RAIRO-Oper. Res. 56 (2022) 3611–3634
https://doi.org/10.1051/ro/2022162

RAIRO Operations Research
www.rairo-ro.org

OPTIMAL INVESTMENT AND REINSURANCE ON SURVIVAL AND GROWTH PROBLEMS FOR THE RISK MODEL WITH COMMON SHOCK DEPENDENCE

Shida Duan[1] and Zhibin Liang[2,*]

Abstract. This paper investigates goal-reaching problems regarding optimal investment and proportional reinsurance with two dependent classes of insurance business, where the two claim number processes are correlated through a common shock component. The optimization problems are formulated in a general form first, and then four criteria including maximum survival probability, minimum expected ruin penalty, minimum (maximum) expected time (reward) to reach a goal are fully discussed. By the technique of stochastic control theory and through the corresponding Hamilton–Jacobi–Bellman equation, the optimal results are derived and analyzed in different cases. In particular, when discussing the maximum survival probability with a target level U beyond the safe level (where ruin can be avoided with certainty once it is achieved), we construct ϵ-optimal (suboptimal) strategies to resolve the inaccessibility of the safe level caused by classical optimal strategies. Furthermore, numerical simulations and analysis are presented to illustrate the influence of typical parameters on the main results.

Mathematics Subject Classification. 60J60, 62P05, 91B28, 91B30, 93E20 .

Received January 7, 2022. Accepted September 16, 2022.

1. Introduction

The topic of reaching a goal has been discussed widely in the past few decades. Associated research started from Dubins and Savage [1], Pestien and Sudderth [2] and continuoued with the work of Browne [3–6]. Karatzas [7] discussed the maximization of reaching a target level in a fixed period of time, and Browne [4] investigated the optimal investment strategies for both survival and growth problems in infinite horizon. More recently, there has been a focus on maximizing the probability of reaching the bequest. See, for instance, Liang and Young [8], Bayraktar and Young [9], Bayraktar *et al.* [10,11].

The discussion on goal-reaching problems lies mainly in two fields including life insurance and non-life insurance. In the area of non-life insurance business and from the perspective of an insurance company, there are several intriguing problems like maximizing the probability of reaching a target wealth level before ruin or minimizing the probability of ruin [12,13]. The initial work can be found in Schmidli [14], Promislow and Young [15], and Luo [16]. Yener [17] further discussed target maximization issues on portfolio strategies regarding

Keywords. Proportional reinsurance, common shock dependence, stochastic control, ϵ-optimal (suboptimal) strategy, Hamilton–Jacobi–Bellman equation.

[1] Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA.

[2] School of Mathematical Sciences and Institute of Finance and Statistics, Nanjing Normal University, Jiangsu 210023, P.R. China.

*Corresponding author: liangzhibin111@hotmail.com

survival and growth problems, where constraints with borrowing are set in the financial market. Considering the influence of common shock, Han *et al.* [18] investigated the optimal proportional reinsurance with constraints of $[0, 1]$ on the retention level to minimize the probability of drawdown. Luo *et al.* [19] considered goal-reaching problems regarding optimal robust investment and proportional reinsurance with penalty on ambiguity, and the uncertainty lied in the drift of the risky asset and the claim process.

In the area of life insurance business and from the perspective of an individual, the objectives mainly discussed involve minimizing the probability of lifetime ruin and maximizing the probability of reaching a bequest goal. Early studies were pioneered by Milevsky and Robinson [20] and Young [21]. After that, many papers adopted constraints on consumption and borrowing. Bayraktar *et al.* [11] seeked the optimal strategies of reaching a bequest goal under the framework that the individual could consume from the investment account and purchase term life insurance. Liang and Young [8] solved two optimization problems of reaching a bequest goal with ambiguity in the return rate of risky asset and the hazard rate of mortality.

Even though a lot of work has been done regarding goal-reaching problems, very few of them considered common shock influence. Typically, insurance businesses are often shown as dependent. For instance, an earthquake, hurricane or tsunami often leads to various insurance claims such as death claims, medical claims and household claims. Therefore, a single event generates claims from different lines of insurance. The so-called common shock risk model is designed to depict such a dependent structure. Research on common shock problems have been extensively discussed in the past years. See, for example, Wang [22]; Yuen *et al.* [23, 24]; Centeno [25]; Bai *et al.* [26]; Yuen *et al.* [27]; Liang and Yuen [28, 29]; Bi *et al.* [30]; Han *et al.* [18]. Centeno [25] studied the optimal excess of loss retention limits for two dependent classes of insurance risks. Under the criterion of maximizing the expected exponential utility, Liang and Yuen [28] considered the optimal reinsurance strategy in a risk model with two dependent classes of insurance business by the variance premium principle. Bi *et al.* [30] considered the problem of optimal reinsurance with two dependent classes of insurance risks in a regime-switching financial market.

Inspired by the above mentioned work, we focus on both survival problems in the danger-zone and growth problems in the safe-region from the perspective of an insurance company. The insurance company involves in two dependent classes of business which correlated with a common shock, and it not only invests in the financial market with multiple risky assets and a risk-less bond but also shares claim risk with an reinsurance company. Thus, the control variables being considered are the investment strategy and the retention level for each business. In this work, we first formulate a general form of portfolio and asset allocation problem under the financial market framework with the verification theorem. Next, the optimal results for both survival and growth problems are discussed in detail under a financial market with one risky asset. Since we constrain the retention levels to be nonnegative, the optimization problems are divided into multiple cases, which makes the problem more challenging. Note that because of the non-cheap cost of reinsurance, there exists a safe level that ruin can be avoided once the wealth level hits it. However, it turns out that the optimal strategy will not help the insurance company reach the safe level when the initial surplus is below it. Motivated by Browne [4], we construct an ϵ-optimal strategy to overcome this dilemma so that the wealth can achieve the safe region with a positive probability.

Based on Browne [3], we extend the model and the problems to the reinsurance industry which can be seen in the following aspects. Firstly, Browne [4] dealt with optimal investment and consumption, while our work discusses optimal investments and proportional reinsurance under a common shock framework. When considering a full reinsurance strategy, the surplus process in our work will degenerate to a process with investments and a constant consumption rate, then the model in Browne [4] can be realized under our framework. From this perspective, our paper is more general. Secondly, Browne [4] set no constraints on the control variables, while we constraint the retention level of reinsurance to be nonnegative, which entails more cases to discuss. Furthermore, we perform numerical examples and provide economic explanations behind those results. In particular, we add Example 5.3 in Section 5 to analyze the economic background and stress the importance of common shock influence.

This paper is organized as follows. In Section 2, we construct usual portfolio and asset allocation problems in a general form and present its Hamilton–Jacobi–Bellman equations with regard to multiple risky assets model. In Section 3, we discuss two survival problems regarding maximum survival probability and minimum expected ruin penalty in different cases. For the maximum problem, when the target level U is set below the safe level, the optimal solutions are obtained explicitly through stochastic control theory and dynamic programming principle; when U is larger than or equal to the safe level, suboptimal strategies are constructed to resolve the inaccessibility of the safe level under classical strategy. In Section 4, two growth problems about the minimum expected time and maximum expected reward to a goal are further discussed. To illustrate the main results, we study numerical examples regarding Section 3.1 and make an analysis in Section 5. Finally, we present a few interesting directions for further research in Section 6.

2. Model and problem formulation

Throughout this paper, we assume that $W_t = (W_{1,t}, ..., W_{d,t}, W_{d+1,t}, W_{d+2,t})$ is a $d+2$ dimensional correlated Brownian motion with $E[W_{i,t}W_{j,t}] = \rho_{ij}t$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$. We assume that $\rho_{ij} \in [0, 1]$ and the matrix (ρ_{ij}) is invertible.

Risk model with common shock. Assume that there are two dependent classes of insurance business which are generated by three claim number processes: let $N_{j,t}, j \in \{1, 2\}$, be the claim number process of only type j business by time t and let N_t be the number of claims that generates both claims by time t. Let $\{Y_j^i\}_{i\in\mathbb{N}}$, $j \in \{1, 2\}$ be two sequence of positive independent and identically distributed (i.i.d) random variables with second moments. The sequence of random variables $\{Y_j^i\}_{i\in\mathbb{N}}$, $j \in \{1, 2\}$ represent the sequence of claim size of the two classes of insurance business, following the distribution functions $F_j(y)$, $j \in \{1, 2\}$. Without loss of generality, we assume that $F_j(y) = 0$, $j \in \{1, 2\}$ for $y \leq 0$ and $0 < F_j(y) \leq 1$, $j \in \{1, 2\}$ for $y > 0$. Then the aggregate claims processes are given by

$$Z_t = Z_{1t} + Z_{2t} = \sum_{j=1}^{2} \sum_{i=1}^{N_{j,t}+N_t} Y_j^i,$$

where $N_{j,t} + N_t, j \in \{1, 2\}$, are the aggregated claim number processes for the class j, and $\{N_{j,t}\}_{t\geq 0}$, $j \in \{1, 2\}$, $\{N_t\}_{t\geq 0}$ are independent Poisson processes with intensity ζ_j, $j \in \{1, 2\}$ and ζ respectively. The counting process $\{N_t\}_{t\geq 0}$ plays the role of the common shock, such as a nature disaster brings claims to both insurance businesses. Note that the dependence of the two classes of business comes from a common shock governed by the counting process $\{N_t\}_{t\geq 0}$.

Following Grandell [31], we make diffusion approximations to $Z_{j,t}, j \in \{1, 2\}$ using Brownian motion risk models given by

$$\hat{Z}_{j,t} = a_j t - b_j W_{d+j,t};$$

$j \in \{1, 2\}$ with $a_j = (\zeta_j + \zeta)E[Y_j]$, $b_j^2 = (\zeta_j + \zeta)E[(Y_j)^2]$. The correlation coefficient of the two standard Brownian motions is $\rho_{d+1,d+2}$.

Risk model with reinsurance. Assume that the insurance company continuously purchases reinsurance for the sake of risk control. Let $(X_t)_{t\geq 0}$ be the wealth process under the control of both reinsurance and investment strategies. Let $q_{j,t}$ be the retention level of claim $j \in \{1, 2\}$ at time t and $\delta(q_{1,t}, q_{2,t})$ be the reinsurance premium rate given the retention level $q_{j,t}$, $j \in \{1, 2\}$.

Following the expected value premium principle, the premium rate c is given by

$$c = (1 + \theta_1)a_1 + (1 + \theta_2)a_2, \tag{2.1}$$

and the reinsurance premium rate $\delta(q_{1,t}, q_{2,t})$ is given by

$$\delta(q_{1,t}, q_{2,t}) = (1 + \eta_1)(1 - q_{1,t})a_1 + (1 + \eta_2)(1 - q_{2,t})a_2, \tag{2.2}$$

where θ_i and η_i, $i \in \{1,2\}$, are the insurer and the reinsurer's safety loading of the two classes of insurance business, respectively. To avoid triviality, we assume $\eta_i \geq \theta_i$ $(i = 1, 2)$.

Risk model with portfolio. Suppose that a financial market consists of a risk-free bond with interest rate $r \geq 0$, and d risky assets with the geometric Brownian motion

$$\frac{dS_{i,t}}{S_{i,t}} = \mu_i \mathrm{d}t + \sigma_i dW_{i,t}, \quad i = 1, 2, ..., d. \tag{2.3}$$

We assume correlation among all the process. For $i, j \in \{1, ..., d\}$, $\rho_{ij} = cov(S_i, S_j)$; for $i \in \{1, ..., d\}$, $j \in \{1, 2\}$, $\rho_{i(d+j)} = cov(S_i, Z_j)$; for $i, j \in \{1, 2\}$, $\rho_{(d+i)(d+j)} = cov(Z_i, Z_j)$. Assume $\mu_i \in \mathbb{R}$ and $\sigma_i \in \mathbb{R}$ be $\mathbb{F}$-progressively measurable processes. Let $\pi_{i,t}$ be the amounts invested in the ith, $i \in \{1, 2, ..., d\}$, risky assets at time t. Then the diffusion wealth process X_t evolves as

$$\begin{aligned} dX_t &= \sum_{i=1}^{d} \pi_{i,t} \frac{dS_{i,t}}{S_{i,t}} + \left(X_t - \sum_{i=1}^{d} \pi_{i,t} \right) r\mathrm{d}t + [c - \delta(q_{1,t}, q_{2,t})]\mathrm{d}t - q_{1,t} d\hat{Z_{1,t}} - q_2, t) d\hat{Z_{2,t}} \\ &= \left[rX_t - \alpha + \sum_{i=1}^{d} \pi_{i,t}(\mu_i - r) + \alpha_1 q_{1,t} + \alpha_2 q_{2,t} \right] \mathrm{d}t + \sum_{i=1}^{d} \pi_{i,t} \sigma_i dW_{i,t} + \sum_{j=1}^{2} q_{j,t} b_j dW_{d+j,t}, \end{aligned} \tag{2.4}$$

where $\alpha_1 = \eta_1 a_1 \geq 0, \alpha_2 = \eta_2 a_2 \geq 0$, $\alpha = (\eta_1 - \theta_1)a_1 + (\eta_2 - \theta_2)a_2 \geq 0$, $X_0 = x$.

