RAIRO-Oper. Res. 56 (2022) 4251–4280
https://doi.org/10.1051/ro/2022145

RAIRO Operations Research
www.rairo-ro.org

AN INVENTORY MODEL WITH UNCERTAIN DEMAND UNDER PRESERVATION STRATEGY FOR DETERIORATING ITEMS

Amalendu Singha Mahapatra[1], Arup Dasgupta[2], Ashok Kumar Shaw[3] and Biswajit Sarkar[4,5,*]

Abstract. The capacity of a firm to accomplish its goals is financially compromised by degeneration of goods. A suitable preservation strategy to reduce degradation is a vital part of the managerial decisions. This study employs preservation technologies under uncertain demand to frame a continuous review inventory model with full back-ordering and the influence of promotional efforts. Survey of existing research finds few models with synchronised optimization over this entire scenario with all factors.The best values of the preservation cost and the two fractions of the cycle period when inventory is kept against the backorder part are determined to lower the total average cost. A mathematical model is built to incorporate these elements and numerical scenarios are presented to compare three possible approaches. In both crisp and fuzzy contexts, the sensitivity of the solution and decision variables concerning various inventory characteristics is investigated. Backorder duration is inversely proportional to the presence of preservation. The coefficient of preservation has a tipping point below which accepting the impact of undamped deterioration becomes more cost-effective. The total cost at the optimal point is more elastic to a reduction in base deterioration rate and relatively inelastic to its increase. Finally, this study proves that the preservation strategy converges over deterioration for the crisp case rather than the fuzzy case. It is expected the fuzzy case can provide better results, however, the crisp case provides lower total cost than the fuzzy case though it is slightly less efficient in per unit cost.

Mathematics Subject Classification. 90-08, 90-10, 90B05, 90B06, 90B99, 90C70.

Received March 20, 2022. Accepted August 23, 2022.

Keywords. Continuous review inventory, deterioration, preservation investment, backlogging, uncertain demand.

[1] Department of Basic Science and Humanities, Techno International New Town, New Town, Rajarhat, 700156 Kolkata, West Bengal, India.

[2] Department of Basic Engineering Sciences, Netaji Subhash Engineering College, Techno City, Garia, Kolkata 700152, West Bengal, India.

[3] Department of Basic Science and Humanities, Budge Budge Institute of Technology, Nishchintapur, Budge Budge, Kolkata, West Bengal 700137, India.

[4] Department of Industrial Engineering, Yonsei University, 50 Yonsei-ro, Sinchon-dong, Seodaemun-gu, Seoul 03722, South Korea.

[5] Center for Transdisciplinary Research (CFTR), Saveetha Dental College, Saveetha Institute of Medical and Technical Sciences, Saveetha University, 162, Poonamallee High Road, Velappanchavadi, Chennai, Tamil Nadu 600077, India.

*Corresponding author: bsbiswajitsarkar@gmail.com

1. Introduction

In the present scenario, limited-time procedures and showcasing are closely connected. In this pandemic circumstance due to Coronavirus episode, special procedures forced immense changes in the stock framework. The worldwide pandemic (COVID-19) has driven commercial entities towards revamping present and future promotional activities to maintain a healthy flow of financial returns. Limited time exertion might zero in on various objectives like further developed arrangements or make better thoughts among customers. Extension of in-store traffic and a higher inspiration of the staff are likewise potential advantages. The goal is to clear extra stock and make a higher income. Therefore, inventory management is critical for every modern, sophisticated firm. Properly managed inventories pay out in a variety of ways, such as direct profits and customer loyalty. In addition, the interwoven linkages among these diverse business aims maintain inventory management's dazzling importance. The continuous review has attracted more attention than the periodic review because of its mathematical approach and capacity to tackle a wide range of problems.

Disintegration is a character portrayed by deterioration, harm, decay, hurt or other change in item quality because of natural issues during caching. A couple of examples are items like battery, semiconductor chips, food assortments, unstable fluids, clinical things, i.e, blood face deterioration and gradually lose potential. A huge issue in the stock framework is to control and stay aware of inventories of decaying merchandise. The objective of stock administration is to further develop profit from the venture by diminishing stock waste and the deteriorated items contribute negatively towards this target. The pace of decay of products can be seen as a dependent variable, subject to control utilizing protection innovation. Organisations have understood that they must strictly manage the deterioration losses. One of the common avenues of control is to improve and upgrade storage processes. The retailers can decrease the pace of decay of items through viable capital input along these avenues and thus keep away from superfluous waste, limit financial misfortunes, and upgrade business efficiency. Such models concerning deterioration control have gotten a lot of consideration and have greater concurrence with the actual inventory circumstances.

Perishable products especially require precise preservation and inventory control in today's dynamic markets. For example, the deterioration of food will damage the goodwill and reputation of the retailer. As a significant reason for stock misfortune, weakening builds the association's expense and this manner diminishes the benefit. The weakening interaction brought about by activities of chemicals, oxidation and microorganisms often rely upon ecological conditions, like temperature, stickiness, and environment. Temperature has a significant impact on deterioration and must be controlled to maintain products quality. The protection innovation, for example, temperature controlling gear and imaginative bundling, can influence the weakening rate and thus defer the crumbling cycle.

When management launches a new product then, they have imperfect knowledge about the demand and other factors related to the product. The management relies on experts' analysis. When such counsel is imprecise then demand or other factors related to the expert opinion can be modeled using fuzzy principles and the corresponding environment is known as fuzzy environment.

A unique methodology to improve demand forecasting, which is a major difficulty of a continuous review inventory model, is offered in this work. This study further examines the effect of fuzzy demand within an infinite time horizon. An optimal operating strategy is sought for a continuous review inventory system to reduce the total layout in a fuzzy setup. The mechanism of full-backordering is assumed to balance the component of loss during absence of inventory. A continuous review inventory system is numerically solved for both crisp and fuzzy instances, as well as an analysis of the best policy. The decision maker's strategies under uncertain demand can benefit strongly from the results obtained here and reduce the total inventory cost based on preservation strategy.

The latest investigation is motivated by the fact that while there has been numerous research on each of the features, the overall interaction has not been exhaustively investigated. Previous studies that regulated deterioration by implementing functional forms of investment in maintenance techniques have ignored two important aspects.

- The maximum suitable investment in safety techniques ought to not be based totally on the fee of deterioration. Techniques are mostly base on selling price (and therefore unit price) and degradation fee are more feasible and are the strength of this paper.
- Protection time values synchronize with the unit of time are not sustainable in opposition to deterioration for all sorts of goods. Preparations or meals require their personal packaging to lessen expiration. Others can boost up the deterioration of numerous easy gadgets withinside the presence of big quantities of rather oxidized products.

Practical considerations, such as attenuated deterioration, demand affected by both fuzzy base and promotional up-scaling, complicate the approach. It has been proposed to consider analytical convexity without approximating the deterioration factor. Because of the infinite time horizon, the continuous domain for cycle length leads to continuous time duration variables. As a result, these aspects of inventory cost minimization form a helpful aim that motivates to tackle this issue vis-a-vis the following research gap:

- For exponential dampening preservation, truncation error is introduced through polynomial approximation.This spills over into the accuracy and analysis of the optimal point. For promotional impact, detached uncertainty in base demand is not modelled.
- Convexity is not examined in domain-wide analytical outcome. It is usually coordinate limited in process and scope and has other parametric barriers which weaken the mathematical basis.
- Choice of time decision variables and arbitrary horizon is often less than ideal reducing managerial flexibility. The analysis of the two phases of the cycle as impacted variously by deterioration, preservation, and demand uncertainty is often not clearly reflected.

In light of these considerations, the originality and contribution of the current study is that it compares a continuous review inventory model, which allows for full exponential depreciation attenuation analysis, to an inventory model. The retention strategy is designed for a more realistic scheme in which demand uncertainty is modeled with a trapezoidal fuzzy base and a term that scales with promotion efforts.

The decision process of *a priori* planning requires the forecasting of demand patterns on the retailer or seller's part. The comprehensive performance of the arrangement can be improved by flexibility in managing resources and operations. The first model hereafter uses the pure deterministic scenario where exact demand information is available to the decision-maker and it serves as a valuable mathematical prototype. The second model incorporates the issue of deterioration control through variable preservation investment and its economic ramifications, whereby this additional budgeting eventually leads to enhanced cost optimization. The third model returns to the original contention of flexibility to tackle uncertainty, by using a fuzzy formulation for the inexactly predicted demand. The three models together provide the researcher a clear and concise look into the mathematical procedure and economic justification of incorporating preservation and fuzziness to cope with deterioration and uncertainty respectively.

The structure of the present article's organization is as follows: the literature review is in Section 2, which goes through earlier research work that helps to frame this work. Section 3 contains the research problem and the assumptions that aid in the planning of the model's outline. The notations used are tabulated. Section 4 discusses the modelling framework and solutions for the crisp and fuzzy environments. The applicability of this paradigm in a real-world situation is clarified using numerical examples in the next Section 5. The inventory system's sensitivity analysis and administrative framework are found in Section 6. In Section 7, avenues for furthering this research as well as its findings are discussed. Finally, in Appendix A, a short introduction to the trapezoidal fuzzy numbers is laid out.

2. Literature review

2.1. Promotional impact

Discounts, credit period, free goods, and after-sale services are some of the common promotional strategies. Practitioners and researchers have investigated the importance of promotional policies. Taleizadeh *et al.* [49]

formulated an inventory system by considering partial backlogging and delayed payment with shortages. Rajan and Uthayakumar [48] considered a deterministic inventory model by considering full backorder and holding cost that grow exponentially under permitted payment delays and promotional effort. Other researchers (Mahapatra *et al.* [15], Soni and Suthar [47], Liao *et al.* [14]) had considered inventory systems with demand sensitive to promotional strategies. Nouri *et al.* [20] analysed the benefit of investing for promotion in a supplier-retailer chain model where demand was stochastic. Recently, Pal [23] established optimal pricing while offering incentive decisions for a supply chain model. Zhou *et al.* [55] formulated an inventory system by calculating joint pricing by considering a deep reinforcement learning algorithm. Mahapatra *et al.* [16] formulated three different types of economic order quantity (EOQ) model by considering time-dependent deterioration rate, promotional effect, preservation technology, and uncertainty learning. An inventory system's promotional activities effects the whole system's sales volume, and hence on the earnings of other members. As a result, in the inventory model, it's important to organize promotional decisions.

2.2. Deterioration

Deterioration has been deeply explored in inventory management. Food spoilage through microbes or oxidation is ubiquitous. Electronics goods storage must account for damage through electrostatic discharge, moisture, and contaminants. Pervin *et al.* [24] determined EOQ model for perishable goods with demand varying with stock level and holding costs that was time-dependent. Pervin *et al.* [25] designed a multi-item inventory model that took into account trade credits, on-demand, and constant rate of deterioration. Barman *et al.* [1] analysed an economic production quantity (EPQ) model with shortages and inflation in a fuzzy environment using fixed rate of deterioration and incorporating time-dependent demand. Roy *et al.* [30] developed a two warehouse, two credit level, and probabilistic setup for deteriorating items. Roy *et al.* [31] proposed an imperfect production system with partial backlog under a credit policy for a deteriorating product. In the practical scenario, it is seen that variable deterioration rates better represent the rate of failure and the life span of various goods. Many practitioners like (Ouyang *et al.* [22], Soni and Patel [46], Zhang *et al.* [54], Mishra *et al.* [18]) worked with this phenomenon but in different directions. By taking into account trade credit policy, partial backlog, and preservation strategy, Shaikh *et al.* [44] developed an inventory control system for deteriorating product by considering time-varying demand. Khan *et al.* [11] analysed a setup with constant deterioration and variable demand pattern along with delayed payment. Recently, Shah *et al.* [43] investigated the case where the goods deteriorate and their demand fluctuates with the selling price by considering the greening effect.

2.3. Preservation investment

Financially, the weakening of items contrarily affects the accounts and brand image of a firm. High deterioration rates are entitled to higher yearly expenses, deficiencies, and lost deals. On this record, business associations are keen on appreciating the reasons for crumbling and creating ways to deal with save their delivered merchandise and increment benefit. Antimicrobial (*e.g.* sulfites in fruits) and antioxidant (*e.g.* butylated hydroxytoluene in edible oils) agents are commonly employed in a wide range of biologically derived products. Recent advances have employed hydrophilic and lipophilic nanosystems for edible preservative coating in food. Clean-room paradigms are used to control deterioration in pharmaceuticals and electronics sectors.

Dye and Hsieh [5] applied preservation procedures to control the deterioration rate for stored goods. Jani *et al.* [10] explored a production inventory system for lifetime-dependent products by using preservation technology. Pervin *et al.* [26] utilised an EPQ system with decaying products, stock preservation technology, and demand that was sensitive to the selling price, with partial backlog. Both preservation and inspection policy are employed in Pervin *et al.* [27] to optimise a quadratic demand vendor-buyer model. Das *et al.* [3] proposed a multi-period credit policy and stock and price varying demand for an item with non-instantaneous spoilage with preservation. Shah *et al.* [42] formulated a model for optimum inventory under stock-dependent rate of depletion with a constant degree of deterioration controlled by preservation investment.

Priyamvada *et al.* [28] took into account the deterioration of products under the effect of cost and stock ward. They limited deterioration using preservation processes. Priyamvada *et al.* [29] proposed a realistic preservation formulation that varies the investment with the unit price and inventory level. The impact of strategic inventories on a supply chain was studied by Saha *et al.* [32] who took into account how two producers or a common retailer and a manufacturer could collaborate through wholesale pricing in the supply chain. Shah *et al.* [41] developed an EPQ model for deterioration products as price sensitive demand to reduce emissions by using green technology and preservation investment. Sarkar *et al.* [34] considered a variable demand-based supply chain model for sustainable manufacturing and investment for autonomation. Recently, Sarkar *et al.* [38] presented a substitutable product manufacturing process through a dual-channel and ensure environmental sustainability by using carbon tax along with carbon cap. Reducing waste and meeting emissions targets are important goals for inventory systems. Sarkar *et al.* [37] introduced a multi-phase production system to regulate defective products and reduce waste through a reworking process in order to optimize profit. Kumar *et al.* [13] designed a supply chain model for manufacturers and consumers that was based on demand and the carbon emissions process. Various sustainability measures were useful investments in keeping with the biodegradable products without preservation philosophy as considered recently by Sarkar *et al.* [35].

