RAIRO-Oper. Res. 56 (2022) 3545–3560
https://doi.org/10.1051/ro/2022166

RAIRO Operations Research
www.rairo-ro.org

DYNAMIC MARKET SHARE THROUGH ADVERTISING AND LOYALTY REWARD PROGRAMS: A CASE STUDY OF OFO VERSUS MOBIKE

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Abstract. The emerging business models have impelled firms' competition in a brand-new and fast-growing market. This study examines such a competition for market share through advertising and loyalty reward programs. Two stages can be distinguished based on the transformation of the competition strategies. These stages can be formulated as a dynamic differential game model and a static Hotelling game model. The operation strategies and performances of two firms are determined and compared *via* analytical studies and case analyses toward a bicycle-sharing program, *i.e.*, Ofo *versus* Mobike. Historical data are used to fit the curve of the market size and to estimate the parameter values of these models. The results show that during the first stage, the difference in market share between these firms gradually decreases (increases) when the advertising response constant is low (high). Meanwhile, in the second stage, these firms offer converse reward strategies.

Mathematics Subject Classification. 90B60, 90B90.

Received February 22, 2021. Accepted September 22, 2022.

1. Introduction

With the development of information technology, firms have explored an increasing number of business models. Particularly, some business models can lead to the establishment of a brand-new and fast-growing market. Given that a business model is easily to copy, competition is inevitable in such a market. For example, free-floating bicycle-sharing programs have emerged in recent years. As a sharing economy model in sustainable urban transportation, two popular Chinese firms, namely, Ofo and Mobike, developed rapidly and account for almost the entire bicycle-sharing market in China. Uber and Lift are another typical example of a shared taxi market in the USA.

This study aims to investigate firms' market competition strategies for a brand-new business model. In practice, the use of this model results in a rapid occupation of the market through large capital investments [1]. Over time, firms alter their enterprise strategy into a profit-seeking one, given that the market becomes relatively stable subsequently and advertising influence attains saturation. Previous studies on market competition were

Keywords. Dynamic market share, loyalty reward programs, advertising, bicycle-sharing program, case study.

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concerned with either market expansion or market maintenance problems. That is, these studies addressed the two stages separately rather than comprehensively. To overcome this drawback, this work investigates the entire competition process and firms' strategy alterations in the two stages. In these stages, firms compete for market share by advertising and then by implementing loyalty reward programs. The following research questions are explored:

(a) How do firms compete through advertising in the first stage? What are their equilibrium advertising strategies? How does advertising affect their market shares and profits?
(b) How do firms compete *via* pricing and loyalty reward programs in the second stage? What are their optimal decision-making strategies? What are the influences of these strategies on their market shares and profits?
(c) Are these two stages linked? How do they influence each other?

This study establishes dynamic differential and static Hotelling game models for these two stages based on the market characteristics and the variations in the investigated firms' strategies. An analysis of these models toward the case of Ofo *versus* Mobike can explain why these firms survive in the market and yet incur losses each day. The observations from these models can offer management recommendations that can guide and benefit both these firms.

The rest of this paper is organized as follows: Section 2 reviews the relevant literature. Section 3 presents the models and fundamental results. Section 4 presents the parameter estimation, a benchmark scenario of symmetric firms, and the case analysis. Section 5 presents the conclusions.

2. Literature review

This study is related to two main streams: dynamic market share and loyalty reward programs.

2.1. Dynamic market share

Advertising is an essential means to attract customers [2]. To compete the share of a new market, firms usually keep advertising in the market growth stage. This time-dependent advertising problem can be characterized by a dynamic model in a discrete- or continuous-time differential form. The continuous-time model is commonly used and can be constructed as a differential game when two or more companies compete in a market [3]. Several models have been proposed in the past half century for problems in duopoly settings. The Nerlove–Arrow model, which uses goodwill stock to describe the dynamics of the impact of advertising on demand, was developed by Nerlove and Arrow [4]. Viscolani and Zaccour [5] incorporated competition into the model to consider the negative impact of competitors' advertisement on goodwill. The Nerlove–Arrow model was extended by Nair and Narasimhan [6] and Bertuzzi and Lambertini [7] by simultaneously considering quality and pricing decisions, respectively. The Nerlove–Arrow model can also describe the impact of other factors such as quality level of a product, on the goodwill stock [8]. The Vidale–Wolfe model depicts the relationship between sale variation and advertising impact. The model was first proposed by Vidale and Wolfe [9] and used in the monopoly setting. Thereafter, this model was extended to address duopoly markets, where the advertising decisions of a company and its rival affect their market shares [10,11]. An extension of the Vidale–Wolfe model is the Lanchester model, which was constructed by Kimball [12] to specifically analyze duopoly market shares. Chalikias and Skordoulis [13] applied the Lanchester model to study the advertising case of Coca-Cola *versus* Pepsi in the Greek market. Jarrar *et al.* [14] introduced a square advertising effect (which differs from the linear advertising effect in the standard Lanchester model) and proposed a numerical approach to obtaining advertising strategies. Considering the online advertising spillover effect, Zhou *et al.* [15] formulated the investment cost function and improved the Lanchester model.

Based on the research method of Coca-Cola *versus* Pepsi [13], the advertising campaigns can be formulated by a dynamic game model. The difference is that this study applies the Vidale–Wolfe model to characterize the impact of advertising on the market share. In this campaign period, we focus only on the advertising strategies and omit other impact factors because the key factor (price) is convenient to determine. Each firm utilizes the following strategy as a reaction when it observes the other's price variation. Thus, in this setting, we can conveniently analyze the impact of optimal advertising investment on the market share and profit.

2.2. Loyalty reward programs

The pricing and reward strategy is known as loyalty reward program, which is widely applied in airline, hotel, credit card, and other industries, particularly those related to travel. Rossi [16] compared the reward program and price reduction strategies by using data on reward programs of a retail gasoline company. He observed that 11% of all consumers are more sensitive to reward values than price variations. The author also demonstrated that reward programs can reduce price competition and hence increase the profit from gasoline sales by 5%. Similar observations have been obtained through analytical studies as well. For example, Kim *et al.* [17] examined a duopoly market and observed that loyalty reward programs can increase market prices. Singh *et al.* [18] studied another asymmetric duopoly market where one firm offers a loyalty reward program and the other decides whether to adopt a similar strategy or lower the price. They investigated a specific type of loyalty reward program that provides price discount as loyalty benefit. The authors demonstrated that under certain conditions, the optimal strategy is to not respond to rivals' similar program. Moreover, Gandomi and Zolfaghari [19] considered users' satisfaction level and used a stochastic product value to depict the heterogeneity in user preferences.

