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EDGE OPEN PACKING SETS IN GRAPHS

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Abstract. In a graph $G = (V, E)$, two edges e_1 and e_2 are said to have a *common edge* if there exists an edge $e \in E(G)$ different from e_1 and e_2 such that e joins a vertex of e_1 to a vertex of e_2 in G. That is, $\langle e_1, e, e_2\rangle$ is either P_4 or K_3 in G. A non-empty set $D \subseteq E(G)$ is an *edge open packing set* of a graph G if no two edges of D have a common edge in G. The maximum cardinality of an edge open packing set is the *edge open packing number* of G and is denoted by $\rho_e^o(G)$. In this paper, we initiate a study on this parameter.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph-theoretic terminology, we refer to [6]. Throughout this paper, graphs are assumed to be connected and non-trivial.

The open neighborhood $N(e)$ of an edge $e \in E$ is the set of all edges adjacent to e in G, while the *closed neighborhood* of e in G is $N[e] = N(e) \cup \{e\}$. The *degree* of an edge $e = uv$ of G is $\deg e = \deg u + \deg v - 2$. A simple bipartite graph with bipartition (X, Y) is (α, β)*-biregular* if every vertex in X has degree α and every vertex in Y has degree β [2].

Coloring is one of the most important research areas in graph theory and umpteen number of coloring parameters have been introduced and well studied by several authors because of its numerous applications in various fields such as coding theory [13], biological networks [10], neural networks [3] and so on. Of these, the concept of injective edge coloring was introduced in [4] and obtained several results on it, we may refer to [5, 11, 12]. In a graph G, three edges e_1, e_2 and e_3 (in this fixed order) are *consecutive* if $e_1 = xy$, $e_2 = yz$ and $e_3 = zu$ for some vertices x, y, z, u (where $x = u$ is allowed). In other words, three edges are consecutive if they form a path or cycle of length three. A coloring, $c : E(G) \to C$, where C is a set of colors, is an *injective edge coloring* (i-edge coloring for short) if the edges e_1, e_2 and e_3 are consecutive in G, then e_1 and e_3 should receive different colors. The *injective edge coloring number* or *injective edge chromatic index* $\chi_i'(G)$ of graph G is the minimum number of colors permitted in an i-edge coloring. This concept is motivated by the problem of

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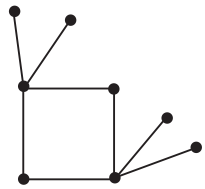

FIGURE 1. The graph G.

assigning channels between the stations in order to avoid secondary interference in the Packet Radio Network (PRN).

It is obvious that an i-edge coloring of G partitioning the edge set $E(G)$ into edge subsets E_i having the property that no three edges in E_i to form a consecutiveness in G. Of course, studying the nature of these subsets is more useful. For instance, suppose we have the position to find the maximum number of transmission lines of the given PRN in which the secondary interference does not occur. This situation can be interpreted by defining the parameter, we call as *edge open packing sets* in graphs as follows.

Definition 1.1. An edge e of a graph G is a *common edge* of two distinct edges e_1, e_2 different from e if e joins a vertex of e_1 to a vertex of e_2. That is, $\langle e_1, e, e_2\rangle$ forms either P_4 or C_3. A non-empty set $D \subseteq E(G)$ is an *edge open packing set* if no two edges in D have a common edge in G. The maximum cardinality among all edge open packings is the *edge open packing number* of G and is denoted by $\rho_e^o(G)$. An edge open packing set of cardinality $\rho_e^o(G)$ is called a ρ_e^o-set of G.

Example 1.2. Consider the graph G given in Figure 1.

It is easy to observe that the set of pendant edges of G forms an edge open packing set with the maximum cardinality and so $\rho_e^o(G) = 4$.

Remark 1.3. It is clear that the induced subgraph $\langle D\rangle$ induced by an edge open packing set is the union of stars.

2. Standard graphs

In this section, we determine the value of the edge open packing number for some standard graphs such as paths, cycles, complete multipartite graphs, wheels and the Petersen graph.

Proposition 2.1. *Let P_n be a path of size $m \geq 2$. Then*

$$\rho_e^o(P_n) = \begin{cases} \frac{m+2}{2} & \text{if } m \equiv 2 (\text{mod } 4) \\ \lceil \frac{m}{2} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $E(P_n) = \{e_1, e_2, \ldots, e_m\}$ such that e_ie_{i+1} are adjacent, where $i = 1, 2, \ldots, m-1$. It is obvious that any two adjacent edges of the paths P_3 or P_4 form an edge open packing set with maximum cardinality and so $\rho_e^o(P_3) = \rho_e^o(P_4) = 2$. Now, consider the following cases for $m \geq 4$.

Case 1. $m \equiv 2(\text{mod } 4)$.

Let $m = 4k + 2$, where $k \geq 1$ is an integer. Consider the set $S = \left\{e_{i+2\lfloor\frac{i-1}{2}\rfloor} : 1 \leq i \leq \frac{m+2}{2}\right\}$. Obviously, no two edges of S have a common edge and so it forms an edge open packing set of P_n and thus $\rho_e^o(P_n) \geq |S| = \frac{m+2}{2}$. On the other hand, let D be a maximal edge open packing set of P_n. Now, for $1 \leq r \leq k$, define $E_r = \{e_i : 4r-3 \leq i \leq 4r\}$ and $E^* = E(P_n) \setminus E_r$. Then, $E_r \cup E^* = E(P_n)$ and $|E^*| = 2$. Obviously, D can have at most two edges from each E_r for $1 \leq r \leq k$ and all the edges from E^*. Therefore, $\rho_e^o(P_n) \leq |D| = 2k + 2 = 2\left(\frac{m-2}{4}\right) + 2 = \frac{m+2}{2}$ and Case 1 is proved.

Case 2. $m \not\equiv 2 (\text{mod } 4)$.

Let $m = 4k + t$, where $t \in \{0, 1, 3\}$ and $k \geq 1$ is an integer. Consider the set $D = \left\{e_{i+2\left\lfloor \frac{i-1}{2} \right\rfloor} : 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil\right\}$. Then D is an edge open packing set of P_n so that $\rho_e^o(P_n) \geq |D| = \left\lceil \frac{m}{2} \right\rceil$. For the other inequality, define $E_j = \{e_i : 4j - 3 \leq i \leq 4j\}$ for $1 \leq j \leq k$ and

$$E_1^* = E(P_n) - E_j = \begin{cases} \phi & \text{if } t = 0 \\ \{e_m\} & \text{if } t = 1 \\ \{e_{m-2}, e_{m-1}, e_m\} & \text{if } t = 3. \end{cases}$$

Now, let D_1 be a maximal edge open packing set of P_n. Then D_1 can contains at most two edges from each E_j $(1 \leq j \leq k)$ and contains at most $\left\lceil \frac{t}{2} \right\rceil$ edges from E_1^*, it follows that $\rho_e^o(P_n) \leq |D_1| = 2k + \left\lceil \frac{t}{2} \right\rceil \leq 2k + \left\lceil \frac{m-4k}{2} \right\rceil \leq \left\lceil \frac{m}{2} \right\rceil$.