2.4. Fuzzy modeling of uncertainty

Apart from promotional efforts, decaying commodities, and preservation technology, demand uncertainty stems from the inventory model's many unknown aspects. However, in real-world settings, the uncertain parameters like demand, various relevant expenses, lead time, preservation cost, may have a higher possibility of deviating from the exact value, leading to a situation in which these uncertain parameters do not follow any probability distribution. Initially, Zadeh [53] first developed the concept of fuzzy set. After that many pioneer's researchers like (Yao *et al.* [52], Glock *et al.* [7], Shah and Soni [39]) captured the impreciseness by developing various fuzzy inventory models. The model analyzed by Garai *et al.* [6] had holding cost that scaled with time and price dependent demand by considering trapezoidal fuzzy numbers. Shah and Patel [40] used preservation technology and developed a model for inventory to reduce rate of spoilage under a cloud fuzzy prescription. Yadav *et al.* [51] considered an flexible production system with a variable pollution control for a fuzzy environment. A learning environment for dense fuzzy demand was employed by De and Mahata [4] under the overlap of order with rework batches. Time varying demand for decaying items was modeled by Kumar and Paikray [12] in crisp and fuzzy formulations with three different scenarios under total backlogging. A multithreaded neural network was efficiently employed by Sarkar *et al.* [36] to tackle a fuzzy inflationary model.

3. Problem definition

3.1. Research problem

The proposed inventory model herein aims to determine the lowest total cost per unit time for the cases of crisp and fuzzy demand by computing the most effective replenishment policy and optimal investment in preservation technology for a deteriorating item. A model without preservation is used as a baseline formulation to compare and contrast with the goal of the model. Figure 1 is an illustrative inventory profile of the model. In the three models, most of the parameters are common while some are specific to the model as required. The assumptions made are itemised below. The decision variables and other specifications are laid out in Table 2.

3.2. Notation

3.3. Assumptions

(1) The demand rate is $D(\rho) = D_0 + d_1\rho$, where D_0 is fixed base demand while $d_1\rho$ measures the effect of promotional activity ρ with d_1 as a positive, constant scaling term. Here, in model-3 it is assumed that

TABLE 1. Analysis of this model in comparison to prior models.

Authors	Demand rate	Deterioration rate	Back-ordering type	Preservation technology	Environment
Salameh *et al.* [33]	Constant	CN	CN	CN	Crisp
Soni and Patel [46]	Time and price dependent	Constant	CN	CN	Crisp & fuzzy
Zhang *et al.* [54]	Price & stock dependent	Constant	CN	CN	Crisp
Das *et al.* [3]	Selling price and stock level dependent	Three-parameter Weibull distribution dependent	Partial	Fully	Crisp
Mahata and Goswami [17]	Constant	CN	Fully	CN	Fuzzy
Shah *et al.* [43]	Price and stock greening effect dependent	Constant	CN	CN	Crisp
Mahapatra *et al.* [16]	Time dependent and uncertainty	Time-dependent	Fully	Partially	Crisp & fuzzy
Sarkar *et al.* [38]	Price and cross-elasticity dependent	CN	CN	CN	Crisp
Priyamvada *et al.* [29]	Price-sensitive demand	Constant	CN	Fully	Crisp
This study	Uncertainty with promotional effort	Constant	Fully	Fully	Crisp & fuzzy

Notes. CN: Contribution Not-Available relative to present analysis.

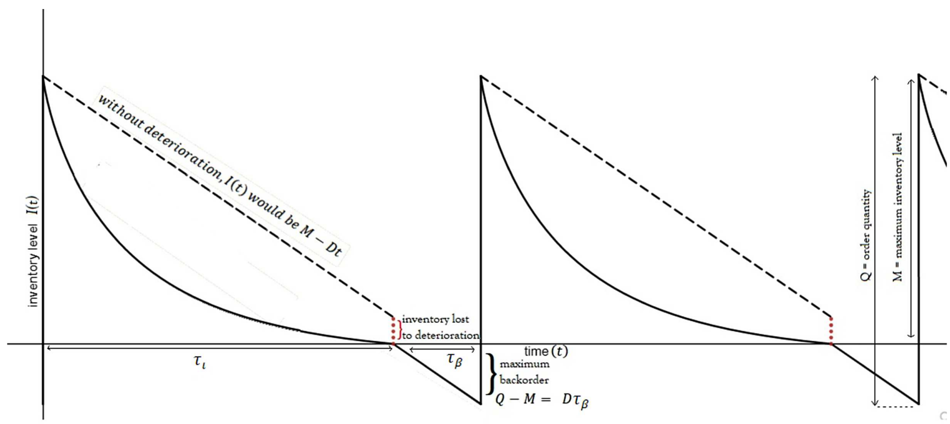

FIGURE 1. Inventory model with backorder and deterioration.

D_0 is a trapezoidal fuzzy number (TpFN) as follows: $\widetilde{D_0} = (D_0 - \Delta_{l2}, D_0 - \Delta_{l1}, D_0 + \Delta_{r1}, D_0 + \Delta_{r2})$, where $0 < \Delta_{l1} < \Delta_{l2} < D_0$; $0 < \Delta_{r1} < \Delta_{r2} < D_0$ and $(\Delta_{r1} - \Delta_{l1} + \Delta_{r2} - \Delta_{l2}) > 0$. The decision makers determine Δ_{l1}, Δ_{l2}, Δ_{r1} and Δ_{r2}.

(2) A single type of item is analyzed in this model, as well as shortages and full backlogs. When replenishment occurs, backlogged demand is initially met.

(3) The constant deterioration rate is θ_0 $(0 < \theta_0 < 1)$. Deterioration is proportional to the inventory. Per unit time, a percentage of goods in the inventory deteriorate. Repair or replenishment of such items does not occur within a cycle.

TABLE 2. Specification notation.

Symbol	Description
Parameters	
Q	order quantity
$D(\rho)$	demand rate per unit time
A	ordering cost (\$/per order)
h	holding cost (\$/per unit)
w	deteriorating cost (\$/per unit)
s	shortage cost (\$/per unit)
k	promotional cost (\$/per unit)
ρ	promotional effort per unit time
θ_0	constant rate at which items in inventory deteriorate ($0 < \theta_0 < 1$)
m	the scale parameter for promotion
a	constant coefficient of preservation
θ	effective deterioration rate, dampened by use of preservation strategy ($\theta = \theta_0 e^{-au}$)
M	maximum positive "inventory level" at time $t = 0$
U	maximum preservation outlay (\$/ per unit time)
$I(t)$	level of inventory at time t
$TAC(\tau_\iota, \tau_\beta)$	total cost in absence of preservation (Model-1)(\$/per unit time)
$TAC(\tau_\iota, \tau_\beta, u)$	total cost with preservation (Model-2) (\$/per unit time)
$\widetilde{TAC}(\tau_\iota, \tau_\beta, u)$	total cost for fuzzy demand with preservation (Model-3) (\$/per unit time)
Decision variables	
τ_ι	time duration for inventory level to fall to zero after replenishment
τ_β	time between zero inventory and replenishment, during which orders are fully backlogged (denoted by negative inventory level)
u	preservation investment as cost per unit time (\$/per time)

(4) The promotional effort cost (PEC) is $k\rho^m$ which collect from the leading results of the cost-of-promotion function, where k and m are constant. (Mahapatra *et al.* [15], Soni and Suthar [47])

(5) The impact of preservation investment u per unit time is that it reduces the effective rate of deterioration and is modeled by a term using a strictly monotonic increasing function, $\theta_0(1 - e^{-au})$ for $u \geq 0$, where $a =$ positive constant. This ensures that there is no impact when $u = 0$ and as u increases, the effect increase but is bounded by θ_0, *i.e.*, the maximum impact as $u \to \infty$ is θ_0 (Shah *et al.* [42]). Realistically, the decision maker has budgetary limits for preservation investment and hence an upper bound $u \leq U$, is a necessary numerical constraint.

(6) The effective deterioration rate after subtracting the impact of the preservation cost is $\theta_0 - \theta_0(1 - e^{-au}) = \theta_0 e^{-au}$, *i.e.*, $\theta \to \theta_0$ if $u \to 0$ (when without preservation) and $\theta \to 0$ if $u \to \infty$ (the limiting case where deterioration fully reduced by preservation).

(7) The lead time is negligible together with infinite replenishment rate and the order quantity is finite.

4. Mathematical model

4.1. Model development

The models and their solution methodology are laid out in this section. The governing differential equations are setup and solved to obtain the total cost function. Necessary and sufficient criterion for convexity and global optimality are imposed on this objective function. Defuzzification is used when fuzzy parameters are employed.

4.2. A Continuous review inventory model with crisp demand and constant deterioration rate (Model-1)

A continuous review EOQ setup is constructed with the above assumptions. An instantaneous replenishment starts the cycle at $t = 0$ and the inventory level jumps to its maximum value, $M = I(0)$. $I(t)$ reduces in time interval $[0, \tau_\iota]$ as units are consumed to fulfill the demand while some are lost to deterioration. All units are used up at $t = \tau_\iota$ so $I(\tau_\iota) = 0$. Backorders are kept during the period $[\tau_\iota, \tau_\beta + \tau_\iota]$ which are fully backlogged to be satisfied from the next replenishment. In Model-1 there is no preservation investment ($u = 0$). Based on these assumptions, the differential equations governing by the following cases:
Case 1 ($0 \leq t \leq \tau_\iota$): Consumption due to demand and loss due to deterioration depletes the inventory, hence the equation governing the inventory level $I(t)$ is

$$\frac{dI(t)}{dt} + \theta I(t) = -D, \quad \text{when} \quad 0 \leq t \leq \tau_\iota. \tag{4.1}$$

The deterioration term $\theta I(t)$ is proportional to the existing on-hand inventory and $\theta = \theta_0$ is a constant in Model-1. In the differential equation (4.1), the boundary condition $I(\tau_\iota) = 0$ is applied to obtain the level of inventory

$$I(t) = \frac{D}{\theta_0}\left(e^{\theta_0(\tau_\iota - t)} - 1\right) \quad \text{when} \;\; 0 \leq t \leq \tau_\iota. \tag{4.2}$$

Using equation (4.2) the highest inventory level M is at $t = 0$ as follows:

$$M = I(0) = \frac{D}{\theta_0}\left(e^{\theta_0 \tau_\iota} - 1\right). \tag{4.3}$$

Now, equation (4.3) is different compared to the case of no deterioration where $M = D\tau_\iota$. It is exactly this difference which gives the amount of inventory lost due to deterioration, as in the buyer's inventory model considered in Wee *et al.* [50], given by

$$M - D\tau_\iota = \frac{D}{\theta_0}\left(e^{\theta_0 \tau_\iota} - 1 - \theta_0 \tau_\iota\right). \tag{4.4}$$

Case 2 ($\tau_\iota \leq t \leq \tau_\beta + \tau_\iota$): Consider shortages during the period, as well as backorders caused by a negative inventory level $I(t)$. The only term is due to demand because orders are entirely backlogged. As a result, the differential equations that follows

$$\frac{dI(t)}{dt} = -D, \quad \text{when} \;\; \tau_\iota \leq t \leq \tau_\beta + \tau_\iota. \tag{4.5}$$

Using the boundary condition $I(\tau_\iota) = 0$; equation (4.5) leads to the following expression for the inventory level

$$I(t) = -D\left(t - \tau_\iota\right), \quad \text{in} \;\; \tau_\iota \leq t \leq \tau_\beta + \tau_\iota. \tag{4.6}$$

This negative inventory level actually implies that the accumulated backorder at time t (in $\tau_\iota \leq t \leq \tau_\beta + \tau_\iota$) is $D\left(t - \tau_\iota\right)$. Here $I\left(\tau_\beta + \tau_\iota\right) = -D\tau_\beta$ is the lowest inventory level and the maximum backorder is therefore $D\tau_\beta$. The ordered quantity Q fulfills this backorder first and the remaining inventory provides the maximum inventory level.

$$Q - D\tau_\beta = M \Rightarrow Q = D\tau_\beta + \frac{D}{\theta_0}\left(e^{\theta_0 \tau_\iota} - 1\right). \tag{4.7}$$

In equation (4.7), the order quantity is larger than that in the classical backorder model without deterioration $[D(\tau_\iota + \tau_\beta)]$, as it has to satisfy the demand together with the items lost to deterioration.

Therefore, the total inventory cost has the following constituents.

(a) Ordering cost (OC) $= A$

(b) Holding cost (HC)

$$= \int_0^{\tau_\iota} \frac{Dh}{\theta_0}\left(e^{\theta_0(\tau_\iota - t)} - 1\right) dt = \frac{Dh}{\theta_0^2}\left(e^{\theta_0\tau_\iota} - \theta_0\tau_\iota - 1\right).$$

(c) Shortage cost (SC) =

$$\int_{\tau_\iota}^{\tau_\beta+\tau_\iota} \Big(-I(t)\Big) s dt = \int_{\tau_\iota}^{\tau_\beta+\tau_\iota} D\left(t-\tau_\iota\right) s dt = \frac{\tau_\beta^2 Ds}{2}\cdot$$

(d) Promotional effort cost (PEC) $= k\rho^m$, where k and m are positive parameters and ρ is the promotional effort.

(e) Cost due to the deteriorated items (w is the per unit deterioration cost) (DC)

$$= \left(M - D\tau_\iota\right) w = \frac{Dw}{\theta_0}\left(e^{\theta_0\tau_\iota} - \theta_0\tau_\iota - 1\right).$$

As a result, by considering deterioration but not preservation costs, the total inventory cost per cycle is as follows:

$$\begin{aligned} TAC(\tau_\iota, \tau_\beta) &= \frac{1}{\tau_\beta + \tau_\iota}\Big[OC + PEC + HC + DC + SC\Big] \\ &= \frac{A + k\rho^m}{\tau_\beta + \tau_\iota} + \frac{D(h+\theta_0 w)}{\theta_0^2}\frac{\left(e^{\theta_0\tau_\iota} - \theta_0\tau_\iota - 1\right)}{\tau_\beta + \tau_\iota} + \frac{Ds\tau_\beta^2}{2(\tau_\beta + \tau_\iota)}\cdot \end{aligned} \tag{4.8}$$

4.2.1. Optimization methodology (Model-1)

The classical optimization process yields the decision variable values for the least total cost ($/cycle). The computational procedure has two parts.
Step 1: Obtain critical point $\left(\tau_\iota^*,\ \tau_\beta^*\right)$ satisfying

$$\frac{\partial TAC}{\partial \tau_\iota} = 0 \qquad \text{and} \qquad \frac{\partial TAC}{\partial \tau_\beta} = 0.$$

Step 2: Verify the convexity of $TAC(\tau_\iota, \tau_\beta)$ by proving that (in the feasible region)

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} > 0 \qquad \text{and} \qquad \frac{\partial^2 TAC}{\partial \tau_\iota^2}\cdot\frac{\partial^2 TAC}{\partial \tau_\beta^2} - \left[\frac{\partial^2 TAC}{\partial \tau_\iota \partial \tau_\beta}\right]^2 > 0.$$

The factors not containing the decision variables in equation (4.8) are collected and grouped for algebraic convenience.

$$TAC\left(\tau_\iota, \tau_\beta\right) = \frac{A_1}{\tau_\beta + \tau_\iota} + \frac{B_1\tau_\beta^2}{\tau_\beta + \tau_\iota} + \frac{C_1\left(e^{\theta_0\tau_\iota} - 1 - \theta_0\tau_\iota\right)}{\tau_\beta + \tau_\iota} \tag{4.9}$$

where

$$A_1 = A + k\rho^m \qquad B_1 = \frac{Ds}{2} \qquad C_1 = \frac{D(h+\theta_0 w)}{\theta_0^2}\cdot$$

