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A NOVEL DYNAMIC DATA ENVELOPMENT ANALYSIS APPROACH WITH PARABOLIC FUZZY DATA: CASE STUDY IN THE INDIAN BANKING SECTOR

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Abstract. Data envelopment analysis (DEA) is a non-parametric approach that measures the efficiency of a decision-making unit (DMU) statically and requires crisp input-output data. However, as a performance analysis tool, DEA overlooks the inter-relationship present among periods, and in many real applications, it is challenging to define the information for variables like customer satisfaction, service quality, etc. in precise form. To fix this, the present paper develops a novel parabolic fuzzy dynamic DEA (PFDDEA) approach that not only measures the system and period fuzzy efficiencies of DMUs by considering the inter-dependence among periods in the presence of undesirable resources but also handles data as parabolic fuzzy numbers (PFNs). It evaluates fuzzy efficiencies in a dynamic environment by distinguishing the role of links as inputs/outputs. In the proposed approach, system fuzzy efficiencies are estimated by solving the proposed PFDDEA models based on the α-cut approach that guarantees the shape of the membership function of the system fuzzy efficiencies obtained at different α-levels as PFNs. Further, an algorithmic approach for measuring period fuzzy efficiencies based on the concept of α-cuts and Pareto's efficiency is developed that leads to the estimation of the shapes of their membership functions. Finally, a relationship has been derived between upper (lower) bound system efficiency and upper (lower) bound period efficiencies at each α-level. To the best of our knowledge, this is the first attempt that dynamically evaluates fuzzy efficiencies (system and period) of DMUs when the data for the inputs/outputs/links are PFNs. To validate the applicability and robustness of the proposed approach, it is applied to eleven Indian banks for two periods 2019-2020 and 2020-2021, including loss due to non-performing assets (NPAs) as an undesirable output and unused assets as a link between periods. Here, NPAs are the bad loans that cease to generate income for the banks. The findings of the study (i) depict the system and period efficiencies as PFNs, (ii) conclude that the Federal Bank (FB) is the most efficient and Punjab National Bank (PNB) is the least efficient bank in the system and all periods, and (iii) provide implications that are highly valuable for bank experts to consider the impact of NPAs and unused assets for improving underperformed banks. These findings indicate that the proposed PFDDEA approach is highly useful for ranking/benchmarking in a dynamic manner keeping in view the presence of uncertain data variables represented as PFNs.

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1. Introduction

Data envelopment analysis (DEA) is a widely used linear programming tool to measure the productivity of homogenous decision-making units (DMUs) that consume multiple inputs to yield multiple outputs. Charnes *et al.* [4] introduced DEA as a non-parametric tool to investigate the efficiency of DMUs utilizing similar types of inputs to produce similar types of outputs. Later, Banker *et al.* [2] developed a model that considered variable returns to scale in DEA, and Tone [39] presented a slacks-based DEA model to measure efficiency by dealing with input excesses and output shortfalls. The dispersal of methods to assess the efficiency and to assist managerial decision-making has been mainly motivated by the development of DEA [26], and it is largely employed technique to estimate efficacy in profit/non-profit organizations like the educational sector, health care sector, financial sector, etc. DEA is valuable compared to other frontier techniques as it does not entail the predetermined weights attached to each input and output. Despite many advantages, DEA possesses two major limitations: (i) it measures the performance of entities statically, *i.e.*, it evaluates efficiency over a particular time and ignores the inter-relationship of periods among each other that leads to misleading efficiency results, and (ii) it necessitates precisely defined input-output data; however, the data for variables like environment pollution, customer satisfaction [50], service quality, social responsibility, and hospital reputation [15], etc. is not always available in a precise form for many real applications.

To deal with the first limitation, DEA was extended to dynamic DEA that incorporates the effect of time while evaluating the efficiency of a DMU. In dynamic DEA, inter-relations between consecutive periods are considered, and the efficiency of a DMU is estimated dynamically to assess intertemporal performance. Färe and Grosskopf [9] introduced dynamic DEA models to measure the intertemporal efficiency of DMUs by considering the presence of intermediates in two consecutive periods. Nemoto and Goto [27] incorporated two types of inputs (variable and quasi-fixed inputs) into dynamic DEA. Tone and Tsutsui [40] developed dynamic DEA models to evaluate intertemporal efficiency in the presence of carry-overs. Kao [21] introduced a relational dynamic DEA model to measure system and period efficiencies in a dynamic environment by assigning the same weights to the intermediates despite the fact whether they are being used as input or output in that particular period. Emrouznejad and Thanassoulis [6] introduced the concept of input-output paths mapped out by operating units over time. Later, it was utilized by Yen and Chiou [47] and Zeinodin and Ghobadi [48] to assess the efficiency of DMUs in the presence of intertemporal input-output dependence. Wanke *et al.* [42] investigated the efficiency of MENA banking structure in a dynamic environment by introducing an input-oriented dynamic network DEA model. The extensive literature on the evolution of dynamic DEA models can be seen in Fallah-Fini *et al.* [8], Ghobadi [12], Soleimani-damaneh [36], Sueyoshi and Sekitani [38], and Mariz *et al.* [26].