□

In the following proposition, we present the value of ρ_e^o for cycles. The proof is similar to that of Proposition 2.1, so we provide the statement alone.

Proposition 2.2. *For the cycles C_n with size $m \geq 3$, we have*

$$\rho_e^o(C_n) = \begin{cases} \frac{m}{2} - 1 & \text{if } m \equiv 2 (\text{mod } 4) \\ \left\lfloor \frac{m}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

Proposition 2.3. *Let $G = G_{n_1, n_2, \ldots, n_k}$ be a complete multipartite graph with partition $(V_1, V_2, \ldots, V_k)$, where $|V_i| = n_i$, $1 \leq i \leq k$ and $n_1 \leq n_2 \leq \ldots \leq n_k$. Then $\rho_e^o(G) = n_k$.*

Proof. Choose an arbitrary vertex $v \in V_i$ for some $1 \leq i \leq k - 1$. Then the set of edges from the vertex v to the vertices belonging to V_k forms an edge open packing set of G and hence $\rho_e^o(G) \geq |V_k| = n_k$. Also, it is clear that any maximal edge open packing set D of G can never contains non-adjacent edges of G and so $|D| \leq n_k$. Therefore, $\rho_e^o(G) = n_k$. □

Proposition 2.4. *For wheels W_n with size m, we have $\rho_e^o(W_n) = \left\lfloor \frac{m}{4} \right\rfloor$.*

Proof. Let $\{v_1, v_2, \ldots, v_{n-1}\}$ be the set of vertices on the rim and let v be the center vertex of W_n. Certainly, the set $D = \{vv_i : 1 \leq i \leq n - 1 \text{ and } i \text{ is odd}\}$ is a maximal edge open packing set of W_n, we get $\rho_e^o(W_n) \geq |D| \geq \left\lfloor \frac{m}{4} \right\rfloor$. Now, consider the sets $E_1 = \{vv_i : 1 \leq i \leq n - 1\}$ and $E_2 = \{v_i v_{i+1} : 1 \leq i \leq n - 2\} \cup \{v_1 v_{n-1}\}$. Then any edge open packing set D' of W_n consisting the edges from exactly one of the sets E_1 and E_2. If $D' \subseteq E_1$, then D' contains at most $\left\lfloor \frac{|E_1|}{2} \right\rfloor$ edges as any two consecutive edges of E_1 have a common edge in W_n. Suppose $D' \subseteq E_2$. Then $|D'| = \rho_e^o(C_{n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{m}{4} \right\rfloor$ and this completes the proof. □

Proposition 2.5. *Let G be the Petersen graph. Then $\rho_e^o(G) = 3$.*

Proof. Consider the Petersen graph given in Figure 2. It is clear that the set $\{x_1, x_2, y_1\}$ forms an edge open packing set of G and so $\rho_e^o(G) \geq 3$. Also, since any maximal edge open packing set D has at most two edges from exactly one of the two cycles (outer and inner cycles) of G and one edge from $\{y_1, y_2, \ldots, y_5\}$. Thus $\rho_e^o(G) \leq |D| \leq 3$. □

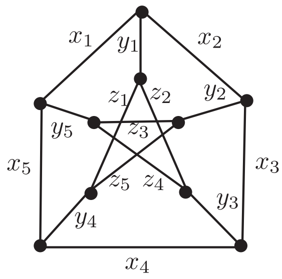

FIGURE 2. The Petersen graph.

3. BOUNDS AND CHARACTERIZATION RESULTS

In this section, we provide some bounds for the edge open packing number in terms of diameter, size, minimum degree, clique number and girth of a graph. Denote the set of edges incident at the vertex v by $N_e(v)$. A graph G is $K_{1,s}$-free graph if it does not contain $K_{1,s}$ as an induced subgraph. If G contains $K_{1,s}$ as an induced subgraph, then $\rho_e^o(G) \geq s$. It is obvious that for any graph G, we have $1 \leq \rho_e^o(G) \leq m$. Furthermore, $\rho_e^o(G) = 1$ if and only if G is complete and $\rho_e^o(G) = m$ if and only if G is a star.

Now, we characterize graphs G for which $\rho_e^o(G) = 2$. Before going to characterize, we prove the following proposition.

Proposition 3.1. *For any graph G, we have $\rho_e^o(G) \geq \left\lceil \frac{\operatorname{diam}(G)}{2} \right\rceil$.*

Proof. Suppose $\operatorname{diam}(G) = k$. In any diametral path P, we can choose at least $\lceil \frac{k}{2} \rceil$ edges with the property that no two edges have a common edge in P and so any maximal edge open packing set of G can have at least $\lceil \frac{k}{2} \rceil$ edges. Thus $\rho_e^o(G) \geq \lceil \frac{k}{2} \rceil = \left\lceil \frac{\operatorname{diam}(G)}{2} \right\rceil$. □

Theorem 3.2. *$\rho_e^o(G) = 2$ if and only if the following conditions are true*

(i) $2 \leq \operatorname{diam}(G) \leq 4$;
(ii) *G is $K_{1,s}$-free, where $s \geq 3$ and;*
(iii) *for any two non-adjacent edges $e_1 = uv$ and $e_2 = xy$ such that e_1 and e_2 have no common edge in G, every vertex in $V(G) \setminus \{u, v, x, y\}$ is adjacent to at least two vertices in the set $\{u, v, x, y\}$.*