Online shopping costs are lower than the costs of visiting offline stores. Hence, consumers in online markets have dynamic preferences and display low sensitivity to differentiation among stores. Considering this issue, Zhang and Wedel [20] indicated through empirical evidence that loyalty reward programs are more effective in online than in offline markets. Meanwhile, Lim and Lee [21] obtained an identical result using a game-theoretic model. Another issue with regard to loyalty reward programs is whether a firm sets a reward expiry date, which affects pricing and profit. Bazargan *et al.* [22] studied this issue and revealed that a firm that has not adopted the policy of reward expiry should increase its price when the rival adopts this policy. A firm can also earn well by adopting the policy of reward expiry if users prefer reward and time simultaneously. If a loyalty reward program can be conveniently replicated by competitors, firms may fall into a competitive pricing trap. Nastasoiu and Vandenbosch [23] proposed that firms should design reward programs based on aspects of personalization, customizable rewards, and additional services, which cannot be emulated conveniently.

This study differs to the above two streams in that we investigate the competition for market share between two firms in a period that spans over the two competitive stages and examine the variations in their competitive strategies. We also provide a case study to demonstrate how to apply our model in real practices.

3. Model

This study proposes two game-theoretic models that examine the research questions stated in the previous section. The first model involves the advertising strategies of two firms in the first stage. The second model includes the pricing and loyalty reward program strategies of the firms in the second stage. We formulate the following hypotheses to simplify the models: (1) We consider that the market is served by only two firms. The decision makers in both the firms are assumed to be rational and possess complete market information. (2) The entire research process is divided into two stages. We divide customers into two types corresponding to the two stages. Customers who avail a firm's service/product in both stages are described as heavy users of that firm. Customers purchasing service in only the second stage are known as light users of that firm. Each customer purchases only one unit of service in a stage. (3) We assume that the demand capacity, $D(t)$, increases over time t in the first stage owing to the impact of market expansion. The demand capacity is constant in the second stage because the market is saturated. Table 1 lists the notations used in our models. (4) In the first stage, the firms are assumed to compete only by advertising, which is the specific research focus. Hence, we assume that the price per unit service of each firm is equal to p to simplify the research. Given that the firms are assumed to be asymmetric, the values of their parameters (except for advertising response constant) differ in the model. (5) In the second stage, the entire market is in a relatively stable state. Therefore, the two firms compete by

TABLE 1. List of notations.

Notation	Definition
X	Subscript for firms, $X \in \{\mathrm{A}, \mathrm{B}\}$ in the model and $\{\mathrm{ofo}, \mathrm{mob}\}$ in the case, where mob represents Mobike.
$D(t)$	Demand capacity at time t.
$\mathrm{T}_1, \mathrm{T}_2$	End times of the first and second stages, respectively.
ρ	Advertising response constant.
δ_X	Churn parameter of X.
p	Price per unit service in the first stage.
r	Discount rate.
$u_X(t)$	X's advertising effort at t, $0 < t < \mathrm{T}_1$.
$x(t)$	A's market share at t, $0 < t < \mathrm{T}_1$ and $x_{\mathrm{T}_1} \equiv x(\mathrm{T}_1)$.
$1 - x(t)$	B's market share at t, $0 < t < \mathrm{T}_1$.
$\pi_{X,1}$	X's profit in the first stage.
$V_{X,1}$	X's value function in the first stage.
η_X^H	Proportion of X's heavy users.
η_X^L	Proportion of X's light users.
p_X	Price per unit service of X in the second stage.
c_X	Rewards per unit service of X.
θ	Base value of the service.
$\pi_{X,2}$	X's profit in the second stage.
$V_{X,2}$	X's value function in the second stage.
λ	Price insensitivity of the users who purchase service from B in the first stage.
$\alpha\lambda$	Price insensitivity of the users who purchase service from A in the first stage, $0 < \alpha \leq 1$.
V_X	X's value function in the entire two-stage period.

pricing and loyalty reward programs. We assume that the price-sensitivities of heavy and light users who are served by the same firm in the first stage are equal, whereas those corresponding to different firms are different.

3.1. Model set-up in the first stage

We study the optimal advertising efforts in the first stage. The market share of each firm depends on the two firms' advertising effort. Two firms' advertising costs are assumed to be quadratic in the advertising effort and hence given by $u_{\mathrm{A}}^2(t)$ and $u_{\mathrm{B}}^2(t)$, respectively. The assumption of quadratic cost function is commonly used in the literature and implies diminishing marginal returns to advertising expenditure [24,25]. Notably, the market share is non-decreasing under a firm's own advertising effort and non-increasing under the competitor's advertising effort. For simplicity, the service cost for both the firms is assumed to be zero. Thus, the market share dynamics of Firm A is given by the following:

$$\dot{x}(t) = \rho u_A(t)\sqrt{1 - x(t)} - \rho u_B(t)\sqrt{x(t)} - \delta_{\mathrm{A}} x(t) + \delta_{\mathrm{B}}(1 - x(t)). \tag{1}$$

This specification is characterized by the square-root feature introduced by Sethi [26]. Accordingly, the two firms' profits at t are given by the following:

$$\pi_{\mathrm{A},1} = px(t)D(t) - u_{\mathrm{A}}^2(t), \tag{2}$$

$$\pi_{\mathrm{B},1} = p(1 - x(t))D(t) - u_{\mathrm{B}}^2(t). \tag{3}$$

Thus, the value functions of the firms are obtained as follows:

$$V_{\mathrm{A},1} = \max_{u_{\mathrm{A}}(t), 0 < t < \mathrm{T}_1} \int_0^{\mathrm{T}_1} e^{-rt}\left(px(t)D(t) - u_{\mathrm{A}}^2(t)\right)\mathrm{d}t, \tag{4}$$

$$V_{\mathrm{B},1} = \max_{u_{\mathrm{B}}(t), 0 < t < \mathrm{T}_1} \int_0^{\mathrm{T}_1} e^{-rt}\left(p(1 - x(t))D(t) - u_{\mathrm{B}}^2(t)\right)\mathrm{d}t. \tag{5}$$

3.2. Model set-up in the second stage

Specifically, the second stage can be regarded as an instant. We study the decisions with regard to the optimal price and optimal rewards of unit service for the two firms. Pricing is not a continuous behavior in this stage. Thus, we must make such decisions after we obtain the market share at T_1. The goal is to achieve the highest profit after T_1 for each firm.

