ANNALES SCIENTIFIQUES DE L'É.N.S.

MOSHÉ FLATO
GEORGES PINCZON
JACQUES SIMON

Non linear representations of Lie groups

Annales scientifiques de l'É.N.S. 4e série, tome 10, nº 3 (1977), p. 405-418
<http://www.numdam.org/item?id=ASENS_1977_4_10_3_405_0>

Ann. scient. Éc. Norm. Sup.
4ᵉ série, t. 10, 1977, p. 405 à 418.

NON LINEAR REPRESENTATIONS OF LIE GROUPS

BY MOSHÉ FLATO
GEORGES PINCZON AND JACQUES SIMON

ABSTRACT. — Non linear representations of Lie groups in Banach spaces and their connection with non linear representations of Lie algebras are studied. Applications to their equivalence with linear representations are given.

Introduction

The aim of this article is to study the non linear actions of a Lie group in a (complex) Banach space. It seems that such a general treatment did not appear up to now. Most of the research connected with this subject is either geometrical, in which case the differentiability conditions are too strong (when the space is not finite dimensional) to apply even for linear representation, or related to the theory of partial differential equations, in which case the Lie group is the real line.

Given a Fréchet space E, we denote by $\mathscr{L}_n(E)$ the space of the n-linear continuous mappings from E^n into E. When E is a Banach space, if we denote by B_r the open ball of radius r, $\mathscr{L}_n(E)$ is a Banach space with the norm $\|f\| = \sup \|f(B_1 \times \ldots \times B_1)\|$. We denote by $\hat{f}$ the polynomial associated with the n-linear mapping f. The set $\mathscr{F}(E)$ of formal power series of the type $\sum_{n \geqq 1} f^n$, with $f^n \in \mathscr{L}_n(E)$, is a complex vector space.

DEFINITION 1. — *A* formal representation (S, E), *of a real Lie group* G, *in a Fréchet space* E, *is a morphism* S *from* G *to the group of the invertible elements, for the composition law in* $\mathscr{F}(E)$, *such that, if* $S_g = \sum_{n \geqq 1} S_g^n$ $(g \in G)$ *and* $\varphi_i \in E$ $(1 \leqq i \leqq n)$, *the mappings* $g \to S_g^n(\varphi_1, \ldots, \varphi_n)$ *are measurable for every* $n \geqq 1$.

The fact that S is a morphism is equivalent to the set of equations

$$(1) \qquad \hat{S}_{gg'}^n = \sum_{1 \leqq p \leqq n} S_g^p \sum_{i_1 + \ldots + i_p = n} \hat{S}_{g'}^{i_1} \otimes \ldots \otimes \hat{S}_{g'}^{i_p}.$$

In particular S^1 is a strongly measurable (and therefore continuous when E is a Banach space) linear representation of G in E; (S^1, E) will be called the *free part* of (S, E).

In Section 1, we prove (Prop. 1) that in Definition 1 we can replace measurable by continuous when E is a Banach space.

DEFINITION 2. — *An* analytic representation (S, E) *of a real Lie group* G *in a Banach space* E, *is a formal representation such that there exists a neighbourhood* V *of the identity in* G *such that* $S_g = \sum_{n \geqq 1} S_g^n$ *is an analytic mapping in a neighbourhood* U_g *of the origin in* E *for every* $g \in V$.

In Section 2, we prove (Lemma 2) that the neighbourhood of the origin in E can be taken independent of g in V. Moreover we prove (Corol. 1) that if the neighbourhood of the origin is taken small enough, V can contain any given bounded set in G. We then introduce the notion of *smooth representation* (Def. 6) and prove for these representations differentiability properties which will be utilized in Section 4.

Given an analytic mapping $A = \sum_{n \geqq 1} A^n$, $A^n \in \mathscr{L}_n(E)$, in a neighbourhood of the origin in the Banach space E, A is an analytic isomorphism around the origin if and only if A^1 is an automorphism of E.

DEFINITION 3. — *Two analytic representations* (S, E) *and* (S′, E′) *of a real Lie group* G *in Banach spaces* E *and* E′ *respectively are* equivalent *if there exists an analytic mapping* $A = \sum_{n \geqq 1} A^n$, A^1 *being an automorphism of* E, *such that the equality* $S'_g = A\, S_g\, A^{-1}$ *of power series holds for every* $g \in G$.

Such an equivalence is obviously an equivalence relation. In Section 5, we prove (Prop. 5) that every analytic representation is equivalent to a smooth representation.

Given two elements $A = \sum_{n \geqq 1} A^n$ and $B = \sum_{n \geqq 1} B^n$ in $\mathscr{F}(E)$, we define a new element $A * B \in \mathscr{F}(E)$ by

$$(A * B)^n = \sum_{1 \leqq p \leqq n} A^p \Big(\sum_{0 \leqq q \leqq p-1} I_q \otimes B^{n-p+1} \otimes I_{p-q-1} \Big) \sigma_n,$$

where σ_n is the symmetrization operator on the projective tensor product $E \hat{\otimes} E \hat{\otimes} \ldots \hat{\otimes} E$ (n times) defined by

$$\sigma_n(\varphi_1 \otimes \ldots \otimes \varphi_n) = \frac{1}{n\,!} \sum_{\sigma \in \mathfrak{S}_n} \varphi_{\sigma(1)} \otimes \ldots \otimes \varphi_{\sigma(n)}$$

for every $n \geqq 1$, $\varphi_1, \ldots, \varphi_n \in E$ and $\mathfrak{S}_n$ is the group of permutations of n elements. The composition law $(A, B) \to A * B$ from $\mathscr{F}(E) \times \mathscr{F}(E)$ to $\mathscr{F}(E)$ is bilinear.

Define $[A, B]_* = A * B - B * A$. We shall see in Section 4 that the complex vector space $\mathscr{F}(E)$ becomes a Lie algebra for this bracket.

DEFINITION 4. — *A formal representation* (S, E) *of a real Lie algebra* $\mathfrak{g}$ *in a Fréchet space* E, *is a linear mapping* $S : \mathfrak{g} \to \mathscr{F}(E)$ *such that if* $x, y \in \mathfrak{g}$

$$S_{[x, y]} = [S_x, S_y]_*$$

In particular (S^1, E) is a linear representation of $\mathfrak{g}$ in E; (S^1, E) will be called the *free part* of (S, E).

In Section 4, we prove (Prop. 7) that it is possible, under a technical assumption, to differentiate a formal representation of a real Lie group G and to get a formal representation of its Lie algebra. When the group representation is smooth, this Lie algebra representation is not only formal in the sense of formal power series, but acts as functions on a subset in the space of the differentiable vectors of the free part of the smooth representation (Prop. 8).