Proof. Assume that $\rho_e^o(G) = 2$. Then by Proposition 3.1 that $\operatorname{diam}(G) \leq 4$. Also, for the graphs of diameter one, that is for complete graphs, the value of ρ_e^o is 1, this implies that here $\operatorname{diam}(G) \geq 2$. Thus (i) follows. Now, if G contains $H \cong K_{1,s}, (s \geq 3)$ as an induced subgraph, then the edges of H forms an edge open packing set of G and so $\rho_e^o(G) \geq 3$, which is a contradiction to our assumption. Therefore, condition (ii) is satisfied. Finally, let $e_1 = uv$ and $e_2 = xy$ be two non-adjacent edges such that e_1 and e_2 have no common edge in G and let $A = \{u, v, x, y\}$. Consider an arbitrary vertex w in $V(G) \setminus A$ and assume that w has at most one neighbor in A. If w is adjacent with exactly one vertex in A, say u, then the set $\{e_1, e_2, wu\}$ will become an edge open packing set of G, we have $\rho_e^o(G) \geq 3$. Suppose w has no neighbor in A. Let $w' \in V(G) \setminus A$ such that $ww' \in E(G)$. If w' has no neighbor in A, then the set $\{e_1, e_2, ww'\}$ is an edge open packing set of G, we have $\rho_e^o(G) \geq 3$. On the other hand, the vertex w' is adjacent with either exactly one vertex or two adjacent vertices in A as G is $K_{1,s}$-free($s \geq 3$). Certainly, the set $\{uw', ww', xy\}$ will form an edge open packing of G and so $\rho_e^o(G) \geq 3$. In all the cases, we arrive at a contradiction to $\rho_e^o(G) = 2$ and this proves condition (iii) is true.

Conversely, a graph G satisfies the conditions from (i) to (iii) stated in the theorem. Let D be any maximal edge open packing set of G. We claim that $|D| \leq 2$. Now, by condition (iii), $\langle D \rangle$ has at most two components. If $\langle D \rangle$ has exactly one component D_1, then by condition (ii), either $D_1 \cong K_{1,1}$ or $D_1 \cong K_{1,2}$. Otherwise, let D_1

and D_2 be two components of $\langle D\rangle$. Then both D_1 and D_2 are isomorphic to $K_{1,1}$ according to the condition (iii) and hence our claim. Thus $\rho_e^o(G) = 2$ follows from the condition (i). □

In the following theorems, we characterize the graphs G for which $\rho_e^o(G) = m-1$ and $\rho_e^o(G) = m-2$.

Theorem 3.3. *Let G be a graph with size $m \geq 3$. Then $\rho_e^o(G) = m-1$ if and only if G is a graph obtained from a star $K_{1,m-1}$ by subdividing exactly one edge of $K_{1,m-1}$ by once.*

Proof. Let D be a ρ_e^o-set of G such that $|D| = m-1$ and let $e = uv \in E-D$. If $\langle D\rangle$ consists of two components, say G_1 and G_2, then two vertices u and v do not lie in the same component. Without loss of generality, assume that $u \in V(G_1)$ and $v \in V(G_2)$. Now, let $e_1 = wu$ and $e_2 = vx$ be two edges in G_1 and G_2 respectively. Certainly, e is a common edge between the edges e_1 and e_2, which is not possible and so $\langle D\rangle$ has exactly one component. Let $H \cong K_{1,m-1}$ be the component of $\langle D\rangle$. As G is connected, either u or v should be a pendant vertex in H and thus G is isomorphic to a graph obtained from a star $K_{1,m-1}$ by subdividing exactly one edge of $K_{1,m-1}$ by once. The converse is obvious. □

Next, we characterize graphs G for which $\rho_e^o(G) = m-2$. For this, we use the following definitions and describe new family of graphs as follows.

Definition 3.4. (i) An edge $e = uv$ has vertices u and v that are saturated by e. Given a set of edges D, denote the set of vertices saturated by edges of D by V_D.

(ii) A non-pendant edge $e = uv$ is a *support edge* of G if either u or v is a support vertex in G.

Observation 3.5. If G is a graph of size at least 3 that is not a star and D is an edge open packing set of G such that $\langle D\rangle$ contains $k \geq 2$ components, then $E-D$ contains at least k edges.

Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be the families of graphs obtained from P_5, P_6 and P_7 respectively by attaching $r(\geq 0)$ number of pendant edges at each support vertex of P_5, P_6 and P_7.
Let $\mathcal{A}_4$ and $\mathcal{A}_5$ be the families of graphs obtained from C_3 and C_4 respectively by attaching $t(\geq 0)$ pendant edges at exactly one vertex of C_3 and exactly one vertex of C_4.
Let $\mathcal{A}_6$ be the family of graphs obtained from a star $K_{1,s}$, where $s \geq 3$ by subdividing exactly two edges of $K_{1,s}$ once at a time.
Let $\mathcal{A}_7$ be the family of graphs obtained from a star $K_{1,s}$, where $s \geq 3$ by attaching two pendant edges at exactly one of the pendant vertices of $K_{1,s}$.

The seven families of graphs are shown in Figure 3.

Theorem 3.6. *Let G be a graph with size m. Then $\rho_e^o(G) = m-2$ if and only if $G \in \cup_{i=1}^{7}\mathcal{A}_i$.*

Proof. Suppose $\rho_e^o(G) = m-2$ and let D be a ρ_e^o-set of G. Then $E-D$ has exactly two edges and so $\langle D\rangle$ contains at most two components follows from Observation 3.5. Let $e_1 = uv$ and $e_2 = xy$ be the edges in $E-D$. We prove this theorem in the following cases.

Case 1. $\langle D\rangle$ has exactly one component.
Let $\langle D\rangle = K_{1,s}$, where $s = m-2$ and assume that e_1 and e_2 are adjacent in G. Without loss of generality, let $v = x$ and let $\langle E-D\rangle = P : (u, e_1, x, e_2, y)$. As G is connected and $|E-D| = 2$, the set V_D should contains at least one vertex but at most two vertices of P.
If V_D contains exactly one of the vertices u, x and y, then none of the vertices u, x and y will become the center vertex of $K_{1,s}$. Otherwise, if any one of the vertices u, x and y is the center vertex of $K_{1,s}$, then the edge(s) on P incident at that center vertex together with D forms an edge open packing set of G, which produces a contradiction to the maximality of D. Therefore, if V_D contains exactly one vertex of P, then it is a pendant vertex of $K_{1,s}$. Further, if $s = 1$, then either G is P_4 or a star, which implies that $\rho_e^o(G) \geq m-1$, a contradiction and this yields that $s \geq 2$. Now, if V_D contains exactly one of u and y, then $G \in \mathcal{A}_1$. Suppose