The total cost per time unit varies continuously with the positive inventory time τ_ι and negative inventory time τ_β. This objective function is minimized using the decision variables τ_ι and τ_β. From equation (4.9), the first order partial derivatives of $TAC\,(\tau_\iota, \tau_\beta)$ are

$$\frac{\partial TAC}{\partial \tau_\iota} = \frac{\theta_0 C_1 (e^{\theta_0 \tau_\iota} - 1)}{\tau_\beta + \tau_\iota} - \frac{(TAC)}{(\tau_\beta + \tau_\iota)} \tag{4.10}$$

$$\frac{\partial TAC}{\partial \tau_\beta} = \frac{2B_1 \tau_\beta}{\tau_\beta + \tau_\iota} - \frac{(TAC)}{(\tau_\beta + \tau_\iota)}. \tag{4.11}$$

The second order partial derivatives of $TAC\,(\tau_\iota, \tau_\beta)$ are

$$\frac{\partial^2 TAC}{\partial \tau_\iota^2} = \frac{2\,(TAC)}{(\tau_\beta + \tau_\iota)^2} - \frac{2\theta_0 C_1 \left(e^{\theta_0 \tau_\iota} - 1\right)}{(\tau_\beta + \tau_\iota)^2} + \frac{\theta_0^2 C_1 e^{\theta_0 \tau_\iota}}{\tau_\beta + \tau_\iota}, \tag{4.12}$$

$$\frac{\partial^2 TAC}{\partial \tau_\iota \partial \tau_\beta} = \frac{2\,(TAC)}{(\tau_\beta + \tau_\iota)^2} - \frac{\theta_0 C_1 \left(e^{\theta_0 \tau_\iota} - 1\right)}{(\tau_\beta + \tau_\iota)^2} - \frac{2B_1 \tau_\beta}{(\tau_\beta + \tau_\iota)^2}, \tag{4.13}$$

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} = \frac{2\,(TAC)}{(\tau_\beta + \tau_\iota)^2} - \frac{4B_1 \tau_\beta}{(\tau_\beta + \tau_\iota)^2} + \frac{2B_1}{\tau_\beta + \tau_\iota}. \tag{4.14}$$

As $e^{\theta_0 \tau_\iota} - 1 - \theta_0 \tau_\iota > 0 \;\; \forall\, \theta_0 \tau_\iota > 0$, hence equation (4.9) gives

$$TAC(\tau_\iota, \tau_\beta) > \frac{B_1 \tau_\beta^2}{\tau_\beta + \tau_\iota} \quad \forall\, \theta_0 \tau_\iota > 0.$$

Using this inequality in equation (4.14), which becomes

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} > \frac{2B_1 \tau_\beta^2}{(\tau_\beta + \tau_\iota)^3} + \frac{2B_1 \left(\tau_\iota - \tau_\beta\right)}{(\tau_\beta + \tau_\iota)^2} = \frac{2B_1 \tau_\iota^2}{(\tau_\beta + \tau_\iota)^3} > 0 \quad \forall\, \theta_0 \tau_\iota > 0. \tag{4.15}$$

The necessary condition for the objective function to attain minimum cost are that the first order partial derivative must be zero (Step-1 above). when sufficient conditions (Step-2 above) are met; setting these partial derivative equal to zero gives the best solution, that is $\frac{\partial TAC}{\partial \tau_\iota} = 0$ and $\frac{\partial TAC}{\partial \tau_\beta} = 0$.

Further, to make certain optimality, the enough situation should be satisfied. Hence forth the corresponding fundamental minors should be positive definite.
The Hessian determinant is

$$H\,(\tau_\iota, \tau_\beta) = \frac{\partial^2 TAC}{\partial \tau_\iota^2} \cdot \frac{\partial^2 TAC}{\partial \tau_\beta^2} - \left[\frac{\partial^2 TAC}{\partial \tau_\iota \partial \tau_\beta}\right]^2.$$

From equations (4.12), (4.13) and (4.14), we get

$$\begin{aligned}(\tau_\beta + \tau_\iota)^4 H\,(\tau_\iota, \tau_\beta) = {} & 2A_1 \left(2B_1 + C_1 {\theta_0}^2 e^{\theta_0 \tau_\iota}\right) \\ & + 2C_1 B_1 \left(e^{\theta_0 \tau_\iota} \left(1 - \theta_0 \tau_\iota\right)^2 + e^{\theta_0 \tau_\iota} - 2\right) \\ & + {\theta_0}^2 {C_1}^2 \left(e^{2\theta_0 \tau_\iota} - 2\theta_0 \tau_\iota e^{\theta_0 \tau_\iota} - 1\right).\end{aligned} \tag{4.16}$$

Simplifying $e^{\theta_0 \tau_\iota} \left(1 - \theta_0 \tau_\iota\right)^2 + e^{\theta_0 \tau_\iota} - 2$ and $e^{2\theta_0 \tau_\iota} - 2\theta_0 \tau_\iota e^{\theta_0 \tau_\iota} - 1$.

$$\begin{aligned}(\tau_\beta + \tau_\iota)^4 H\,(\tau_\iota, \tau_\beta) \; & > \; 2A_1 \left(2B_1 + C_1 {\theta_0}^2 e^{\theta_0 \tau_\iota}\right) + C_1 B_1 \left(\theta_0 \tau_\iota\right)^4 + \frac{{\theta_0}^2 {C_1}^2 \left(\theta_0 \tau_\iota\right)^4}{4} \\ & \Rightarrow H\,(\tau_\iota, \tau_\beta) > 0 \;\forall\; \theta_0 \tau_\iota > 0.\end{aligned} \tag{4.17}$$

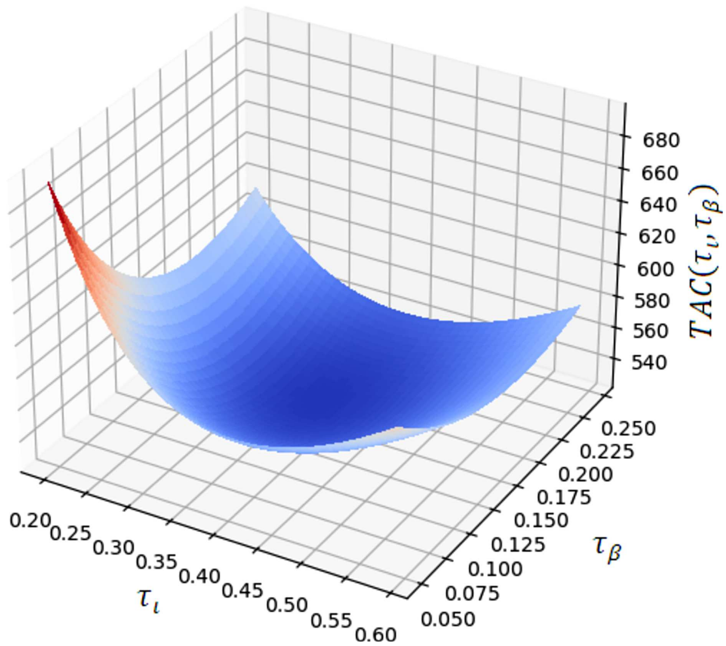

FIGURE 2. Objective function convexity (Model-1).

Equations (4.15) and (4.17) imply a positive definite Hessian. Therefore, the cost function $TAC(\tau_\iota, \tau_\beta)$ is convex and this nature is again proved graphically in Figure 2.

From equation (4.10) and (4.11), it has a unique global minima in the feasible region at $\left(\tau_\iota^*,\ \tau_\beta^*\right)$ satisfying (critical point)

$$\frac{\partial TAC}{\partial \tau_\iota} = \frac{\theta_0 C_1 \left(e^{\theta_0 \tau_\iota} - 1\right)}{\tau_\beta + \tau_\iota} - \frac{(TAC)}{(\tau_\beta + \tau_\iota)} = 0, \tag{4.18}$$

$$\frac{\partial TAC}{\partial \tau_\beta} = \frac{2B_1 \tau_\beta}{\tau_\beta + \tau_\iota} - \frac{(TAC)}{(\tau_\beta + \tau_\iota)} = 0. \tag{4.19}$$

Comparing equation (4.18) and (4.19)

$$TAC\left(\tau_\iota^*,\ \tau_\beta^*\right) = \theta_0 C_1 \left(e^{\theta_0 \tau_\iota^*} - 1\right) = 2B_1 \tau_\beta^*. \tag{4.20}$$

Hence, equations (4.9) and (4.20) give

$$2B_1 \tau_\beta^* = \frac{A_1}{\tau_\beta^* + \tau_\iota^*} + \frac{B_1 {\tau_\beta^*}^2}{\tau_\beta^* + \tau_\iota^*} + \frac{C_1 \left(e^{\theta_0 \tau_\iota^*} - 1 - \theta_0 \tau_\iota^*\right)}{\tau_\beta^* + \tau_\iota^*}. \tag{4.21}$$

Simplifying equation (4.21), one can get

$$\tau_\beta^* = \sqrt{{\tau_\iota^*}^2 + \frac{A_1 + C_1 \left(e^{\theta_0 \tau_\iota^*} - 1 - \theta_0 \tau_\iota^*\right)}{B_1}} - \tau_\iota^*. \tag{4.22}$$

Equation (4.20) implies

$$\tau_\iota^* = \frac{1}{\theta_0} \log\left\{\frac{2B_1 \tau_\beta^*}{\theta_0 C_1} + 1\right\}. \tag{4.23}$$

The iterative numerical solution of equation (4.22) and 4.23 yields the appropriate values of τ_β^* and τ_ι^* for optimization.

From equations (4.20) and (4.7), the minimum total cost per unit time and the economic order quantity are

$$TAC\left(\tau_\iota^*,\ \tau_\beta^*\right) = 2B_1\tau_\beta^*, \qquad Q^* = D\tau_\beta^* + \frac{D\left(e^{\theta_0\tau_\iota^*}-1\right)}{\theta_0}. \tag{4.24}$$

From equation (4.4), effective rate of loss (deterioration per unit time averaged over a cycle) is

$$= \frac{D\left(e^{\theta_0\tau_\iota^*}-1-\theta_0\tau_\iota^*\right)}{\theta_0(\tau_\beta+\tau_\iota)} = \frac{Q^*}{\tau_\beta+\tau_\iota} - D.$$

4.3. A continuous review inventory model by considering deterioration items and preservation effects (Model-2)

Proper maintenance strategies reduce the impact of inventory system degradation. As fewer items are lost, this translates to savings. The less items you lose, the more you save. However, investments in such technologies should be included in the overall cost. In this subsection, an optimal point is found between these two conflicting economic effects.

In this model, preservation strategies cost is used by the retailer as an additional investment of u per unit time and this reduces the rate at which items deteriorate by $\theta_0(1-e^{-au})$, where the coefficient of preservation is $a>0$ (as in Shah *et al.* [42]). This is a strictly monotonic increasing function of the preservation investment, bounded above by θ_0, and below by 0.

The effective deterioration coefficient is (for $u>0$)

$$\theta = \theta_0 - \theta_0\left(1-e^{-au}\right) = \theta_0 e^{-au} \quad \Rightarrow \quad 0<\theta<\theta_0<1. \tag{4.25}$$

Now, Model-1 is extended to incorporate the preservation technology. This variable has two effects as the preservation cost adds to the total cost, but dampens the deterioration through θ. Solving the differential equations (4.1) and (4.5) for $I(t)$ using the boundary condition $I(\tau_\iota)=0$; the inventory level $I(t)$ with the consideration of preservation investment are as follows:

$$I(t) = \frac{D}{\theta}\left(e^{\theta(\tau_\iota-t)}-1\right) \qquad in\ 0\le t\le\tau_\iota, \tag{4.26}$$

$$I(t) = -D\left(t-\tau_\iota\right) \qquad in\ \tau_\iota\le t\le\tau_\beta+\tau_\iota. \tag{4.27}$$

Equations (4.26) and (4.27) give the inventory at any point in the cycle $(0\le t\le\tau_\beta+\tau_\iota)$. Maximum inventory is at $I(0)$ and maximum backorder is at $I(\tau_\beta+\tau_\iota)$.

$$M = I(0) = \frac{D}{\theta}\left(e^{\theta\tau_\iota}-1\right)$$

$$\text{and} \qquad Q - M = -I(\tau_\iota+\tau_\beta) = D\tau_\beta$$

$$Q = \frac{D}{\theta}\left(e^{\theta\tau_\iota}-1\right) + D\tau_\beta. \tag{4.28}$$

Over a complete cycle, number of items lost to deterioration is (as in equation (4.4))

$$M - D\tau_\iota = \frac{D}{\theta}\left(e^{\theta\tau_\iota}-1-\theta\tau_\iota\right). \tag{4.29}$$

The total cost per cycle in this inventory model now contains the preservation cost u per unit time. Hence, the preservation cost per cycle (PC) is $(\tau_\iota+\tau_\beta)u$.

The other cost components are as defined in Model-1. Thus, the total inventory cost per cycle with additional preservation cost and deterioration dampened by preservation is as follows:
$TAC(\tau_\iota, \tau_\beta, u) = \frac{1}{\tau_\beta+\tau_\iota}$(Ordering cost+Holding cost+Shortage cost+Promotional effect cost+Deterioration cost+ Preservation cost),
i.e.,

$$TAC(\tau_\iota, \tau_\beta, u) = u + \frac{A + k\rho^m}{\tau_\beta + \tau_\iota} + \frac{D(h+\theta w)}{\theta^2}\frac{\left(e^{\theta\tau_\iota} - \theta\tau_\iota - 1\right)}{\tau_\beta + \tau_\iota} + \frac{Ds\tau_\beta^2}{2(\tau_\beta + \tau_\iota)}. \tag{4.30}$$

The total cost per cycle time varies continuously with the positive inventory time τ_ι, negative inventory time τ_β, and the preservation investment u. This is the objective function to be minimized using decision variables τ_ι, τ_β, and u.

4.3.1. Optimization methodology (Model-2)

In this study, the positive inventory time τ_ι, negative inventory time τ_β, and preservation investment u that minimizes TAC are found. This investment reduces the deterioration to a certain degree, but beyond the point at which the benefits of the preservation outweigh the costs, the cost of preserving the investment might be more than the savings from reducing the number of deteriorated items.

The model aims to strike a balance between the marginal cost of preservation investment and the deteriorated quantity. This subsection chalks out the classical optimization pathway step by step for the projected scenario to derive the optimal solution as follows:

Step 1: To achieve least total cost under this model, necessary conditions are to be satisfied, i.e, the first order partial derivatives must vanish at the optimum point $(\tau_\iota^*, \tau_\beta^*, u^*)$.

$$\frac{\partial TAC}{\partial \tau_\iota} = 0, \qquad \frac{\partial TAC}{\partial \tau_\beta} = 0, \qquad \frac{\partial TAC}{\partial u} = 0.$$

Step 2: Convexity of $TAC(\tau_\iota, \tau_\beta, u)$ in the feasible region is investigated at this step through the application of the sufficient conditions involving the second order partial derivatives.