Definition 2.1 (Admissible set). $\boldsymbol{\pi} = (\pi_{1,t}, ..., \pi_{d,t}, q_{1,t}, q_{2,t})_{t\geq 0}^{\top}$ is an admissible control if it is $\mathbb{F} - adapted$ and meets the integrability condition that

$$\int_0^T \left(\sum_{j=1}^{2} q_{j,s}^2 + \sum_{i=1}^{d} \pi_{i,s}^2 \right) \mathrm{d}s < \infty$$

for every $T < \infty$, $\mathbb{P} - a.s.$ The set of all admissible control is denoted by $\mathcal{A}$.

General problem formulation. Denote the first time hitting the point z under the control $\boldsymbol{\pi}_t = (\pi_{1,t}, ..., \pi_{d,t}, q_{1,t}, q_{2,t})_{t\geq 0}^{\top}$ as $\tau_z^{\boldsymbol{\pi}} := \inf\{t > 0 : X_t^{\boldsymbol{\pi}} = z\}$. Set a lower level L which could be interpreted as the bankruptcy level and an upper level U which could be interpreted as the target level for the wealth process. The first time to escape from the interval (L, U) with $L < X_0 < U$ is defined as $\tau^{\boldsymbol{\pi}} := \min\{\tau_L^{\boldsymbol{\pi}}, \tau_U^{\boldsymbol{\pi}}\}$. We will write $\tau^{\boldsymbol{\pi}}$ as τ in the following context for simplicity. Consider the value function

$$V(x) = \sup_{\boldsymbol{\pi} \in \mathcal{A}} \mathbb{E}_x \left[\int_0^{\tau} g(X_t) e^{-\int_0^t \lambda(X_s)\mathrm{d}s} \mathrm{d}t + h(X_\tau) e^{-\int_0^\tau \lambda(X_s)\mathrm{d}s} \right], \tag{2.5}$$

in which $\mathbb{E}_x(\cdot) := \mathbb{E}(\cdot | X_0 = x)$, $\lambda(x)$ is a nonnegative continuous function, $h(x)$ and $g(x)$ are real bounded and continuous functions.

For any function $v(x) \in \mathcal{C}^2(L, U)$, the operater is defined as

$$\mathcal{L}_{\boldsymbol{\pi}} v(x) = v_x(rx - \alpha) + v_x \boldsymbol{\mu}^\top \boldsymbol{\pi} + \frac{1}{2} v_{xx} \boldsymbol{\pi}^\top \boldsymbol{\Omega} \boldsymbol{\pi}, \tag{2.6}$$

where v_x and v_{xx} are partial derivatives of $v(x)$, $\boldsymbol{\pi} = (\pi_1, ..., \pi_d, q_1, q_2)^\top$, $\boldsymbol{\mu} := (\mu_1 - r, ..., \mu_d - r, \alpha_1, \alpha_2)^\top$, and the positive semi-definite covariance matrix

$$\boldsymbol{\Omega} := \begin{pmatrix} cov(S_1,S_1) & \cdots & cov(S_1,S_d) & cov(S_1,Z_1) & cov(S_1,Z_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ cov(S_d,S_1) & \cdots & cov(S_d,S_d) & cov(S_d,Z_1) & cov(S_d,Z_2) \\ cov(Z_1,S_1) & \cdots & cov(Z_1,S_d) & cov(Z_1,Z_1) & cov(Z_1,Z_2) \\ cov(Z_2,S_1) & \cdots & cov(Z_2,S_d) & cov(Z_2,Z_1) & cov(Z_2,Z_2) \end{pmatrix} \tag{2.7}$$

$$= \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_d\rho_{1d} & \sigma_1 b_1 \rho_{1(d+1)} & \sigma_1 b_2 \rho_{1(d+2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_1\sigma_d\rho_{1d} & \cdots & \sigma_d^2 & \sigma_d b_1 \rho_{d(d+1)} & \sigma_d b_2 \rho_{d(d+2)} \\ \sigma_1 b_1 \rho_{1(d+1)} & \cdots & \sigma_d b_1 \rho_{d(d+1)} & b_1^2 & b_1 b_2 \rho_{(d+1)(d+2)} \\ \sigma_1 b_2 \rho_{1(d+2)} & \cdots & \sigma_d b_2 \rho_{d(d+2)} & b_1 b_2 \rho_{(d+1)(d+2)} & b_2^2 \end{pmatrix}. \tag{2.8}$$

Now we give the verification theorem for the stochastic control problem (2.5).

Theorem 2.1. *Suppose $v(x) \in \mathcal{C}^2(L,U)$ satisfies the following properties: for any $x \in (L,U)$,*

1. *$v(x)$ is concave, meaning that its derivatives satisfy $v_x(x) > 0, v_{xx}(x) < 0$;*
2. *$v(L) = h(L), v(U) = h(U)$;*
3. *$-\lambda(x)v(x) + \mathcal{L}_{\boldsymbol{\pi}} v(x) + g(x) \le 0$ for all $\boldsymbol{\pi} \in \mathbb{R}^d \times [0,+\infty)^2$;*
4. *$-\lambda(x)v(x) + \mathcal{L}_{\boldsymbol{\pi}^*} v(x) + g(x) = 0$ for some function $\boldsymbol{\pi}^*(x) \in \mathbb{R}^d \times [0,+\infty)^2$;*
5. *for $\boldsymbol{\pi}^*(x)$ in condition 3, the following feedback stochastic differential equation has a unique strong solution:*

$$\begin{aligned} dX_t^* = &\left[rX_t^* - \alpha + \sum_{i=1}^d \pi_i^*(X_t^*)(\mu_i - r) + \alpha_1 q_1^*(X_t^*) + \alpha_2 q_2^*(X_t^*)\right]\mathrm{d}t \\ &+ \sum_{i=1}^d \sigma_i \pi_i^*(X_t^*) dW_{i,t} + \sum_{j=1}^2 b_j q_j^*(X_t^*) dW_{d+j,t} \end{aligned} \tag{2.9}$$

where $X_0 = x$ and $\left(\boldsymbol{\pi}^*(X_t^*)\right)_{t\ge 0} \in \mathcal{A}$.

Then, $v = V$ and $\left(\boldsymbol{\pi}^*(X_t^*)\right)_{t\ge 0}$ is an optimal control.

Proof. See Appendix A. □

Then, the HJB equation follows as

$$\begin{aligned} 0 &= \sup_{\boldsymbol{\pi} \in \mathbb{R}^d \times [0,+\infty)^2} \left[-\lambda(x)v(x) + \mathcal{L}_{\boldsymbol{\pi}} v(x) + g(x)\right] \\ &= -\lambda(x)v(x) + g(x) + v_x(rx - \alpha) + \sup_{\boldsymbol{\pi} \in \mathbb{R}^d \times [0,+\infty)^2} \left[v_x \boldsymbol{\mu}^\top \boldsymbol{\pi} + \frac{1}{2} v_{xx} \boldsymbol{\pi}^\top \boldsymbol{\Omega} \boldsymbol{\pi}\right], \end{aligned} \tag{2.10}$$

with the boundary conditions $v(L) = h(L)$ and $v(U) = h(U)$.
Note that

$$\sup_{\boldsymbol{\pi} \in \mathbb{R}^d \times [0,+\infty)^2} \left\{v_x \boldsymbol{\mu}^\top \boldsymbol{\pi} + \frac{1}{2} v_{xx} \boldsymbol{\pi}^\top \boldsymbol{\Omega} \boldsymbol{\pi}\right\} \tag{2.11}$$

is obtained at

$$\hat{\boldsymbol{\pi}} = -\frac{v_x}{v_{xx}} \boldsymbol{\Omega}^{-1} \boldsymbol{\mu}. \tag{2.12}$$

Substitute $\hat{\boldsymbol{\pi}}$ into the HJB equation we get

$$0 = -\lambda(x)v(x) + g(x) + v_x(rx - \alpha) - \frac{1}{2}\frac{v_x^2}{v_{xx}}\boldsymbol{\mu}^\top \boldsymbol{\Omega}^{-1}\boldsymbol{\mu},$$

$L < x < U$ with boundary conditions $v(L) = h(L), v(U) = h(U)$.

Remark 2.1. From (2.4), we can see that when the initial wealth level $x > \frac{\alpha}{r}$, ruin can be avoided by simply choosing a policy which invests only in the risk-free bond and transfers all of the claim risk to the reinsurance company. To be more specific, if we choose $\boldsymbol{\pi} = (\pi_1, ..., \pi_d, q_1, q_2)^\top = 0$, the surplus will follow a deterministic differential equation $dX_t = (rX_t - \alpha)\mathrm{d}t$ for $X_0 = x > \frac{\alpha}{r}$, which implies that the wealth process will experience an exponential growth. Thus, $P(\tau_{x-\epsilon} = \infty) = 1, a.s.$ for any $\epsilon > 0$. On the other hand, when the initial wealth level $x < \frac{\alpha}{r}$, for any admissible control $\pi \in \mathcal{A}$, the insurance company will inevitably face a possibility of bankruptcy because of the stochastic factors existed in the wealth process (2.4). Therefore, $\frac{\alpha}{r}$ is called the "safe point" or "safe level" since the company can avoid ruin once it is achieved. Obviously, the survival problem is interesting in the so-called "danger-zone" with $x < \frac{\alpha}{r}$ and the growth problem is meaningful in the so-called "safe-region" with $x > \frac{\alpha}{r}$.

Remark 2.2. As we can see from the form of the maximizers in (2.12), the discussion of the constraints ($\hat{q}_1 \geq 0$ and $\hat{q}_2 \geq 0$) requires an explicit expression of the last two rows of $\boldsymbol{\Omega}^{-1}$, which is difficult to be derived in the $(d+2) \times (d+2)$ dimensional matrix. Fortunately, when $\rho_{i(d+1)} = 0$ and $\rho_{i(d+2)} = 0$, $i \in \{1, ..., d\}$, (implying that the financial market is independent with the two aggregate claims), the last two rows of $\boldsymbol{\Omega}^{-1}$ will be able to be figured out (with the nonzero elements coming from the inverse of the 2×2 matrix in the lower right corner of $\boldsymbol{\Omega}$). From this point of view, the discussion of $d+2$ dimensional framework with d risky assets being independent with the two aggregate claims will be able to be accomplished and it has no much difference with the situation when $d = 1$ under this independent structure, which would be a special case (with $\rho_{12} = \rho_{13} = 0$) in the detailed discussions below. Therefore, we focus on $d = 1$ dimensional financial framework with dependent structure (meaning that the financial market with one risky asset is dependent with the two aggregate claims) from now on.

Special problem formulation. Consider a financial market with one risky asset ($d = 1$).