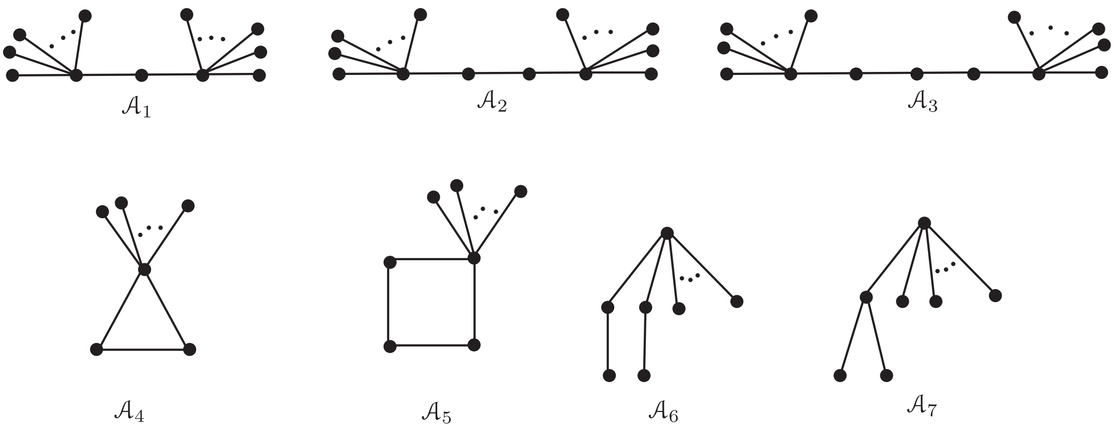

FIGURE 3. Families from $\mathcal{A}_1$ to $\mathcal{A}_7$.

$x \in V_D$ and $s = 2$. Then G is isomorphic to the graph obtained from the star $K_{1,3}$ by subdividing exactly one edge of $K_{1,3}$ by once and so $\rho_e^o(G) = m-1$ follows from Theorem 3.3. This is a contradiction and hence $s \geq 3$. Therefore, $G \in \mathcal{A}_7$ when $s \geq 3$ and $x \in V_D$.

Suppose V_D contains two vertices of P. Now, we first claim that $x \notin V_D$. If not, then either $D \cup \{xu\}$ or $D \cup \{xy\}$ forms an edge open packing set according as $u \in V_D$ or $y \in V_D$. This is a contradiction and hence $u, y \in V_D$. If both u and y are the pendant vertices of $K_{1,s}$, then $G \in \mathcal{A}_4$ and $G \in \mathcal{A}_5$ according as $s = 1$ and $s \geq 2$. If any one of u and y is the center vertex of $K_{1,s}$, then $G \in \mathcal{A}_4$.

Now, let us consider that the edges $e_1 = uv$ and $e_2 = xy$ are independent in $\langle E - D\rangle$. Since G is connected and D is edge open packing, the component $K_{1,s}$ of $\langle D\rangle$ should consist of exactly one vertex from e_1 and one vertex from e_2, and both which cannot be the center vertex of $K_{1,s}$. Certainly, these two vertices are the pendant vertices of $K_{1,s}$ and in which case $G \in \mathcal{A}_6$.

Case 2. $\langle D\rangle$ has two components.

Let $H_1 = K_{1,a}$ and $H_2 = K_{1,b}$ be the components of $\langle D\rangle$, where $a + b = m - 2$. Since G is connected and no two edges of D have a common edge in $E - D$, it follows that e_1 and e_2 should be adjacent in $\langle E - D\rangle$. Without loss of generality, let $x = v$. Certainly, $x \notin V_D$ as D is an edge open packing set of G and so exactly one of the vertices of u and y belongs to H_1 and the other vertex in H_2. Suppose $u \in V(H_1)$ and $y \in V(H_2)$. Then $G \in \mathcal{A}_1$, when both u and y are the center vertices of H_1 and H_2 respectively. If both u and y are the pendant vertices of H_1 and H_2, then

(i) $G \in \mathcal{A}_1$ when $a = b = 1$.
(ii) $G \in \mathcal{A}_2$ when $a = 1$ and $b > 1$ or $a > 1$ and $b = 1$.
(iii) $G \in \mathcal{A}_3$ when $a > 1$ and $b > 1$.

Finally, if u is the center vertex of H_1 and y is a pendant vertex of H_2, then we have

(i) $G \in \mathcal{A}_1$ when $a \geq 1$ and $b = 1$.
(ii) $G \in \mathcal{A}_2$ when $a \geq 1$ and $b > 1$.

Conversely, assume that $G \in \cup_{i=1}^{7}\mathcal{A}_i$. Since $\rho_e^o(G) = m$ only when G is a star and $\rho_e^o(G) = m-1$ only when G is a graph stated in Theorem 3.3, which implies that $\rho_e^o(G) \leq m-2$ if $G \in \cup_{i=1}^{7}\mathcal{A}_i$.

If $G \in \mathcal{A}_1$, then the set of all pendant edges is an edge open packing set of G so that $\rho_e^o(G) = m-2$. Suppose $G \in \mathcal{A}_2 \cup \mathcal{A}_4$. Then the set of all pendant edges with exactly one of its support edges forms an edge open packing set of G and thus $\rho_e^o(G) = m-2$. Suppose $G \in \mathcal{A}_3 \cup \mathcal{A}_5$. Then the set of all pendant and support edges is an edge open packing set of G and so $\rho_e^o(G) = m-2$. If $G \in \mathcal{A}_6 \cup \mathcal{A}_7$, then the edges in $K_{1,s}$ forms an edge open packing set of G and hence $\rho_e^o(G) = m-2$.

□

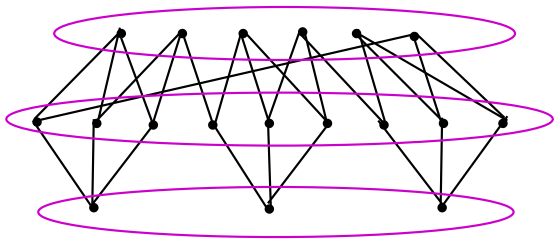

FIGURE 4. A graph in $\psi_{3,18}$.

In the following proposition, we present an upper bound for ρ_e^o in terms of the size and the minimum degree.

Proposition 3.7. *Let G be a graph of size m. Then $\rho_e^o(G) \leq \frac{m}{\delta(G)}$.*

Proof. Let D be a ρ_e^o-set of G and let $uv \in D$. Since D is an edge open packing set, it follows that $E-D$ contains at least $(\delta-1)$-edges incident with u or v and which holds for every edge in D. Thus $|E(G)-D| \geq |D|(\delta(G)-1)$ and hence the result follows. □

Next, we characterize r-regular graphs which attain the bound in the above proposition. For this purpose, we construct a family of r-regular graphs $\psi_{r,n}$ as follows.