To deal with the second limitation, Sengupta [35] introduced the concept of fuzziness in DEA. Zerafat Angiz *et al.* [49] presented a fuzzy DEA approach for performance evaluation by maximizing the membership functions of inputs/outputs that retained the fuzziness of the model. Fuzzy DEA models were initially classified into four groups by Hatami-Marbini *et al.* [16] and later been expanded to six groups by Emroujnezad *et al.* [7] which are (i) tolerance approach, (ii) α-cut approach, (iii) fuzzy ranking approach, (iv) possibility approach, (v) fuzzy arithmetic approach, and (vi) fuzzy random/type-2 fuzzy set approach. Kao and Liu [22] presented fuzzy DEA models to evaluate efficiency with the help of the α-cut approach and also estimated membership functions of the fuzzy efficiencies. Peykani *et al.* [30] utilized possibility, necessity and credibility measures to develop an adjustable fuzzy DEA model which can consider the different optimistic-pessimistic attitudes of a decision-maker. Sahil *et al.* [34] presented a fuzzy DEA model to evaluate efficiency in the presence of parabolic fuzzy inputs and outputs by using the α-cut approach. The other studies on fuzzy DEA can be seen in Gholizadeh and Fazlollahtabar [12], Gholizadeh *et al.* [14], Puri and Yadav [32], and Wang and Chin [41].

The notion of imprecision in the form of fuzzy data has also been introduced in dynamic DEA by Gholizadeh *et al.* [13]. They designed a dynamic DEA model, an analytical-descriptive research method for measuring the dynamic efficiencies of 27 investment corporations in the Tehran stock exchange during 2004-2008. They first utilized the fuzzy DEA model to evaluate the efficiency of each corporation when input-output data are in terms of LR fuzzy numbers. Further, as a dynamic framework, the efficiency of a firm was measured 24 times each 3 months to analyze the performance of a decision-making unit (DMU) in different time periods. This is dynamic efficiency measurement as it includes time entity as a variant. The dynamic efficiency discussed in Gholizadeh *et al.* [13] is based on window analysis, which measures efficiency that changes over time. It is apparent that the window analysis considers the time effect; however, it ignores intertemporal elements like carry-over activities between consecutive periods that are mostly present in the production process of a DMU in practical situations. In the same notion, Kordrostami and Noveiri [25] and Peykani *et al.* [31] have developed fuzzy dynamic DEA approaches based on window analysis and the Malmquist DEA index to measure efficiency that too ignore the carry-overs. Ghobadi et al. [11] developed fuzzy dynamic DEA models to measure performance when data are available in the form of LR fuzzy numbers. Olfat *et al.* [29] presented fuzzy dynamic DEA models to deal with trapezoidal interval type-2 fuzzy data and evaluated the system and period efficiency of 28 Iranian airports over two periods. Hasani and Mokhtari [15] developed a hybrid fuzzy DEA model to measure the system efficiency as well as period efficiencies of Iranian hospitals in a dynamic environment comprising two periods in the presence of undesirable inputs/outputs. Zhou *et al.* [50] evaluated the system and period efficiencies of 20 suppliers over three periods in the dynamic environment with customer satisfaction (desirable output) as a triangular fuzzy number. Some other extensions of fuzzy dynamic DEA models are Jafarian-Moghaddam and Ghoseiri [19], Khodaparasti and Reza Maleki [23], Olfat and Pishdar [28], Soltanzadeh and Omrani [37], Yaghoubi and Amiri [46], and Yen and Chiou [47].

Moreover, in the production process of many real applications, resources (inputs/outputs) might be undesirable (bad) [17] that need to be minimized in a dynamic environment. For example, CO2 emissions in electric power industries [44], non-performing loans (NPLs) in the banking sector [3], etc. The extensive literature on dynamic DEA models dealing with undesirable resources can be found in studies like Amowine *et al.* [1], Tone and Tsutsui [40], Woo *et al.* [43], and Xie *et al.* [45].

Despite many extensions made in dynamic DEA, such as fuzzy dynamic DEA, the existing approaches still possess some limitations/research gaps that need to be addressed to enhance their applicability in practical situations. In particular, the existing literature on fuzzy dynamic DEA

– lacks in handling data for inputs/outputs/links as parabolic fuzzy numbers (PFNs) and fails to distinguish between the role of links as inputs or outputs by assigning similar weights to links that are used as input and/or output.
– has not presented any methodology based on the α-cut approach and Pareto's efficiency concept to estimate system and period fuzzy efficiencies of a DMU when data is in terms of PFNs and to predict the shapes of their membership functions.
– lacks in presenting any application with PFN data and analyzing the impact of undesirable output (*e.g.*, loss due to NPAs in the banking sector) and good link (*e.g.*, unused assets in the banking sector) on the system and periods' fuzzy efficiencies.