In Section 5, we make the passage from the Lie algebra to the Lie group (integrability).

DEFINITION 5. — *An analytic representation* (S, E) *of a real Lie group* G *in a Banach space* E *is called* banal *if it is equivalent to a linear representation.*

In Section 6, we prove that any analytic representation of a semi-simple Lie group in a finite-dimensional space is banal, and that any analytic representation of a nilpotent Lie group, the free part of which is a non trivial unitary irreductible representation, is banal. The first result, which was a conjecture of Palais and Smale, was initially proved by Guillemin and Sternberg [4], by different technics.

As one can see from the definitions, the representations we consider leave the origin invariant. We did not look at a more general situation. Many propositions given for analytic representations have their counter part in formal representations. We did not give them here to avoid a too heavy formulation.

1. Formal representations

Given a Fréchet (resp. Banach) space E, we denote by $\hat{\otimes}^n$ E the projective tensor product of E by itself n-times; $E_n = \bigoplus_{i=1}^{n} (\hat{\otimes}^i E)$ is a Fréchet (resp. Banach) space. We define the (algebraic) vector space $\tilde{E} = \bigcup_{n \geq 1} E_n$. The set L ($\tilde{E}$) of all linear endomorphisms of $\tilde{E}$, leaving E_n invariant for every $n \geq 1$ and continuous on E_n, is a complex algebra.

We shall keep the same notation for an element in $\mathscr{L}_n$ (E) and its canonical identification with an element of $\mathscr{L}(\hat{\otimes}^n E, E)$.

We define a mapping $\Lambda : \mathscr{F}(E) \to L(\tilde{E})$ in the following way : if $A = \sum_{n \geq 1} A^n$;

$$\Lambda(A)(\varphi_1 \otimes \ldots \otimes \varphi_n) = \sum_{1 \leq p \leq n} \sum_{i_1 + \ldots + i_p = n} A^{i_1} \otimes \ldots \otimes A^{i_p} (\sigma_n (\varphi_1 \otimes \ldots \otimes \varphi_n)),$$

for every $n \geq 1$ and $\varphi_1, \ldots, \varphi_n$ in E.

One easily sees that $\Lambda(AB) = \Lambda(A)\Lambda(B)$ and that the mapping Λ is one to one.

Given a formal representation (S, E) of a real Lie group G, the mapping $g \to \tilde{S}_g = \Lambda(S_g)$ is a homomorphism from G into the group of the invertible elements in L ($\tilde{E}$). ($\tilde{S}$, $\tilde{E}$) will be called the *linear representation associated with* (S, E).

PROPOSITION 1. — *Given a formal representation* (S, E), *of a real Lie group* G *in a Banach space* E, $S_g = \sum_{n \geq 1} S_g^n$, *the mappings* $(g, \varphi) \to S_g^n(\varphi)$ *are continuous from* $G \times E^n$ to E. *Moreover, if* E *is finite dimensional, the mappings* $g \to S_g^n$ *from* G *to* $\mathscr{L}_n$ (E) *are analytic.*

Proof. — (S^1, E) is a continuous representation (resp. an analytic representation when E is finite dimensional). Suppose that the result is true for $1 \leq p \leq n-1$. Since $\tilde{S}_{E_n}$ is a measurable and therefore continuous (resp. analytic) representation of G, the continuity (resp. analyticity) of S^n results from the identity

$$S_g^n(\varphi_1, \ldots, \varphi_n) = \tilde{S}_g(\varphi_1 \otimes \ldots \otimes \varphi_n) - \sum_{2 \leqq p \leqq n} \sum_{i_1+\ldots+i_p=n} S_g^{i_1} \otimes \ldots \otimes S_g^{i_p}(\sigma_n(\varphi_1 \otimes \ldots \otimes \varphi_n)).$$

LEMMA 1. — *Given a formal representation* (S, E) *of a real Lie group* G *in a Banach space* E, $S_g = \sum_{n \geqq 1} S_g^n$, *suppose that there exists an open neighbourhood* V_n *of the identity in* G *such that the function* $(g, \varphi) \to R_g^n(\varphi) \equiv S_{g^{-1}}^1 S_g^n(\varphi)$ *is* C^∞ *from* $V_n \times E^n$ *to* E, *for every* $n \geqq 1$. *Then, if we denote by* E_∞ *the Fréchet space of the differentiable vectors for the free part of* (S, E), *the function* $(g, \varphi) \to S_g^n(\varphi)$ *is* C^∞ *from* $G \times E_\infty^n$ *to the Fréchet space* E_∞. *Consequently* $S_g^n \in \mathscr{L}_n(E_\infty)$ *and* S_g *restricted to* E_∞ *defines a formal representation of* G *on* E_∞.

Proof. — The result holds for $n = 1$ ([6], Prop. 1.2). Suppose it is true for $1 \leqq p \leqq n-1$. Take $\varphi = (\varphi_1, \ldots, \varphi_n)$ in E_∞^n. The function

$$(g, g', \varphi) \to A_{g,g'}^n(\varphi) = \sum_{2 \leqq p \leqq n-1} S_g^p \sum_{i_1+\ldots+i_p=n} S_{g'}^{i_1} \otimes \ldots \otimes S_{g'}^{i_p}(\sigma_n(\varphi_1 \otimes \ldots \otimes \varphi_n)),$$

is C^∞ from $G \times G \times E_\infty^n$ to E_∞. From identity (1) we have

$$A_{g,g'}^n(\varphi) = S_{gg'}^n(\varphi) - S_g^1 S_{g'}^n(\varphi) - S_g^n(\otimes^n S_{g'}^1(\varphi)).$$

Multiplying on the left by $S_{(gg')^{-1}}^1$, we get that the mapping,

$$(g, g', \varphi) \to R_{gg'}^n(\varphi) - R_g^n(\varphi) - S_{g'^{-1}}^1 R_g^n(\otimes^n S_{g'}^1(\varphi)),$$

is C^∞ from $G \times G \times E_\infty$ to E_∞. Choose now an open neighbourhood V_n' of the identity in G such that $V_n'^2 \subset V_n$. Then the function

$$(g, g', \varphi) \to B_{g,g'}^n = S_{g'^{-1}}^1 R_g^n(\varphi)$$

is C^∞ from $V_n' \times V_n' \times E_\infty$ to E and therefore takes its values in E_∞. Therefore $R_g^n \in \mathscr{L}_n(E_\infty)$ and, deriving B^n with respect to the variable g', we get that $g \to R_g^n(\varphi)$ is C^∞ from V_n' to E_∞ for every $\varphi \in E_\infty$. Consequently $g \to S_g^n(\varphi) = S_g^1 R_g^n(\varphi)$ is C^∞ from V_n' to E_∞. It results then from relation (1) that $g \to S_g^n(\varphi)$ is C^∞ from G to E_∞. Then obviously $(g, \varphi) \to S_g^n(\varphi)$ is C^∞ from $G \times E_\infty^n$ to E_∞.