Let $V(G) = X \cup Y \cup Z$ such that no two vertices of X or Y or Z are adjacent, where $|X| = \frac{n}{2} - \frac{n}{2r}$, $|Y| = \frac{n}{2}$ and $|Z| = \frac{n}{2r}$. Consider $(r, r-1)$-biregular graph with the vertex sets X and Y in which the degree of each vertex in X is r and the degree of each vertex in Y is $r-1$. The existence of a biregular graph is possible by the condition that $|X|r = |Y|(r-1)$. Now, join each vertex in Z to r vertices of Y such that $N(u) \cap N(v) = \phi$ for any two vertices $u, v \in Z$ and let G be the resultant graph. It is obvious that G is an r-regular graph with n vertices such that $n \equiv 0 (\text{mod } 2r)$ (Fig. 4).

Theorem 3.8. *Let G be an r-regular graph. Then $\rho_e^o(G) = \frac{m}{r}$ if and only if $G \in \psi_{r,n}$.*

Proof. Assume that $G \in \psi_{r,n}$. Obviously, the set of edges connecting the vertices of Y to the vertices of Z in the above construction forms an edge open packing set of G so that $\rho_e^o(G) \geq \frac{n}{2r}r = \frac{m}{r}$. Thus $\rho_e^o(G) = \frac{m}{r}$ follows from Proposition 3.7.

Conversely, let G be an r-regular graph of order n such that $\rho_e^o(G) = \frac{m}{r}$. Suppose D is a ρ_e^o-set of G with $H \cong K_{1,t} (t < r)$ as one of its component, then $|E-D| \geq \rho_e^o(G)r - \rho_e^o(G) + r - t$ and which in turns that $\rho_e^o(G) \leq \frac{m-(r-t)}{r} < \frac{m}{r}$, a contradiction. Therefore, every component of $\langle D \rangle$ is isomorphic to $K_{1,r}$ and there are $\frac{n}{2r}$ copies of such components in $\langle D \rangle$. Let Y and Z be two sets of pendant vertices and support vertices of $\langle D \rangle$ respectively. Then $|Y| = \frac{n}{2}$ and $|Z| = \frac{n}{2r}$. Since D is a ρ_e^o-set, no two vertices of Y are adjacent and no two vertices of Z are adjacent. Let X be the set of vertices in $V - V_D$. Since G is r-regular and every vertex in Z has degree r, it follows that each vertex in X is adjacent with r vertices in Y alone and thus $G \in \psi_{r,n}$. □

Theorem 3.9. *Let G be a graph with the size and the clique number ω. Then $\rho_e^o(G) \leq m - \frac{\omega(\omega-1)}{2} + 1$. Furthermore, $\rho_e^o(G) = m - \frac{\omega(\omega-1)}{2} + 1$ if and only if G is either K_n or a graph obtained from K_n by attaching any number of pendant edges at exactly one vertex of K_n.*

Proof. Let H be a maximum clique and D be a maximal edge open packing set of G. Then D contains at most one edge from H and hence we achieved the inequality.

Suppose $\rho_e^o(G) = m - \frac{\omega(\omega-1)}{2} + 1$. Then D contains exactly one edge from H and all the edges from $F = E(G) \setminus E(H)$ (Note that this set may be empty). We first claim that every vertex that lies outside of H is pendant. Suppose not, Let u, v, w be the vertices of G such that $u \in V(H)$, $v, w \in V \setminus V(H)$ and $uv, vw \in E(G)$.

As $F \subseteq D$, the set D consists both the edges uv, vw, it follows that no edge of H belong to D, which is not true. Thus, every vertex outside of H is pendant. Also, since $F \subseteq D$, F itself is an edge open packing set of G. Thus $\langle F \rangle$ is a star and hence either $G \cong K_n$ or a graph obtained from K_n by attaching any number of pendant edges at exactly one vertex of K_n. Conversely, if $G \cong K_n$, then $\rho_e^o(G) = 1 = m - \frac{\omega(\omega-1)}{2} + 1$. Suppose G is a graph obtained from K_n by attaching any number of pendant edges at exactly one vertex of K_n. Then the set of pendant edges of G together with exactly one edge of K_n, that is, adjacent with these pendant edges forms an edge open packing set of G and thus $\rho_e^o(G) = m - \frac{\omega(\omega-1)}{2} + 1$. □

Proposition 3.10. *If G is a graph with $g(G) \geq 3$, then $\rho_e^o(G) \geq \left\lceil \frac{g(G)}{2} \right\rceil - 1$. Moreover, the equality holds if and only if G is either complete or a cycle of size m with $m \not\equiv 0 (\mathrm{mod}\ 4)$.*

Proof. Consider a cycle C in G such that the length of C is equal to $g(G)$. Then by Proposition 2.2, we have $\rho_e^o(G) \geq \rho_e^o(C) \geq \left\lceil \frac{g(G)}{2} \right\rceil - 1$.

Suppose $\rho_e^o(G) = \left\lceil \frac{g(G)}{2} \right\rceil - 1$. If $g(G) = 3$, then $\rho_e^o(G) = 1$ and thus G is complete. Assume that $g(G) \geq 4$. Suppose G is not a cycle. Let e be an edge incident at the vertex on C, say v and label $e_1, e_2, \ldots, e_{g(G)}$ to the edges of C such that e_1 and e_2 are incident at v. Then the set $D = \left\{ e_{i+2\lfloor \frac{i-1}{2} \rfloor} : 1 \leq i \leq \left\lceil \frac{g(G)}{2} \right\rceil - 1 \right\} \cup \{e\}$ is an edge open packing of G and thus $\rho_e^o(G) \geq |D| = \left\lceil \frac{g(G)}{2} \right\rceil - 1 + 1 = \left\lceil \frac{g(G)}{2} \right\rceil$, a contradiction and hence G is a cycle. Now, by Proposition 2.2, we have G is a cycle with $g(G) \not\equiv 0 (\mathrm{mod}\ 4)$. The converse is just a verification. □

Definition 3.11 ([1])**.** A subset S of E is called a 2-edge packing if $N[e] \cap N[f] = \phi$ for all $e, f \in S$. The maximum cardinality of a 2-edge packing in G is called the 2-edge packing number of G and is denoted by $P_2'(G)$.