Denote $\boldsymbol{\mu} = (\mu - r, \alpha_1, \alpha_2)^\top$, $\boldsymbol{\pi} = (\pi_t, q_{1,t}, q_{2,t})_{t\geq 0}^\top$ and the covariance matrix

$$\boldsymbol{\Omega} = \begin{pmatrix} \sigma^2 & \sigma b_1\rho_{12} & \sigma b_2\rho_{13} \\ \sigma b_1\rho_{12} & b_1^2 & b_1 b_2\rho_{23} \\ \sigma b_2\rho_{13} & b_1 b_2\rho_{23} & b_2^2 \end{pmatrix}.$$

Then the wealth process is given by

$$dX_t = [rX_t - \alpha + \pi_t(\mu - r) + \alpha_1 q_{1,t} + \alpha_2 q_{2,t}]\,\mathrm{d}t + \pi_t\sigma dW_{1,t} + \sum_{j=1}^{2} q_{j,t} b_j dW_{1+j,t}. \tag{2.13}$$

Accordingly, the generator of value function is reduced to

$$\mathcal{L}_{\boldsymbol{\pi}} v(x) = v_x(rx - \alpha + \boldsymbol{\mu}^\top\boldsymbol{\pi}) + \frac{1}{2}v_{xx}\boldsymbol{\pi}^\top\boldsymbol{\Omega}\boldsymbol{\pi}. \tag{2.14}$$

Define

$$p(\pi, q_1, q_2) = v_x\boldsymbol{\mu}^\top\boldsymbol{\pi} + \frac{1}{2}v_{xx}\boldsymbol{\pi}^\top\boldsymbol{\Omega}\boldsymbol{\pi}, \tag{2.15}$$

its maximizer is $\hat{\boldsymbol{\pi}} = -\frac{v_x}{v_{xx}}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}$. Since

$$\boldsymbol{\Omega}^{-1} = \kappa_1 \begin{pmatrix} \frac{1-\rho_{23}^2}{\sigma^2} & -\kappa_2 & -\kappa_4 \\ -\kappa_2 & \frac{1-\rho_{13}^2}{b_1^2} & -\kappa_3 \\ -\kappa_4 & -\kappa_3 & \frac{1-\rho_{12}^2}{b_2^2} \end{pmatrix} \quad ,$$

where

$$\begin{cases} \kappa_1 = \dfrac{1}{(1-\rho_{12}^2)(1-\rho_{13}^2) - (\rho_{12}\rho_{13} - \rho_{23})^2}, \\ \kappa_2 = \dfrac{\rho_{12} - \rho_{13}\rho_{23}}{\sigma b_1}, \\ \kappa_3 = \dfrac{\rho_{23} - \rho_{12}\rho_{13}}{b_1 b_2}, \\ \kappa_4 = \dfrac{\rho_{13} - \rho_{12}\rho_{23}}{\sigma b_2}, \end{cases} \tag{2.16}$$

and by denoting

$$\boldsymbol{m} = \boldsymbol{\Omega}^{-1}\boldsymbol{\mu} = \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} = \kappa_1 \begin{pmatrix} \frac{1-\rho_{23}^2}{\sigma^2}(\mu - r) - \kappa_2\alpha_1 - \kappa_4\alpha_2 \\ -\kappa_2(\mu - r) + \frac{1-\rho_{13}^2}{b_1^2}\alpha_1 - \kappa_3\alpha_2 \\ -\kappa_4(\mu - r) - \kappa_3\alpha_1 + \frac{1-\rho_{12}^2}{b_2^2}\alpha_2 \end{pmatrix}, \tag{2.17}$$

we have

$$\hat{\boldsymbol{\pi}} = -\frac{v_x}{v_{xx}}\boldsymbol{m}. \tag{2.18}$$

Substituting (2.18) into the HJB equation

$$0 = -\lambda(x)v(x) + g(x) + v_x(rx - \alpha) + \sup_{\boldsymbol{\pi}\in\mathbb{R}\times[0,+\infty)^2} \left\{ v_x\boldsymbol{\mu}^\top\boldsymbol{\pi} + \frac{1}{2}v_{xx}\boldsymbol{\pi}^\top\Sigma\Sigma^\top\boldsymbol{\pi} \right\}, \tag{2.19}$$

we obtain

$$\begin{aligned} 0 &= -\lambda(x)v(x) + g(x) + v_x(rx - \alpha) - \frac{1}{2}\frac{v_x^2}{v_{xx}}\boldsymbol{\mu}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{\mu} \\ &= -\lambda(x)v(x) + g(x) + v_x(rx - \alpha) + \frac{1}{2}\frac{v_x^2}{v_{xx}}\kappa_1 \left[-\frac{1-\rho_{23}^2}{\sigma^2}(\mu - r)^2 - \frac{1-\rho_{13}^2}{b_1^2}\alpha_1^2 \right. \\ &\quad \left. -\frac{1-\rho_{12}^2}{b_2^2}\alpha_2^2 + 2(\kappa_2\alpha_1 + \kappa_4\alpha_2)(\mu - r) + 2\kappa_3\alpha_1\alpha_2 \right], \end{aligned} \tag{2.20}$$

with the boundary conditions $v(L) = h(L), v(U) = h(U)$ for $x \in (L, U)$.

In the next two sections, we will discuss about two different scenarios. In Section 3, we discuss about survival problems, in which the initial wealth of an insurance company is below the safe level. In Section 4, we discuss about growth problems, in which the initial wealth of an insurance company is above the safe level.

3. Optimal results for survival problems

In this section, we discuss the optimal investment and reinsurance problems under two criteria within the "danger-zone" with $x < \frac{\alpha}{r}$, where the insurance company has a possibility of ruin.

3.1. Maximize the probability of reaching a goal before ruin

Now, we formulate the following problem: for any initial wealth point $x \in (L, \frac{\alpha}{r})$, we aim to obtain the optimal policy which maximizes the probability of hitting the level U before L. Let

$$V_1(x : L, U) = \sup_{\boldsymbol{\pi} \in \mathcal{A}} \mathbb{P}_x\left(\tau_L^{\boldsymbol{\pi}} > \tau_U^{\boldsymbol{\pi}}\right) = \sup_{\boldsymbol{\pi} \in \mathcal{A}} \mathbb{E}_x\left[I_{\{\tau_L^{\boldsymbol{\pi}} > \tau_U^{\boldsymbol{\pi}}\}}\right]. \tag{3.1}$$

Remark 3.1. Recognize that the problem in this subsection is the special case of (2.5) with $\lambda = 0, g = 0$ and $h(x) = \frac{x-L}{U-L}$, so that $h(U) = 1, h(L) = 0$.

Remark 3.2. Constrain the retention level q_1 and q_2 within $[0, +\infty)$, and the investment strategy $\pi \in \mathbb{R}$. From (2.16)–(2.18) and the condition that $\frac{v_x}{v_{xx}} < 0$, the sign of q_i, $i \in \{1, 2\}$ are related to the signs of κ_1, $-\kappa_2(\mu - r) + \frac{1-\rho_{13}^2}{b_1^2}\alpha_1 - \kappa_3\alpha_2$ and $-\kappa_4(\mu - r) - \kappa_3\alpha_1 + \frac{1-\rho_{12}^2}{b_2^2}\alpha_2$. We will discuss the sign of q_i, $i \in \{1, 2\}$ from the perspective of comparing α_1. Without loss of generality, suppose $\kappa_1 > 0$ (the discussion of the other direction can be obtained in the same way) and set

$$\begin{cases} \varsigma_1 = \dfrac{b_1^2}{(1-\rho_{13}^2)}\left[\kappa_2(\mu - r) + \kappa_3\alpha_2\right], \\ \varsigma_2 = \dfrac{1}{\kappa_3}\left[-\kappa_4(\mu - r) + \dfrac{1-\rho_{12}^2}{b_2^2}\alpha_2\right]. \end{cases} \tag{3.2}$$

Thus, we split the problem into following three cases.

Case A. $\kappa_1 > 0$ and $\kappa_3 > 0$,

For $\varsigma_1 < \varsigma_2$, we have

$$\begin{cases} \text{Case 1}: & \alpha_1 \geq \varsigma_2, & \Rightarrow \hat{q}_1 > 0, \hat{q}_2 \leq 0; \\ \text{Case 2}: & \varsigma_1 < \alpha_1 < \varsigma_2, & \Rightarrow \hat{q}_1 > 0, \hat{q}_2 > 0; \\ \text{Case 3}: & \alpha_1 \leq \varsigma_1, & \Rightarrow \hat{q}_1 \leq 0, \hat{q}_2 > 0. \end{cases}$$

For $\varsigma_1 > \varsigma_2$, we have

$$\begin{cases} \text{Case 1}: & \alpha_1 > \varsigma_1, & \Rightarrow \hat{q}_1 > 0, \hat{q}_2 < 0; \\ \text{Case 2}: & \varsigma_2 \leq \alpha_1 \leq \varsigma_1, & \Rightarrow \hat{q}_1 \leq 0, \hat{q}_2 \leq 0; \\ \text{Case 3}: & \alpha_1 < \varsigma_2, & \Rightarrow \hat{q}_1 < 0, \hat{q}_2 > 0. \end{cases}$$

For $\varsigma_1 = \varsigma_2$, we have

$$\begin{cases} \text{Case 1}: & \alpha_1 > \varsigma_1 = \varsigma_2, & \Rightarrow \hat{q}_1 > 0, \hat{q}_2 < 0; \\ \text{Case 2}: & \alpha_1 = \varsigma_1 = \varsigma_2, & \Rightarrow \hat{q}_1 = 0, \hat{q}_2 = 0; \\ \text{Case 3}: & \alpha_1 < \varsigma_1 = \varsigma_2, & \Rightarrow \hat{q}_1 < 0, \hat{q}_2 > 0. \end{cases}$$

Case B. $\kappa_1 > 0$ and $\kappa_3 < 0$.

For $\varsigma_1 < \varsigma_2$, we have

$$\begin{cases} \text{Case 1}: & \alpha_1 > \varsigma_2, & \Rightarrow \hat{q}_1 > 0, \hat{q}_2 > 0; \\ \text{Case 2}: & \varsigma_1 < \alpha_1 \leq \varsigma_2, & \Rightarrow \hat{q}_1 > 0, \hat{q}_2 \leq 0; \\ \text{Case 3}: & \alpha_1 \leq \varsigma_1, & \Rightarrow \hat{q}_1 \leq 0, \hat{q}_2 < 0. \end{cases}$$

For $\varsigma_1 > \varsigma_2$, we have

$$\begin{cases} \text{Case 1}: & \alpha_1 > \varsigma_1, & \Rightarrow \hat{q}_1 > 0, \hat{q}_2 > 0; \\ \text{Case 2}: & \varsigma_2 < \alpha_1 \leq \varsigma_1, & \Rightarrow \hat{q}_1 \leq 0, \hat{q}_2 > 0; \\ \text{Case 3}: & \alpha_1 \leq \varsigma_2, & \Rightarrow \hat{q}_1 < 0, \hat{q}_2 \leq 0. \end{cases}$$

For $\varsigma_1 = \varsigma_2$, we have

$$\begin{cases} \text{Case 1}: & \alpha_1 > \varsigma_1 = \varsigma_2, \quad \Rightarrow \hat{q}_1 > 0, \hat{q}_2 > 0; \\ \text{Case 2}: & \alpha_1 \le \varsigma_1 = \varsigma_2, \quad \Rightarrow \hat{q}_1 \le 0, \hat{q}_2 \le 0. \end{cases}$$

Case C. $\kappa_1 > 0$ and $\kappa_3 = 0$,

let $\nu = \frac{\kappa_2 b_1^2}{1-\rho_{1d+2}^2}(\mu_1 - r)$, $\beta = \frac{\kappa_4 b_2^2}{1-\rho_{1d+1}^2}(\mu_1 - r)$, we have

$$\begin{cases} \text{Case 1}: & \alpha_1 > \nu, \quad \alpha_2 > \beta, \quad \Rightarrow \hat{q}_1 > 0, \hat{q}_2 > 0; \\ \text{Case 2}: & \alpha_1 \le \nu, \quad \alpha_2 \le \beta, \quad \Rightarrow \hat{q}_1 \le 0, \hat{q}_2 \le 0; \\ \text{Case 3}: & \alpha_1 > \nu, \quad \alpha_2 \le \beta, \quad \Rightarrow \hat{q}_1 > 0, \hat{q}_2 \le 0; \\ \text{Case 4}: & \alpha_1 \le \nu, \quad \alpha_2 > \beta, \quad \Rightarrow \hat{q}_1 \le 0, \hat{q}_2 > 0. \end{cases}$$

We will only discuss **Case A** with $\varsigma_1 < \varsigma_2$ in detail, and other cases can be deduced similarly.

The discussion differs in $U < \frac{\alpha}{r}$ and $U \ge \frac{\alpha}{r}$. So we split the problem into two situations in the next two subsections. Specifically, in Section 3.1.1 when the goal level $U < \frac{\alpha}{r}$, we derive the explicit expressions for the value function and the corresponding optimal investment and proportional reinsurance strategy through classical stochastic control theory. In addition, we will show that it is impossible to realize the goal of reaching the safe level before hitting L under this optimal strategy. In Section 3.1.2 when the target level $U \ge \frac{\alpha}{r}$, we construct an ϵ-optimal strategy so that the wealth can achieve $\frac{\alpha}{r}$ with a maximum probability of J_δ, where δ is uniquely determined by x and ϵ.

3.1.1. The case with $U < \frac{\alpha}{r}$

When $U < \frac{\alpha}{r}$, the corresponding explicit optimal results can be obtained as follows.