Without advertising, the whole market (for which the two firms compete) can be regarded as relatively stable. That is, the two firms compete with each other for relatively stable market shares through pricing and loyalty reward programs. For a user who has purchased service from Firm A in the first stage, two options are available when he/she wishes to purchase the same service again: buy the service from the same firm or do so from the other firm. When the user selects the former option, he becomes a heavy user of A and can be rewarded because of the repurchase behavior. If he selects the latter option, he receives no reward because he becomes a light user of B. The key considerations for him to select are the service price and the rewards he can gain. If the money he would spend after being rewarded is more than that he would spend on B, he may select the latter option. That is, being a heavy user causes him to spend less, indicating that he is willing to purchase the same service from A. Similarly, a user who purchased service from B also has two options. Therefore, the price and loyalty reward policies adopted by the firms affect their subsequent market shares significantly. This competition can be graphically described as a competition for a ready-made industry. According to the above descriptions, we can conveniently obtain the following equations describing the firms' market shares:

$$x_{\mathrm{T}_1}\eta_{\mathrm{A}}^{H} + x_{\mathrm{T}_1}\eta_{\mathrm{B}}^{L} = x_{\mathrm{T}_1}, \tag{6}$$

$$(1 - x_{\mathrm{T}_1})\,\eta_{\mathrm{B}}^{H} + (1 - x_{\mathrm{T}_1})\,\eta_{\mathrm{A}}^{L} = 1 - x_{\mathrm{T}_1}, \tag{7}$$

where x_{T_1} represents the market share of A at T_1, *i.e.*, $x(\mathrm{T}_1)$.

We use the Hotelling model to describe the competition for market share in the second stage. The Hotelling model is commonly used in studies on loyalty reward programs [17]. In accordance with these articles, λ (or $\alpha\lambda$) is a positive parameter and represents unit transport cost in the standard Hotelling model. That is, λ indicates the importance of these users' preference toward firms compared with their price preference. λ is also regarded as the unimportance of price in these users' selection decision. α indicates the difference in users' behavior preference between the two firms. It results from the difference in user experiences between the two firms in the first stage. Note that $\alpha = 1$ depicts a special case with symmetric users. According to the Hotelling model, users who purchase service from A in the first stage, *i.e.*, the market share of A at T_1, can be regarded as a line segment. These users are uniformly distributed in the interval $[0, x_{\mathrm{T}_1}]$, with the firms positioned at the two ends. As shown in Figure 1, the left endpoint is for A, whereas the right endpoint is for B. If a user continues to purchase A's service, he becomes a heavy user of A. His surplus is $\theta - \omega\alpha\lambda - p_{\mathrm{A}} + c_{\mathrm{A}}$, where θ represents the base value of the service and ω is his location. Note that this user gains a reward c_{A} from A. If a user opts for B, he becomes a light user of B, and his surplus is $\theta - (x_{\mathrm{T}_1} - \omega)\,\alpha\lambda - p_{\mathrm{B}}$. A special point exists on the line, which represents a marginal user. This user features indifference between purchasing A's service again and switching to B. The left side of this special point on the line segment is the market share of A's heavy users, whereas the right side can be considered to be that of B's light users. Thus, this special point is an equilibrium point and is depicted in equation (8).

$$\theta - x_{\mathrm{T}_1}\eta_{\mathrm{A}}^{H}\alpha\lambda - p_{\mathrm{A}} + c_{\mathrm{A}} = \theta - x_{\mathrm{T}_1}\eta_{\mathrm{B}}^{L}\alpha\lambda - p_{\mathrm{B}}. \tag{8}$$

Similarly, we can obtain another equation from an analysis of B's heavy users and A's light users:

$$\theta - (1 - x_{\mathrm{T}_1})\,\eta_{\mathrm{B}}^{H}\lambda - p_{\mathrm{B}} + c_{\mathrm{B}} = \theta - (1 - x_{\mathrm{T}_1})\,\eta_{\mathrm{A}}^{L}\lambda - p_{\mathrm{A}}. \tag{9}$$

Thus, the profit functions of the firms in the second stage are as follows:

$$\pi_{\mathrm{A},2} = p_{\mathrm{A}}x_{\mathrm{T}_1}\eta_{\mathrm{A}}^{H}D_{\mathrm{T}_1} + p_{\mathrm{A}}\,(1 - x_{\mathrm{T}_1})\,\eta_{\mathrm{A}}^{L}D_{\mathrm{T}_1} - c_{\mathrm{A}}x_{\mathrm{T}_1}\eta_{\mathrm{A}}^{H}D_{\mathrm{T}_1}, \tag{10}$$

$$\pi_{\mathrm{B},2} = p_{\mathrm{B}}\,(1 - x_{\mathrm{T}_1})\,\eta_{\mathrm{B}}^{H}D_{\mathrm{T}_1} + p_{\mathrm{B}}x_{\mathrm{T}_1}\eta_{\mathrm{B}}^{L}D_{\mathrm{T}_1} - c_{\mathrm{B}}\,(1 - x_{\mathrm{T}_1})\,\eta_{\mathrm{B}}^{H}D_{\mathrm{T}_1}. \tag{11}$$

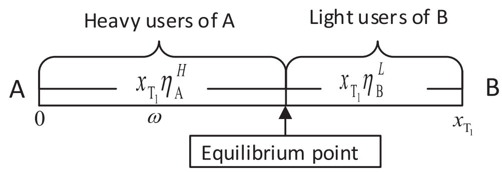

FIGURE 1. Variation in A's market share at T_1.

3.3. Model solution and analysis

This two-stage model can be solved by backward induction. We first address the second stage and then solve the first stage, with the outcome of the second stage functioning as a salvage function. Lemma 1 characterizes the equilibrium results in the second stage.

Lemma 1. *In the second stage, the prices and rewards for unit service of the two firms are given by the following:*

$$p_A = \lambda\left(1 - x_{T_1}\right), \tag{12}$$
$$p_B = \alpha\lambda x_{T_1}, \tag{13}$$
$$c_A = \lambda\left(\left(1 - x_{T_1}\right) - \alpha x_{T_1}\right), \tag{14}$$
$$c_B = \lambda\left(\alpha x_{T_1} - \left(1 - x_{T_1}\right)\right). \tag{15}$$

The proportions of the firms' heavy and light users are given by the following:

$$\eta_A^H = \frac{1}{2} + \frac{p_B - (p_A - c_A)}{2x_{T_1} \cdot \alpha\lambda}, \tag{16}$$
$$\eta_A^L = \frac{1}{2} + \frac{p_B - c_B - p_A}{2\left(1 - x_{T_1}\right) \cdot \lambda}, \tag{17}$$
$$\eta_B^H = \frac{1}{2} + \frac{p_A - (p_B - c_B)}{2\left(1 - x_{T_1}\right) \cdot \lambda}, \tag{18}$$
$$\eta_B^L = \frac{1}{2} + \frac{p_A - c_A - p_B}{2x_{T_1} \cdot \alpha\lambda}. \tag{19}$$

Proof. We obtain equations (16)–(19) for η_A^H, η_A^L, η_B^H, and η_B^L, respectively, by calculating equations (6)–(9) simultaneously. We obtain the following by substituting equations (16)–(19) into equations (10) and (11) and taking the partial derivatives of the resulting $\pi_{A,2}$ and $\pi_{B,2}$ with respect to p_X, c_X, and $X \in \{A, B\}$:

$$p_A = \frac{(1+\alpha)\,p_B + 2c_A - \alpha c_B + \alpha\lambda}{2(1+\alpha)}, \tag{20}$$
$$c_A = \frac{2p_A - x_{T_1} \cdot \alpha\lambda - p_B}{2}, \tag{21}$$
$$p_B = \frac{(1+\alpha)\,p_A + 2\alpha c_B - c_A + \alpha\lambda}{2(1+\alpha)}, \tag{22}$$
$$c_B = \frac{2p_B - \left(1 - x_{T_1}\right) \cdot \lambda - p_A}{2}. \tag{23}$$