In $TAC(\tau_\iota, \tau_\beta, u)$ there are two time duration variables and one preservation cost variable. First the convexity of $TAC(\tau_\iota, \tau_\beta \, |u)$ is considered for any particular feasible value of the preservation cost and then the convexity of $TAC(u \, |\tau_\iota, \tau_\beta)$ is considered for any particular feasible values of the time duration variables.

The process for checking convexity of $TAC(\tau_\iota, \tau_\beta|u)$ is similar to the derivation in Model-1. Note that, here $\theta = \theta_0 e^{-au}$ contains the third decision variable u. For any particular given feasible value of $u = u^*$, this cost function is checked for convexity.

Hence, in the following analysis, equation (4.30) is rewritten and the compact algebraic form of $TAC(\tau_\iota, \tau_\beta|u^*)$ is as follows:

$$TAC(\tau_\iota, \tau_\beta|u^*) = u^* + \frac{A_1}{\tau_\beta + \tau_\iota} + \frac{B_1\tau_\beta^2}{\tau_\beta + \tau_\iota} + \frac{C_1(e^{\theta\tau_\iota} - 1 - \theta\tau_\iota)}{\tau_\beta + \tau_\iota}. \tag{4.31}$$

$$\text{where} \quad \theta = \theta_0 e^{-au^*} \Rightarrow C_1 = \frac{D\,(h+\theta w)}{\theta^2} = \frac{D\left(h + w\theta_0 e^{-au^*}\right)e^{2au^*}}{\theta_0^2}$$

$$\text{and} \qquad A_1 = A + k\rho^m \qquad B_1 = \frac{Ds}{2}.$$

The first order partial derivatives of $TAC(\tau_\iota, \tau_\beta|u^*)$ with respect to τ_ι and τ_β from equation (4.31) are

$$\frac{\partial TAC}{\partial \tau_\iota} = \frac{(u^* - TAC)}{(\tau_\beta + \tau_\iota)} + \frac{\theta C_1(e^{\theta\tau_\iota} - 1)}{\tau_\beta + \tau_\iota}$$
$$\frac{\partial TAC}{\partial \tau_\beta} = \frac{(u^* - TAC)}{(\tau_\beta + \tau_\iota)} + \frac{2B_1\tau_\beta}{\tau_\beta + \tau_\iota}.$$

The second order partial derivatives of $TAC(\tau_\iota, \tau_\beta|u^*)$ with respect to τ_ι and τ_β are computed to check the nature of the Hessian matrix for positive definiteness as follows.

$$\frac{\partial^2 TAC}{\partial \tau_\iota^2} = \frac{2\left(TAC - u^*\right)}{\left(\tau_\beta + \tau_\iota\right)^2} - \frac{2\theta C_1\left(e^{\theta\tau_\iota} - 1\right)}{\left(\tau_\beta + \tau_\iota\right)^2} + \frac{\theta^2 C_1 e^{\theta\tau_\iota}}{\tau_\beta + \tau_\iota} \tag{4.32}$$

$$\frac{\partial^2 TAC}{\partial \tau_\iota \partial \tau_\beta} = \frac{2\left(TAC - u^*\right)}{\left(\tau_\beta + \tau_\iota\right)^2} - \frac{\theta C_1\left(e^{\theta\tau_\iota} - 1\right)}{\left(\tau_\beta + \tau_\iota\right)^2} - \frac{2B_1\tau_\beta}{\left(\tau_\beta + \tau_\iota\right)^2} \tag{4.33}$$

$$\frac{\partial^2 TAC}{\partial \tau_\beta^2} = \frac{2\left(TAC - u^*\right)}{\left(\tau_\beta + \tau_\iota\right)^2} - \frac{4B_1\tau_\beta}{\left(\tau_\beta + \tau_\iota\right)^2} + \frac{2B_1}{\tau_\beta + \tau_\iota}\cdot \tag{4.34}$$

It is interesting to note that the above equations imply

$$\frac{\partial TAC}{\partial \tau_\iota} + \frac{\partial TAC}{\partial \tau_\beta} = -(\tau_\beta + \tau_\iota)\frac{\partial^2 TAC}{\partial \tau_\iota \partial \tau_\beta}\cdot$$

At the optimal point where the *lhs* terms are zero, the mixed second derivative with respect to τ_ι, τ_β is therefore zero.

In the feasible range, $1 - \theta\tau_\iota + e^{\theta\tau_\iota} > 0$. Hence equation (4.31) implies

$$TAC(\tau_\iota, \tau_\beta|u^*) - u^* > \frac{B_1\tau_\beta^2}{\tau_\beta + \tau_\iota}\cdot$$

Substituting this inequality in equation (4.34) shows

$$\frac{\partial^2\left(TAC\left(\tau_\iota, \tau_\beta|u^*\right)\right)}{\partial \tau_\beta^2} > \frac{2B_1\tau_\iota^2}{\left(\tau_\beta + \tau_\iota\right)^3} > 0 \quad \forall\ \theta\tau_\iota > 0. \tag{4.35}$$

The Hessian determinant of $TAC(\tau_\iota, \tau_\beta|u^*)$ is

$$H\left(\tau_\iota, \tau_\beta|u^*\right) = \left\{\frac{\partial^2 TAC\left(\tau_\iota, \tau_\beta|u^*\right)}{\partial \tau_\iota^2}\right\}\left\{\frac{\partial^2 TAC\left(\tau_\iota, \tau_\beta|u^*\right)}{\partial \tau_\beta^2}\right\} - \left\{\frac{\partial^2 TAC\left(\tau_\iota, \tau_\beta|u^*\right)}{\partial \tau_\iota \partial \tau_\beta}\right\}^2.$$

Using the second order partial derivatives from equations (4.32), (4.33), and (4.34)

$$\begin{aligned}\left(\tau_\beta + \tau_\iota\right)^4 H\left(\tau_\iota, \tau_\beta|u^*\right) = {} & 2C_1B_1\left(e^{\theta\tau_\iota}\left(1 - \theta\tau_\iota\right)^2 + e^{\theta\tau_\iota} - 2\right) + 2A_1\left(2B_1 + C_1\theta^2 e^{\theta\tau_\iota}\right) \\ & + \theta^2 {C_1}^2\left(e^{2\theta\tau_\iota} - 2\theta\tau_\iota e^{\theta\tau_\iota} - 1\right).\end{aligned} \tag{4.36}$$

In the feasible region, where $\theta\tau_\iota > 0$, the terms $e^{\theta\tau_\iota}\left(1 - \theta\tau_\iota\right)^2 + e^{\theta\tau_\iota} - 2$ and $e^{2\theta\tau_\iota} - 2\theta\tau_\iota e^{\theta\tau_\iota} - 1$ can be simplified to obtain the following inequality from equation (4.36)

$$\left(\tau_\beta + \tau_\iota\right)^4 H\left(\tau_\iota, \tau_\beta|u^*\right) > 2A_1\left(2B_1 + C_1\theta^2 e^{\theta\tau_\iota}\right) + C_1B_1\left(\theta\tau_\iota\right)^4 + \frac{\theta^2 {C_1}^2\left(\theta\tau_\iota\right)^4}{4} > 0 \quad \forall \quad \theta\tau_\iota > 0. \tag{4.37}$$

Hence, from equation (4.35) and (4.37), it can be concluded that $TAC(\tau_\iota, \tau_\beta|u^*)$ is convex in feasible range of τ_ι, τ_β for any particular feasible value of $u = u^*$.

Similarly, the convexity of $TAC(u|\tau_\iota, \tau_\beta)$ for any particular feasible values of τ_ι, τ_β is proved analytically. Rearrange equation (4.30) for calculating the partial derivative of the function $TAC(u|\tau_\iota, \tau_\beta)$ with respect to u, while noting that $\theta = \theta_0 e^{-au}$ and $\tau_\iota = \tau_\iota^*,\ \tau_\beta = \tau_\beta^*$.

$$TAC\left(u|\tau_\iota^*, \tau_\beta^*\right) = u + H_1 + \left[\frac{D_{1h}}{\theta^2} + \frac{D_{1w}}{\theta}\right]\left(e^{\theta\tau_\iota^*} - 1 - \theta\tau_\iota^*\right), \tag{4.38}$$

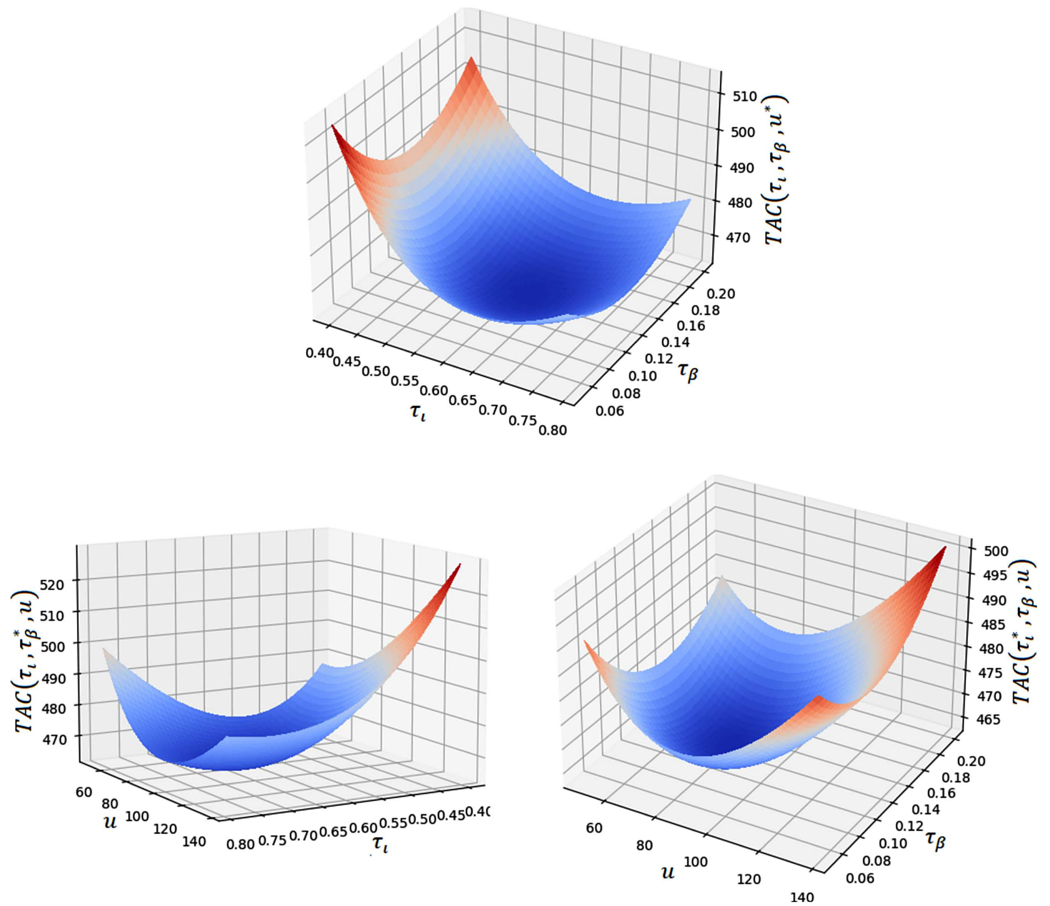

FIGURE 3. Objective function convexity (Model-2).

where $\frac{(A+k\rho^m)}{\tau_\beta^*+\tau_\iota^*} + \frac{Ds{\tau_\beta^*}^2}{2(\tau_\beta^*+\tau_\iota^*)} = H_1$, $\frac{Dh}{\tau_\beta^*+\tau_\iota^*} = D_{1h}$, $\frac{Dw}{\tau_\beta^*+\tau_\iota^*} = D_{1w}$.

Computing second order partial derivative with respect to u of equation (4.38) (using chain rule) gives

$$\frac{\partial^2 TAC}{\partial u^2} = \frac{a^2 D_{1h}}{\theta^2}\left\{\left(\theta^2{\tau_\iota^*}^2 - 3\theta\tau_\iota^* + 4\right)e^{\theta\tau_\iota^*} - \theta\tau_\iota^* - 4\right\} + \frac{a^2 D_{1w}}{\theta}\left\{\left(\theta^2{\tau_\iota^*}^2 - \theta\tau_\iota^* + 1\right)e^{\theta\tau_\iota^*} - 1\right\}. \quad (4.39)$$

Simplifying $\left(\theta^2{\tau_\iota^*}^2 - 3\theta\tau_\iota^* + 4\right)e^{\theta\tau_\iota^*} - \theta\tau_\iota^* - 4$ and $\left(\theta^2{\tau_\iota^*}^2 - \theta\tau_\iota^* + 1\right)e^{\theta\tau_\iota^*} - 1$ in equation (4.39) leads to

$$\frac{\partial^2 TAC\left(u|\tau_\iota^*, \tau_\beta^*\right)}{\partial u^2} > \frac{a^2 D_{1h}}{\theta^2}\left\{\frac{\theta^3{\tau_\iota^*}^3}{6}\left(\theta^2{\tau_\iota^*}^2 + 1\right)\right\} + \frac{a^2 D_{1w}}{\theta}\left\{\theta^3{\tau_\iota^*}^3\right\} > 0 \quad \forall\ \theta\tau_\iota^* > 0. \quad (4.40)$$

Hence $TAC\left(u|\tau_\iota^*, \tau_\beta^*\right)$ is convex in the feasible range of u for any particular feasible values of $\tau_\iota = \tau_\iota^*,\ \tau_\beta = \tau_\beta^*$. The convex nature of the objective function is further illustrated below which is shown in Figure 3.