Theorem 3.1. *In Case A with* $\varsigma_1 < \alpha_1 < \varsigma_2$*, we have* $\hat{q}_1 > 0, \hat{q}_2 > 0$*. The value function is*

$$V_1(x : L, U) = \frac{(\alpha - rx)^{\frac{u}{r}+1} - (\alpha - rL)^{\frac{u}{r}+1}}{(\alpha - rU)^{\frac{u}{r}+1} - (\alpha - rL)^{\frac{u}{r}+1}}, \tag{3.3}$$

and the optimal strategy is

$$\boldsymbol{\pi}^*(x) = \begin{pmatrix} \pi^* \\ q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} \hat{\pi} \\ \hat{q}_1 \\ \hat{q}_2 \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} \frac{r}{u}\left(\frac{\alpha}{r} - x\right), \tag{3.4}$$

where

$$\begin{aligned} \alpha &= (\eta_1 - \theta_1)a_1 + (\eta_2 - \theta_2)a_2 \ge 0, \\ u &= \frac{1}{2}\boldsymbol{\mu}^\top \boldsymbol{\Omega}^{-1}\boldsymbol{\mu} = \frac{1}{2}\kappa_1[\frac{1-\rho_{23}^2}{\sigma^2}(\mu - r)^2 + \frac{1-\rho_{13}^2}{b_1^2}\alpha_1^2 + \frac{1-\rho_{12}^2}{b_2^2}\alpha_2^2 \\ &\quad - 2(\kappa_2\alpha_1 + \kappa_4\alpha_2)(\mu - r) - 2\kappa_3\alpha_1\alpha_2] > 0. \end{aligned} \tag{3.5}$$

Proof. In this case, $\varsigma_1 < \alpha_1 < \varsigma_2$ means that $\hat{q}_1 > 0, \hat{q}_2 > 0$. According to (2.18) and (2.20) with $\lambda = 0, g = 0$, the value function satisfies

$$(rx - \alpha)v_x - u\frac{v_x^2}{v_{xx}} = 0, \quad \text{for} \quad L < x < U < \frac{\alpha}{r}, \tag{3.6}$$

with the boundary conditions $v(L) = 0$, $v(U) = 1$, which admits a solution of (3.3). It is easy to verify that $v_x > 0$, $v_{xx} < 0$ and v satisfies the Verification Theorem 2.1. So we conclude that $v(x)$ is indeed the value function, and the optimal strategy can be obtained directly from (2.18). □

Remark 3.3 (Safe level unattainable)**.** Note that under the optimal strategy π^*, the wealth process X_t^* satisfies

$$dX_t^* = (\alpha - rX_t^*)\mathrm{d}t + (\alpha - rX_t^*)\left(\frac{\sigma m_0}{u}dW_{1,t} + \frac{b_1 m_1}{u}dW_{2,t} + \frac{b_2 m_2}{u}dW_{3,t}\right), \quad \text{for} \quad t \le T^*, \tag{3.7}$$

where $T^* = \min\{\tau_L^*, \tau_U^*\}$. As we can see from (3.4) and (3.7), when the wealth value approaches $\frac{\alpha}{r}$, the optimal strategy π^* together with the increment of the wealth dX_t^* approaches 0, which indicates that the company prefers to choose a "timid" strategy and this in turn shut off the drift and the variance of the wealth process. Therefore, $\frac{\alpha}{r}$ is an attracting but inaccessible barrier for the process X_t^* since the wealth can never cross from the danger zone to the safe region under the optimal strategy (3.4).

The other two cases with $\varsigma_1 < \varsigma_2$ can be deduced in similar lines by simply changing the notations. Hence, we only discuss one of them in the following theorem.

Theorem 3.2. *In Case A with $\varsigma_1 < \varsigma_2 \le \alpha_1$, the value function is given by*

$$\tilde{V}_1(x : L, U) = \frac{(\alpha - rx)^{\frac{\tilde{u}}{r}+1} - (\alpha - rL)^{\frac{\tilde{u}}{r}+1}}{(\alpha - rU)^{\frac{\tilde{u}}{r}+1} - (\alpha - rL)^{\frac{\tilde{u}}{r}+1}}, \tag{3.8}$$

and the optimal strategy is

$$\boldsymbol{\pi}^*(x) = \begin{pmatrix} \pi^* \\ q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} \tilde{m}_0 \\ \tilde{m}_1 \\ \tilde{m}_2 \end{pmatrix} \frac{r}{\tilde{u}} \left(\frac{\alpha}{r} - x\right), \tag{3.9}$$

where

$$\tilde{u} = \frac{1}{2(1-\rho_{12}^2)}\left[\frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha_1^2}{b_1^2} - 2\frac{\rho_{12}}{\sigma b_1}\alpha_1(\mu - r)\right] > 0, \tag{3.10}$$

and

$$\begin{aligned} \tilde{m}_0 &= \frac{1}{1-\rho_{12}^2}\left[\frac{\mu - r}{\sigma^2} - \frac{\rho_{12}\alpha_1}{\sigma b_1}\right], \\ \tilde{m}_1 &= \frac{1}{1-\rho_{12}^2}\left[-\frac{\rho_{12}(\mu - r)}{\sigma b_1} + \frac{\alpha_1}{b_1^2}\right] > 0, \\ \tilde{m}_2 &= 0. \end{aligned} \tag{3.11}$$

Proof. In this case, we have $\hat{q}_1 > 0, \hat{q}_2 \le 0$. Hence $q_2^* = 0$ and the maximizer of (2.15) is

$$\begin{cases} \tilde{\pi} = -\dfrac{v_x}{v_{xx}}\tilde{m}_0, \\ \tilde{q_1} = -\dfrac{v_x}{v_{xx}}\tilde{m}_1. \end{cases} \tag{3.12}$$

Inserting (3.12) into the generator (2.14) yields

$$0 = (rx - \alpha)v_x - \tilde{u}\frac{v_x^2}{v_{xx}}, \tag{3.13}$$

with the boundary conditions $v(L) = 0$ and $v(U) = 1$. Through typical calculation and Theorem 2.1, we obtain the value function as (3.8). Accordingly, we have

$$\tilde{q_1} = \frac{1}{1-\rho_{12}^2}\left[-\frac{\rho_{12}(\mu - r)}{\sigma b_1} + \frac{\alpha_1}{b_1^2}\right]\frac{r}{\tilde{u}}\left(\frac{\alpha}{r} - x\right).$$

Let $\varsigma_3 = \frac{b_1\rho_{12}}{\sigma}(\mu - r)$. From $\varsigma_1 < \varsigma_2$, we have $\frac{\mu - r}{\sigma}\rho_{13} < \frac{\alpha_2}{b_2}$. Together with $\kappa_1 > 0$ and $\kappa_3 > 0$, we have

$$\varsigma_1 - \varsigma_3 = b_1\left(\frac{\rho_{12}\rho_{13} - \rho_{23}}{1-\rho_{13}^2}\right)\left(\frac{\mu - r}{\sigma}\rho_{13} - \frac{\alpha_2}{b_2}\right) > 0,$$

which implies that $\alpha_1 \ge \varsigma_2 > \varsigma_1 > \varsigma_3$. So we verify that $\tilde{q_1} > 0$, and the optimal strategy is (3.9). ☐

Remark 3.4. Similarly, the safe level $\frac{\alpha}{r}$ is inaccessible under the optimal strategies obtained in Theorem 3.2. Therefore, the value function and the optimal strategy derived by the classical methodology are only applicable when $U < \frac{\alpha}{r}$. Furthermore, as is shown in Remark 2.1, once the wealth exceeds the safe level, the goal level $U(> \frac{\alpha}{r})$ can be achieved almost surely by simply choose a policy of $\boldsymbol{\pi} = (\pi, q_1, q_2) = (0,0,0)$. Thus, for $x < \frac{\alpha}{r}$ and $U \geq \frac{\alpha}{r}$, techniques to determine the optimal strategy and the maximum probability of reaching $U(= \frac{\alpha}{r})$ before hitting L are crucial, which will be discussed in the following subsection.

3.1.2. The case with $U \geq \frac{\alpha}{r}$

As described in Remark 3.3, $\boldsymbol{\pi}^* \to (0,0,0)$ when $x \uparrow \frac{\alpha}{r}$. Intuitively, when the wealth approaches the boundary of the safe region, the company gets increasingly cautious so as not to lose the chances of getting there. However, this behavior in turn shuts off the drift and the volatility term of the wealth process, causing the wealth to never cross the safe level with the strategy derived in (3.4). Inspired by Browne [4], we solve this problem by constructing an ϵ-optimal strategy.

Definition 3.1 (ϵ-optimal strategy)**.** For any $\delta > 0$, define

$$\boldsymbol{\pi}^*_\delta(x) = \begin{cases} \boldsymbol{\pi}^*(x), & x \leq \dfrac{\alpha}{r} - \delta; \\ \boldsymbol{\pi}^*(\dfrac{\alpha}{r} - \delta), & x > \dfrac{\alpha}{r} - \delta, \end{cases} \tag{3.14}$$

in which $\boldsymbol{\pi}^*$ is the optimal strategy as we derived in (3.4). Let $J_\delta\left(x_0; L, \frac{\alpha}{r}\right)$ be the probability of reaching U before hitting $\frac{\alpha}{r}$ under $\boldsymbol{\pi}^*_\delta$, starting from an initial wealth level $x_0 < \frac{\alpha}{r}$. For an $\epsilon > 0$, assume that there is a $\delta > 0$ satisfying

$$J_\delta\left(x_0; L, \frac{\alpha}{r}\right) = V_1\left(x_0; L, \frac{\alpha}{r}\right) - \epsilon,$$

then we call $\boldsymbol{\pi}^*_\delta$ the ϵ- optimal strategy.

Next, we discuss the ϵ-optimal strategy of Case A with $\varsigma_1 < \alpha_1 < \varsigma_2$. So, we have

$$\boldsymbol{\pi}^*_\delta(x) = (m_0, m_1, m_2)^\top \left(\frac{r}{u}\right) \max\left\{\frac{\alpha}{r} - x, \delta\right\}.$$

Substituting $U = \frac{\alpha}{r}$ into (3.3), we obtain

$$V_1\left(x; L, \frac{\alpha}{r}\right) = 1 - \left(\frac{\alpha - rx}{\alpha - rL}\right)^{\frac{u}{r}+1}.$$

The objective is to find a policy satisfying $J_\delta = V_1\left(x_0; L, \frac{\alpha}{r}\right) - \epsilon$ for any $\epsilon > 0$ and $L < x_0 < \frac{\alpha}{r}$. Clearly, the key is to figure out $\delta = \delta(x_0, \epsilon)$.

Define the wealth process under the ϵ-optimal strategy as X_t^δ with drift function $\mu_\delta(x)$ and volatility function $\sigma^2_\delta(x)$ given by

$$\mu_\delta(x) = rx - \alpha + (\mu - r, \alpha_1, \alpha_2), \quad \boldsymbol{\pi}^*_\delta = \max\left\{\alpha - rx, rx - \alpha + 2r\delta\right\},$$

and

$$\begin{aligned}\sigma^2_\delta(x) &= \pi^2_\delta\sigma^2 + q^2_{1\delta}b_1^2 + q^2_{2\delta}b_2^2 + 2q_{1\delta}q_{2\delta}\rho_{23}b_1b_2 + 2q_{1\delta}\pi_\delta\rho_{12}b_1\sigma + 2q_{2\delta}\pi_\delta\rho_{13}b_2\sigma \\ &= \max\left\{\frac{2(\alpha - rx)^2}{u}, \frac{2r^2\delta^2}{u}\right\}.\end{aligned}$$

Referring to Browne [4], the scale density function for this new process is defines by

$$s_\delta(y) = \exp\left\{-\int^y \frac{2\mu_\delta(x)}{\sigma_\delta(x)}\mathrm{d}x\right\}. \tag{3.15}$$

For $y \le \frac{\alpha}{r} - \delta$, we have

$$s_\delta(y) = \exp\left\{-\int^y 2\frac{\alpha - rx}{2\frac{(\alpha-rx)^2}{u}}\mathrm{d}x\right\} = (\alpha - ry)^{\frac{u}{r}};$$

for $y > \frac{\alpha}{r} - \delta$,

$$s_\delta(y) = \exp\left\{\int^{\frac{\alpha}{r}-\delta} \frac{u}{r}\frac{d(\alpha - rx)}{\alpha - rx} - \int_{\frac{\alpha}{r}-\delta}^{y} \frac{2(rx - \alpha + 2r\delta)}{\frac{2r^2\delta^2}{u}}\mathrm{d}x\right\} = (r\delta)^{\frac{u}{r}} e^{\frac{u}{2r}}\sqrt{2\pi}\varphi\left(\frac{y + 2\delta - \frac{\alpha}{r}}{\sqrt{\frac{r\delta^2}{u}}}\right),$$

where $\varphi\left(\frac{y+2\delta-\frac{\alpha}{r}}{\sqrt{\frac{r\delta^2}{u}}}\right) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{(y+2\delta-\frac{\alpha}{r})^2}{\frac{\delta^2}{u}}\right\}$ is the density function of the standard normal distribution.