Q.E.D.

This lemma will be utilized in Section 4.

2. Analytic representations

LEMMA 2. — *Given an analytic representation* (S, E) *of a real Lie group* G *in a Banach space* E, *there exist a neighbourhood* V *of the identity in* G, $a > 0$ *and* $r > 0$, *such that the function* $\varphi \to S_g(\varphi) = \sum_{n \geqq 1} S_g^n(\varphi)$ *is analytic in the ball* B_r *in* E, *and* $\| S_g^n \| \leqq a^n$ $(n \geqq 1)$ *for every* g *in* V.

Proof. – Denote by U an open neighbourhood of the identity of G where $\sum_{n \geqq 1} S_g^n$ has a radius of convergence $r_g \neq 0$ for every $g \in U$.

$1/r_g = \limsup \| S_g^n \|^{1/n}$ is lower semi-continuous on U. Choose a symmetric neighbourhood U_1 of the identity in G such that $U_1 . U_1 \subset U$.

There exist a non empty open set U_2 in U_1 and $\lambda \geqq 0$ such that $r_g \geqq \lambda$ on U_2. Then, if $0 < \mu < \lambda$ and $g \in U_2$, define $A_g = \sum_{n \geqq 1} \mu^n \| S_g^n \| < +\infty$. The mapping $g \mapsto A_g$ being lower semi-continuous on U_2, there exist a non empty open set $U_3 \subset U_2$ and $K > 0$ such that $A_g \leqq K$ for every $g \in U_3$. Therefore $\| S_g^n \| \leqq K \mu^{-n}$ for every $g \in U_3$. By modifying K and μ we have the same result in a non empty set U_4 in U_3^{-1}. Therefore the result holds on $U_5 = U_4 \cup U_4^{-1}$ which is symmetric.

It results from (1) that $\| S_{gg'}^n \| \leqq \sum_{1 \leqq p \leqq n} \| S_g^p \| \sum_{i_1 + \ldots + i_p = n} \| S_{g'}^{i_1} \| \ldots \| S_{g'}^{i_p} \|$. And, on the other hand, there exists $L > 0$ such that $L^n \geqq \sum_{1 \leqq p \leqq n} \sum_{i_1 + \ldots + i_p = n} a(i_1, \ldots, i_p)$ where $a(i_1, \ldots, i_p) = 1$. Therefore if $g \in V = U_5 . U_5$, we have $\|S_g^n\| \leqq a^n$ with $a = LK^2$.

Q.E.D.

PROPOSITION 2. – *Given an analytic representation* (S, E) *of a real Lie group* G *in a Banach space* E *and a left invariant Riemannian metric d on* G, *there exists* $C > 0$ *such that* $\| S_g^n \| \leqq C^{n(d(g, e)+1)}$ *for every* $g \in G$ *and every* $n \geqq 1$.

The existence of a left invariant Riemannian metric on G is straightforward [3] (take a ds^2 at the identity and translate it).

Proof. – We saw, in Lemma 2, that there exist a neighbourhood V of the identity in G and $a > 0$ such that $\| S_g^n \| \leqq a^n$ for every $n \geqq 1$ and $g \in V$. Choose $M > 0$ and $\omega > 0$ such that $\| S_g^1 \| \leqq M e^{\omega d(g, e)}$ for every $g \in G$ ([3], § 2, Lemma 1). We can choose d in such a way that the unit ball around the identity is contained in V. Given $g \in G$, define m as the smallest integer such that $d(g, e) < m$. Then ([3], § 1), there exists $g' \in G$ such that $d(g, g') < 1$ and $d(g', e) < m-1$.

Since

$$S_g^n = S_{g'g'^{-1}g}^n = \sum_{1 \leqq p \leqq n} S_{g'}^p \Big(\sum_{i_1 + \ldots + i_p = n} S_{g'^{-1}g}^{i_1} \otimes \ldots \otimes S_{g'^{-1}g}^{i_p} \Big) \sigma_n,$$

we have

$$\| S_g^n \| \leqq (a L)^n \Big(M e^{\omega (m-1)} + \sum_{1 \leqq p \leqq n} \| S_{g'}^p \| \Big),$$

L being the same as in the proof of Lemma 2. Define $C = \sup (2 a L, M e^{\omega})$.

Suppose that for $d(g', e) < m-1$ $(m \geqq 2)$ we have $\| T_{g'}^n \| \leqq C^{n(m-1)}$ for every $n \geqq 1$. We then have

$$\| S_g^n \| \leqq 2^n (a L)^n C^{n(m-1)} \leqq C^{nm}.$$

Therefore

$$\| S_g^n \| \leqq C^{n(d(g, e)+1)}.$$

Q.E.D.

COROLLARY 1. — *Given an analytic representation* (S, E) *of a real Lie group* G, *then for every bounded subset* V *in* G, *there exists* $r > 0$ *such that* S_g *is analytic in the ball* B_r *for every* $g \in V$.

COROLLARY 2. — *Given an analytic representation* (S, E) *of a compact Lie group* G, *there exists* $r > 0$ *such that* S_g *is analytic in the ball* B_r *for every* $g \in G$.

Given $r > 0$, we denote by $\mathscr{H}_r(E)$ the subset of the formal power series $f = \sum_{n \geqq 1} f^n$ on the Banach space E such that $\|f\|_r = \sum_{n \geqq 1} r^n \|f^n\| < +\infty$. $\mathscr{H}_r(E)$ is a Banach space for the norm $\| \ \|_r$ ([1], 3.1.2). The mapping $F_r : \mathscr{H}_r(E) \times B_r \to E$ defined by $F_r(f, \varphi) = f(\varphi)$ is obviously C^∞.

DEFINITION 6. — *A* smooth representation (S, E) *of a real Lie group* G *in a Banach space* E *is an analytic representation of* G *in* E *such that there exist* $r > 0$ *and an open neighbourhood* V *of the identity in* G *such that, if* (S^1, E) *is the free part of* (S, E), *the mapping* $g \to R_g = S^1_{g^{-1}} S_g$ *is* C^∞ *from* V *to* $\mathscr{H}_r(E)$.