Solving joint equations (20)–(23), we can conveniently obtain equations (12)–(15). □

In the first stage, we can obtain the market share at T_1 for a specified initial market share. An analysis of equations (12)–(15) reveals that the optimal price and rewards of unit service in the second stage are directly related to the market share at T_1, which links the two stages. We can study the pricing and rewarding strategies for a specified market share at T_1. When the relationship between the firms' market shares and their users' price insensitivity satisfies the equality $\alpha = \frac{1-x_{T_1}}{x_{T_1}}$, their rewards of unit service are both zero. That is, the firms would not implement loyalty reward programs. When the relationship violates this equality, *e.g.*, $\alpha > \frac{1-x_{T_1}}{x_{T_1}}$, B must offer rewards, whereas A's rewards per service are negative. Negative rewards imply that A provides no reward to its heavy users. Rather, A should set a higher price for the service for its heavy users than that for its light users. That is, A provides a discount to its light users. Such a discount can be regarded as a promotion for new users. Proposition 1 summarizes the above analysis.

Proposition 1. *On the basis of the market share at T_1 in the first stage, the two firms would implement different strategies for heavy/light users in the second stage as follows:*

(a) A should reward its heavy users, and B should provide a discount to its light users if $\alpha < \frac{1-x_{T_1}}{x_{T_1}}$.

(b) A should provide a discount to its light users, and B should reward its heavy users if $\alpha > \frac{1-x_{T_1}}{x_{T_1}}$.

(c) The two firms should neither reward heavy users nor discount light users if $\alpha = \frac{1-x_{T_1}}{x_{T_1}}$.

From Lemma 1, we can conveniently achieve Proposition 2 by comparing the service prices of the firms. When the users of the two firms feature identical price insensitivity, *i.e.*, $\alpha = 1$, A should adopt a low price (from Prop. 2) for its light users. However, heavy users of A would pay a high price (from Prop. 1) in the second stage because the market share of A is higher than that of B at T_1. Hence, a high price can offset the loss of old users. In contrast, B should adopt a high price and a loyalty reward program. Note that the conditions in Proposition 1 are consistent with those in Proposition 1. The comprehensive consideration of the rewarding and pricing strategies under different circumstances results in Proposition 3. This noteworthy observation shows that the two firms set an identical price for similar user groups in the first stage. This is because loyalty reward programs can adjust the pricing strategy among different user groups. Hence, users served by a firm in the first stage are charged the same price in the second stage irrespective of the firm they opt for after T_1.

Proposition 2. *In the second stage, the two firms' service prices can be compared as follows:*
(a)$p_A > p_B$ *if* $\alpha < \frac{1-x_{T_1}}{x_{T_1}}$.
(b)$p_A < p_B$ *if* $\alpha > \frac{1-x_{T_1}}{x_{T_1}}$.
(c)$p_A = p_B$ *if* $\alpha = \frac{1-x_{T_1}}{x_{T_1}}$.

Proposition 3. *Users served by a firm in the first stage confront an identical price in the second stage irrespective of whether they become the same firm's heavy users or shift to the other firm.*

Apart from the rewarding and pricing strategies, we can analyze the market share and performance of the firms in the second stage. Performance is discussed by comparing the profits of the two firms. These results are determined from Lemma 1 and summarized in Lemma 2. A noteworthy finding is that irrespective of how the firms spilt the market and the price insensitivity of the firms' users, the two firms equally split the market and earn equally in the second stage. This is because the actual price for a group is equal in both stages owing to the adjustment mechanism of the loyalty reward programs. However, the earnings depend on these influence factors. A larger λ and/or α can result in a large profit for both the firms, implying that when users prefer a low price, the firms earn a low profit.

Lemma 2. *In the second stage, two firms equally split the market and earn equally, i.e.,* $\pi_{A,2} = \pi_{B,2} = \frac{\lambda D_{T_1}}{2}\left(\alpha x_{T_1}^2 + (1-x_{T_1})^2\right)$.

Proposition 4. *In the second stage, two firms' profits (a) increase monotonically with respect to x_{T_1} if $x_{T_1} > \frac{1}{1+\alpha}$; (b) decrease monotonically with respect to x_{T_1} if $x_{T_1} < \frac{1}{1+\alpha}$.*

Proposition 4 is determined from Lemma 2 and describes the firms' preference for x_{T_1} in the second stage. We identify a threshold, *i.e.*, $x_{T_1} = \frac{1}{1+\alpha}$. If A's market share at T_1 is larger than this threshold, the two firms prefer a high market share for A at T_1. Therefore, B should reduce its advertising effort and permit A to exert effort. Moreover, this threshold increases as the difference in price insensitivity between the two firms' users increases. Hence, the two firms prefer a high x_{T_1}. If A's market share at T_1 is less than this threshold, the two firms prefer to equally share the market.

Next, we provide the solution for the second stage. Combined with the second stage, the firms' objective functionals in the entire period comprising the two stages are given by

$$V_A = \max_{u_A(t),0<t<T_1} \int_0^{T_1} e^{-rt}\left(px(t)D(t) - u_A^2(t)\right)dt + \int_{T_1}^{T_2} e^{-rt}\frac{\lambda D_{T_1}}{2}\left(\alpha x_{T_1}^2 + (1-x_{T_1})^2\right)dt, \tag{24}$$

$$V_B = \max_{u_B(t),0<t<T_1} \int_0^{T_1} e^{-rt}\left(p(1-x(t))D(t) - u_B^2(t)\right)dt + \int_{T_1}^{T_2} e^{-rt}\frac{\lambda D_{T_1}}{2}\left(\alpha x_{T_1}^2 + (1-x_{T_1})^2\right)dt. \tag{25}$$