Therefore, the scale density function can be written as

$$s_\delta(y) = \begin{cases} (\alpha - ry)^{\frac{u}{r}}, & y \le \dfrac{\alpha}{r} - \delta; \\ (r\delta)^{\frac{u}{r}} e^{\frac{u}{2r}}\sqrt{2\pi}\varphi\left(\dfrac{y + 2\delta - \frac{\alpha}{r}}{\sqrt{\frac{r\delta^2}{u}}}\right), & y > \dfrac{\alpha}{r} - \delta. \end{cases} \tag{3.16}$$

As explained in Browne [4], J_δ can be expressed by the scale density function as

$$J_\delta\left(x_0; L, \frac{\alpha}{r}\right) = \frac{\int_L^{x_0} s_\delta(y)\mathrm{d}y}{\int_L^{\frac{\alpha}{r}} s_\delta(y)\mathrm{d}y}.$$

Since we have

$$\int_L^{x_0} s_\delta(y)\mathrm{d}y = \frac{(\alpha - rL)^{\frac{u}{r}+1}}{u + r}V\left(x_0; L, \frac{\alpha}{r}\right),$$
$$\int_L^{\frac{\alpha}{r}} s_\delta(y)\mathrm{d}y = \frac{(\alpha - rL)^{\frac{u}{r}+1}}{u + r}\left[1 - \left(\frac{r\delta}{\alpha - rL}\right)^{\frac{u}{r}+1}\right] + r^{\frac{u}{r}}\delta^{\frac{u}{r}+1}e^{\frac{u}{2r}}\sqrt{\frac{2\pi r}{u}}\left[\Phi\left(2\sqrt{\frac{u}{r}}\right) - \Phi\left(\sqrt{\frac{u}{r}}\right)\right],$$

where Φ is the distribution function of the standard normal distribution. Thus, we get

$$J_\delta\left(x_0; L, \frac{\alpha}{r}\right) = V_1\left(x_0; L, \frac{\alpha}{r}\right)\left[1 + \delta^{\frac{u}{r}+1}\cdot H(u, r, \alpha)\right]^{-1},$$

where

$$H(u, r, \alpha) = \left(\frac{r}{\alpha - rL}\right)^{\frac{u}{r}+1}\left\{(1 + \frac{u}{r})e^{\frac{u}{2r}}\sqrt{\frac{2\pi r}{u}}\left[\Phi\left(2\sqrt{\frac{u}{r}}\right) - \Phi\left(\sqrt{\frac{u}{r}}\right)\right] - 1\right\}.$$

By setting $J_\delta = V_1\cdot\left(1 + \delta^{\frac{u}{r}+1}H\right)^{-1} = V_1 - \epsilon$, we get

$$\delta(x_0, \epsilon) = \left(\frac{\epsilon}{H(u, r, \alpha)[V_1(x_0; L, \frac{\alpha}{r}) - \epsilon]}\right)^{\frac{r}{u+r}}.$$

To summarize, we present the following Theorem 3.3.

Theorem 3.3. *The ϵ-optimal strategy for Case A with $\varsigma_1 < \alpha_1 < \varsigma_2$ is given by*

$$\boldsymbol{\pi}_\delta^*(x) = \begin{cases} \boldsymbol{\pi}^*(x) = (\pi^*, q_1^*, q_2^*)^\top, & L < x \le \dfrac{\alpha}{r} - \delta; \\ \dfrac{r}{u}\left(\dfrac{\epsilon}{H(u, r, \alpha)[V_1(x_0; L, \frac{\alpha}{r}) - \epsilon]}\right)^{\frac{r}{u+r}}(m_0, m_1, m_2)^\top, & x > \dfrac{\alpha}{r} - \delta, \end{cases} \tag{3.17}$$

where

$$\delta(x_0,\epsilon) = \left(\frac{\epsilon}{H(u,r,\alpha)[V_1(x_0;L,\frac{\alpha}{r})-\epsilon]}\right)^{\frac{r}{u+r}},$$

and

$$V_1\left(x_0;L,\frac{\alpha}{r}\right) = 1-\left(\frac{\alpha-rx_0}{\alpha-rL}\right)^{\frac{u}{r}+1}.$$

$\boldsymbol{\pi}^*_\delta(x)$ is an ϵ-optimal strategy for maximizing the probability of reaching the safe level $\frac{\alpha}{r}$ before hitting the lower boundary L with initial wealth level $x_0 \in (L,\frac{\alpha}{r})$.

Following the same lines, the ϵ-optimal strategy for other cases are straightforward. Thus, we present the optimal results for Case A with $\varsigma_1 < \varsigma_2 \le \alpha_1$ directly in the following theorem.

Theorem 3.4. *The ϵ-optimal strategy for Case A with $\varsigma_1 < \varsigma_2 \le \alpha_1$ is given by*

$$\boldsymbol{\pi}^*_\delta(x) = \begin{cases} \boldsymbol{\pi}^*(x) = (\pi^*, q_1^*, q_2^*)^\top, \quad L < x \le \dfrac{\alpha}{r} - \delta; \\ \dfrac{r}{\tilde{u}}\left(\dfrac{\epsilon}{H(\tilde{u},r,\alpha)[\tilde{V}_1(x_0;L,\frac{\alpha}{r})-\epsilon]}\right)^{\frac{r}{\tilde{u}+r}} (\tilde{m}_0,\tilde{m}_1,\tilde{m}_2)^\top, \quad x > \dfrac{\alpha}{r} - \delta, \end{cases} \tag{3.18}$$

where

$$\delta(x_0,\epsilon) = \left(\frac{\epsilon}{H(\tilde{u},r,\alpha)[\tilde{V}_1(x_0;L,\frac{\alpha}{r})-\epsilon]}\right)^{\frac{r}{\tilde{u}+r}},$$

and

$$\tilde{V}_1\left(x_0;L,\frac{\alpha}{r}\right) = 1-\left(\frac{\alpha-rx_0}{\alpha-rL}\right)^{\frac{\tilde{u}}{r}+1}.$$

3.2. Minimize the discounted penalty of bankruptcy

Suppose that there is a penalty M to be paid by the insurer when bankruptcy happens. Thus, it is natural to concern about how to minimize the penalty when the wealth level hits the ruin level.

Set a constant $\lambda > 0$ as the discounted rate, so the penalty of hitting the ruin level L is $Me^{-\lambda\tau_L^{\boldsymbol{\pi}}}$. Clearly, the objective is to minimize $\mathbb{E}_x[e^{-\lambda\tau_L^{\boldsymbol{\pi}}}]$ and we have the value function as

$$V_2(x) = \inf_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}_x[e^{-\lambda\tau_L^{\boldsymbol{\pi}}}]. \tag{3.19}$$

The next steps are to figure out the explicit expressions of the value function and optimal policy which minimize the expected discounted penalty.

Similarly, we focus on **Case A** with $\varsigma_1 < \varsigma_2$ in the context below. Other cases can be deduced in the same way.

Theorem 3.5. *In Case A with $\varsigma_1 < \alpha_1 < \varsigma_2$, the value function is given by*

$$V_2(x) = \left(\frac{\alpha-rx}{\alpha-rL}\right)^{\gamma^+}, \quad \text{for} \quad L \le x < \frac{\alpha}{r}, \tag{3.20}$$

and the optimal strategy is

$$\boldsymbol{\pi}^*(x) = \begin{pmatrix}\pi^*\\ q_1^*\\ q_2^*\end{pmatrix} = \begin{pmatrix}m_0\\ m_1\\ m_2\end{pmatrix}\left(\frac{1}{1-\gamma^+}\right)\left(\frac{\alpha}{r}-x\right), \tag{3.21}$$

where m_0, m_1, m_2 are defined in (2.17), $\gamma^+ = \frac{1}{2r}\left[(u+\lambda+r)+\sqrt{E}\right]$, $E = (u+\lambda+r)^2 + 4ru$, $u > 0$ and $\alpha > 0$ are defined in (3.5).

Proof. See Appendix B. ☐

It is not difficult to get the optimal results for the other case following the same methodology.

Theorem 3.6. *In Case A with $\varsigma_1 < \varsigma_2 \le \alpha_1$, the value function is*

$$\tilde{V}_2(x) = \left(\frac{\alpha - rx}{\alpha - rL}\right)^{\tilde{\gamma}^+}, \quad \text{for} \quad L \le x < \frac{\alpha}{r}, \tag{3.22}$$

and the optimal strategy is

$$\boldsymbol{\pi}^*(x) = \begin{pmatrix} \pi^* \\ q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} \tilde{m}_0 \\ \tilde{m}_1 \\ \tilde{m}_2 \end{pmatrix} \left(\frac{1}{1-\tilde{\gamma}^+}\right)\left(\frac{\alpha}{r} - x\right), \tag{3.23}$$

where $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3$ are defined in (3.11), and $\tilde{u} > 0$ is defined in (3.10), $\tilde{\gamma}^+ = \frac{1}{2r}\left[(\tilde{u} + \lambda + r) + \sqrt{\tilde{E}}\right]$, $\tilde{E} = (\tilde{u} + \lambda - r)^2 + 4r\tilde{u}$.

4. Optimal results for growth problems

Suppose that the safe level $\frac{\alpha}{r}$ has been achieved, which means that ruin can be avoided with certainty. Thus, the insurer concerns about the time to meet a target level or the reward of reaching the target. In the following subsections, we will discuss two problems including minimizing the expected time to reach the goal and maximizing the expected reward once the target level is achieved.

4.1. Minimize the expected time to reach a goal

Let the initial wealth $X_0 = x$ satisfy $\frac{\alpha}{r} < x < U$. Define a stopping time $\tau_U^{\boldsymbol{\pi}} := \inf\{t > 0, X_t^{\boldsymbol{\pi}} = U\}$, and the objective is described as

$$V_3(x) = \inf_{\boldsymbol{\pi} \in \mathcal{A}} \mathbb{E}_x[\tau_U^{\boldsymbol{\pi}}]. \tag{4.1}$$

We will still focus on **Case A** with $\varsigma_1 < \varsigma_2$ and find the explicit expressions of the value function and optimal policy regarding problem (4.1).

Theorem 4.1. *In Case A with $\varsigma_1 < \alpha_1 < \varsigma_2$, the value function for the problem (4.1) is*

$$V_3(x) = \frac{1}{r+u} \ln\left(\frac{rU - \alpha}{rx - \alpha}\right), \quad \text{for} \quad \frac{\alpha}{r} < x < U, \tag{4.2}$$

and the optimal strategy is

$$\boldsymbol{\pi}^*(x) = \begin{pmatrix} \pi^* \\ q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} \left(x - \frac{\alpha}{r}\right), \tag{4.3}$$

where m_0, m_2, m_3 are defined in (2.17), $u > 0$ and $\alpha > 0$ are defined in (3.5).

Proof. See Appendix C. ☐

Theorem 4.2. *In Case A with $\varsigma_1 < \varsigma_2 \le \alpha_1$, the value function is given by*

$$\tilde{V}_3(x) = \frac{1}{r+\tilde{u}} \ln\left(\frac{rU - \alpha}{rx - \alpha}\right), \quad \text{for} \quad \frac{\alpha}{r} < x < U, \tag{4.4}$$

and the optimal strategy is

$$\boldsymbol{\pi}^*(x) = \begin{pmatrix} \pi^* \\ q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} \tilde{m}_0 \\ \tilde{m}_1 \\ \tilde{m}_2 \end{pmatrix} \left(x - \frac{\alpha}{r}\right), \tag{4.5}$$

where $\tilde{m}_0, \tilde{m}_1, \tilde{m}_2$ are defined in (3.11) where we can see that $\tilde{m}_2 = 0$. $\tilde{u} > 0$ is defined in (3.10).

Other cases can be analyzed in similar lines. So we dismiss them here.

4.2. Maximize the expected discounted reward of reaching a goal

In this subsection, we concern about how to maximize the expected reward of reaching the goal. The objective is thereby

$$V_4(x) = \sup_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}_x \left[e^{-\lambda \tau_U^{\boldsymbol{\pi}}} \right]. \tag{4.6}$$

Again, we focus on **Case A** with $\varsigma_1 < \varsigma_2$ and give the associated optimal results in the following theorems.