PROPOSITION 3. — *Given a smooth representation* (S, E) *of a real Lie group* G *in a Banach space* E, *we denote by* (S^1, E) *its free part. There exist an open neighbourhood* V *of the identity of* G *and* $r > 0$ *such that:*

(1) *The functions*

$$(g, \varphi) \to R_g(\varphi) \equiv S^1_{g^{-1}} S_g \varphi$$

and

$$(g, \varphi) \to L_g(\varphi) \equiv S_g S^1_{g^{-1}} \varphi,$$

are C^∞ *from* $V \times B_r$ *to* E.

(2) *The function* $g \to R^n_g = S_{g^{-1}}^{1} S^n_g$ *is* C^∞ *from* V *into* $\mathscr{L}_n(E)$ *for every* $n \geqq 1$.

Proof. — (1) There exist a symmetric neighbourhood V of the identity in G and $r > 0$ such that the mapping $g \to R_g$ is C^∞ from V to $\mathscr{H}_r(E)$. Therefore $(g, \varphi) \to R_g(\varphi) = F_r(R_g, \varphi)$ is C^∞ from $V \times B_r$ to E.

Consider the mapping A: $\mathscr{H}_r(E) \times E \times B_r \to E$ defined for all $R \in \mathscr{H}_r(E)$, $\varphi \in E$, $\psi \in B_r$, by $A(R, \varphi, \psi) = R(\psi) - \varphi$. It is a C^∞ function and we have the derivative $D^3_{(I, 0, 0)} A = I$. Therefore, by the implicit functions theorem, there exist $r' > 0$ and a C^∞ function $u : \mathscr{B}^r_{r'} \times B_{r'} \to E$ ($\mathscr{B}^r_{r'}$ is the ball of radius r' around I in $\mathscr{H}_r(E)$) such that $R(u(R, \varphi)) = \varphi$. Now we choose V small enough such that $g \to R_{g^{-1}}$ is C^∞ from V to $\mathscr{B}^r_{r'}$. We have $R_{g^{-1}}(u(R_{g^{-1}} \varphi)) = \varphi$. So, if r' is taken small enough, the function $(g, \varphi) \to S_g S^1_{g^{-1}} \varphi$ is C^∞ from $V \times B_{r'}$ to E.

(2) The mapping $R \to R^n$ is C^∞ from $\mathscr{H}_r(E)$ into $\mathscr{L}_n(E)$, so that $g \to R^n_g$ is C^∞ from V to $\mathscr{L}_n(E)$.

Q.E.D.

DEFINITION 7. — *Given an analytic representation* (S, E) *of* G *in a Banach space* E, a differentiable vector *of* (S, E) *is a vector* $\varphi \in E$ *such that* $g \to S_g(\varphi)$ *is* C^∞ *from a neighbourhood of the identity in* G *into* E.

PROPOSITION 4. — *Given a smooth representation* (S, E) *of a real Lie group* G *in a Banach space* E, *there exist a neighbourhood* V *of the identity in* G *and* $r > 0$ *such that, if* E_∞ *is the space of the differentiable vectors of the free part* (S^1, E) *of* (S, E):

(1) $E_\infty \cap B_r$ *is the set of the differentiable vectors of* (S, E) *contained in* B_r.

(2) *If* $\varphi \in E_\infty \cap B_r$ *the mapping* $g \to S_g(\varphi)$ *is* C^∞ *from* V *to the Fréchet space* E_∞.

Proof. — Choose a bounded symmetric neighbourhood V of the identity in G and $r' > 0$ satisfying the conclusions of proposition 3. Suppose that $\| S_g^1 \| \leqq M$ if $g \in V$. Take $r = M^{-1} r'$.

(*a*) Suppose that $\varphi \in E_\infty \cap B_r$. Then $g \to S_g \varphi = L_g S_g^1 \varphi$ is C^∞ from V to E.

(*b*) By proposition 2, since V is bounded, there exists $\lambda > 0$ such that $S_g(\varphi) \in B_r$ if $g \in V$ and $\varphi \in B_\lambda$. Suppose now that $\varphi \in B_\lambda$ is a differentiable vector for (S, E). The mapping $g \to S_g^1(\varphi) = R_{g^{-1}} S_g(\varphi)$ is C^∞ on V, hence $\varphi \in E_\infty \cap B_\lambda$. Then, the mapping $g \to S_{g'} S_g(\varphi) = S_{g'g}(\varphi)$ is C^∞ from a neighbourhood V′ of the identity such that $V'.V' \subset V$ into E if $\varphi \in B_\lambda$ so $S_g(E_\infty \cap B_\lambda) \subset E_\infty$ if $g \in V'$.

The mapping $(g, g') \to S_g^1 S_{g'} \varphi = R_{g^{-1}} S_{gg'} \varphi$ is C^∞ from $V' \times V'$ to E if φ is in $E_\infty \cap B_\lambda$. Therefore, deriving this function k times with respect to g, we get that $g' \to d S_{x_1}^1 \ldots d S_{x_k}^1 S_{g'}(\varphi)$ is C^∞ from V′ to E for every $k \geqq 0$, $x_1, \ldots, x_k$ in the Lie algebra of G ($d S_x^1$ being the expression of the differential of the representation S^1 on the element x in the Lie algebra). By definition of the topology of E_∞, this means that the mapping $g \to S_g(\varphi)$ is C^∞ from V′ to the Fréchet space E_∞.

Q.E.D.

3. Smoothing of analytic representations

PROPOSITION 5. — *Given an analytic representation* (S, E) *of a real Lie group* G *in a Banach space* E *and a compact subgroup* K *of* G, *there exists a smooth representation* (S′, E), *equivalent to* (S, E), *such that the restriction of* S′ *to* K *is linear.*

Proof. — Suppose that $S_g = \sum_{n \geqq 1} S_g^n$ is defined and analytic on B_r ($r > 0$) for every g in an open bounded neighbourhood V of the identity containing K. Choose an open symmetric set V′ containing K such that $V'.V' \subset V$ and a C^∞ function with compact support in V′ such that $\int_G f(g)\, dg = 1$. If $\varphi \in B_r$ define

$$D_g(\varphi) = \int_G f(gg') S_{g'}^1 S_{g'^{-1}}(\varphi)\, dg'$$

for $g \in V'$. It results from proposition 2 that $\| S_{g^{-1}}^1 S_g^n \| \leqq C^{n(d(g,e)+1)}$. Therefore, for $r > 0$ small enough and $g \in V'$, the mapping $g \to D_g$ is C^∞ from V′ to $\mathscr{H}_r(E)$. Given an open neighbourhood V″ of the identity containing K such that $V'' V'' \subset V'$, there exists $r' > 0$ such that $S_g^1 D_{g'} = D_{gg'^{-1}} S_g$ on $B_{r'}$, if g, g' are in V″. Define $T_g = D_e S_g D_e^{-1}$; since $T_{g^{-1}}^1 T_g = D_g D_e^{-1}$, (T, E) is a smooth representation of G. More precisely, there exists $r'' > 0$ such that $g \to T_{g^{-1}}^1 T_g$ is C^∞ from V″ to $\mathscr{H}_{r''}(E)$.