These objective functionals and the constraint condition of equation (1) constitute the differential game model, which is an optimal control problem. Each functional consists of two integrals, which represent the profits in the first and second stages. The first integral describes the impact of the advertising effort decision on the state of market share and profit. Although the second integral does not evidently contain advertising effort, it is affected by the state of market share at the end of the first stage. This state, *i.e.*, x_{T_1}, also depends on the advertising effort decision in the first stage. Thus, these integrals must be solved comprehensively to ensure an optimum outcome for the entire period. However, this optimal control problem cannot be solved by analytical methods. This is because the function of t, *i.e.*, $D(t)$, appears in the first integral, which also adds the second integral. Although $D(t)$ degenerates to a constant, we cannot provide an analytical solution *via* the Hamilton–Jacobi–Bellman (HJB) equation or a Hamiltonian system. Therefore, we adopt the method of numerical calculation to address this optimal control problem. The HJB equations of the firms are

$$-\frac{\partial V_A}{\partial t} = \max_{u_A}\left\{\left(\rho u_A\sqrt{1-x} - \rho u_B\sqrt{x} - \delta_A x + \delta_B(1-x)\right)\frac{\partial V_A}{\partial x} + \left(pxD - ku_A^2\right)\right\} - rV_A, \tag{26}$$

and

$$-\frac{\partial V_B}{\partial t} = \max_{u_B}\left\{\left(\rho u_A\sqrt{1-x} - \rho u_B\sqrt{x} - \delta_A x + \delta_B(1-x)\right)\frac{\partial V_B}{\partial x} + \left(p(1-x)D - ku_B^2\right)\right\} - rV_B, \tag{27}$$

with the terminal conditions

$$V_A(T_1, x) = \frac{\lambda D_{T_1}}{2r}\left(1 - e^{-r(T_2-T_1)}\right)\left(\alpha x^2 + (1-x)^2\right), \tag{28}$$

$$V_B(T_1, x) = \frac{\lambda D_{T_1}}{2r}\left(1 - e^{-r(T_2-T_1)}\right)\left(\alpha x^2 + (1-x)^2\right). \tag{29}$$

Thus, we obtain the first optimal conditions:

$$u_A^* = \frac{\rho\sqrt{1-x}}{2k}\cdot\frac{\partial V_A}{\partial x}, \tag{30}$$

$$u_B^* = -\frac{\rho\sqrt{x}}{2k}\cdot\frac{\partial V_B}{\partial x}. \tag{31}$$

In the numerical computation, we use MATLAB's pdepe() function for solving the HJB equations (26) and (27).

TABLE 2. Market capacity data.

Date	Mar 2016	Nov 2016	Dec 2016	Jan 2017	Feb 2017	Mar 2017	Feb 2018
Market capacity (unit: 100 million)	0	0.14845	0.14964	0.17948	0.2353	0.30125	0.94

4. Case analysis

In this section, we describe our analysis of the case. First, we estimate the values of the parameters used in the model. Second, we investigate a benchmark scenario of symmetric firms. Finally, we analyze the case in detail by considering the advertising strategy and performance of the firms.

4.1. Case specification and Parameter estimation

Two bicycle-sharing firms, Ofo and Mobike, are considered to illustrate the practical significance of this study. Ofo and Mobike have recently achieved notable success in the bicycle-sharing market in China. These firms form a pair of competitors. When they enter a city, they can rapidly develop the bicycle-sharing market. In their early stage of business expansion, the firms must increase daily business volume to obtain additional financing. Ofo and Mobike implemented programs to win over users regardless of the costs. For example, they offered free rides in the first month and a monthly card for 1 RMB. The prices have returned to normal because the market is relatively stable at present. The main work of the two firms has also shifted to the maintenance of existing users. The strategies of advertising, pricing, and loyalty reward programs and the performances of the two firms are discussed in this section. Through these discussions, we can provide reasons for the financial crisis encountered by Ofo and Mobike recently. Accordingly, we offer recommendations for Ofo and Mobike and summarize several managerial insights.

This subsection uses historical data and empirical investigation to estimate the parameters in the model. First, we use available data on the bicycle-sharing market to fit the curve of market capacity, *i.e.*, $D(t)$. These data are listed in Table 2. We assume that the market originated in March 2016 and that the capacity at that time was zero. The data in the other months are from two research reports by BigData-Research (BDR) company of China (www.BigData-Research.com). The data collection period ends in February 2018 owing to the firms' strategy transfer and the market saturation beyond this month. That is, T_1 is February 2018. Each of the monthly data is transformed to an annual measuring unit. We use a Gaussian function $D(t) = a_0 e^{-\left(\frac{t-a_2}{a_1}\right)^2}$ to fit this capacity and the result is $a_0 = 0.9651$, $a_1 = 1.053$, and $a_2 = 2.161$ with SSE $= 0.001163$ and $R^2 = 0.9979$. The result illustrates that the bicycle-sharing market displays a rapidly increasing trend in the first stage, particularly in the middle of the first stage.

Second, we estimate the parameter in the second stage, *i.e.*, the price insensitivity of the users. For Mobike this parameter, λ, is estimated through a user survey. We let users specify their firm and price preferences. The survey is of 300 users, and it reports that $\lambda = 48.07/51.93$. Note that the prices and market shares in equations (8) and (9) differ in the units of measurement. These variables must be normalized to one. Thus, the final value of $\lambda = p \times 48.07/51.93$.

The difference in price insensitivity between the two firms' users, α, is computed using an intermediate variable, *i.e.*, the satisfaction level of user experience. de Haan *et al.* [27] revealed that user satisfaction has a significant and positive impact on customer retention based on data on 93 companies in 18 industries. In our study, the satisfaction level is measured by an index system that is presented in BDR reports. BDR conducted a survey of 5,326 bicycle-sharing users on the basis of this index system and a total of 4,682 valid samples were obtained. However, the reports do not present the weight of each index element. Therefore, we adopt the analytic hierarchy process (AHP) [28] to evaluate these weights. 12 experts are invited to score the relative importance between two elements, and seven experts' scores pass the consistency assessments. After computing

TABLE 3. Parameter values of the case.

Parameter	T_1	T_2	r	ρ	δ_{ofo}	δ_{mob}	$x(0)$	a_0	a_1	a_2	p	λ	α
Value	2	4	0.03	$(0, 0.5]$	0.26	0.32	0.56079	0.9651	1.053	2.161	12	$0.92567 \times p$	0.99194

the satisfaction values of Ofo and Mobike, we estimate α by the equation $\alpha = \frac{\text{Ofo'ssatisfaction}}{\text{Mobike'ssatisfaction}}$. The results are as follows: Ofo'ssatisfaction=4.47892, Mobike'ssatisfaction=4.51532, and $\alpha = 0.99194$.

In the first stage, the churn and ρ must be estimated. A report by Trustdata (www.itrustdata.com) reveals that Ofo and Mobike's user retention rates are approximately 74% and 68%, respectively, during the period from July 2016 to March 2017. Thus, we adopt these data to estimate the churn in the first stage, *i.e.*, $\delta_{\text{ofo}} = 26\%$ and $\delta_{\text{mob}} = 32\%$. Note that these values are similar to the driver churn of Uber, *i.e.*, over 30% [29]. This information verifies the data further by using Trustdata because Uber is also a shared-economy firm in the transportation industry.

Although ρ can be estimated using equation (1), we cannot obtain data on the advertising costs. The square root of these costs represents the corresponding advertising effort. Therefore, we refer to previous studies that consider the effect of advertising and involve ρ. The most related articles are He *et al.* [30] and He *et al.* [31], both of which use the dynamics of market share with the square-root feature and churn to depict a retail market oligopoly. These articles set the parameter values as $\rho = 0.5$ and $r = 0.03$. We adopt this setting and vary ρ within the interval $(0, 0.5]$ to investigate its impact on the firms' strategy and performance.