Theorem 4.3. *In Case A with $\varsigma_1 < \eta_1 < \varsigma_2$, the value function is in the form of*

$$V_4(x) = \left(\frac{\alpha - rx}{\alpha - rU}\right)^{\gamma^-}, \quad \text{for} \quad \frac{\alpha}{r} < x < U, \tag{4.7}$$

and the optimal strategy is

$$\boldsymbol{\pi}^*(x) = \begin{pmatrix} \pi^* \\ q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} \left(\frac{1}{1-\gamma^-}\right)\left(x - \frac{\alpha}{r}\right), \tag{4.8}$$

where m_0, m_1, m_2 are defined in (2.17), and $\tilde{\gamma}^- = \frac{1}{2r}\left[(u+\lambda+r) - \sqrt{E}\right]$, $E = (u+\lambda-r)^2 + 4ru$. Also, $u > 0$ and $\alpha > 0$ are defined in (3.5).

Proof. See Appendix D. □

Along the same lines, by modifying the related parameters m_0, m_1, m_2, u and γ, we can make a conclusion for the other case in the following theorem.

Theorem 4.4. *In Case A with $\varsigma_1 < \varsigma_2 \le \alpha_1$, the value function for the problem (4.6) is given by*

$$\tilde{V}_4(x) = \left(\frac{\alpha - rx}{\alpha - rU}\right)^{\tilde{\gamma}^-}, \quad \text{for} \quad \frac{\alpha}{r} < x < U, \tag{4.9}$$

and the associated optimal strategy is

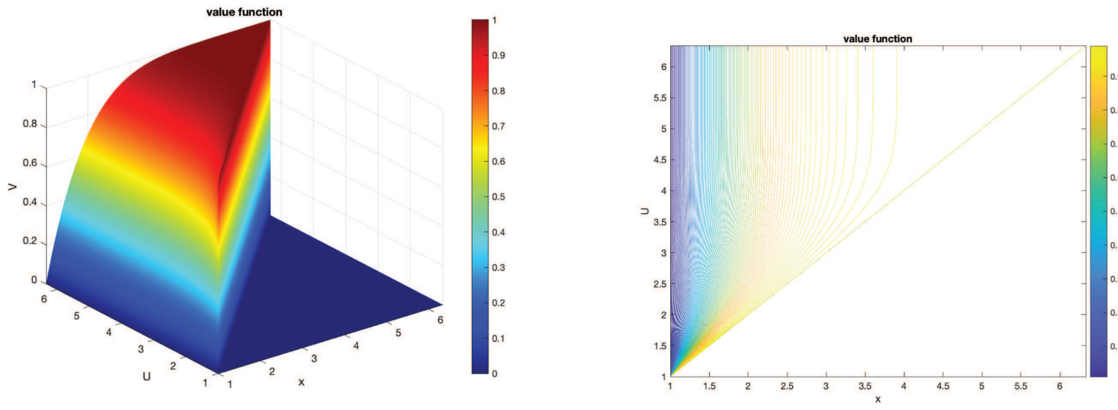
$$\boldsymbol{\pi}^*(x) = \begin{pmatrix} \pi^* \\ q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} \tilde{m}_0 \\ \tilde{m}_1 \\ \tilde{m}_2 \end{pmatrix} \left(\frac{1}{1-\tilde{\gamma}^-}\right)\left(x - \frac{\alpha}{r}\right), \tag{4.10}$$

where $\tilde{m}_0, \tilde{m}_1, \tilde{m}_2$ are defined in (3.11), and $\tilde{\gamma}^- = \frac{1}{2r}\left[(\tilde{u}+\lambda+r) - \sqrt{\tilde{E}}\right]$, $\tilde{u} > 0$ is defined in (3.10), $\tilde{E} = (\tilde{u}+\lambda-r)^2 + 4r\tilde{u}$.

5. Numerical example

In this section, some numerical examples are given to illustrate the influence of typical parameters on the value function and optimal strategies. We first analyze the optimal results in Section 3.1.1, and also the common shock influence on q_i^*, $i \in \{1, 2\}$. Then we focus on the analysis of the ϵ-optimal strategy in Section 3.1.2.

Assume that the claim size of two classes of business Y_1 and Y_2 follow the exponential distribution with parameters 3 and 4. In the following examples, the values of some parameters are given as: $r = 0.05, \mu = 0.1, \sigma = 0.2, \rho_{12} = 0.3, \rho_{13} = 0.4, \zeta = 2, \zeta_1 = 3, \zeta_2 = 4, L = 1, \theta_1 = \theta_2 = 0.2, \eta_1 = \eta_2 = 0.3$. By calculation, we obtain $a_1 = 5/3, b_1 = 10/9, a_2 = 3/2, b_2 = 3/4$. Thus, we get $u = -0.2431, , m_0 = -0.4445, m_1 = 0.3358, m_2 = 0.7566$, which implies that $\pi^* < 0$ and $q_1^* > 0, q_2^* > 0$. These results correspond with our expectation in Case A with $\varsigma_1 < \alpha_1 < \varsigma_2$.



(a) (b)

FIGURE 1. Influence of U and x on value function.

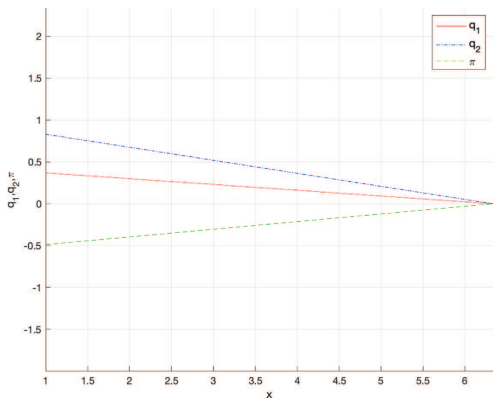


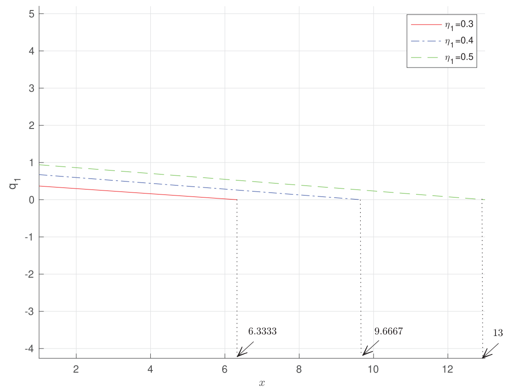
FIGURE 2. Influence of x on optimal strategies.

5.1. Impact of parameters on the value function and optimal strategies

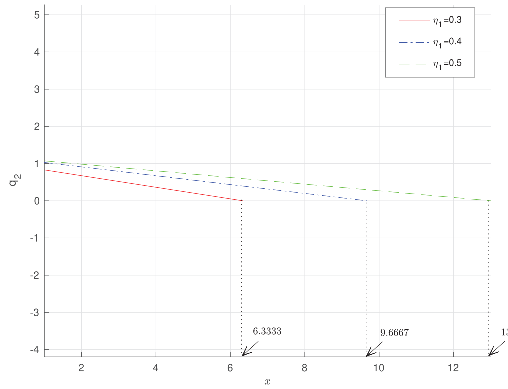
Example 5.1. Based on the results of Theorem 3.1, we illustrate the influence of U and x on the value function and the optimal strategy. Set the initial wealth level x satisfying $L < x < U < \alpha/r$. In this case, $\kappa_1 > 0$, $\kappa_3 > 0$, $\varsigma_1 < \alpha_1 < \varsigma_2$ and $u > 0$. The safe level in this example is $\alpha/r = 6.3333$. The results are shown in Figures 1 and 2.

From Figures 1a and 1b, we can see that fixing $U < \alpha/r = 6.3333$, the value function is a concave and increasing function regarding to the initial wealth level $x < U$. In particular, fixing U in Figure 1b, it is clear that a greater value of x yields a greater value of V. This coincides with our intuition that the more initial wealth the company has, the less risk of ruin it faces.

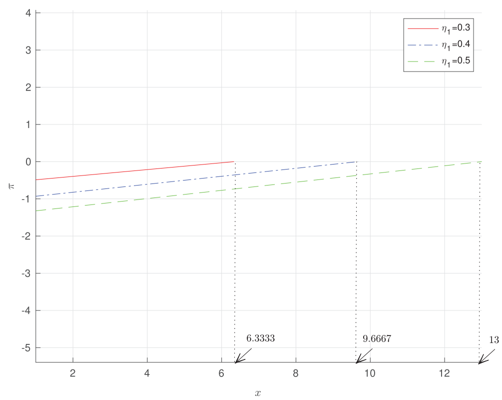
From Figure 2, we can see that as x increases to $\alpha/r = 6.3333$, the optimal strategy approaches 0 which indicates that the company invests less and less capital into the financial market and retains less and less shares of each claim. Practically, this timid strategy is reasonable because the company does not want to lose the chances of crossing the safe level when $x \uparrow \alpha/r$. However, this policy in turn shuts off the drift and the volatility terms of the wealth process so that it would never cross the barrier and reach the safe region. Therefore, when



(a)



(b)



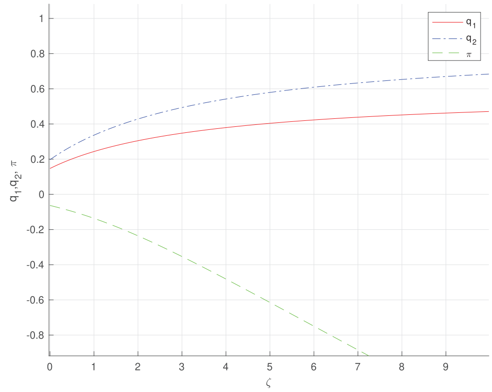
(c)

FIGURE 3. The influence of η_1 on the optimal strategies. (a) q_1^*. (b) q_2^*. (c) π^*.

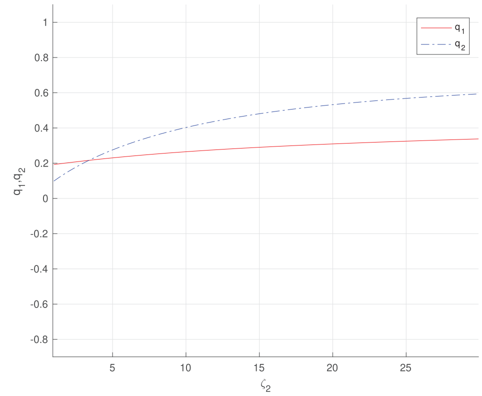
the target goal U is lager than the safe level, we need to discuss an appropriate strategy to realize the goal of reaching the safe region before hitting L, which will be shown in Section 5.2.

Example 5.2. In this example, we present the influence of the safety loading η_1 on the optimal strategies. Fix $\eta_2 = 0.3$ and consider three cases of $\eta_1 = 0.3, \eta_1 = 0.4, \eta_1 = 0.5$. The safe level becomes $6.3333, 9.6667$ and 13 respectively. The optimal investment and proportional reinsurance strategies are given in Figure 3.

From Figures 3a and 3b, we can see that q_1 and q_2 decrease as x increases, and the greater value of η_1, the greater value of the safe-level α/r. Fixing the initial wealth x, it can be seen that q_1 and q_2 increase as η_1 increases. This corresponds with our expectation because the larger value of the safe loading yields the higher payment of the reinsurance premium. Thus, the insurance company would rather retain more claims and transfer less to the reinsurance company. From Figure 3c, we can see that the optimal investment strategy increases as x increases, which means that the insurance company sells less risky asset when the initial wealth x increases. Furthermore, given a fixed initial wealth level x, it is reasonable to see that π decreases when the safe loading increases because the company needs to sell more risky asset to complement a higher reinsurance premium and to ensure that they can undertake more claims.



(a)



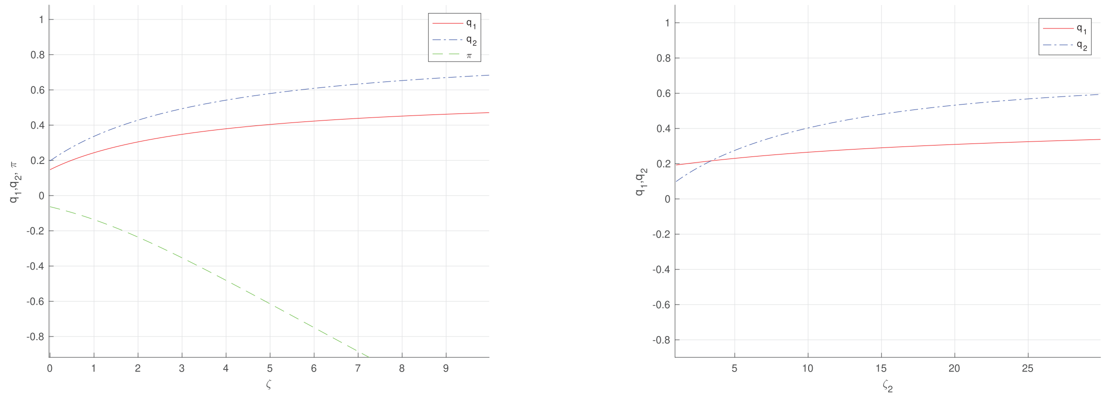
(b)

FIGURE 4. Sensitivity of optimal strategies on the common shock.