Theorem 3.12. *For any graph G, we have $P_2'(G) \leq \rho_e^o(G) \leq \Delta(G) P_2'(G)$.*

Proof. Let D be a 2-edge packing set of G. Since no two edges in D have a common edge in G, the set D is an edge open packing set of G so that $P_2'(G) \leq \rho_e^o(G)$. Further, any maximal edge open packing set D' of G such that $\langle D' \rangle$ has at most $P_2'(G)$ components and each component has at most $\Delta(G)$ edges, it follows that $\rho_e^o(G) \leq \Delta(G) P_2'(G)$. □

Definition 3.13 ([14])**.** For all $1 \leq i \leq n-1$,

(i) An alternate triangular snake $A(T_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternately) to a new vertex v_i. That is every alternate edge of a path is replaced by C_3.
(ii) An alternate quadrilateral snake $A(QS_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i, u_{i+1} (alternately) to a new vertex v_i, w_i respectively and then joining v_i and w_i. That is every alternate edge of a path is replaced by C_4.

Theorem 3.14. *Given any two positive integers $a, b \geq 1$ and an integer $k \geq 3$ with $\left\lceil \frac{b}{k} \right\rceil \leq a \leq b \leq ka$, there exists a graph G such that $P_2'(G) = a$, $\rho_e^o(G) = b$ and $\Delta(G) = k$.*

Proof. If $a = b = 1$, then G is a complete graph on $k+1$ vertices. If $a, b > 1$ and $b = k$, then G is obtained from $K_{1,b}$ by subdividing exactly a edges in $K_{1,b}$. Suppose $a, b > 1$ and $b \neq k$. Then consider the following cases.

Case 1. $b < k$.

Consider a star $K_{1,k}$ with the center vertex u and let $v_1, v_2, \ldots, v_k$ be the pendant vertices of $K_{1,k}$. Form a clique by making the adjacency between every pair of vertices of $\{u, v_1, v_2, \ldots, v_{k-b+1}\}$ and attach exactly one pendant vertex w_i to the vertex $v_{k-b+1+i}$ for all $1 \leq i \leq a-1$. Let G be the resultant graph. For instance, a graph G with $a = 3$, $b = 5$ and $k = 8$ is given in Figure 5.

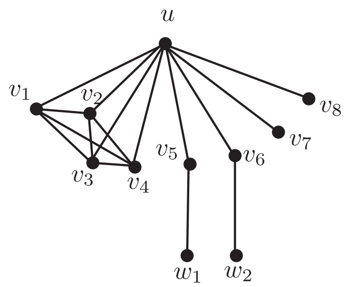

FIGURE 5. A graph G with $a = 3$, $b = 5$ and $k = 8$.

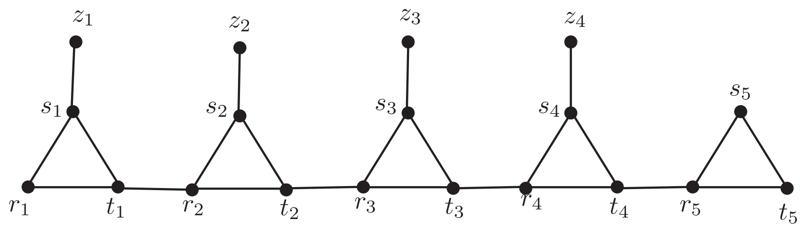

FIGURE 6. A graph G_1 with $a = 5$ and $b = 9$.

Now, consider the sets $S_1 = \{w_i v_{(k-b+1+i)} : 1 \le i \le a-1\} \cup \{v_1 v_2\}$ and $S_2 = \{uv_{(k-b+i)} : 1 \le i \le b\}$. It is easy to see that S_1 and S_2 are the 2-edge packing set and edge open packing set of G respectively and thus $P_2'(G) \ge |S_1| = a - 1 + 1 = a$ and $\rho_e^o(G) \ge |S_2| = b$. Suppose D is a maximal 2-edge packing set of G. If D consisting an edge e of G which is incident at u, then we cannot add any edge to D and so $|D| = 1$ in this case. If D consisting no edge incident at u, then D can include at most one edge from the clique, say v_1v_2, and all the pendant edges that are not incident at u. Thus $|D| \le 1 + (a-1) = a$. Let D_1 be a maximal edge open packing set of G. Then D_1 has at most one edge from the clique. If D_1 has no edge from the clique, then D_1 has exactly two edges, one from the set of pendant edges not incident at u and the other one is adjacent to the chosen pendant edge. Suppose D_1 has one edge from the clique, say e. If e is not incident at u, then D_1 should contains all the pendant edges that are not incident at u, and in this case, we have $|D_1| = 1 + (a-1) = a$. On the other hand, if e is incident at u, then D_1 can contains all the edges incident at u not belonging in the clique and so $|D_1| = 1 + (b-1) = b$.

Case 2. $b > k$.

Now, we construct a graph G in the following subcases based on the values of k as follows.

Subcase 2.1. $k = 3$.

If $a \le b \le 2a$, then construct an alternate triangular snake $A(T_{2a})$ and name the vertices in ith triangle by $C_3^i : (r_i, s_i, t_i)$ for all $1 \le i \le a$. Introduce the set of vertices $\{z_j : 1 \le j \le b-a\}$ and join z_j to s_j for all $1 \le j \le b-a$. Let G_1 be the resultant graph. Figure 6. shows a graph G_1 with $a = 5$ and $b = 9$.
Define the sets $R_1 = \{s_i t_i : 1 \le i \le a\}$ and $R_2 = \{z_j s_j : 1 \le j \le b-a\}$. Now, the sets R_1 and $R_1 \cup R_2$ are 2-edge packing and edge open packing of G_1 respectively so that $P_2'(G_1) \ge a$ and $\rho_e^o(G_1) \ge a + b - a = b$. For all $1 \le i \le b-a$, let $A_i = \langle\{z_i, s_i, r_i, t_i, r_{i+1}\}\rangle$ and for all $b - a < j \le a-1$, let $B_j = \langle\{s_j, r_j, t_j, r_{j+1}\}\rangle$ and let $C = \langle\{s_a, r_a, t_a\}\rangle$ (Note that the subgraphs B_j and C of G_1 become null graphs when $b = 2a$ and B_j alone null graph when $b = 2a-1$). Then any maximal 2-edge packing set has at most one edge from each A_i, each B_j and from C, so we get $P_2'(G_1) \le (b-a) + (2a-b-1) + 1 = a$. Also, any maximal edge open packing set contains at most two edges from each A_i and at most $(2a-b)$ edges from each $\cup_{j=b-a}^{a-1} B_j \cup C$. Therefore, $\rho_e^o(G_1) \le 2(b-a) + (2a-b) = b$.