According to the BDR reports, Ofo and Mobike's market shares in 2016 are 51.2% and 40.1%, respectively. Therefore, we approximate the initial market share as $x(0) = 51.2/(51.2 + 40.1)$. The service price in the first stage is set to be $p = 12$ because the firms always provide a monthly price of 1 RMB. As previously mentioned in Section 2, the durations of the two stages are set to be two years. The parameter values of the case are summarized in Table 3.

4.2. Benchmark solution of symmetric firms

We mainly conduct numerical analysis on the advertising efforts of the firms and variations in their market shares over time. To describe the result intuitively, we first consider a scenario where the firms are symmetric. In this scenario, the parameter values are set as $\delta_A = \delta_B = 0.3$, $x(0) = 0.5$, and $\alpha = 1$. We vary ρ to 0.05, 0.2, 0.35, and 0.5. The other values are identical to those in Table 3. Furthermore, we introduce the upper bound constraints $u_A \leq 3$ and $u_B \leq 3$ into the model to ensure the existence of an equilibrium solution. In this scenario, the two firms' equilibrium market shares, advertising efforts (as shown in Fig. 2), and profits in each stage are equal. Therefore, both the firms equally share the market in the entire period comprising the two stages, and this state remains unaltered over time. However, the firms must invest in advertising to compete for users. As ρ increases, the firms' adversiting efforts increase, and hence, their profits decrease. This is because ρ depicts the effect of advertising on market competition. Therefore, the firms must exert additional effort when ρ is high. We also observe that a similar effort displays an increasing trend first and then a decrease to zero, over time. When ρ is high, *i.e.*, $\rho = 0.5$, the firms exert their best efforts and hence, attain the upper bounds in the early part of the first stage. Thus, the Nash equilibrium solution is non-existent in reality. This observation can partly account for the realistic advertising campaign between Ofo and Mobike. When the firms observe a high advertising effect and balanced power and state of competition, an intense advertising competition occurs. Without the boundary constraint, Ofo and Mobike must invest their entire capital in advertising and hence incur losses. The observations in the symmetric scenario are summarized in Observation 1.

Observation 1. *When two firms are symmetric, (a) they share the market equally and earn equally; (b) the firms' advertising efforts are also identical, and these first increase and then decrease to zero over time; (c) a high value of ρ results in a high advertising effort.*

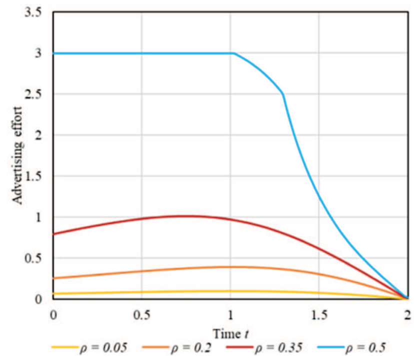

FIGURE 2. Advertising efforts in symmetric scenario.

4.3. Solution and performance of the case

In this subsection, we consider the relationship between the decision variables (advertising efforts) and state variables (market shares) over time when the estimated parameter values (listed in Tab. 3) are adopted. The figures in this subsection consider two representative scenarios: a low advertising effect ($\rho = 0.05$) and a high advertising effect ($\rho = 0.5$).

Figures 3 and 4 display the advertising efforts and market shares, respectively, over time. We observe that when the firms are asymmetric, their strategies and resulting performances differ. Ofo's advertising effort and resulting market share are larger than those of Mobike's. This is because Ofo initially presents a higher market share and a lower churn than Mobike. Therefore, Ofo dominates in the initial part of the first stage. Its dominant position encourages Ofo to exert higher effort than Mobike to maintain the competitive advantage. This finding is repeated in Observation 2. Figure 4 illustrates that when the advertising effect is low, the difference between the two firms' market shares gradually decreases over time. When the advertising effect is high, the difference between the two firms' market shares increases over time. The reason is that a low advertising effect implies that a large advertising effort cannot result in a large increase in market share. In this scenario, the firms cannot invest substantially in advertising and therefore, compete peacefully in advertising. Moreover, the advertising effort displays a trend wherein it first increases and then decreases. A high advertising effect inspires Ofo to exert additional effort in advertising over time because the firm observes that an additional effort results in an increase in its market share. Thus, Ofo's effort increases over time in the entire first stage. In contrast, Mobike undertakes a low effort, which reduces to zero at a certain time during the first stage. An observation reveals that Mobike's advertising has negligible effect on market share. Therefore, Mobike exits the advertising campaign, and its market share decreases over time in the entirety of the first stage. The above observation is summarized in Observation 3.

Observation 2. *If the firm presents a dominant position initially, it exerts more effort than its competitor.*

Observation 3. *In the first stage, a significantly low ρ results in a gradual decrease in the difference between the two firms' market shares. Meanwhile, a significantly high value results in a gradual increase in the difference, and the firm that is dominant initially remains dominant during the entire stage, and its advantage enhances.*

The market shares at the end of the first stage and the performance of the firms as ρ varies are listed in Table 4. The result reveals that as the advertising effect increases, Ofo's market share increases, whereas Mobike's market share decreases. Table 4 shows a noteworthy observation: a significantly high advertising effect ($\rho = 0.5$) results in the highest total profits of the firms in the two stages, although Ofo incurs a high advertising cost and Mobike presents a low market share. The reason is that as the advertising effect increases, the profit in the first stage decreases, whereas that in the second stage increases. Moreover, the increase in volume in the

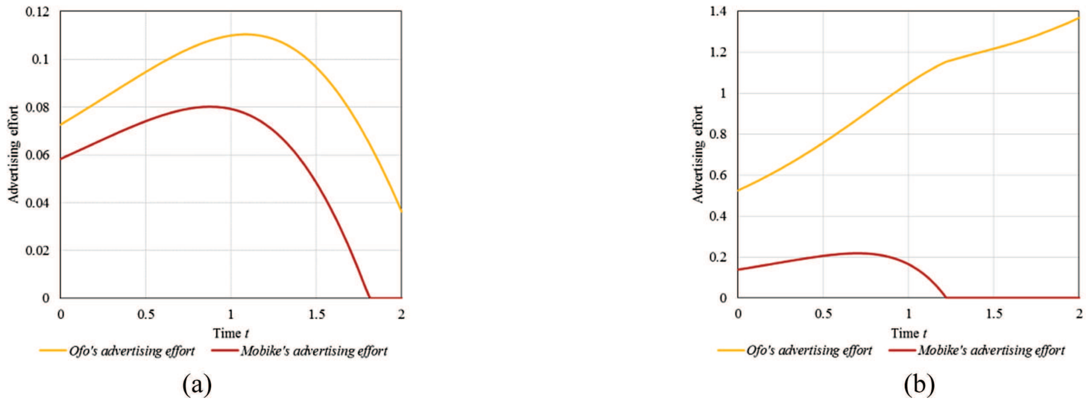

FIGURE 3. Advertising efforts in the case.