These results prove the validity of the optimization process. Solving the equations obtained by setting the first partial derivatives of equation (4.30) with respect to the three decision variables (τ_ι, τ_β and u) to zero gives the optimal values for a continuous review inventory model considering deterioration and preservation effects and crisp demand (Model-2).

$$\frac{\partial TAC}{\partial \tau_\iota} = \frac{(u - TAC)}{\tau_\beta + \tau_\iota} + \frac{D\,(h + \theta w)}{\theta}\frac{\left(e^{\theta\tau_\iota} - 1\right)}{\tau_\beta + \tau_\iota} \tag{4.41}$$

$$\frac{\partial TAC}{\partial \tau_\beta} = \frac{(u - TAC)}{\tau_\beta + \tau_\iota} + \frac{Ds\tau_\beta}{\tau_\beta + \tau_\iota} \tag{4.42}$$

$$\frac{\partial TAC}{\partial u} = 1 + \frac{d}{d\theta}\left[\frac{D(h + \theta w)}{\theta^2}\frac{\left(e^{\theta\tau_\iota} - 1 - \theta\tau_\iota\right)}{\tau_\beta + \tau_\iota}\right]\frac{d\theta}{du}. \tag{4.43}$$

Equating the expressions from equations (4.41), (4.42) and (4.43) to zero and computing the optimal solutions

$$TAC(\tau_\iota^*, \tau_\beta^*, u^*) - u^* = \frac{D\,(h + \theta^* w)\left(e^{\theta^*\tau_\iota^*} - 1\right)}{\theta^*} = Ds\tau_\beta^* \tag{4.44}$$

$$(e^{\theta^*\tau_\iota^*} - 1 - \theta^*\tau_\iota^*)(2h + w\theta^*) + \frac{(\tau_\beta^* + \tau_\iota^*)\theta^{*2}}{aD} - \theta^*\tau_\iota^*(e^{\theta^*\tau_\iota^*} - 1)(h + w\theta^*) = 0. \tag{4.45}$$

Using equation (4.30) and solving equation (4.44) gives

$$\tau_\iota^* = \frac{1}{\theta^*}\log\left\{\frac{\theta^* s\tau_\beta^*}{h + \theta^* w} + 1\right\} \tag{4.46}$$

$$\tau_\beta^* = \sqrt{\tau_\iota^{*2} + \frac{2(A + k\rho^m)}{Ds} + \frac{2\,(h + \theta^* w)\,(e^{\theta^*\tau_\iota^*} - 1 - \theta^*\tau_\iota^*)}{s\theta^{*2}}} - \tau_\iota^*\ . \tag{4.47}$$

Equations (4.45), (4.46), and (4.47) provide the numerical values of $\tau_\iota^*, \tau_\beta^*, u^*$ (where $\theta^* = \theta_0 e^{-au^*}$) when solved iteratively. The inventory system's minimum total cost per unit time together with Q^* is subsequently calculated using equations (4.28) and (4.44) as

$$TAC^* = u^* + \tau_\beta^* Ds\ , \qquad Q^* = \frac{D}{\theta^*}(e^{\theta^*\tau_\iota^*} - 1) + D\tau_\beta^*\ . \tag{4.48}$$

As in Model-1, by equation (4.29), number of items lost due to deterioration per unit time at the optimal point is

$$\frac{D\left(e^{\theta^*\tau_\iota^*} - 1 - \theta^*\tau_\iota^*\right)}{\theta^*(\tau_\beta^* + \tau_\iota^*)} = \frac{Q^*}{\tau_\beta^* + \tau_\iota^*} - D.$$

4.4. A continuous review inventory model with fuzzy demand by considering deterioration items and preservation effects (Model-3)

Incorporating the effect of the promotional effort, in the previous models the annual demand has been taken in the form $D(\rho) = D_0 + d_1\rho$. In this subsection, fuzzy demand is considered where D_0 is a trapezoidal fuzzy number, i.e, $\widetilde{D_0} = TpFN\,(D_0 - \Delta_{l2}, D_0 - \Delta_{l1}, D_0 + \Delta_{r1}, D_0 + \Delta_{r2})$. This enhances the flexibility in modelling real scenarios.

In the present article, the function principle and signed distance defuzzification procedure (see Jaggi *et al.* [9], Sharma and Govindaluri [45]) are considered.

Now, the membership function of $\widetilde{D_0}$ is as follows:

$$\mu_{\widetilde{D_0}}(x) = \left\{\begin{array}{cc} \frac{x-(D_0-\Delta_{l2})}{(D_0-\Delta_{l1})-(D_0-\Delta_{l2})} & \text{in } \ (D_0 - \Delta_{l2}) \le x \le (D_0 - \Delta_{l1}) \\ 1 & \text{in } \ (D_0 - \Delta_{l1}) \le x \le (D_0 + \Delta_{r1}) \\ \frac{(D_0+\Delta_{r2})-x}{(D_0+\Delta_{r2})-(D_0+\Delta_{r1})} & \text{in } \ (D_0 + \Delta_{r1}) \le x \le (D_0 + \Delta_{r2}) \\ 0 & \text{otherwise} \end{array}\right\}. \tag{4.49}$$

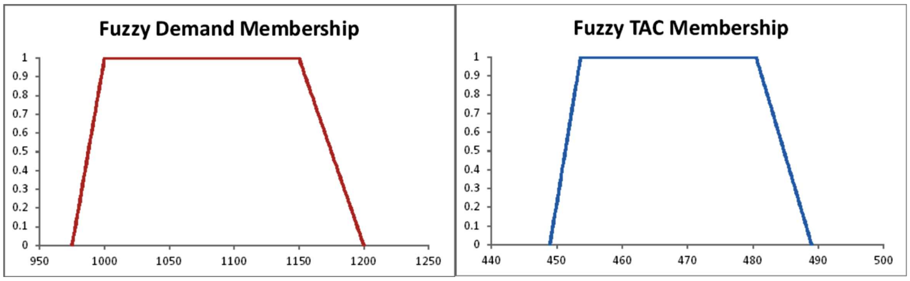

FIGURE 4. Fuzziness of demand parameter and objective function.

With this fuzzy demand, equation (4.30) is transformed to

$$\widetilde{TAC}\left(\tau_{\iota},\tau_{\beta},u\right)=\frac{\widetilde{D_0}+d_1\rho}{\tau_{\beta}+\tau_{\iota}}\left[\frac{(h+\theta w)\left(e^{\theta\tau_{\iota}}-1-\theta\tau_{\iota}\right)}{\theta^2}+\frac{s\tau_{\beta}^2}{2}\right]+\frac{(A+k\rho^m)}{\tau_{\beta}+\tau_{\iota}}+u. \tag{4.50}$$

Then by the function principle (Mahata and Goswami [17]), as the demand has a $TpFN$ base value, the above expression reveals that the total cost per unit time becomes a $TpFN$ as shown in Figure 4.

The parameters of this $TpFN$ are themselves real valued functions. For any feasible values of $\tau_{\iota},\tau_{\beta},u$ this holds:

$$\begin{aligned}
TAC_{l2}\left(\tau_{\iota},\tau_{\beta},u\right) &\le TAC_{l1}\left(\tau_{\iota},\tau_{\beta},u\right) \;\le\; TAC_{r1}\left(\tau_{\iota},\tau_{\beta},u\right) \;\le\; TAC_{r2}\left(\tau_{\iota},\tau_{\beta},u\right)\\
\widetilde{TAC}\left(\tau_{\iota},\tau_{\beta},u\right) &= TpFN\left(TAC_{l2},TAC_{l1},TAC_{r1},TAC_{r2}\right) \qquad \text{where}\\
TAC_{l2}(\tau_{\iota},\tau_{\beta},u) &= \frac{(D_0-\Delta_{l2})+d_1\rho}{\tau_{\beta}+\tau_{\iota}}\left[\frac{(h+\theta w)\left(e^{\theta\tau_{\iota}}-1-\theta\tau_{\iota}\right)}{\theta^2}+\frac{s\tau_{\beta}^2}{2}\right]+\frac{(A+k\rho^m)}{\tau_{\beta}+\tau_{\iota}}+u\\
TAC_{l1}\left(\tau_{\iota},\tau_{\beta},u\right) &= \frac{(D_0-\Delta_{l1})+d_1\rho}{\tau_{\beta}+\tau_{\iota}}\left[\frac{(h+\theta w)\left(e^{\theta\tau_{\iota}}-1-\theta\tau_{\iota}\right)}{\theta^2}+\frac{s\tau_{\beta}^2}{2}\right]+\frac{(A+k\rho^m)}{\tau_{\beta}+\tau_{\iota}}+u\\
TAC_{r1}\left(\tau_{\iota},\tau_{\beta},u\right) &= \frac{(D_0+\Delta_{r1})+d_1\rho}{\tau_{\beta}+\tau_{\iota}}\left[\frac{(h+\theta w)\left(e^{\theta\tau_{\iota}}-1-\theta\tau_{\iota}\right)}{\theta^2}+\frac{s\tau_{\beta}^2}{2}\right]+\frac{(A+k\rho^m)}{\tau_{\beta}+\tau_{\iota}}+u\\
TAC_{r2}\left(\tau_{\iota},\tau_{\beta},u\right) &= \frac{(D_0+\Delta_{r2})+d_1\rho}{\tau_{\beta}+\tau_{\iota}}\left[\frac{(h+\theta w)\left(e^{\theta\tau_{\iota}}-1-\theta\tau_{\iota}\right)}{\theta^2}+\frac{s\tau_{\beta}^2}{2}\right]+\frac{(A+k\rho^m)}{\tau_{\beta}+\tau_{\iota}}+u.
\end{aligned}$$

Applying the median formula, the total cost per time unit in Model-3 is estimated as

$$\begin{aligned}
\text{Median }\left(\widetilde{TAC}\left(\tau_{\iota},\tau_{\beta},u\right)\right) &= \frac{1}{4}\left[TAC_{l2}\left(\tau_{\iota},\tau_{\beta},u\right)+TAC_{l1}\left(\tau_{\iota},\tau_{\beta},u\right)+TAC_{r1}\left(\tau_{\iota},\tau_{\beta},u\right)+TAC_{r2}\left(\tau_{\iota},\tau_{\beta},u\right)\right]\\
&= \frac{D_0+d_1\rho}{\tau_{\beta}+\tau_{\iota}}\left[\frac{(h+\theta w)\left(e^{\theta\tau_{\iota}}-1-\theta\tau_{\iota}\right)}{\theta^2}+\frac{s\tau_{\beta}^2}{2}\right]+\frac{(A+k\rho^m)}{\tau_{\beta}+\tau_{\iota}}+u\\
&+\frac{(\Delta_{r1}-\Delta_{l1}+\Delta_{r2}-\Delta_{l2})}{4\left(\tau_{\beta}+\tau_{\iota}\right)}\times\left[\frac{(h+\theta w)}{\theta^2}\left(e^{\theta\tau_{\iota}}-1-\theta\tau_{\iota}\right)+\frac{s\tau_{\beta}^2}{2}\right].
\end{aligned} \tag{4.51}$$

Thus, the median estimate of the total cost per time unit for Model-3 (fuzzy demand with preservation) exists as the sum of two parts. The first term is identical to the total cost per time unit for Model-2 (crisp demand with preservation).

$$\text{Median}\ \left(\widetilde{TAC}\left(\tau_\iota,\tau_\beta,u\right)\right)=TAC\left(\tau_\iota,\tau_\beta,u\right)+FC\left(\tau_\iota,\tau_\beta,u\right)$$

where

$$FC\left(\tau_\iota,\tau_\beta,u\right)=\frac{\left(\Delta_{r1}-\Delta_{l1}+\Delta_{r2}-\Delta_{l2}\right)}{4\left(\tau_\beta+\tau_\iota\right)}\left[\frac{(h+\theta w)\left(e^{\theta\tau_\iota}-1-\theta\tau_\iota\right)}{\theta^2}+\frac{s\tau_\beta^2}{2}\right]. \tag{4.52}$$

4.4.1. Optimization methodology (Model-3)

Lemma 4.1. *The non-negative weighted sum of convex functions (over the same interval) is itself convex. (Boyd and Vandenberghe [2]).*

The convexity of $FC(\tau_\iota,\tau_\beta,u)$ as given by equation (4.52) is investigated. $\frac{\partial^2 FC(u|\tau_\iota,\tau_\beta)}{\partial u^2}$ and the Hessian of $FC(\tau_\iota,\tau_\beta|u)$ are constructed. As in Model-2; the terms are grouped.

For any particular given feasible value of $u=u^*$ (hence $\theta=\theta_0e^{-au^*}$ below) in equation (4.52)

$$FC(\tau_\iota,\tau_\beta|u^*)=\frac{B_2\tau_\beta^2}{\tau_\beta+\tau_\iota}+\frac{C_2\left(-1-\theta\tau_\iota+e^{\theta\tau_\iota}\right)}{\tau_\beta+\tau_\iota} \tag{4.53}$$

where $\Delta_{var}=(\Delta_{r1}-\Delta_{l1}+\Delta_{r2}-\Delta_{l2})$ and

$$B_2=\frac{\Delta_{var}s}{8}\quad,\qquad C_2=\frac{\Delta_{var}(h+\theta w)}{4\theta^2}\cdot$$

From equation (4.53), the first order partial derivatives of $FC(\tau_\iota,\tau_\beta|u^*)$ with respect to τ_ι and τ_β are

$$\frac{\partial FC}{\partial\tau_\iota}=\frac{-(FC)}{\tau_\beta+\tau_\iota}+\frac{\theta C_2(e^{\theta\tau_\iota}-1)}{\tau_\beta+\tau_\iota}$$
$$\frac{\partial FC}{\partial\tau_\beta}=\frac{-(FC)}{\tau_\beta+\tau_\iota}+\frac{2B_2\tau_\beta}{\tau_\beta+\tau_\iota}\cdot$$

The second order partial derivatives of $FC(\tau_\iota,\tau_\beta|u^*)$ with respect to τ_ι and τ_β then become

$$\frac{\partial^2 FC}{\partial\tau_\iota^2}=\frac{2(FC)}{(\tau_\beta+\tau_\iota)^2}-\frac{2\theta C_2(e^{\theta\tau_\iota}-1)}{(\tau_\beta+\tau_\iota)^2}+\frac{\theta^2C_2e^{\theta\tau_\iota}}{\tau_\beta+\tau_\iota} \tag{4.54}$$
$$\frac{\partial^2 FC}{\partial\tau_\iota\partial\tau_\beta}=\frac{2(FC)}{(\tau_\beta+\tau_\iota)^2}-\frac{\theta C_2(e^{\theta\tau_\iota}-1)}{(\tau_\beta+\tau_\iota)^2}-\frac{2B_2\tau_\beta}{(\tau_\beta+\tau_\iota)^2} \tag{4.55}$$
$$\frac{\partial^2 FC}{\partial\tau_\beta^2}=\frac{2(FC)}{(\tau_\beta+\tau_\iota)^2}-\frac{4B_2\tau_\beta}{(\tau_\beta+\tau_\iota)^2}+\frac{2B_2}{\tau_\beta+\tau_\iota}\cdot \tag{4.56}$$

In equation (4.53), the inequality $e^{\theta\tau_\iota}-\ \theta\tau_\iota-1>0\ \forall\ \theta\tau_\iota>0$ gives

$$FC(\tau_\iota,\tau_\beta|u^*)=\frac{B_2\tau_\beta^2}{\tau_\beta+\tau_\iota}+\frac{C_2\left(-1-\theta\tau_\iota+e^{\theta\tau_\iota}\right)}{\tau_\beta+\tau_\iota}>\frac{B_2\tau_\beta^2}{\tau_\beta+\tau_\iota}\cdot$$

Substituting this in equation (4.56)

$$\frac{\partial^2 FC}{\partial\tau_\beta^2}>\frac{2B_2\tau_\iota^2}{(\tau_\beta+\tau_\iota)^3}\Rightarrow\frac{\partial^2(FC(\tau_\iota,\tau_\beta|u^*))}{\partial\tau_\beta^2}>0. \tag{4.57}$$

The Hessian determinant of $FC(\tau_\iota, \tau_\beta|u^*)$ is

$$H(\tau_\iota, \tau_\beta|u^*) = \left\{\frac{\partial^2 FC(\tau_\iota, \tau_\beta|u^*)}{\partial \tau_\iota^2}\right\}\left\{\frac{\partial^2 FC(\tau_\iota, \tau_\beta|u^*)}{\partial \tau_\beta^2}\right\} - \left\{\frac{\partial^2 FC(\tau_\iota, \tau_\beta|u^*)}{\partial \tau_\iota \partial \tau_\beta}\right\}^2.$$

Substituting the expressions for the second order partial derivatives from equations (4.54), (4.55), and (4.56)

$$(\tau_\beta + \tau_\iota)^4 H(\tau_\iota, \tau_\beta|u^*) = 2C_2B_2(e^{\theta\tau_\iota}(1-\theta\tau_\iota)^2 + e^{\theta\tau_\iota} - 2) + \theta^2 {C_2}^2(e^{2\theta\tau_\iota} - 2\theta\tau_\iota e^{\theta\tau_\iota} - 1). \tag{4.58}$$

For any particular feasible value of $u = u^*$ this can be simplified as in equation (4.36).

$$(\tau_\beta + \tau_\iota)^4 H(\tau_\iota, \tau_\beta|u^*) > C_2B_2(\theta\tau_\iota)^4 + \frac{\theta^2 {C_2}^2(\theta\tau_\iota)^4}{4} > 0 \;\forall\; \theta\tau_\iota > 0. \tag{4.59}$$

Thus, from equations (4.57) and (4.59); $FC(\tau_\iota, \tau_\beta|u^*)$ is convex in feasible range of τ_ι, τ_β for any particular feasible value of $u = u^*$.

