 N \times G/M \times G/\Gamma$ is ergodic. Recall that the stabilizer in $\Gamma_1 N$ of a point in the non-trivial M-orbit is a conjugate of $\Gamma_1 N \cap A$. We claim that this is not compact. For any $\gamma \in \Gamma_1$ there is $n \in N$ with $\gamma n \in A$. Since $\Gamma_1 \not\subset KN$ we can find such a product $\gamma n \notin K$. But any compact subgroup of A is actually contained in K, so $\Gamma_1 N \cap A$ is not compact. Since the stabilizer in $\Gamma_1 N$ on the conull orbit is not compact, it follows that the product G-action $G/\Gamma_1 N \times G/M$ has a conull

orbit with a non-compact stabilizer. By Moore's theorem [14], it follows that $G/\Gamma_1 N \times G/M \times G/\Gamma$ is ergodic, providing the desired contradiction.

It remains to consider the case in which $\Gamma_1 \subset KN$. The group KN is transitive on the non-trivial M-orbit M/A, and as a KN-space this is isomorphic to

$$KN/KN \cap A = KN/K.$$

Since K is compact and Γ_1 is discrete, Γ_1 has infinitely many ergodic components on M/A and hence infinitely many ergodic components on G/M. As above, it follows that Γ_0 is not ergodic on G/M and so $G/M \times G/\Gamma_0$ is not ergodic. However, reasoning as above, $G/M \times G/M \times G/\Gamma$ is ergodic, and so the existence of a measure-class preserving G-map $G/M \times G/M \times G/\Gamma \to G/M \times G/\Gamma_0$ is a contradiction. This completes the proof of the theorem.

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