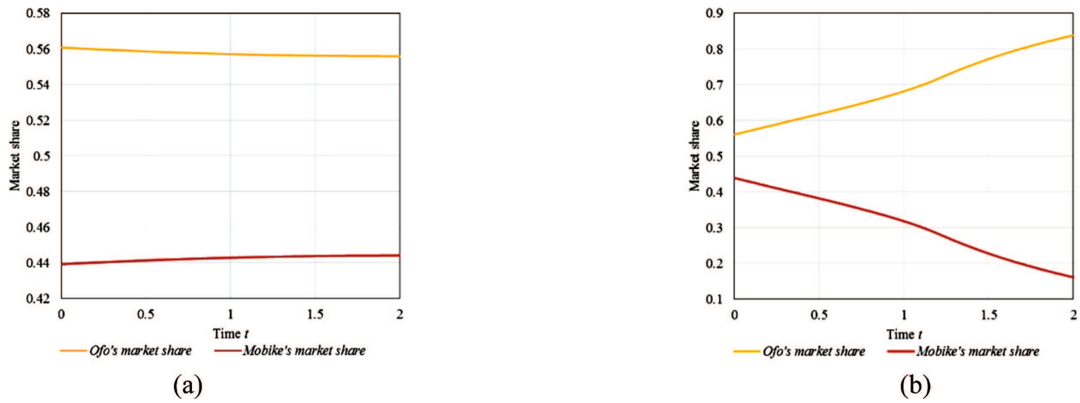

FIGURE 4. Market shares in the case.

second stage is larger than the decrease in volume in the first stage. Note that $\alpha = 0.99194$, indicating that the firms prefer a high x_{T_1} from the perspective of profit maximization in the second stage (recall Prop. 4). x_{T_1} is also determined by the consideration of profit in the first stage. Thus, it cannot achieve the largest value because it reduces the profit in the first stage. The value of x_{T_1} results from the comprehensive optimization of the two stages. In addition, Ofo performs better than Mobike. Observation 4 summarizes the above analysis.

Observation 4. *In the case, (a) the dominant firm (Ofo) earns more than its competitor (Mobike); (b) as ρ increases, the firms' profits in the first stage decrease and those in the second stage increase; Ofo's total profit in the two stages increases, and Mobike's total profit first decreases and then increases.*

The result in Table 4 demonstrates that the firms can make a profit regardless of ρ. The reported scenario of incurring losses is caused by the excessive delivery of bicycles. By the end of December 2017, the total number of shared bicycles in Beijing was 2.35 million, whereas the demand was 0.43 million. In that year, the number of bicycles ordered by Ofo and Mobike were 17.8 and 15.6 million, with unit costs of approximately 500 and 1000 RMB, respectively. The firms' financing volumes were approximately 8.05 and 8 billion RMB. Thus, both the firms' cost of bicycles purchased, *i.e.*, 8.9 and 15.6 billion RMB, exceeded the financing volumes.

Rationally allocating bicycle quantity in cities and keeping the bicycles in good shape and in a usable condition are critical for the firms. Therefore, the two firms transform the competition strategy in the second stage.

TABLE 4. Performances of the case with different ρ values.

ρ	x_{T_1}	V_{ofo}	$V_{ofo,1}$	$V_{ofo,2}$	V_{mob}	$V_{mob,1}$	$V_{mob,2}$
0.05	0.556	9.852	5.030	4.823	8.899	4.076	4.823
0.1	0.560	9.840	5.008	4.831	8.875	4.044	4.831
0.15	0.569	9.826	4.972	4.853	8.835	3.982	4.853
0.2	0.588	9.825	4.918	4.908	8.777	3.870	4.908
0.25	0.625	9.872	4.815	5.057	8.699	3.643	5.057
0.3	0.684	10.045	4.646	5.399	8.607	3.208	5.399
0.35	0.738	10.436	4.606	5.830	8.571	2.741	5.830
0.4	0.779	10.874	4.648	6.226	8.654	2.429	6.226
0.45	0.812	11.212	4.615	6.597	8.795	2.198	6.597
0.5	0.839	11.479	4.552	6.927	8.953	2.026	6.927

According to the model's result, both the firms can survive in the second stage. This observation can explain the real state of Mobike. However, Ofo was on the verge of bankruptcy in 2019. The reason is reported as ineffective management of the firm and high damage rate of its bicycles.

4.4. Sensitivity analysis of the case

In this subsection, we present the sensitivity analysis. We adopt two values of ρ (0.05 and 0.2) to ensure the existence of an equilibrium solution. The basic setting of the parameters is equal to that presented in Table 3. We vary p and δ_{ofo} and examine their impacts. The results are illustrated in Figure 5 and Table 5. Figure 5a shows that the combination of ρ and p significantly affects the trend of the firms' market shares. Ofo's market share 1) decreases gradually over time when the combination has a low value, 2) first decreases and then increases over time when the combination has a medium value, and 3) always increases over time when the combination has a high value. Moreover, a threshold of the combination is observed. When the values of the combination are above (conversely, below) this threshold, Ofo's market share at the end of the first stage is larger (conversely, less) than its initial share. Observation 5 highlights the abovementioned findings. Figures 5c and 5e demonstrate that a high value of ρ and/or p results in high advertising efforts by Ofo, a trend similar to that in the symmetric scenario. However, this high value results in Mobike's high advertising efforts in the early part of the first stage and low efforts in the latter part of the first stage. The top half of Table 5 shows an increasing trend of the two firms' profits as p increases. However, the bottom half of Table 5 shows that Ofo should reduce its customer churn for profit-seeking. This finding is repeated in Observation 6. The influence of Ofo's churn on the market share and effort is illustrated in Figures 5b, 5d, and 5f. We observe that Ofo's churn features a converse influence, *i.e.*, a high churn value of Ofo results in its low market shares and advertising efforts.