Consider $FC(u|\tau_\iota, \tau_\beta)$ for any particular feasible value of τ_ι, τ_β in equation (4.52), for $\tau_\iota = \tau_\iota^*, \tau_\beta = \tau_\beta^*$ let

$$FC(u|\tau_\iota^*, \tau_\beta^*) = H_2 + \left[\frac{D_{2h}}{\theta^2} + \frac{D_{2w}}{\theta}\right](e^{\theta\tau_\iota^*} - 1 - \theta\tau_\iota^*) \tag{4.60}$$

where $\Delta_{var} = (\Delta_{r1} - \Delta_{l1} + \Delta_{r2} - \Delta_{l2})$ and

$$H_2 = \frac{\Delta_{var}}{4(\tau_\beta^* + \tau_\iota^*)}\left(\frac{s{\tau_\beta^*}^2}{2}\right) \quad , \quad D_{2h} = \frac{\Delta_{var} h}{4(\tau_\beta^* + \tau_\iota^*)} \quad , \quad D_{2w} = \frac{\Delta_{var} w}{4(\tau_\beta^* + \tau_\iota^*)}.$$

The first and second order partial derivatives of $FC(u|\tau_\iota^*, \tau_\beta^*)$ with respect to u are computed from equation (4.60) (where $\theta = \theta_0 e^{-au}$).

$$\frac{\partial FC}{\partial u} = a\theta\left[\tau_\iota^*(1 - e^{\theta\tau_\iota^*})\left\{\frac{D_{2h}}{\theta^2} + \frac{D_{2w}}{\theta}\right\} + \left(e^{\theta\tau_\iota^*} - 1 - \theta\tau_\iota^*\right)\left\{\frac{2D_{2h}}{\theta^3} + \frac{D_{2w}}{\theta^2}\right\}\right]$$

$$\frac{\partial^2 FC}{\partial u^2} = \frac{a^2 D_{2h}}{\theta^2}\left\{(\theta^2{\tau_\iota^*}^2 - 3\theta\tau_\iota^* + 4)e^{\theta\tau_\iota^*} - \theta\tau_\iota^* - 4\right\} + \frac{a^2 D_{2w}}{\theta}\left\{(\theta^2{\tau_\iota^*}^2 - \theta\tau_\iota^* + 1)e^{\theta\tau_\iota^*} - 1\right\}. \tag{4.61}$$

The exponential terms in equation (4.61) are comparable to those in equation (4.39) and are simplified similarly.

$$\frac{\partial^2 FC(u|\tau_\iota^*, \tau_\beta^*)}{\partial u^2} > \frac{a^2 D_{2h}}{\theta^2}\left\{\frac{\theta^3{\tau_\iota^*}^3}{6}(\theta^2{\tau_\iota^*}^2 + 1)\right\} + \frac{a^2 D_{2w}}{\theta}\left\{\theta^3{\tau_\iota^*}^3\right\} > 0 \quad \forall\; \theta\tau_\iota^* > 0. \tag{4.62}$$

Therefore $FC(u|\tau_\iota^*, \tau_\beta^*)$ is convex in feasible range of u for any particular feasible value of $\tau_\iota = \tau_\iota^*,\ \tau_\beta = \tau_\beta^*$.

Since, Median $(\widetilde{TAC}(\tau_\iota, \tau_\beta, u)) = TAC(\tau_\iota, \tau_\beta, u) + FC(\tau_\iota, \tau_\beta, u)$, hence the convexity of Median $(\widetilde{TAC}(\tau_\iota, \tau_\beta, u))$ can be determined from the convexity of $TAC(\tau_\iota, \tau_\beta, u)$ and $FC(\tau_\iota, \tau_\beta, u)$.

As per the analysis of both $TAC(\tau_\iota, \tau_\beta, u)$ and $FC(\tau_\iota, \tau_\beta, u)$ shown in equations (4.37), (4.40), (4.59), and (4.62); from the above lemma it is concluded that in this model, Median $(\widetilde{TAC}(\tau_\iota, \tau_\beta, u))$ is itself convex and can be optimized under Model-3 by solving the following equations for τ_ι, τ_β, and u.

$$\frac{\partial \text{Median } (\widetilde{TAC}(\tau_\iota, \tau_\beta, u))}{\partial \tau_\iota} = \frac{\partial TAC(\tau_\iota, \tau_\beta, u)}{\partial \tau_\iota} + \frac{\partial FC(\tau_\iota, \tau_\beta, u)}{\partial \tau_\iota} = 0$$

$$\frac{\partial \text{Median } (\widetilde{TAC}(\tau_\iota, \tau_\beta, u))}{\partial \tau_\beta} = \frac{\partial TAC(\tau_\iota, \tau_\beta, u)}{\partial \tau_\beta} + \frac{\partial FC(\tau_\iota, \tau_\beta, u)}{\partial \tau_\beta} = 0$$

$$\frac{\partial \text{Median } (\widetilde{TAC}(\tau_\iota, \tau_\beta, u))}{\partial u} = \frac{\partial TAC(\tau_\iota, \tau_\beta, u)}{\partial u} + \frac{\partial FC(\tau_\iota, \tau_\beta, u)}{\partial u} = 0.$$

The iteration scheme for optimization of $Median(\widetilde{TAC}(\tau_\iota, \tau_\beta, u))$ is obtained from these three equations as follows

$$\tau_{f\iota} = \frac{1}{\theta_f} \log\left(\frac{\theta_f \tau_{f\beta} s}{h + \theta_f w} + 1\right) \tag{4.63}$$

$$\tau_{f\beta} = \left[\tau_{f\iota}^2 + \frac{8(A + k\rho^m)}{s(4D + (\Delta_{r1} - \Delta_{l1} + \Delta_{r2} - \Delta_{l2}))} + \frac{2(h + \theta_f w)\left(e^{\theta_f \tau_{f\iota}} - 1 - \theta_f \tau_{f\iota}\right)}{s{\theta_f}^2}\right]^{\frac{1}{2}} - \tau_{f\iota} \tag{4.64}$$

$$u_f = \frac{\log(\theta_0) - \log(\theta_f)}{a}$$

where

$$(2h + \theta_f w + \theta_f h \tau_{f\iota}) + \Big[(\theta_f \tau_{f\iota} - 1)(\theta_f w + h) - h\Big] e^{\theta_f \tau_{f\iota}} - \frac{4\Big(\tau_{f\iota} + \tau_{f\beta}\Big){\theta_f}^2}{a\Big(4D + (\Delta_{r1} - \Delta_{l1} + \Delta_{r2} - \Delta_{l2})\Big)} = 0. \tag{4.65}$$

As in Model-2, using the optimal time duration values $\tau_{f\iota}^*$, $\tau_{f\beta}^*$ and the optimal preservation cost per unit time u_f^* obtained from the above scheme (Eqs. (4.63), (4.64), and (4.65)) with $\theta_f^* = \theta_0 e^{-a u_f^*}$, gives the optimal results for the Model-3. The optimal TAC and Q for Model-3 are

$$Q_f^* = \left(D + \frac{\Delta_{r1} - \Delta_{l1} + \Delta_{r2} - \Delta_{l2}}{4}\right)\left[\tau_{f\beta}^* + \frac{e^{\theta_f^* \tau_\iota^*} - 1}{\theta_f^*}\right] \tag{4.66}$$

$$\text{Median } (\widetilde{TAC_f^*}) = u_f^* + s\tau_{f\beta}^*\left(D + \frac{\Delta_{r1} - \Delta_{l1} + \Delta_{r2} - \Delta_{l2}}{4}\right). \tag{4.67}$$

5. Numerical analysis

5.1. Input parameters

A numerical example is employed to illustrate the models developed here. A decision maker may extract useful insights by analysing the results. Numerical values of the parameters used (Mahapatra *et al.* [15]) here are given below: $A = 80$, $h = 0.4$, $D_0 = 1000$, $d_1 = 20$, $k = 10$, $m = 2$, $\rho = 2.5$, $w = 6$, $s = 3$, $\Delta_{l2} = 75$, $\Delta_{l1} = 50$, $\Delta_{r1} = 100$, $\Delta_{r2} = 150$ and $\theta_0 = 0.15$. The preservation technology investment (u) dampens the deterioration rate as $w(u) = \theta_0 e^{-au}$, where the positive parameter a is a coefficient measuring the dampening of deterioration rate for every unit of such investment u. The value of a is considered as $a = 0.02$ in this study. The upper bound of the preservation investment is $U = 200$ where $u \leq U$. The values of the decision variables: time duration for inventory level to fall to zero after replenishment (τ_ι), time between zero inventory and replenishment that is fully backlogged, (τ_β) and preservation cost (u) for Model-1, Model-2 and Model-3 and corresponding order quantity and total cost at the optimal point are tabulated (Tab. 3).

The column $\frac{Q^*}{\tau_\iota^* + \tau_\beta^*} - D$ gives the number of units lost per unit time over a cycle as a measure of the effective rate of deterioration. In Table 3, the per unit cost per unit time at the optimal point is $\frac{TAC^*}{Q^*} = 0.907, 0.587, 0.585$ for Models-1,2,3, respectively, which proves that the fuzzy formulation (Model-3) is the most efficient on a per unit basis even though it has a slightly higher TAC^* than Model-2.

Figure 5 shows the variation in TAC^* for various cycle lengths ($\tau_\iota^* + \tau_\beta^*$). It exhibits two distinct phases. In the first phase, for very small cycle lengths, the impact of deterioration is not prominent and all three models yield similar results.

TABLE 3. Optimal solutions of Model-1, Model-2, and Model-3.

Model	τ_ι^*	τ_β^*	u^*	Q^*	$\frac{Q^*}{\tau_\iota^*+\tau_\beta^*} - D$	$TAC(\tau_\iota^*, \tau_\beta^*, u^*)$(per cycle)
Model-1	0.37	0.17	–	578.18	20.73	524.40
Model-2	0.62	0.12	81.10	787.98	08.17	462.84
Model-3	0.62	0.12	82.03	801.47	08.17	468.46

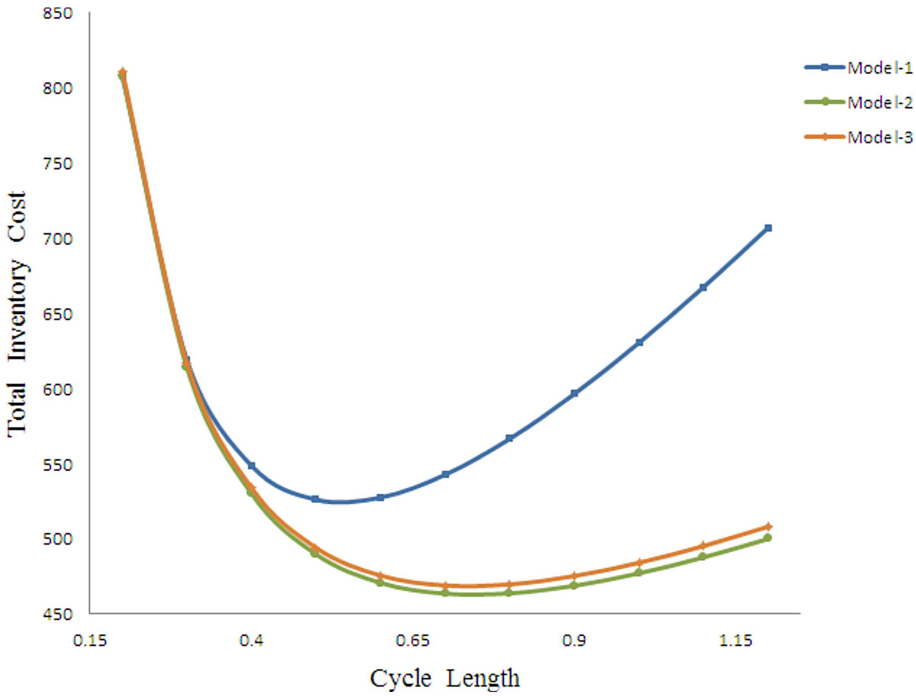

FIGURE 5. Variation in optimum total cost with cycle length.

As cycle length increases, the impact of deterioration and the economy of preservation becomes clear. The second phase has the total cost of Model-1 (without preservation) rising sharply beyond its global minima (at $\tau_\iota^* + \tau_\beta^* = 0.54$) whereas the other two models attain lower global optimal costs at longer cycle lengths (around $\tau_\iota^* + \tau_\beta^* = 0.74$). The change in TAC^* is much more gentle in Model-2 and Model-3 as cycle length continues to increase.

A substantial fraction of the total inventory cost goes into preservation investment in Model-2 and Model-3. The percentage investment in preservation is shown in Figure 6. For longer cycle lengths, Model-3 has a slightly lower fraction of the cost under this head.