Define

$$A(\varphi) = \int_K T^1_{k^{-1}} T_k(\varphi)\, dk, \quad \varphi \in B_{r''},$$

and $S'_g = AT_g A^{-1}$. We have

$$T^1_{g^{-1}} AT_g(\varphi) = \int_K T^1_{(kg)^{-1}} T_{kg}(\varphi)\, dk.$$

Suppose that g is taken in an open neighbourhood U of the identity in G, such that $K.U \subset V''$: the mapping $g \to T^1_{g^{-1}} A\, T_g$ is C^∞ from U to $\mathscr{H}_{r''}(E)$. Therefore (S′, E) is a smooth representation of G. Obviously the restriction of S′ to K is linear and (S′, E) is equivalent to (T, E) and therefore to (S, E).

Q.E.D.

PROPOSITION 6. — *Given an analytic representation* (S, E) *of* G *in a Banach space* E, *there exist* $r > 0$ *and a neighbourhood* V *of the identity in* G *such that the mapping* $(g, \varphi) \to S_g(\varphi)$ *is continuous from* $V \times B_r$ *to* E.

Proof. — (S, E) is equivalent to a smooth representation (S′, E), and since the latter has the desired property, so does (S, E).

4. Passage from the Lie group to the Lie algebra

Given a Fréchet space E, we define a linear mapping $d\Lambda: \mathscr{F}(E) \to L(\tilde{E})$ as following: if $A = \sum_{p \geq 1} A^p$ and $\varphi_1, \ldots, \varphi_n \in E$,

$$(4) \quad d\Lambda(A)(\varphi_1 \otimes \ldots \otimes \varphi_n) = \sum_{1 \leq p \leq n} \Big(\sum_{0 \leq q \leq p-1} I_q \otimes A^{n-p+1} \otimes I_{p-q-1} \Big)(\sigma_n(\varphi_1 \otimes \ldots \otimes \varphi_n)),$$

for every $n \geq 1$.

We easily see that $d\Lambda([A, B]_*) = [d\Lambda(A), d\Lambda(B)]$ and the linear mapping $d\Lambda$ is one to one.

Consequently the bracket $[\ ,\ ]_*$ defines a Lie algebra structure on $\mathscr{F}(E)$.

Given a formal representation (S, E) of a real Lie algebra $\mathfrak{g}$ in E, the mapping $x \to \tilde{S}_x = d\Lambda(S_x)$ is a homomorphism from $\mathfrak{g}$ into the Lie algebra of the associative algebra $L(\tilde{E})$. $(\tilde{S}, \tilde{E})$ will be called the *linear representation* associated with (S, E).

Given a linear continuous representation (S, E) of a real Lie group G, we denote by (dS, E_∞) its differential defined on the space E_∞ of the differentiable vectors.

PROPOSITION 7. — *Let* (S, E) *be a formal representation of a real Lie groups* G, *in a Banach space* E *such that, if* $S_g = \sum_{n \geq 1} S^n_g$, *there exists an open neighbourhood* V *of the identity of* G, *where* $g \to R^n_g = S^1_{g^{-1}} S^n_g$ *is* C^∞ *from* V *to* $\mathscr{L}_n(E)$.

Define now for $x \in \mathfrak{g}$, *the Lie algebra of* G,

$$dS_x^n = \frac{d}{dt}(R^n_{\exp tx})_{t=0} \qquad (n \geqq 2).$$

The mapping

$$x \to dS_x = dS_x^1 + \sum_{n \geqq 2} dS_x^n$$

from $\mathfrak{g}$ *into* $\mathscr{F}(E_\infty)$ *defines a formal representation* $(d\,S, E_\infty)$ *of* $\mathfrak{g}$ *in the Fréchet space* E_∞. *This representation of* $\mathfrak{g}$ *will be called the* differential *of* (S, E).

Proof. — The linear representation $(\tilde{S}, \tilde{E})$ of G associated with (S, E) is continuous when restricted to E_n. Then, by Lemma 1 if $\varphi_1, \ldots, \varphi_n$ are in E_∞, $\varphi_1 \otimes \ldots \otimes \varphi_n$ is a differentiable vector for $(\tilde{S}, E_n)$ and we have :

$$(5) \quad d\tilde{S}_x(\varphi_1 \otimes \ldots \otimes \varphi_n) = \sum_{1 \leqq p \leqq n} \Big(\sum_{0 \leqq q \leqq p-1} I_q \otimes dS_x^{n-p+1} \otimes I_{p-q-1} \Big)(\sigma_n(\varphi_1 \otimes \ldots \otimes \varphi_n),$$

for every $x \in \mathfrak{g}$.

Since $d\tilde{S}_x$ and $d\,S_x^1$ are linear in $x \in \mathfrak{g}$, we find by induction, using (5), that $d\,S_x^n$ is linear in $x \in \mathfrak{g}$ for every $n \geqq 1$.

Suppose now that $d\,S_x^p(\varphi_1 \otimes \ldots \otimes \varphi_p) \in E_\infty$ if $\varphi_1, \ldots, \varphi_p$ are in E_∞ and $1 \leqq p \leqq n-1$ (this is obviously true for $p = 1$). Since

$$\begin{aligned} &dS_x^n(\varphi_1 \otimes \ldots \otimes \varphi_n) \\ &\quad = d\tilde{S}_x(\varphi_1 \otimes \ldots \otimes \varphi_n) - \sum_{2 \leqq p \leqq n} \Big(\sum_{0 \leqq q \leqq p-1} I_q \otimes dS_x^{n-p+1} \otimes I_{p-q-1} \Big) \sigma_n(\varphi_1 \otimes \ldots \otimes \varphi_n), \end{aligned}$$

we have $d\,S_x^n(\varphi_1 \otimes \ldots \otimes \varphi_n) \in E_\infty$. Hence, since $d\,S_x^n \in \mathscr{L}_n(E)$ for $n \geqq 2$, we have $d\,S_x^n \in \mathscr{L}_n(E_\infty)$. It is well known that $d\,S_x^1 \in \mathscr{L}(E_\infty)$. Therefore $d\,S_x \in \mathscr{F}(E_\infty)$. We have $d\Lambda(d\,S_x) = d\tilde{S}_x$ and therefore

$$d\Lambda(dS_{[x,y]}) = d\tilde{S}_{[x,y]} = [d\tilde{S}_x, d\tilde{S}_y] = [d\Lambda(dS_x), d\Lambda(dS_y)] = d\Lambda([dS_x, dS_y]_*),$$

Since $d\Lambda$ is one to one we have $d\,S_{[x,y]} = [d\,S_x, d\,S_y]_*$.