Observation 5. *In the case, consider the value combination of p and δ_{ofo}: (a) Ofo's market share in the first stage decreases gradually over time when the value combination of ρ and p is low; (b) Ofo's market share in the first stage first decreases and then increases over time when the value combination is medium; (c) Ofo's market share in the first stage increases over time when the value combination is high; (d) Moreover, a threshold of the value combination is observed. Above (conversely, below) this threshold, Ofo's market share at the end of the first stage is larger (conversely, less) than its initial share.*

Observation 6. *In the case, the two firms should set a high price in the first stage and exert more effort on a low churn than the other to seek profit.*

5. Conclusions

In this study, we analyze a dynamic market share problem involving advertising and loyalty reward program strategies. The competition case between two bicycle-sharing firms, Ofo and Mobike, is also investigated. This

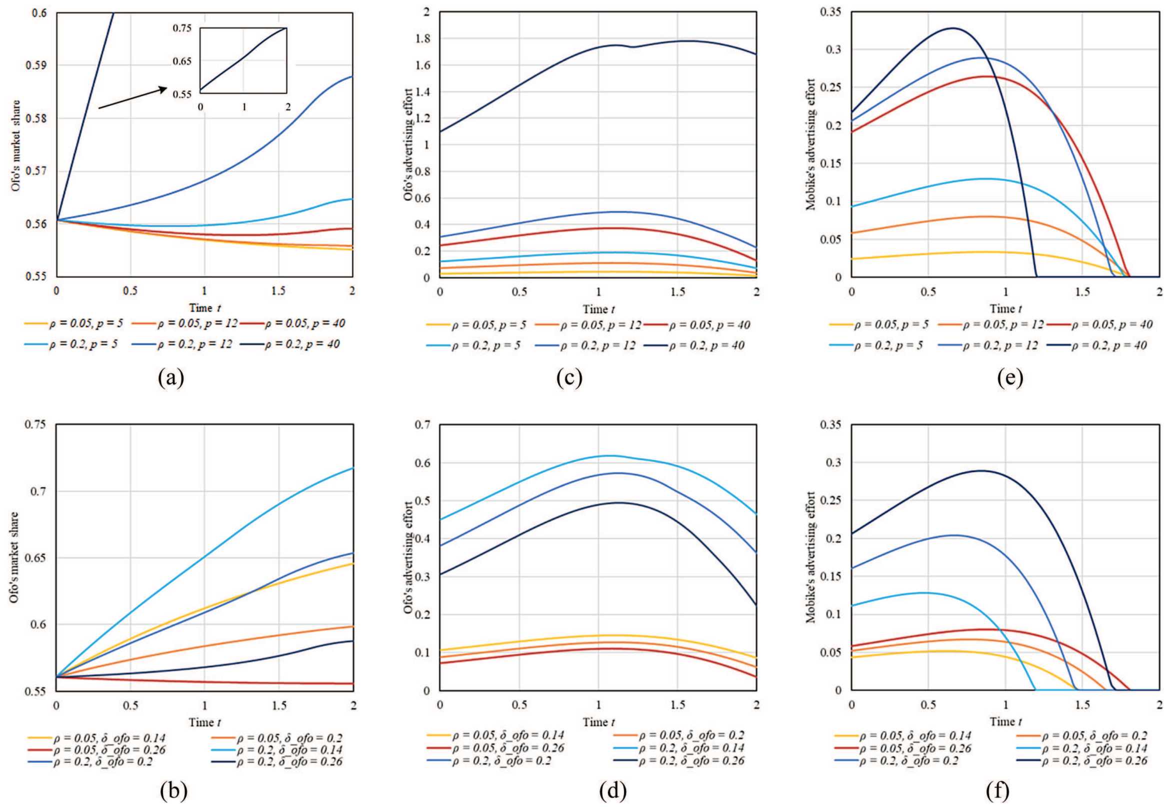

FIGURE 5. Sensitivity of the case (p and δ_{ofo}).

TABLE 5. Performance of the sensitivity of the case.

ρ	p	x_{T_1}	V_{ofo}	$V_{\text{ofo},1}$	$V_{\text{ofo},2}$	V_{mob}	$V_{\text{mob},1}$	$V_{\text{mob},2}$
0.05	5	0.555	4.106	2.098	2.009	3.710	1.701	2.009
	12	0.556	9.852	5.030	4.823	8.899	4.076	4.823
	40	0.559	32.806	16.709	16.098	29.600	13.502	16.098
0.2	5	0.565	4.096	2.078	2.018	3.689	1.671	2.018
	12	0.588	9.825	4.917	4.908	8.777	3.869	4.908
	40	0.751	35.255	15.411	19.844	28.615	8.770	19.844
ρ	δ_{ofo}	x_{T_1}	V_{ofo}	$V_{\text{ofo},1}$	$V_{\text{ofo},2}$	V_{mob}	$V_{\text{mob},1}$	$V_{\text{mob},2}$
0.05	0.14	0.646	10.826	5.665	5.162	8.670	3.509	5.162
	0.2	0.599	10.283	5.338	4.945	8.752	3.807	4.945
	0.26	0.556	9.852	5.030	4.823	8.899	4.076	4.823
0.2	0.14	0.717	11.127	5.475	5.652	8.621	2.970	5.652
	0.2	0.654	10.400	5.193	5.206	8.643	3.436	5.206
	0.26	0.588	9.825	4.917	4.908	8.777	3.869	4.908

competition is divided into two stages. In the first stage, the firms compete for market share through advertising. In the second stage, they adopt either a loyalty reward program to retain old users or the discount promotion strategy to attract new users. To investigate how the firms compete and what strategies they adopt, we establish a dynamic advertising model for the first stage and compare the advertising strategy of the firms and its impact on market share. For the second stage, a static Hotelling model is proposed, which is followed by an analysis of both the firms' pricing and reward/promotion strategies, profits, and market shares. These analyses result in several noteworthy results with important managerial insights.

First, the firm that presents a dominant position invests more in advertising and earns more than its competitor in the first stage and the two stages in general. In the first stage, a significantly low value of ρ causes a gradual decrease in the difference between the two firms' market shares, whereas a significantly high value results in a gradual increase in the difference. In the second stage, both the firms equally split the market and earn equally owing to the adjustment mechanism of loyalty reward programs.

Second, based on the market shares at the end of the first stage, a threshold of the difference in price insensitivity of users between the firms (α) is determined. Neither of the firms can conduct a loyalty reward program if α is equal to this threshold. Otherwise, they can conduct different loyalty reward programs in the second stage. Below this threshold, the dominant firm should reward its heavy users, and its competitor should provide a discount for its light users. Above this threshold, the dominant firm should provide a discount for its light users, and its competitor should reward its heavy users. Therefore, a firm should respond to the reward strategy of its rival with a converse strategy.

Finally, according to the equilibrium solution of our model, the firms can earn if they are rational. The reported scenario that Ofo and Mobike incurred losses is mainly caused by over-advertising, excessive delivery of bicycles, price war in the first stage, and ineffective management of the firms.

We have analyzed firms' competition for market share through advertising and loyalty reward programs. However, certain noteworthy problems motivate us to conduct further investigation. First, we disregarded service quality in this study. In this case, large parts of bicycles break and must be repaired over time. This scenario directly affects service quality and user experience. Therefore, future research can focus on competition in service quality [32]. Second, Ofo and Mobike are reported to be encountering problems related to capital. Thus, the consideration of capital constraints or pre-specified budgets [33] is another issue that is worth investigating in the future.

Acknowledgements. This work is supported by the National Social Science Foundation of China (19CGL002). Thanks to the reviewers and editors for the valuable suggestions.

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