Example 5.3 (Common shock influence)**.** In this example, we investigate the impact of common shock on the optimal strategies. Fix the initial wealth $x = 3$. The common shock influence comes from ρ_{23}, which equals $\frac{\zeta E[Y_1]E[Y_2]}{\sqrt{(\zeta_1+\zeta)E[(Y_1)^2](\zeta_2+\zeta)E[(Y_2)^2]}}$. We can see from this formula that the intensity parameter ζ of the common Poisson counting process plays the role of common shock influence. This result will be shown in Figures 4a and 4b as below. (In Fig. 4a, fix $\zeta_1 = 3$, $\zeta_2 = 4$, $\mu = 0.1$, $\sigma = 0.2$; in Fig. 4b, fix $\zeta = \zeta_1 = 4$, $\mu = \sigma = 0.2$.)

Combining the insurance background with the above figures, we have the following conclusions:

– From the formula of ρ_{23}, we see that as ζ increases (fixing all other parameters), ρ_{23} increases to $\frac{E[Y_1]E[Y_2]}{\sqrt{E[(Y_1)]^2E[(Y_2)^2]}}$, meaning that the common shock influence enhances. This case is shown in Figure 4a. On the other hand, as ζ_j, $j \in \{1, 2\}$ increases (fixing all other parameters), ρ_{23} decreases, meaning that the common shock influence weakens. We present this case in Figure 4b using ζ_2 as an illustration.
– It is shown in Figure 4a that as ζ increases, q_j, $(j \in \{1, 2\})$ increases. It means that when the two insurance claims get highly correlated, the insurance company tends to increase the rentention level of both claims, transferring less proportion to the reinsurance company. This is because as ζ gets higher, the volatility $b_j^2 = (\zeta_j + \zeta)E[(Y_j)^2]$, $j \in \{1, 2\}$, of the diffusion term of both claim processes increases, indicating that the risk of both claims increases, and thus the reinsurance premium gets higher. Note that the objective here is to maximize the probability of reaching a goal before ruin, which means that comparing with risks, the profit is what matters the decisions. Therefore, it is reasonable that the insurance company prefers to maintain more proportion of claim loss for the sake of risk control. Also, we see that the increase of ζ leads to the decrease of π. The company adopts a bolder strategy (selling more risky asset to gain certain capital circulation) when facing more insurance risk.
– On the other hand, Figure 4b shows that the growth of ζ_2 entails a stronger performance of the growth of q_2 than q_1. Since the growth of ζ_2 weakens the common shock influence, it makes sense that ζ_2 has more impact on q_2 than on q_1. This phenomenon matches our intuition since the insurance company should take more actions on q_2 than on q_1 when the risk of the second claim increases, and thus further confirms the influence of the common shock.



(a) (b)

FIGURE 5. Suboptimal value function and strategies.

5.2. Suboptimal strategies

As shown in Figure 2, when $x \uparrow \frac{\alpha}{r}$, we have $\pi^* \to 0$. Therefore, it is important to discuss the ϵ-optimal strategies which make the wealth process cross the safe level before ruin with positive possibility. We discuss numerical simulations for the suboptimal strategies in this subsection.

Example 5.4. In this example, the suboptimal value function J_δ and suboptimal strategy π^*_δ are illustrated in Figure 5. For convenience, we set $L = 0$. Assume $\epsilon = 0.1$, so $J_\delta(x_0) = V(x_0; 0, \alpha/r) - \epsilon$. The first figure shows the relation between J_δ and x_0; the second figure shows the relation between the optimal ϵ-strategies and x, given the starting initial wealth value $x_0 = 2$.

From Figure 5a, we can verify that J_δ is an increasing function w.r.t. x and the asymptote is $J_\delta = 0.9$. This is a nature consequence since $\epsilon = 0.1$. Figure 5b shows that the ϵ-optimal strategies $q_{1\delta}$, $q_{2\delta}$ and π_δ do not get to 0 but remain in constant levels when x approaches the safe level $\alpha/r = 6.3333$. Although the suboptimal strategy π^*_δ is quite timid near the safe level, the drift and the volatility terms of the wealth process are not shut off. Thus, α/r is attainable under this suboptimal strategy.

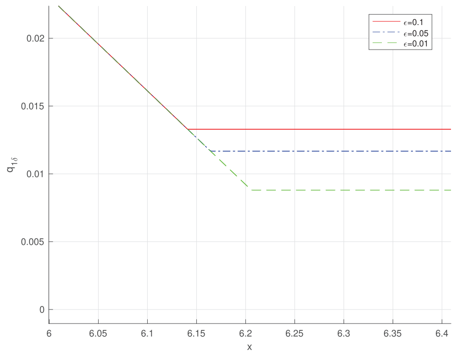
Example 5.5. Fix other parameters , we discuss the influence of ϵ on the suboptimal strategies with cases of $\epsilon = 0.1, \epsilon = 0.05$ and $\epsilon = 0.01$ respectively, given the starting initial wealth value $x_0 = 2$. The results are shown in Figure 6.

It is readily to observe from $J_\delta(x_0) = V(x_0; 0, \alpha/r) - \epsilon$ that J_δ gets greater when ϵ gets smaller. Meanwhile, from Figures 6a and 6b, we can see that the smaller of the value of ϵ, the lower of the constant part of $q_{1\delta}$ and $q_{2\delta}$. It makes sense because the company retains less proportion of the claims if it has greater probability of realizing the goal. Figure 6c shows that the level of the constant part of the suboptimal investment strategy gets higher as ϵ decreases, which is reasonable because the insurance company would sell less risky asset to cover the claim.

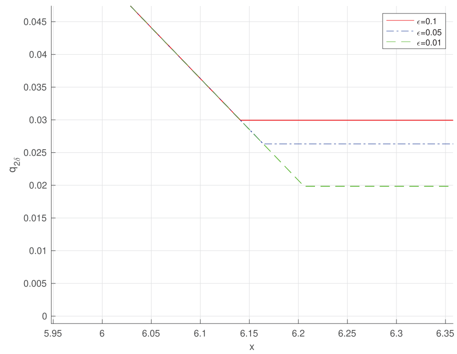
6. FURTHER DISCUSSION AND CONCLUSION

6.1. Futher discussion

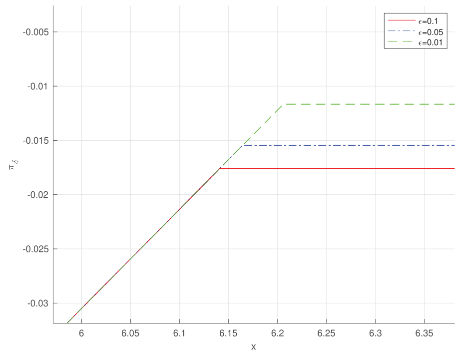
For the future research, there are several appealing directions to explore and investigate. See for example, in Sections 3 and 4, we constrain the reinsurance retention level q_1 and q_2 to be nonnegative but not within $[0, 1]$.



(a)



(b)



(c)

FIGURE 6. The influence of ϵ on optimal strategies. (a) $q^*_{1\delta}$. (b) $q^*_{2\delta}$. (c) π^*_δ.

That is because adding $[0,1]$ constraint involves solving a Free Boundary Problem system. In the problem of maximizing the probability of hitting the goal before ruin, we derived that $\hat{\boldsymbol{\pi}} = -\frac{v_x}{v_{xx}}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu} = -\frac{2(rx-\alpha)}{\boldsymbol{\mu}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}$. Since $-\frac{2(rx-\alpha)}{\boldsymbol{\mu}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}} > 0$, whether q_i, $i \in \{1,2\}$ lies in $[0,1]$ depends on the value of $\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}$. For the situation with only one risky asset in the financial market ($d=1$), let

$$\boldsymbol{\Omega}^{-1}\boldsymbol{\mu} = \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}.$$

Then $0 \le q_i \le 1 \Leftrightarrow 0 \le -\frac{2(rx-\alpha)}{\boldsymbol{\mu}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}\beta_{i+1} \le 1$. In the case of $\beta_{i+1} > 0$, $0 \le q_i \le 1 \Leftrightarrow \frac{\alpha}{r} - \frac{\boldsymbol{\mu}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}{2r\beta_{i+1}} \le x \le \frac{\alpha}{r}$. Note that q_i decreases w.r.t. x when $\beta_{i+1} > 0$ and vice versa. Considering all of combinations of the cases, we come to the most general situation with $\beta_{i+1} > 0$ consisting of three cases to be considered: $q_1^* = q_2^* = 1$; $q_1^* \in [0,1], q_2^* = 1$; $q_1^* = \hat{q}_1, q_2^* = \hat{q}_2$. Substitute the corresponding cases into the HJB equation (2.19), and transfer the problem into a Free Boundary Problem, we come up with a nonlinear ODE system, which is a primary difficulty at present. The cases we mentioned here are just with the condition of $\beta_{i+1} > 0$, $i \in \{1,2\}$. The other conditions also need to be fully discussed. Once this is figured out, the discussion under one of the four objectives in this paper will be enough to be presented as a separate paper. We are working on this and hope to get a satisfied result in the future soon.

In addition, it is meaningful to add model uncertainty and ambiguity aversion into the model and discuss the robust optimal problem. Furthermore, since the risk-less asset is always set with a constant interest rate in goal-reaching problems, it will be much more general if it can be modified as a stochastic process, like in the regime-switching framework. All these problems will be more challenging, but also more meaningful and realistic to be discussed.

6.2. Conclusion

In this paper, we discuss the optimal investment and proportional reinsurance strategies regarding goal-reaching problems for an insurance company. We investigate two dependent classes of insurance business with common shock. There are four objectives regarding the survival and growth problems: maximizing the probability of reaching the safe region before hitting the lower bound; minimizing the expected discounted penalty of bankruptcy; minimizing the expected time to reach a goal; maximizing the expected discounted reward of reaching a goal. Under the multidimensional financial market framework, we derive the Hamilton–Jacobi–Bellman equation to the general problem and give the detailed analysis and optimal results to the model in one-dimensional financial market. More importantly, we solve the dilema that the safe level can never be achieved under the strategy evolved by classical methodology by constructing an ϵ-optimal strategy, so that the wealth can achieve the safe region with positive probability. In addition, we investigate the explicit expressions of optimal results in several cases to ensure that the reinsurance proportions are nonnegative. Finally, numerical examples are presented to analyse those results in detail.

Appendix A. Proof of Theorem 2.1

Proof. We complete the proof in two steps by showing $v \geq V$ and $v \leq V$.

Step 1. Given any $\boldsymbol{\pi} = (\pi_{1,t}, ..., \pi_{d,t}, q_{1,t}, q_{2,t})^{\top}_{t\geq 0} \in \mathcal{A}$ and $(X_t)_{t\geq 0}$ be the corresponding wealth process given in (2.4). Define a non-decreasing sequence of stopping times $\{\tau_n\}_{n=1}^{\infty}$ by

$$\tau_n = \inf\left\{ t \geq 0 : \int_0^t v_x^2(X_s)\left(\sum_{j=1}^{2} q_{j,s}^2 b_j^2 + \sum_{i=1}^{d} \pi_{i,s}^2\sigma_i^2\right) \mathrm{d}s \geq n \right\}, \quad n \geq 1.$$

Then for $n \to \infty$, $\tau \wedge \tau_n \to \tau$. Applying *Itô*'s formula to $e^{-\int_0^t \lambda(X_s)\mathrm{d}s} v(X_t)$ for $0 \leq t \leq \tau \wedge \tau_n$ and adding $\int_0^{\tau\wedge\tau_n} g(X_t) e^{-\int_0^t \lambda(X_s)\mathrm{d}s}\mathrm{d}t$ on both sides yield

$$\begin{aligned} &e^{-\int_0^{\tau\wedge\tau_n} \lambda(X_s)\mathrm{d}s} v(X_{\tau\wedge\tau_n}) + \int_0^{\tau\wedge\tau_n} g(X_t) e^{-\int_0^t \lambda(X_s)\mathrm{d}s}\mathrm{d}t \\ &= v(x) + \int_0^{\tau\wedge\tau_n} e^{-\int_0^t \lambda(X_s)\mathrm{d}s}\left[-\lambda(X_t)v(X_t) + \mathcal{L}_{\boldsymbol{\pi}} v(X_t) + g(X_t)\right]\mathrm{d}t \\ &\quad + \int_0^{\tau\wedge\tau_n} e^{-\int_0^t \lambda(X_s)\mathrm{d}s}\left[v_x(X_t)\left(\sum_{i=1}^{d} \pi_{i,t}\sigma_i dW_{i,t} + \sum_{j=1}^{2} q_{j,t} b_j dW_{d+j,t}\right)\right]. \end{aligned} \tag{A.1}$$

By taking the conditional expectation of both sides, the last term of the RHS vanishes because of the definition of τ_n. Let $n \to \infty$, $\tau \wedge \hat{\tau}_n \to \tau$ so we get

$$\begin{aligned} &\mathbb{E}_x\left[e^{-\int_0^{\tau} \lambda(X_s)\mathrm{d}s} v(X_\tau) + \int_0^{\tau} g(X_t) e^{-\int_0^t \lambda(X_s)\mathrm{d}s}\mathrm{d}t \right] \\ &= v(x) + \mathbb{E}_x\left[\int_0^{\tau} e^{-\int_0^t \lambda(X_s)\mathrm{d}s}\left[-\lambda(X_t)v(X_t) + \mathcal{L}_{\boldsymbol{\pi}} v(X_t) + g(X_t)\right]\mathrm{d}t\right]. \end{aligned} \tag{A.2}$$

From condition 2, $v(X_\tau) = h(X_\tau)$. From condition 3, the second term of the RHS is non-positive. Hence

$$v(x) \geq \mathbb{E}_x \left[e^{-\int_0^\tau \lambda(X_s)\mathrm{d}s} h(X_\tau) + \int_0^\tau g(X_t) e^{-\int_0^t \lambda(X_s)\mathrm{d}s} \mathrm{d}t \right],$$

for all $\boldsymbol{\pi} \in \mathcal{A}$ and $x \in (L, U)$. Therefore, take supremum on both sides yields $v(x) \geq V(x)$.