6. SENSITIVITY ANALYSIS

6.0.1. Effect of unit shortage cost (s)

The length of the backorder phase τ_β increases with the decrease in shortage cost s and there is a small saving in TAC. It becomes profitable to keep larger optimal backorder in a cycle. In Model-2, preservation technology

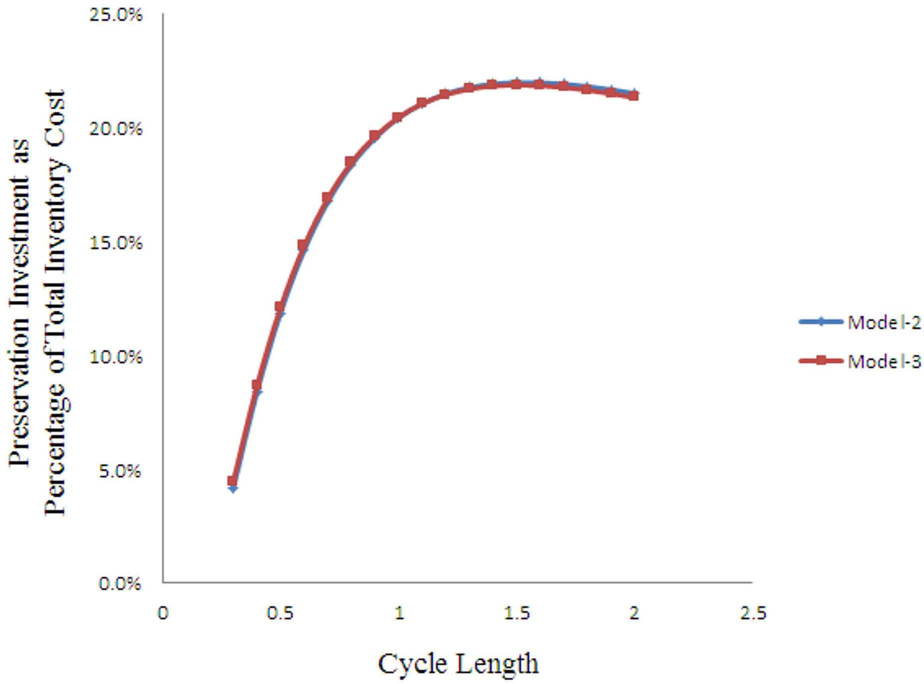

FIGURE 6. Preservation investment percentage with respect to cycle length.

ensures that TAC is much less elastic. The fuzzy values mimic the movement of the crisp values. The fuzziness of demand is balanced out by a small increase in EOQ without much impact on the TAC.

6.0.2. Impact of unit holding cost (h)

The reduction of holding cost has a proportionately greater impact than an increase of similar magnitude. In sensitivity computation, h complements s as this time τ_ι is more affected. The preservation investment allows greater variation in τ_ι, τ_β, and a much larger decrease in TAC and increases in EOQ when h is reduced as bigger orders become optimal. The median fuzzy output shows a displacement in line with the crisp boundaries set by the decision-maker.

6.0.3. Coefficient of preservation (a)

When the coefficient of preservation (a) increases, it leads to a higher EOQ and larger positive inventory time (τ_ι) while still reducing TAC. Model-3 uses a larger investment u to compensate for the fuzziness in demand. There is a change of optimal pattern below $a = 0.017$. Here, u decreases as the system reach a different optimal strategy with more items lost as preservation investment has a sharp reduction in efficiency towards dampening deterioration.

6.0.4. Effect of unit deterioration cost (w)

The positive inventory time and EOQ both increase to take advantage of reduced w and even with more items lost, the TAC is reduced in Model-1. When preservation is included, the effect on TAC and EOQ is smaller. The preservation investment fluctuates in parallel with w and almost completely shields the time duration variables from its effects. Model-3 is marginally less affected in this case than Model-2.

6.0.5. Effect of ordering cost (A)

The ordering cost A has a consistent impact across all three models. There is a small but significant increase in the magnitude of change of positive inventory time and EOQ, but a decrease in the change of backorder time τ_β, and TAC when preservation is introduced. Both these movements are marginally less in Model-3.

6.0.6. Effect of constant deterioration rate (θ_0)

The TAC increases as θ_0 increases but the decrease is more pronounced as this parameter is reduced. The same nature of change occurs in the preservation investment u and for the chosen example this nullifies the effect on the time duration variables. In Model-1 (without this preservation variable), when deterioration increases it is profitable to keep larger backorders, and for low deterioration, larger EOQ and τ_ι are optimal.

6.0.7. Effect of promotional effort (ρ)

A larger value of ρ (promotional effort) increases the TAC, EOQ, and all three decision variables as it directly increases demand. Without preservation, backorder time duration τ_β fluctuates marginally more but with preservation, positive inventory duration changes by a larger margin. For increased demand and reasonably effective preservation, it is optimal to order more and satisfy direct demand rather than keep bigger back orders. The Model-3 shows marginally more stability on a point-to-point basis than Model-2.

The above analysis is graphically reinforced in Figure 7 which separately explores the sensitivity of each of the decision variables τ_ι, τ_β (for Model-1 and Model-2), u (for Model-2), and the objective function ($TAC(\tau_\iota, \tau_\beta)$ and $TAC(\tau_\iota, \tau_\beta, u)$). Each curve represents the effect of changing one of the parameters among unit shortage cost (s), unit holding cost (h), coefficient of preservation (a), unit deterioration cost (w), ordering cost (A), constant deterioration rate (θ_0), and promotional effort (ρ). The movement of the decision variables and the objective function, as they attain new optimal values are shown when individual parameters change by $-50\%, -25\%, +25\%, +50\%$ about their values as specified in the numerical example above while other parameters remain unchanged.

6.1. Managerial insights

Distilling the impact of parameter variation on decision variables and objective function (Tab. 3, Figs. 5, 6, and 7) leads to comprehension of trends beyond the numerical results. The following points provide managerial guidelines for the decision makers:

- In the absence of preservation (Model-1), promotional activities can help to mitigate adverse customer impact during stockouts as it is profitable to keep larger backorders to balance deterioration.
- In the presence of preservation, the decision maker need not be conservative with the predicted cycle length as the optimal cost only rises gradually beyond the global minima (Model-2, Model-3).
- Shorter cycle lengths are sometimes forced due to real-life limitations (storage constraints, erratic demand patterns, supply limitations). This renders the preservation investment uneconomical.
- The fraction of total cost optimally invested in preservation does not remain constant. It appears to increase in step with the cycle length, but then attains a maximum level around 22% beyond which it diminishes very gradually. This observation may provide a budgetary clue for the decision makers.
- The preservation investment almost completely shields the time duration variables from the rate and cost of deterioration. When the preservation investment is reasonably effective, decision makers can reduce the total cost and it is optimal to order more and satisfy direct demand during positive inventory rather than keep bigger backorders.
- There is a change of optimal pattern when preservation effectiveness falls below a threshold and here the decision makers should reduce the preservation investment as the system reaches a different optimal strategy despite more items lost to deterioration.

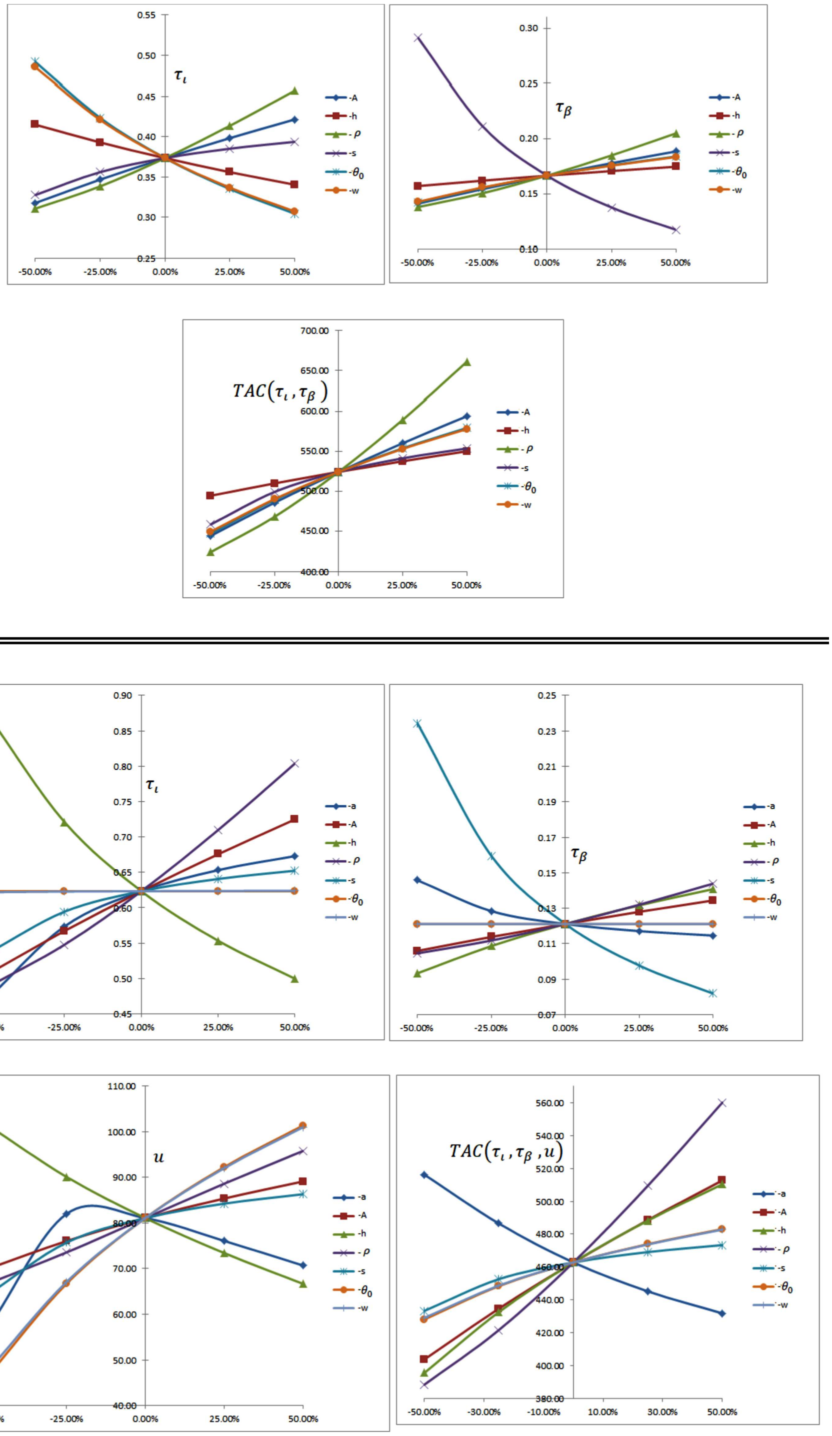

FIGURE 7. Sensitivity analysis: Model-1 parameters (top three: $\tau_\iota, \tau_\beta, TAC(\tau_\iota, \tau_\beta)$) Model-2 parameters (bottom four: $\tau_\iota, \tau_\beta, u, TAC(\tau_\iota, \tau_\beta, u)$).

7. Conclusions

This study expanded the existing models and solutions on EOQ with preservation cost to control deteriorating inventory levels. The per unit time cost was considered with both the period of on-hand inventory and the time with stockout condition in a reorder sequence to determine the best reorder size and cycle length, while the expected total average cost was minimized against the preservation investment and the time duration variables in the presence of promotional effort. The effect of deterioration increased the total cost as items were lost. Model-2 was expanded by modelling uncertainty in demand to generate Model-3. A useful formulation was developed to obtain the ideal values for the reorder process and the cost of preservation technology where the decision makers had imprecise information about the demand. This model illustrated the impact of preservation efforts to control the spoilage and the decision makers could compute the optimal investment for this purpose. The signed distance method was applied in the model with uncertain demand to defuzzify the fuzzy cost function. In both Model-2 and Model-3, an algorithm emerged that identified the optimal solutions from the analytical results presented therein.

A numerical setup was investigated to validate the developed models and examine their sensitivity against key parameters to determine the specific effect on the model. In the classical inventory model, shortage and holding costs had correlated effect on shortage time and positive inventory duration respectively. Analysis revealed that the preservation investment skewed this variation in favour of positive inventory duration. Effective promotional effort induced larger optimal order quantity without significant impact on costs thus economic benefits could be obtained by boosting the promotional effort. The total cost showed higher sensitivity to decrease in deterioration rate than to its increase.

The following aspects can be expanded for future work. Several avenues present themselves to augment this study, *e.g.*, replenishment that takes finite time, deteriorated items being reworked or substituted, uncertainty and randomness in other facets, multi-item inventory and learning effects, and a deterioration rate depending on the expiration date. The paradigm of renewable energy from the animal fat waste as deteriorated items (Habib *et al.* [8]) may provide an alternative model to recoup some of the deterioration impact.

This model can be extended to consider measures to reduce emissions, such as carbon offsets and CO_2 quotas. Some of the cost parameters may be sensitive to carbon emissions levels. It is possible to study the environmental impact of degradation and conservation technologies for sustainable supply chains through such inclusions. Analysis of the effects of non-linearity and time dependent operating costs, the presence of incomplete items, prepayments, trade credit, and economic policy inflation [31] are some of the possible generalizations of the presented models.

Appendix A.

The preliminary mathematical ideas of trapezoidal fuzzy numbers are introduced. The signed distance process of defuzzification is defined.

Fuzzy set $\widetilde{P}$: a set of ordered pairs defined using a membership function $\mu_{\widetilde{P}} : \Pi \to [0,1]$ as $\widetilde{P} = \{(x, \mu_{\widetilde{P}}(x)) : x \in \Pi\}$. Π is the universal set. For a trapezoidal fuzzy set $\widetilde{P}$ we use four crisp numbers, $p_1 < p_2 \leq p_3 < p_4$ to define the membership function.

$$\mu_{\widetilde{P}}(x) = \left\{ \begin{array}{cc} \frac{x-p_1}{p_2-p_1} & \text{if } \; p_1 \leq x \leq p_2 \\ 1 & \text{if } \; p_2 \leq x \leq p_3 \\ \frac{p_4-x}{p_4-p_3} & \text{if } \; p_3 \leq x \leq p_4 \\ 0 & \text{otherwise} \end{array} \right\}.$$

If $p_2 = p_3$, this becomes a triangular fuzzy set.