Q.E.D.

Definition 8. — *Given a continuous linear representation* (U, E) *of a real Lie group* G, *in a Banach space* E, *an* analytic representation (T, E_∞) of the Lie algebra $\mathfrak{g}$ of G compatible with (U, E) *is a formal representation of* $\mathfrak{g}$ *in the Fréchet space* E_∞ *of the differentiable vectors of* (U, E) *such that the free part* (T^1, E_∞) *is equal to* $(d\,U, E_\infty)$, $T_x^n \in \mathscr{L}_n(E)$ $(n \geqq 2)$ *and* $\sum_{n \geqq 2} T_x^n$ is *analytic around the origin in* E *for every* $x \in \mathfrak{g}$.

Proposition 8. — *Given a smooth representation* (S, E) *of a real Lie group* G *in a Banach space* E, *the differential* $(d\,S, E)$ *of* (S, E) *is an analytic representation of the Lie algebra* $\mathfrak{g}$ *of* G, *compatible with the free part of* (S, E). *Moreover, there exists* $r > 0$ *such that* $d\,S_x$ *is a mapping from* $E_\infty \cap B_r$ *to* E_∞ *for every* $x \in \mathfrak{g}$.

Proof. — (dS, E_∞) is a formal representation of the Lie algebra $\mathfrak{g}$ of G. By Proposition 4 there exist an open neighbourhood V of the identity in G and $r > 0$ such that the mapping $(g, g') \to R_{g^{-1}} S_{gg'} \varphi$ is C^∞ from $V \times V$ to E for every $\varphi \in E_\infty \cap B_r$.

But $R_{g^{-1}} S_{gg'} \varphi = S^1_g S_{g'} \varphi$. Therefore $g \to S^1_g (d/dt) (S_{\exp tx} \varphi)_{t=0}$ is C^∞ from V into E, and $(d/dt) (S_{\exp tx} \varphi)_{t=0} \in E_\infty$. Since $g \to R_g$ is C^∞ from V to $\mathscr{H}_r(E)$, the series $I + \sum_{n \geqq 2} d S^n_x$ is in $\mathscr{H}_r(E)$. Since

$$\frac{d}{dt}(S_{\exp tx} \varphi)_{t=0} = dS^1_x \varphi + \sum_{n \geqq 2} \widehat{dS^n_x}(\varphi) \in E_\infty,$$

dS_x is an operator defined on $E_\infty \cap B_r$ with values in E_∞.

Q.E.D.

Remark. — A formal representation (S, E) of a real Lie algebra $\mathfrak{g}$ in a Fréchet space E can easily be extended to a representation (S′, E) of the complexified $\mathfrak{g}_c$ of $\mathfrak{g}$. If moreover (S, E) is an analytic representation of $\mathfrak{g}$ in a Banach space E, the series $\sum_{n \geqq 2} S'^n_x$ defines an analytic mapping around the origin in E for every $x \in \mathfrak{g}_c$.

5. Passage from the Lie algebra to the Lie group

PROPOSITION 9. — *Let* G *be a connected and simply connected real Lie group, and* (U, E) *a continuous linear representation of* G *in a Banach space* E. *We denote by* E_∞ *the Fréchet space of its differentiable vectors. Given a formal representation* (S, E_∞) *of the Lie algebra* $\mathfrak{g}$ *of* G *on* E_∞ *such that, if* $x \in \mathfrak{g}$ *and* $S_x = \sum_{n \geqq 1} S^n_x$, $S^1_x = dU_x$ *and* $S^n_x \in \mathscr{L}_n(E_\infty)$ [*resp.* $S^n_x \in \mathscr{L}_n(E_\infty) \cap \mathscr{L}_n(E)$] *for* $n \geqq 2$, *there exists a unique formal representation* (T, E_∞) (*resp.* (T, E)) *of* G *in* E_∞ (*resp.* E) *such that, if* $T_g = \sum_{n \geqq 1} T^n_g$ *and* $\varphi_1, \ldots, \varphi_n$ *are in* E_∞, *the mapping* $g \to T^n_g(\varphi_1, \ldots, \varphi_n)$ *from* G *to* E *has a derivative and*

$$S^n_x(\varphi_1, \ldots, \varphi_n) = \frac{d}{dt}(T^n_{\exp tx}(\varphi_1, \ldots, \varphi_n) \big|_{t=0}.$$

Proof. — (1) Consider first the case where $G = \mathbf{R}$. As usual, if $x \in \mathfrak{g}$ is chosen, we shall denote by t the element $\exp tx$.

Define by induction

$$T^1_t = U_t \quad \text{and} \quad T^n_t = \int_0^t U_{t-s} \sum_{2 \leqq p \leqq n} S^p_x \sum_{i_1 + \ldots + i_p = n} T^{i_1}_s \otimes \ldots \otimes T^{i_p}_s \, ds \circ \sigma_n.$$

Denote $\mathscr{L}_n = \mathscr{L}_n(E)$ (resp. $\mathscr{L}_n(E_\infty) \cap \mathscr{L}_n(E)$). We have $T^n_t \in \mathscr{L}_n$ and the mapping $t \to R^n_t = U_{-t} T^n_t$ from $\mathbf{R}$ to $\mathscr{L}_n$ has a derivative. Define

$$R_t = \sum_{n \geqq 1} R^n_t \quad \text{and} \quad A_t = \sum_{n \geqq 2} U_{-t} S^n_x(\otimes^n U_t).$$

For every $n \geqq 1$, we have

$$\text{(6)} \qquad \frac{dR^n_t}{dt} = \sum_{2 \leqq p \leqq n} U_{-t} S^n_x(\otimes^n U_t) \sum_{i_1 + \ldots + i_p = n} R^{i_1}_t \otimes \ldots \otimes R^{i_p}_t \circ \sigma_n.$$

Which we shall write more concisely

$$\frac{dR_t}{dt} = A_t \circ R_t. \tag{7}$$

This equation has only one solution in $\mathscr{F}(E)$ with R_0 given. Indeed, $d\,R_t^1/dt = 0$ and hence $R_t^1 = R_0$; by induction, we see that the right hand side of (6), for a given n, contains only the applications R_t^p with $p < n$.