Step 2. Replacing $\boldsymbol{\pi}(X_t)$ in the first step with $\boldsymbol{\pi}^*(X_t^*)$, we have

$$v(x) = \mathbb{E}_x \left[e^{-\int_0^\tau \lambda(X_s^*)\mathrm{d}s} h(X_\tau^*) + \int_0^\tau g(X_t^*) e^{-\int_0^t \lambda(X_s^*)\mathrm{d}s} \mathrm{d}t \right],$$

which implies that $v(x) \leq V(x)$.

□

Appendix B. Proof of Theorem 3.5

Proof. This objective deals with a maximization problem. We have

$$V_2(x) = -\sup_{\boldsymbol{\pi} \in \mathcal{A}} \left\{ -\mathbb{E}_x \left[e^{-\lambda \tau_L^{\boldsymbol{\pi}}} \right] \right\}.$$

Thus, let $\lambda(x) = \lambda, g(x) = 0$ and $h(L) = -1$. Substitute $-V_2$ into (2.19) and revert it back to V_2, we get the HJB equation as follows

$$(rx - \alpha)v_x - u\frac{v_x^2}{v_{xx}} - \lambda v = 0, \quad \text{for} \quad L < x < \frac{\alpha}{r} \tag{B.1}$$

with boundary condition $v(L) = 1$ and $v_x < 0, v_{xx} > 0$. There are two solutions to the non-linear second-order differential equation (B.1) with the form of $C(\alpha - rx)^{\gamma^+}$ and $D(\alpha - rx)^{\gamma^-}$, where $C = (\alpha - rL)^{-\gamma^+}$ and $D = (\alpha - rL)^{-\gamma^-}$ are two constants obtained by the boundary condition $v(L) = 1$. γ^+ and γ^- are roots of the quadratic equation

$$\tilde{Q}(\gamma) = \gamma^2 r - \gamma(u + \lambda + r) + \lambda = 0.$$

Thus, we have

$$\gamma^+ = \frac{1}{2r}\left[(u + \lambda + r) + \sqrt{E}\right],$$

and

$$\gamma^- = \frac{1}{2r}\left[(u + \lambda + r) - \sqrt{E}\right],$$

where the discriminant $E = (u+\lambda-r)^2+4ru$ is positive clearly. To determine which root satisfies the conditions $v_x < 0$ and $v_{xx} > 0$, we need to verify the sign of both γ^+ and γ^-. From the quadratic equation above, we can see that $\gamma^+ \cdot \gamma^- = \frac{\lambda}{r} > 0$. Since $\gamma^+ > 0$ is obvious from its expression, both γ^+ and γ^- are positive constants. In addition, from

$$\gamma^+ - 1 = \frac{1}{2r}\left[(u + \lambda - r) + \sqrt{(u + \lambda - r)^2 + 4ru}\right] > 0,$$

we have $\gamma^+ > 1$; from

$$\gamma^- - 1 = \frac{1}{2r}\left[(u + \lambda - r) - \sqrt{(u + \lambda - r)^2 + 4ru}\right] < 0,$$

we have $\gamma^- < 1$. Thus v_{xx} becomes positive when substituting γ^+. Furthermore, it is easy to check that $V_2(x)$ satisfies the Theorem 2.1 and is indeed the value function. The optimal policy can be obtained by (2.18). □

Appendix C. Proof of Theorem 4.1

Proof. Since $V_3(x) = -\sup_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}_x[-\tau_U^{\boldsymbol{\pi}}]$ in this case, we substitute $-V_3, \lambda(x) = 0, g(x) = -1$, and $h(U) = 0$ into (2.19) and revert it back to V_3. Then, we get

$$(rx-\alpha)v_x - u\frac{v_x^2}{v_{xx}} + 1 = 0, \quad \text{for} \quad \frac{\alpha}{r} < x < U \tag{C.1}$$

with boundary condition $v(U) = 0$. In this problem, note that $v_x < 0, v_{xx} > 0$. The value functions and the optimal policy are derived in (4.2) and (4.3), respectively. These optimal results satisfy the Verification Theorem 2.1 and the proof is readily to be found in Browne [4]. □

Appendix D. Proof of Theorem 4.3

Proof. The proof is identical to the Theorem 3.5. More simply, we directly substitute $\lambda(x) = \lambda$, $g(x) = 0$ and $h(U) = 1$ into (2.19), and then we get

$$(rx-\alpha)v_x - u\frac{v_x^2}{v_{xx}} - \lambda v = 0, \quad \text{for} \quad \frac{\alpha}{r} < x < U \tag{D.1}$$

with boundary condition $v(U) = 1$. In addition, $v_x > 0, v_{xx} < 0$. In this case, the two solutions to the non-linear second-order differential equation (D.1) are $C(\alpha - rx)^{\gamma^+}$ and $D(\alpha - rx)^{\gamma^-}$, where $C = (\alpha - rU)^{-\gamma^+}$ and $D = (\alpha - rU)^{-\gamma^-}$ are two constants obtained by the boundary condition $v(U) = 1$. γ^+ and γ^- are roots of the quadratic equation

$$\tilde{Q}(\gamma) = \gamma^2 r - \gamma(u + \lambda + r) + \lambda = 0.$$

We have verified previously that the root $\gamma^- < 1$, which satisfies the conditions $v_x > 0$ and $v_{xx} < 0$. Furthermore, it is easy to check that $V_4(x)$ satisfies the Theorem 2.1 and is indeed the value function of the problem (4.6). The optimal strategy can be obtained by (2.18). □

Acknowledgements. This research of Zhibin Liang and Shida Duan was supported by the National Natural Science Foundation of China (Grant No. 12071224).

References

[1] L.E. Dubins and L.J. Savage, How to Gamble if You Must: Inequalities for Stochastic Processes, 1965 edition. McGraw-Hill, New York (1965). 1976 edition. Dover, New York (1976).

[2] V.C. Pestien and W.D. Sudderth, Continuous-time red and black: How to control a diffusion to a goal. *Math. Oper. Res.* **10** (1985) 599–611.

[3] S. Browne, Optimal investment policies for a firm with random risk process: Exponential utility and minimizing the probability of ruin. *Math. Oper. Res.* **20** (1995) 937–958.

[4] S. Browne, Survival and growth with a liability: Optimal portfolio strategies in continuous time. *Math. Oper. Res.* **22** (1997) 468–493.

[5] S. Browne, Reaching goals by a deadline: Digital options and continuous-time active portfolio management. *Adv. Appl. Probab.* **31** (1999) 551–577.

[6] S. Browne, Beating a moving target: Optimal portfolio strategies for outperforming a stochastic benchmark. *Finance Stoch.* **3** (1999) 275–294.

[7] I. Karatzas, Adaptive control of a diffusion to a goal, and a parabolic Monge-Ampre-type equation. *Asian J. Math* **1** (1997) 295–313.

[8] X. Liang and V.R. Young, Reaching a bequest goal with life insurance. *ASTIN Bull.: J. IAA* **50** (2020) 187–221.

[9] E. Bayraktar and V.R. Young, Optimally investing to reach a bequest goal. *Insur.: Math. Econ.* **70** (2016) 1–10.

[10] E. Bayraktar, S.D. Promislow and V.R. Young, Purchasing life insurance to reach a bequest goal. *Insur.: Math. Econ.* **58** (2014) 204–216.

[11] E. Bayraktar, S.D. Promislow and V.R. Young, Purchasing term life insurance to reach a bequest goal while consuming. *SIAM J. Financial Math.* **7** (2016) 183–214.

[12] X. Han, Z. Liang, K. Yuen and Y. Yuan, Minimizing the probability of absolute ruin under ambiguity aversion. *Appl. Math. Optim.* **84** (2021) 2495–2525.
[13] D. Li and V.R. Young, Optimal reinsurance to minimize the probability of ruin under ambiguity. *Insur.: Math. Econ.* **87** (2019) 143–152.
[14] H. Schmidli, On minimizing the ruin probability of investment and reinsurance. *J. Appl. Probab.* **12** (2002) 890–907.
[15] D. Promislow and V.R. Young, Minimizing the probability of ruin when claims follow Brownian motion with drift. *N. Am. Actuar. J.* **9** (2005) 109–128.
[16] S. Luo, Ruin minimization for insurers with borrowing constraints. *N. Am. Actuar. J.* **12** (2008) 143–174.
[17] H. Yener, Maximizing survival, growth and goal reaching under borrowing constraints. *Quant. Finance* **15** (2015) 2053–2065.
[18] X. Han, Z. Liang and C. Zhang, Optimal proportional reinsurance with common shock dependence to minimize the probability of drawdown. *Ann. Actuar. Sci.* **13** (2019) 268–294.
[19] S. Luo, M. Wang and W. Zhu, Maximizing a robust goal-reaching probability with penalization on ambiguity. *J. Comput. Appl. Math.* **348** (2019) 261–281.
[20] M.A. Milevsky and C. Robinson, Self-annuitization and ruin in retirement with discussion. *N. Am. Actuar. J.* **4** (2000) 112–129.
[21] V.R. Young, Optimal investment strategy to minimize the probability of lifetime ruin. *N. Am. Actuar. J.* **8** (2004) 105–126.
[22] S. Wang, Aggregation of correlated risk portfolios: Models and algorithms. *Proc. Casualty Actuar. Soc.* **85** (1998) 848–939.
[23] K.C. Yuen, J. Guo and X. Wu, On a correlated aggregate claim model with Poisson and Erlang risk process. *Insur.: Math. Econ.* **31** (2002) 205–214.
[24] K.C. Yuen, J. Guo and X. Wu, On the first time of ruin in the bivariatecompound Poisson model. *Insur.: Math. Econ.* **38** (2006) 298–308.
[25] M. Centeno, Dependent risks and excess of loss reinsurance. *Insur.: Math. Econ.* **37** (2005) 229–238.
[26] L. Bai, J. Cai and M. Zhou, Optimal reinsurance policies for an insurer with abivariate reserve risk process in a dynamic setting. *Insur.: Math. Econ.* **53** (2013) 664–670.
[27] K.C. Yuen, Z. Liang and M. Zhou, Optimal proportional reinsurance with common shock dependence. *Insur.: Math. Econ.* **64** (2015) 1–13.
[28] Z. Liang and K.C. Yuen, Optimal dynamic reinsurance with dependent risks: Variance premium principle. *Scand. Actuar. J.* **2016** (2016) 18–36.
[29] X. Liang and V.R. Young, Minimizing the probability of lifetime ruin: Two riskless assets with transaction costs. *ASTIN Bull.: J. IAA* **49** (2019) 847–883.
[30] J. Bi, Z. Liang and K.C. Yuen, Optimal mean variance investment/reinsurance with common shock in a regime-switching market. *Math. Methods Oper. Res.* **90** (2019) 109–135.
[31] J. Grandell, Aspects of Risk Theory. Springer-Verlag, New York (1991).