The arithmetical operations of two non-negative TpFN, $\widetilde{P} = TpFN(p_1, p_2, p_3, p_4)$ and $\widetilde{N} = TpFN(\nu_1, \nu_2, \nu_3, \nu_4)$

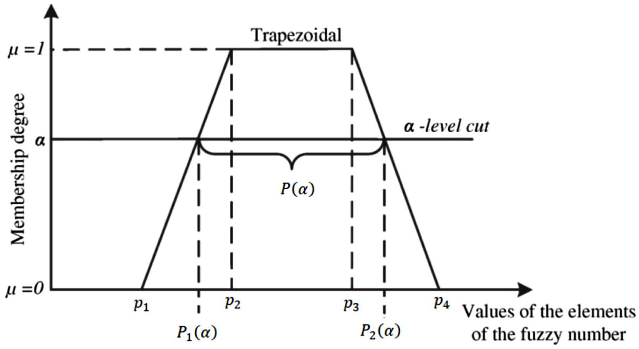

FIGURE A.1. A trapezoidal fuzzy number and its α-cut.

are defined by (Sharma and Govindaluri [45])

$$\begin{aligned}
\widetilde{P} \oplus \widetilde{Q} &= (p_1+\nu_1, p_2+\nu_2, p_3+\nu_3, p_4+\nu_4) \\
\widetilde{P} \odot \widetilde{Q} &= (p_1\nu_1, p_2\nu_2, p_3\nu_3, p_4\nu_4) \\
\widetilde{P} \ominus \widetilde{Q} &= (p_1-\nu_4, p_2-\nu_3, p_3-\nu_2, p_4-\nu_1) \\
\widetilde{P} \oslash \widetilde{Q} &= \left(\frac{p_1}{\nu_4}, \frac{p_2}{\nu_3}, \frac{p_3}{\nu_2}, \frac{p_4}{\nu_1}\right); \nu_1, \nu_2, \nu_3, \nu_4 \neq 0 \\
\kappa \otimes \widetilde{Q} &= (\kappa\nu_1, \kappa\nu_2, \kappa\nu_3, \kappa\nu_4) \quad \text{if } \kappa \geq 0 \\
\kappa \otimes \widetilde{Q} &= (\kappa\nu_4, \kappa\nu_3, \kappa\nu_2, \kappa\nu_1) \quad \text{if } \kappa < 0 \quad 0.
\end{aligned}$$

In a fuzzy set, the α-cut is the crisp set of those elements whose membership value is not less than α. The smallest and the largest elements of this set are denoted as the left α-cut and the right α-cut, respectively. Where, α-cut $= \{(\mu_{\widetilde{P}}(x) \geq \alpha : x \in \Pi)\},\ \alpha \in [0,1]$. For $\widetilde{P} = TpFN(p_1, p_2, p_3, p_4)$ this implies

$$(\alpha\text{-cut})\ P_\alpha = [p_1 + (p_2-p_1)\alpha\ ,\ p_4 - (p_4-p_3)\alpha] = [P_1(\alpha), P_2(\alpha)]$$
$$(\text{left } \alpha\text{-cut})\ P_1(\alpha) = p_1 + (p_2-p_1)\alpha\ \ , \qquad (\text{right } \alpha\text{-cut})\ P_2(\alpha) = p_4 - (p_4-p_3)\alpha.$$

This is shown in Figure A.1.

$Median(\widetilde{P})$, the defuzzified estimate of $\widetilde{P}$ is given by the signed distance, defined as

$$d\left(\widetilde{P}, 0\right) = \frac{1}{2}\int_0^1 \left[P_1(\alpha) + P_2(\alpha)\right]\, d\alpha = \frac{1}{4}(p_1+p_2+p_3+p_4) = Median(\widetilde{P}).$$

Acknowledgements. The authors express their gratitude to Professor Nita H. Shah (Gujarat University, Ahmedabad) for providing valuable suggestions. This work is supported by National Research Foundation of Korea (NRF) grant, funded by the Korea Government (MSIT) (NRF-2020R1F1A1064460).

Conflicts of interest. There is no conflict of interest as per the authors.
Availability of code, data and material. Inapplicable.
Authors contributions. The authors made equal contribution in this research.

Abbreviation

TpFN	:	Trapezoidal fuzzy number
EOQ	:	Economic order quantity
PEC	:	Promotional effort cost
OC	:	Ordering cost
HC	:	Holding cost
SC	:	Shortage cost
DC	:	Cost due to the deteriorated items
PC	:	Preservation cost

References

[1] H. Barman, M. Pervin, S.K. Roy and G.W. Weber, Back-ordered inventory model with inflation in a cloudy-fuzzy environment. *J. Ind. Manag. Optim.* **17** (2021) 1913.

[2] S.P. Boyd and L. Vandenberghe, Convex Optimization. Cambridge University Press, New York (2004).

[3] S.C. Das, A.K. Manna, M.S. Rahman, A.A. Shaikh and A.K. Bhunia, An inventory model for non-instantaneous deteriorating items with preservation technology and multiple credit periods-based trade credit financing via particle swarm optimization. *Soft Comput.* **25** (2021) 5365–5384.

[4] S.K. De and G.C. Mahata, A profit jump inventory model for imperfect quality items with receiving reparative batch and order overlapping in dense fuzzy environment. *Revue d'Orthopdie Dento-Faciale* **55** (2021).

[5] C.Y. Dye and T.P. Hsieh, An optimal replenishment policy for deteriorating items with effective investment in preservation technology. *Eur. J. Oper. Res.* **218** (2012) 106–112.

[6] T. Garai, D. Chakraborty and T.K. Roy, Fully fuzzy inventory model with price-dependent demand and time varying holding cost under fuzzy decision variables. *J. Intell. Fuzzy Syst.* **36** (2019) 3725–3738.

[7] C.H. Glock, K. Schwindl and M.Y. Jaber, An EOQ model with fuzzy demand and learning in fuzziness. *Int. J. Serv. Oper. Manag.* **12** (2012) 90–100.

[8] M.S. Habib, M. Omair, M.B. Ramzan, T.N. Chaudhary, M. Farooq and B. Sarkar, A robust possibilistic flexible programming approach toward a resilient and cost-efficient biodiesel supply chain network. *J. Clean. Prod.* **366** (2022) 132752.

[9] C.K. Jaggi, A. Sharma and R. Jain, Fuzzification of EOQ model under the condition of permissible delay in payments. *Int. J. Strateg. Decis. Sci.* **3** (2012) 1–19.

[10] M.Y. Jani, M.R. Betheja, U. Chaudhari and B. Sarkar, Optimal investment in preservation technology for variable demand under trade-credit and shortages. *Mathematics* **9** (2021) 1301.

[11] M.A.A. Khan, A.A. Shaikh, G. Panda, I. Konstantaras and L.E. Cárdenas-Barrón, The effect of advance payment with discount facility on supply decisions of deteriorating products whose demand is both price and stock dependent. *Int. Trans. Oper. Res.* **27** (2020) 1343–1367.

[12] B.A. Kumar and S.K. Paikray, Cost optimization inventory model for deteriorating items with trapezoidal demand rate under completely backlogged shortages in crisp and fuzzy environment. *RAIRO: OR* **56** (2022) 1969–1994.

[13] S. Kumar, M. Sigroha, K. Kumar and B. Sarkar, Manufacturing/remanufacturing based supply chain management under advertisements and carbon emission process. *RAIRO: OR* (2022).

[14] J.J. Liao, K.N. Huang, K.J. Chung, S.D. Lin, S.T. Chuang and H.M. Srivastava, Optimal ordering policy in an economic order quantity (EOQ) model for non-instantaneous deteriorating items with defective quality and permissible delay in payments. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A: Mat.* **114** (2020) 1–26.

[15] A.S. Mahapatra, B. Sarkar, M.S. Mahapatra, H.N. Soni and S.K. Mazumder, Development of a fuzzy economic order quantity model of deteriorating items with promotional effort and learning in fuzziness with a finite time horizon. *Inventions* **4** (2019) 36.

[16] A.S. Mahapatra, M.S. Mahapatra, B. Sarkar and S.K. Majumder, Benefit of preservation technology with promotion and time-dependent deterioration under fuzzy learning. *Expert Syst. Appl.* **201** (2022) 117169.

[17] G.C. Mahata and A. Goswami, Fuzzy inventory models for items with imperfect quality and shortage backordering under crisp and fuzzy decision variables. *Comput. Ind. Eng.* **64** (2013) 190–199.

[18] U. Mishra, L.E. Cárdenas-Barrón, S. Tiwari, A.A. Shaikh and G. Treviño-Garza, An inventory model under price and stock dependent demand for controllable deterioration rate with shortages and preservation technology investment. *Ann. Oper. Res.* **254** (2017) 165–190.

[19] I. Moon, W.Y. Yun and B. Sarkar, Effects of variable setup cost, reliability, and production costs under controlled carbon emissions in a reliable production system. *Eur. J. Ind. Eng.* **16** (2022) 371–397.

[20] M. Nouri, S.M. Hosseini-Motlagh and M. Nematollahi, Proposing a discount policy for two-level supply chain coordination with periodic review replenishment and promotional efforts decisions. *Oper. Res.* **21** (2021) 365–398.

[21] B. Oryani, A. Moridian, B. Sarkar, S. Rezania, H. Kamyab and M.K. Khan, Assessing the financial resoure curse hypothesis in Iran: The novel dynamic ARDL approach. *Resour. Pol.* **78** (2022) 102899.

[22] L.Y. Ouyang, C.T. Chang and J.T. Teng, An EOQ model for deteriorating items under trade credits. *J. Oper. Res. Soc.* **56** (2005) 719–726.

[23] B. Pal, Optimal pricing and offering reward decisions in a competitive closed-loop dual-channel supply chain with recycling and remanufacturing. *RAIRO: OR* **56** (2022) 1763-1780.

[24] M. Pervin, S.K. Roy and G.W. Weber, An integrated inventory model with variable holding cost under two levels of trade-credit policy. *Numer. Algebra, Control Optim.* **8** (2018) 169.

[25] M. Pervin, S. K. Roy and G. W. Weber, Multi-item deteriorating two-echelon inventory model with price-and stock-dependent demand: A trade-credit policy. *J. Ind. Manag. Optim.* **15** (2019) 1345.

[26] M. Pervin, S.K. Roy and G.W. Weber, Deteriorating inventory with preservation technology under price and stock-sensitive demand. *J. Ind. Manag. Optim.* **16** (2020) 1585.

[27] M. Pervin, S.K. Roy and G.W. Weber, An integrated vendor-buyer model with quadratic demand under inspection policy and preservation technology. *Hacet. J. Math. Stat.* **49** (2020) 1168–1189.

[28] P. Priyamvada, R. Rini, A. Khanna and C.K. Jaggi, An inventory model under price and stock dependent demand for controllable deterioration rate with shortages and preservation technology investment: revisited. *Opsearch* **58** (2021) 181–202.

[29] P. Priyamvada, R. Rini and C.K. Jaggi, Optimal inventory strategies for deteriorating items with price-sensitive investment in preservation technology. *RAIRO: OR* **56** (2022) 601–617.

[30] S.K. Roy, M. Pervin and G.W. Weber, A two-warehouse probabilistic model with price discount on backorders under two levels of trade-credit policy. *J. Ind. Manag. Optim.* **16** (2020) 553.

[31] S.K. Roy, M. Pervin and G.W. Weber, Imperfection with inspection policy and variable demand under trade-credit: A deteriorating inventory model. *Numer. Algebra, Control Optim.* **10** (2020) 45.

[32] S. Saha, I.E. Nielsen and I. Moon, Strategic inventory and pricing decision for substitutable products. *Comput. Ind. Eng.* **160** (2021) 107570.

[33] M.K. Salameh, N.E. Abboud, A.N. El-Kassar and R.E. Ghattas, Continuous review inventory model with delay in payments. *Int. J. Prod. Econ.* **85** (2003) 91–95.

[34] B. Sarkar, B.K. Dey, M. Sarkar and S.J. Kim, A smart production system with an autonomation technology and dual channel retailing. *Comp. Indust. Eng.* **173** (2022) 108607.

[35] B. Sarkar, B. Ganguly, S. Pareek and L.E. Cárdenas-Barrón, A three-echelon green supply chain management for biodegradable products with three transportation modes. *Comp. Ind. Eng.* **174** (2022) 108727.

[36] A. Sarkar, R. Guchhait and B. Sarkar, Application of the artificial neural network with multithreading within an inventory model under uncertainty and inflation. *Int. J. Fuzzy Syst.* **24** (2022) 1–15.

[37] B. Sarkar, J. Joo, Y. Kim, H. Park and M. Sarkar, Controlling defective items in a complex multi-phase manufacturing system. *RAIRO: OR* **56** (2022).

[38] B. Sarkar, S. Kar, K. Basu and R. Guchhait, A sustainable managerial decision-making problem for a substitutable product in a dual-channel under carbon tax policy. *Comp. Ind. Eng.* **172** (2022) 108635.

[39] N.H. Shah and H.N. Soni, Continuous review inventory model with fuzzy stochastic demand and variable lead time. *Int. J. Appl. Ind. Eng.* **1** (2012) 7–24.

[40] N. Shah and M. Patel, Reducing the deterioration rate of inventory through preservation technology investment under fuzzy and cloud fuzzy environment, in *Predictive Analytics*, Edited by V. Kumar and M. Ram. CRC Press, Boca Raton (2021) 65–80.

[41] N. Shah, E. Patel and K. Rabari, EPQ model to price-sensitive stock dependent demand with carbon emission under green and preservation technology investment. *Econ. Comput. Econ. Cybern. Stud. Res.* **56** (2022).

[42] N.H. Shah, K. Rabari and E. Patel, Inventory and preservation investment for deteriorating system with stock-dependent demand and partial backlogged shortages. *Yugosl. J. Oper. Res.* **31** (2021) 181–192.

[43] N.H. Shah, K. Rabari and E. Patel, Greening efforts and deteriorating inventory policies for price-sensitive stock-dependent demand. *Int. J. Syst. Sci.: Oper. Logist.* **1–7** (2022).

[44] A.A. Shaikh, G.C. Panda, S. Sahu and A.K. Das, Economic order quantity model for deteriorating item with preservation technology in time dependent demand with partial backlogging and trade credit. *Int. J. Logist. Syst. Manag.* **32** (2019) 1.

[45] S.K. Sharma and S.M. Govindaluri, An analytical approach for EOQ determination using trapezoidal fuzzy function. *Int. J. Procure. Manag.* **11** (2018) 356–369.

[46] H.N. Soni and K.A. Patel, Joint pricing and replenishment policies for non-instantaneous deteriorating items with imprecise deterioration free time. *Comput. Ind. Eng.* **66** (2013) 944–951.

[47] H.N. Soni and D.N. Suthar, Pricing and inventory decisions for non-instantaneous deteriorating items with price and promotional effort stochastic demand. *J. Control. Decis.* **6** (2019) 191–215.

[48] R. Sundara Rajan and R. Uthayakumar, Analysis and optimization of an EOQ inventory model with promotional efforts and back ordering under delay in payments. *J. Manag. Anal.* **4** (2017) 159–181.

[49] A.A. Taleizadeh, D.W. Pentico, M.S. Jabalameli and M. Aryanezhad, An EOQ model with partial delayed payment and partial backordering. *Omega* **41** (2013) 354–368.

[50] H.M. Wee, M.C. Lee, J.C. Yu and C.E. Wang, Optimal replenishment policy for a deteriorating green product: Life cycle costing analysis. *Int. J. Prod. Econ.* **133** (2011) 603–611.

[51] D. Yadav, R. Singh, A. Kumar and B. Sarkar, Reduction of pollution through sustainable and flexible production by controlling by-products. *J. Environ. Inf.* **40** (2022) 106–124.

[52] J.S. Yao, S.C. Chang and J.S. Su, Fuzzy inventory without backorder for fuzzy order quantity and fuzzy total demand quantity. *Comput. Oper. Res.* **27** (2000) 935–962.

[53] L.A. Zadeh, Fuzzy sets. *Inf. Control* **8** (1965) 338–353.

[54] J. Zhang, Y. Wang, L. Lu and W. Tang, Optimal dynamic pricing and replenishment cycle for non-instantaneous deterioration items with inventory-level-dependent demand. *Int. J. Prod. Econ.* **170** (2015) 136–145.
[55] Q. Zhou, Y. Yang and S. Fu, Deep reinforcement learning approach for solving joint pricing and inventory problem with reference price effects. *Expert Syst. Appl.* **195** (2022) 116564.