Define $T_t = U_t\,R_t = \sum_{n \geqq 1} T_t^n$. If t, t' are real numbers define $f(t) = U_{-t}\,T_{t+t'}$ and $g(t) = U_{-t}\,T_t\,T_{t'}$. The functions f and g satisfy equation (7) with $f(0) = g(0) = T_{t'}$. Therefore $T_{t+t'} = T_t\,T_{t'}$ and (T, E_∞) [resp. (T, E)] has the required properties.

(2) Let us now return to the general case.

Consider the linear representation $(\tilde{S}, \tilde{E}_\infty)$ of $\mathfrak{g}$ associated with (S, E_∞). We have $\tilde{S}\,|_{(E_\infty)_1} = d\,U$. Suppose that $\mathscr{S}^{n-1} = \tilde{S}\,|_{(E_\infty)_{n-1}}$ is the restriction to $(E_\infty)_{n-1}$ of the differential of a continuous linear representation $(\mathscr{T}^{n-1}, (E_\infty)_{n-1})$ [resp. $(\mathscr{T}^{n-1}, E_{n-1})$] of G on $(E_\infty)_{n-1}$ (resp. E_{n-1}). Now, $\mathscr{S}^n = \tilde{S}\,|_{(E_\infty)_n}$ is an extension [5] of $\mathscr{S}^{n-1}$ by $d(\hat{\otimes}^n U)$, the 1-cocycle of the extension being

$$\tau_x^n = \sum_{1 \leqq p \leqq n-1} \Big(\sum_{0 \leqq q \leqq p-1} I_q \otimes S_x^{n-p+1} \otimes I_{p-q-1} \Big) \circ \sigma_n$$

(this follows from formula (4)) and $\tau_x^n \in \mathscr{E}$ with $\mathscr{E} = \mathscr{L}(\hat{\otimes}^n E_\infty, (E_\infty)_{n-1})$ [resp. $\mathscr{E} = \mathscr{L}(\otimes^n E_\infty, (E_\infty)_{n-1}) \cap \mathscr{L}(\otimes^n E, E_{n-1})$].

Now, it is proved in ([5], Prop. 6.1) that there exists a unique 1-cocycle $g \to F_g^n$ from G into $\mathscr{E}$ such that $g \to F_g^n(b)$ is C^∞ from G to $(E)_{n-1}$ and $(d/dt)\,(F^n_{\exp tx}(b))_{t=0} = \tau_x^n(b)$ for every $x \in \mathfrak{g}$ (this was proved in [4] when the spaces are Banach spaces but the proof remains valid, in any locally convex quasi complete topological vector space, without changes).

Therefore $(\mathscr{T}^n, (E_\infty)_n)$ [resp. $(\mathscr{T}^n, E_n)$] is defined by

$$\mathscr{T}_g^n(a+b) = \mathscr{T}_g^{n-1}(a) + \hat{\otimes}^n U_g(b) + F_g^n(b)$$

for $a \in (E_\infty)_{n-1}$ and $b \in \hat{\otimes}^n E_\infty$.

We obviously have $d\mathscr{T}^n = \mathscr{S}^n$ on $(E_\infty)_n$.

We can now define a unique linear representation $(\mathscr{T}, \tilde{E}_\infty)$ [resp. $(\mathscr{T}, \tilde{E})$] such that the restriction to $(E_\infty)_n$ (resp. E_n) is $\mathscr{T}^n$.

It results from the first part that, if $g = \exp tx$,

$$\mathscr{T}_g = \Lambda(T_g) \qquad \text{with} \quad T_g = \sum_{n \geqq 1} T_g^n,\ T_g^1 = U_g$$

and

$$T^n_{\exp tx} = \int_0^t U_{\exp(t-s)x} \sum_{2 \leqq p \leqq n} S_x^p \sum_{i_1+\ldots+i_p=n} T^{i_1}_{\exp s} \otimes \ldots \otimes T^{i_p}_{\exp s}\,ds \circ \sigma_n \qquad \text{for} \quad n \geqq 2.$$

Choose now a neighbourhood V of the identity in G such that V.V is a normal neighbourhood.

Then, if g, g' are in V, we have

$$\Lambda(T_{gg'}) = \mathscr{T}_{gg'} = \mathscr{T}_g \mathscr{T}_{g'} = \Lambda(T_g)\Lambda(T_{g'}) = \Lambda(T_g T_{g'})$$

Since Λ is one-to-one we have $T_{gg'} = T_g T_{g'}$ if g and g' are in V. It results from ([2], Chap. II, § 7, Th. 3) that T can be extended from V to G, and this extension has the required properties.

Q.E.D.

PROPOSITION 10. — *Given a continuous linear representation* (U, E) *of a connected and simply connected Lie group* G *in a Banach space* E *and an analytic representation* (S, E_∞) *of its Lie algebra* $\mathfrak{g}$, *compatible with* (U, E), *the formal representation* (T, E) *of* G *defined by Proposition* 9 *is analytic. Moreover, if* $\varphi \in E_\infty$ *and* $x \in \mathfrak{g}$, *the mapping* $t \to T_{\exp tx}(\varphi)$ *has a derivative around zero and* $(d/dt)\,(T_{\exp tx}(\varphi))_{t=0} = S_x \varphi$.

Proof. — Since we can restrict ourselves to a normal neighbourhood of G, it is sufficient to prove the proposition for $G = \mathbf{R}$.

Let us choose $x \in \mathfrak{g}$ and as usual make the identification $t = \exp tx$.

If $S_x = \sum_{n \geq 1} S_x^n$, we define $A = \sum_{n \geq 2} S_x^n$ and $A_t = U_{-t} \circ A \circ U_t$. We choose $a > 0$ and $r > 0$ such that A_t is analytic on B_r for every $|t| < a$. We denote by $\mathscr{B}$ the Banach space of the bounded continuous functions from $]-a, a[$ into E with the norm $\|f\| = \sup_{|t|<a} \|f_t\|$. We define the mapping $\mathscr{A}$ from the ball W_r, of radius r in $\mathscr{B}$, into $\mathscr{B}$ by, if $f \in W_r$, $\mathscr{A}(f)_t = A_t(f_t)$. This mapping is analytic from W_r into $\mathscr{B}$.

The mapping $B: W_r \times E \to \mathscr{B}$, defined by $B(f, \varphi)_t = f_t - \varphi - \int_0^t \mathscr{A}(f)_s\, ds$, is analytic. Moreover $B(0, 0) = 0$ and $D^1 B_{(0,0)} = I_{\mathscr{B}}$. It results from the implicit functions theorem ([1], 5.6.7) that there exist an open neighbourhood V of the origin in E and a unique analytic mapping $u: V \to \mathscr{B}$ such that $B(u(\varphi), \varphi) = 0$. This means that $u(\varphi)_t = \varphi + \int_0^t A_s(u(\varphi)_s)\, ds$. Therefore the function $t \to u(\varphi)_t$ is C^1 from $]-a, a[$ into E and $du(\varphi)_t/dt = A_t(u(\varphi)_t)$ with the initial condition $u(\varphi)_0 = \varphi$. Since u is analytic from V to $\mathscr{B}$ the mapping $\varphi \to u(\varphi)_t$ is analytic from V to E for every $|t| < a$. It results then from the first part of the proof of Proposition 9 that T_t is analytic on V and $u(\varphi)_t = U_{-t} T_t(\varphi)$ if $|t| < a$.