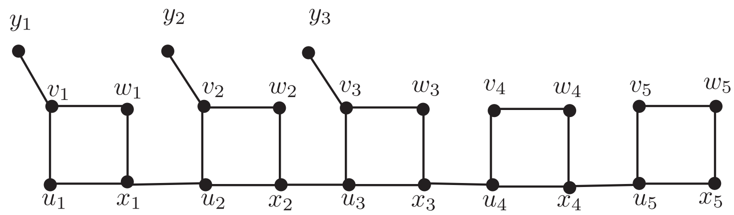

FIGURE 7. A graph G_2 with $a = 5$ and $b = 13$.

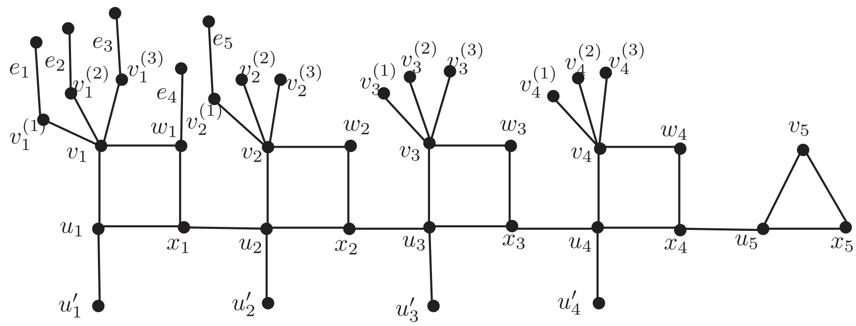

FIGURE 8. A graph G with $a = 10$, $b = 21$ and $k = 5$.

Suppose $2a \leq b \leq 3a$. Then construct an alternate quadrilateral snake $A(QS_{2a})$ and denote the vertices in ith cycle by $C_4^i : (u_i, v_i, w_i, x_i)$ for all $1 \leq i \leq a$. Introduce the set of vertices $\{y_j : 1 \leq j \leq b - 2a\}$ and join y_j to v_j for all $1 \leq j \leq b - 2a$. Let G_2 be the resultant graph. Figure 7. illustrates a graph G_2 with $a = 5$ and $b = 13$.

Now, the set $\{v_i w_i : 1 \leq i \leq a\}$ is a 2-edge packing set of G_2 and the set $\{v_i y_i : 1 \leq i \leq b - 2a\} \cup \{v_i w_i, v_i u_i : 1 \leq i \leq a\}$ is an edge open packing set of G_2 so that $P'_2(G_2) \geq a$ and $\rho_e^o(G_2) \geq b - 2a + 2a = b$. One can easily verify that any maximal 2-edge packing set has at most a edges and any maximal edge open packing set has at most b edges in G_2.

Subcase 2.2. $k \geq 4$.

Now, we construct a graph G that satisfies the given requirement as follows. Let $s = \left\lceil \frac{b}{k} \right\rceil$ and denote $R = b - k(s-1)$. Consider the alternate quadrilateral snake $A(QS_t)$, where $t = 2(s-1)$. For all $1 \leq i \leq s-1$, denote the ith cycle of $A(QS_t)$ by $C_4^i : (u_i, v_i, w_i, x_i, u_i)$. Attach $(k-2)$-pendant edges at each v_i, say $v_i v_i^{(j)}$, where $1 \leq i \leq s-1$ and $1 \leq j \leq k-2$. Introduce $s-1$ vertices $u'_1, u'_2, \ldots, u'_{s-1}$ and join each u'_i to the vertex u_i for all $1 \leq i \leq s-1$. Let H be the resultant graph.

If $R = 1$, construct the cycle $C_3 : (u_s, v_s, x_s, u_s)$ and join the vertex x_{s-1} of H to u_s. Let H_1 be the resultant graph and define $B_1 = \left\{ v_i^{(1)}, v_i^{(2)}, \ldots, v_i^{(k-2)}, w_i : 1 \leq i \leq s-1 \right\}$ in H_1.

If $R = 2$, construct the cycle $C_3 : (u_s, v_s, x_s, u_s)$ and join the vertex x_{s-1} of H to u_s. Further introduce the vertices u'_s and $v_s^{(1)}$. Now, join u'_s to u_s and join $v_s^{(1)}$ to v_s. Let H_2 be the resultant graph and define $B_2 = \left\{ v_i^{(1)}, v_i^{(2)}, \ldots, v_i^{(k-2)}, w_i : 1 \leq i \leq s-1 \right\} \cup \left\{ v_s^{(1)} \right\}$ in H_2.

If $R \geq 3$, construct the cycle $C_4^s : (u_s, v_s, w_s, x_s, u_s)$ and join the vertex x_{s-1} to u_s. Moreover, attach $b - k(s-1) - 2$ pendant edges at v_s and name the pendant vertices incident at v_s by $v_s^{(j)}$, where $1 \leq j \leq b - k(s-1) - 2$. Introduce the vertex u'_s and join it to the vertex u_s. Address H_3 be the resultant graph and define $B_3 = \left\{ v_i^{(1)}, v_i^{(2)}, \ldots, v_i^{(k-2)}, w_i : 1 \leq i \leq s-1 \right\} \cup \left\{ v_s^{(j)} : 1 \leq j \leq b - k(s-1) - 2 \right\}$ in H_3.

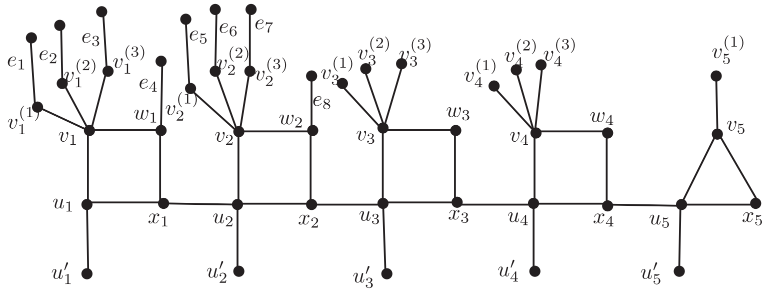

FIGURE 9. A graph G with $a = 13$, $b = 22$ and $k = 5$.

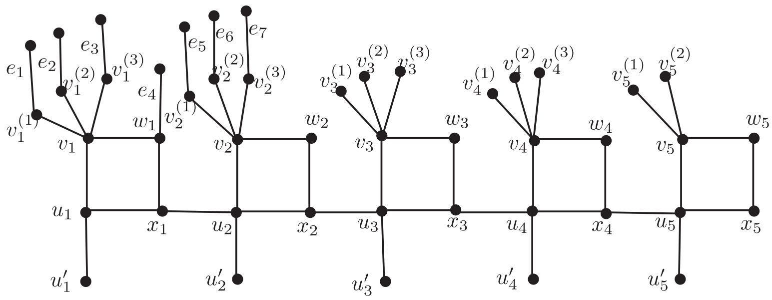

FIGURE 10. A graph G with $a = 12$, $b = 24$ and $k = 5$.

Now, choose any $a - s$ number of vertices from either B_1 or B_2 or B_3 according as $R = 1$ or $R = 2$ or $R = 3$ and attach exactly one pendant edge at each of these $(a - s)$ vertices, say $e_1, e_2, \ldots, e_{a-s}$ and let G be the resultant graph. Graphs that are exhibiting $R = 1$, $R = 2$ and $R \geq 3$ respectively are given in Figures 8–10. Define $E_1 = \{e_1, e_2, \ldots, e_{a-s}\}$. For $R = 1$, the set $\cup_{i=1}^{s-1} N_e(v_i) \cup \{u_s v_s\}$ forms an edge open packing set of G and so $\rho_e^o(G) \geq \left|\cup_{i=1}^{s-1} N_e(v_i) \cup \{u_s v_s\}\right| = \left(\sum_{i=1}^{s-1} \deg\, v_i\right) + 1 = k(s-1) + 1 = b - R + 1 = b - 1 + 1 = b$ and the set $\{u_i u_i' : 1 \leq i \leq s-1\} \cup E_1 \cup \{u_s v_s\}$ is a 2-edge packing set of G so that $P_2'(G) \geq (s-1) + (a-s) + 1 = a$.
For $R = 2$, the set $\cup_{i=1}^{s-1} N_e(v_i) \cup \left\{u_s v_s, v_s v_s^{(1)}\right\}$ forms an edge open packing set of G so that $\rho_e^o(G) \geq \left|\cup_{i=1}^{s-1} N_e(v_i) \cup \left\{u_s v_s, v_s v_s^{(1)}\right\}\right| = \left(\sum_{i=1}^{s-1} \deg\, v_i\right) + 2 = k(s-1) + 2 = b - R + 2 = b - 2 + 2 = b$ and the set $\{u_i u_i' : 1 \leq i \leq s\} \cup E_1$ is a 2-edge packing set of G so that $P_2'(G) \geq s + (a - s) = a$.
For $R \geq 3$, the set $\cup_{i=1}^{s} N_e(v_i)$ is an edge open packing set of G so that $\rho_e^o(G) \geq |\cup_{i=1}^{s} N_e(v_i)| = \left(\sum_{i=1}^{s-1} \deg\, v_i\right) + \deg\, v_s = k(s-1) + b - k(s-1) = b$ and the set $\{u_i u_i' : 1 \leq i \leq s\} \cup E_1$ is a 2-edge packing of G so that $P_2'(G) \geq s + (a - s) = a$.
Let A be the set of edges of G such that $A = E[A(QS_t)]$ and $B = E(G) \setminus A$. Then any maximal edge open packing D of G can have at most $2(s-1)$-edges from A and at most $b - 2(s-1)$ edges from B and thus $\rho_e^o(G) \leq |D| = b$. Also, one can easily verify that any 2-edge packing set consisting at most a edges from G. Therefore, $\rho_e^o(G) = b$ and $P_2'(G) = a$. □

4. CONCLUSION AND SCOPE

In this paper, we have introduced an edge open packing number $\rho_e^o(G)$ and determined the exact value of ρ_e^o for some standard graphs. Also, we have obtained several bounds on it and some characterizations were done. Even though, there is a wide scope for further research on this topic in the following directions.

(1) Characterize graphs G for which $\rho_e^o(G) = \frac{m}{\delta(G)}$.
(2) Characterize graphs G for which
   (i) $\rho_e^o(G) = P_2'(G)$.
   (ii) $\rho_e^o(G) = \Delta(G)P_2'(G)$.
(3) Obtain sharp bounds for the value of ρ_e^o for the family of trees.
(4) Studying the effect of removal of a vertex or an edge from a graph G is of practical importance. This kind of studies can be carried out in many graph-theoretic parameters, one can see [7–9]. Similar to that, one can analyze, how the value of ρ_e^o is changed when a vertex or an edge is removed from G.

Conflicts of interest. The authors have no conflicts of interest to declare that are relevant to the content of this article.

References

[1] S. Arumugam and J. Sithara, Fractional edge domination in graphs. *Appl. Anal. Discrete Math.* **3** (2009) 359–370.
[2] A.S. Asration and C.J. Casselgren, On interval edge colorings of (α, β)-biregular bipartite graphs. *Discrete Math.* **307** (2007) 1951–1956.
[3] L. Barenboim, R. Drucker, O. Zatulovsky and E. Levi, Memory allocation for neural networks using graph coloring, in 23rd International Conference on Distributed Computing and Networking (2022) 232–233.
[4] D.M. Cardoso, J.O. Cerdeira, J.P. Cruz and C. Dominic, Injective edge chromatic index of a graph. Preprint: arXiv:1510.02626 (2015).
[5] D.M. Cardoso, J.O. Cerdeira, J.P. Cruz and C. Dominic, Injective edge coloring of graphs. *Filomat* **33** (2019) 6411–6423.
[6] G. Chartrand and L. Lesniak, Graphs and Digraphs, 4th edition. CRC Press, Boca Raton (2005).
[7] M. Chellali, T.W. Haynes and L. Volkmann, k-independence stable graphs upon edge removal. *Discuss. Math. Graph Theory* **30** (2010) 265–274.
[8] P.J.P. Grobler and C.M. Mynhardt, Domination parameters and edge-removal-critical graphs. *Discrete Math.* **231** (2001) 221–239.
[9] S. Khelifi and M. Chellali, Double domination critical and stable graphs upon vertex removal, *Discuss. Math. Graph Theory* **32** (2012) 643–657.
[10] S. Khor, Application of graph colouring to biological networks. *IET Syst. Biol.* **4** (2010) 185–192.
[11] A. Kostochka, A. Raspaud and J. Xu, Injective edge-coloring of graphs with given maximum degree. *Eur. J. Comb.* **96** (2021) 103335.
[12] Z. Miao, Y. Song and G. Yu, Note on injective edge-coloring of graphs. *Discrete Appl. Math.* **310** (2022) 65–74.
[13] G. Raeisi and M Gholami, Edge coloring of graphs with applications in coding theory. *China Commun.* **18** (2021) 181–195.
[14] S.K. Vaidya and N.B. Vyas, Product cordial labeling for alternate snake graphs. *Malaya J. Mat.* **2** (2014) 188–196.