Q.E.D.

6. Examples of banal representations

PROPOSITION 11. — *Every analytic representation of a connected real semi-simple Lie group in a complex finite dimensional vector space is banal.*

Proof. — We denote by (S, E) the analytic representation of the group G. By Proposition 5 we can suppose that (S, E) is smooth and that $(d\,S, E)$ is an analytic representation of its Lie algebra $\mathfrak{g}$, compatible with the free part (S^1, E) of (S, E). The representation $(d\,S, E)$ can be extended to a representation of the complexified $\mathfrak{g}_c$ of $\mathfrak{g}$. In view of Proposition 10, this representation of $\mathfrak{g}_c$ can be exponentiated to a unique analytic representation (S′, E) of the connected and simply connected Lie group G′ the Lie algebra of which is $\mathfrak{g}_c$. By Proposition 5, this representation is equivalent to a smooth representation S″ linear on the maximal compact subgroup K of G′.

Since the representation (d S″, E) of $\mathfrak{g}_c$ is linear on the Lie algebra $\mathfrak{k}$ of K, its associated linear representation ($d\tilde{S}''$, $\tilde{E}$) satisfies $d\tilde{S}''_k = \sum_{n \geqq 1} \otimes^n d\,S''^1_k$ for every $k \in \mathfrak{k}$.

Now, $\mathfrak{g}_c$ being the complexified of $\mathfrak{k}$, we have $d\tilde{S}''_x = \sum_{n \geqq 1} \hat{\otimes}^n d\,S''^1_x$. This means that (S″, E) is a linear representation of G′ equivalent to (S′, E). Therefore (S, E) is banal.

Q.E.D.

Lemma 3. — *Let* (S, E) *be a formal representation of a connected nilpotent real Lie group* G *in a Fréchet space* E *such that its free part* (S^1, E) *is a non constant representation of* G *by homotheties. Then, the kernel of* (S, E) *is equal to the kernel of* (S^1, E).

Proof. — It is sufficient to prove that the kernel of (S^1, E) is a subgroup of the kernel of (S, E).

We denote by $(G^i)_{i=1,\ldots,k}$ ($G^1 = G$ and $G^k = \{ e \}$) the central decreasing series of G. G^2 is contained in the kernel H of (S^1, E). Take $p \geqq 2$, and suppose that $S\,|_{G}\,p+1$ is constant. If $g \in G$ and $g' \in G^p$ we have $S_g\, S_{g'} = S_{g'}\, S_g$, hence $S^n_{gg'} = S^n_{g'g}$ ($n \geqq 1$). S^1 is constant on G^p. Suppose now that S^q is constant on G^p (hence equals to zero) for $1 \leqq q \leqq n-1$. We then get from relation (1) that $S^1_g\; S^n_{g'} = S^n_{g'} \otimes^n S^1_g$. Since $S^1_g = \lambda_g\, I\, (\lambda_g \in C)$, we have $(\lambda_g - \lambda^n_g)\; S^n_{g'} = 0$ hence $S^n_{g'} = 0$. This proves that $S_g = I$ for every $g \in G^2$. Take now g and g' in G.

We have $S_g\, S_{g'} = S_{g'}\, S_g$. Suppose that H is a periodicity group of S^p for $1 \leqq p \leqq n-1$ (this is true for $p = 1$). It then results from relation (1) for $g \in G$ and $g' \in H$, that $(\lambda_g - \lambda^n_g)\; S^n_{g'} = 0$, and hence $S^n_{g'} = 0$. Therefore H is in the kernel of (S, E).

Proposition 12. — *Every analytic representation of a connected nilpotent real Lie group in a Hilbert space such that the free part is unitary, irreducible and non-trivial, is banal.*

Proof. — Denote by (S, E) an analytic representation of the nilpotent group G such that the hypotheses of the proposition are satisfied.

We denote by C the largest analytic subgroup of G on which S^1 is represented by homotheties. This representation of C is not constant. The kernel H of S^1 restricted to C is such that $C/H \cap C$ is compact since S^1 is unitary. It results from corollary 2 and Lemma 3 that there exists $r > 0$ such that S_c is analytic on B_r for every $c \in C$. We can therefore define $A = \int_{C/H \cap C} S^1_{C^{-1}}\, S_C\; d\,C$. The representation $S'_g = A\, S_g\, A^{-1}$ satisfies $S'_c = S^1_c$ for every $c \in C$. Therefore, it follows from relation (1), since $S'_c\, S'_g = S'_g\, S'_c$, that $S^1_c\, S'^2_g = S'^2_g \otimes^2 S^1_c$. Therefore $(\lambda_c - \lambda^2_c)\;\; S'^2_g = 0$, and $S'^2_g = 0$. Suppose that $S'^p_g = 0$ for $2 \leqq p \leqq n-1$. From relation (1) we get that $(\lambda_c - \lambda^n_c)\, S'^n_g = 0$, so that $S'^n_g = 0$. Therefore (S′, E) is a linear representation equivalent to (S, E).

Q.E.D.

REFERENCES

[1] N. Bourbaki, *Variétés différentielles et analytiques* (*Fascicule de Résultats*, Hermann, Paris, 1967)
[2] C. Chevalley, *Theory of Lie groups* I (Princeton University Press, Princeton, 1946).
[3] L. Garding, *Vecteurs analytiques dans les représentations de groupes de Lie* (*Bull. Soc. Math. France*, t. 88, 1960, p. 73-93).
[4] V. Guillemin and S. Sternberg, *Remarks on a Paper of Hermann* (*Trans. Amer. Math. Soc.*, 130, 1968, p. 110-116).
[5] G. Pinczon and J. Simon, *Extensions of Representations and Cohomology* (*Rep. on* Math. Phys. Tobe published).
[6] N. S. Poulsen, *On* C^{∞}*-vectors and Intertwining Bilinear forms for Representations of Lie groups* (*Journ. Func. Anal.*, t. 9, 1972, p. 87-128).

(Manuscrit reçu le 31 mars 1977.)

M. Flato, G. Pinczon and J. Simon,
Physique Mathématiques
Faculté des Sciences Mirande
21000 Dijon.