To overcome these research gaps, the present study is mainly focused on developing a novel parabolic fuzzy dynamic DEA (PFDDEA) approach that (i) handles the data for inputs/outputs/links in the form of PFNs, (ii) examines a dynamic structure connected through links in the presence of undesirable outputs, (iii) presents an approach that distinguishes the role of links as inputs or outputs and uses different weights for links when used as input or output, (iv) evaluates system interval efficiencies based on α-cut approach which are further used to analyze the shape of the membership function of the system fuzzy efficiencies as PFN, (v) presents an algorithmic approach to obtain period fuzzy efficiencies when data is in terms of PFNs and to estimate the shapes of their membership functions, and (vi) derives a relationship between upper (lower) bound system efficiency and upper (lower) bound period efficiencies at α-level, *i.e.*, define some conditions under which upper (lower) bound system efficiency can be expressed as a linear combination of upper (lower) bound period efficiencies at level α. Further, to prove the applicability and effectiveness of the proposed PFDDEA approach, an application to the Indian banking sector is presented, and implications are discussed to assist experts/decision-makers in policy formulations.

The present paper is systematized as follows. Section 2 presents some basic definitions of fuzzy set theory. In Section 3, PFDDEA models are proposed to evaluate system fuzzy efficiency, followed by an algorithm to measure period efficiencies at different α-levels. Section 4 presents a relationship between the system and period efficiencies. Section 5 includes an application of the proposed approach to the Indian banking sector. Section 6 provides theoretical, managerial and policy implications of the present study, and Section 7 concludes the findings of the present study.

2. Preliminaries

This section presents some fundamental definitions of fuzzy set theory.

Definition 1. [51] Let X be a universe of discourse. Then, a fuzzy set $\tilde{A}$ in X is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}} : X \to [0,1]$ is the membership function of $\tilde{A}$ and $\mu_{\tilde{A}}(x)$ represents the membership degree of x being in $\tilde{A}$.

Definition 2. [51] Let $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ be a fuzzy set in universe of discourse X. Then, support of $\tilde{A}$ denoted by $S(\tilde{A})$ is defined as

$$S(\tilde{A}) = \{x|\mu_{\tilde{A}}(x) \geq 0\}$$

Definition 3. [18,24] An α-cut of a fuzzy set $\tilde{A}$ in X, denoted by $\tilde{A}_\alpha$, is defined as $\tilde{A}_\alpha = \{x \in X|\mu_{\tilde{A}}(x) \geq \alpha\}$ where $\alpha \in (0,1]$, *i.e.*, a crisp set containing all those $x \in X$ whose membership degree is either greater than or equal to α.

Definition 4. [51] A fuzzy set $\tilde{A}$ in universe of discourse X is said to be convex if and only if

$$\mu_{\tilde{A}}(\lambda x' + (1-\lambda)x'') \geq min(\mu_{\tilde{A}}(x'), \mu_{\tilde{A}}(x'')), \text{ for all } x', x'' \in X \text{ and } \lambda \in [0,1].$$

Definition 5. [51] Let $\tilde{A}$ be a fuzzy set in universe of discourse X. Then, it is said to be normal if $\mu_{\tilde{A}}(x') = 1$ for some $x' \in X$.

Definition 6. [51] A fuzzy set $\tilde{A}$ is called a fuzzy number if it is both normal and convex.

Definition 7. [20] A fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ is called PFN with its membership function $\mu_{\tilde{A}}(x)$ defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} \left(\dfrac{x-a_1}{a_2-a_1}\right)^2, & a_1 \leq x < a_2, \\ 1, & x = a_2, \\ \left(\dfrac{a_3-x}{a_3-a_2}\right)^2, & a_2 < x \leq a_3, \\ 0, & otherwise, \end{cases} \tag{2.1}$$

Remark 1. [20] An α-cut $(A)_\alpha$ for a PFN $\tilde{A} = (a_1, a_2, a_3)$ is defined as an interval given by $(A)_\alpha = [a_1 + \sqrt{\alpha}(a_2 - a_1), a_3 + \sqrt{\alpha}(a_2 - a_3)]$.

3. Proposed parabolic fuzzy dynamic DEA model

Assume there are n DMUs and each DMU consumes m inputs to yield s desirable outputs along with h undesirable outputs over T number of periods, and two consecutive periods are connected through q number of links as shown in Figure 1. Let $\tilde{X}_{ij} = \sum_{t=1}^{T} \tilde{X}_{ij}^{(t)}$ be the i^{th} parabolic fuzzy input of a system, where $\tilde{X}_{ij}^{(t)} = (\tilde{X}_{1j}^{(t)}, \tilde{X}_{2j}^{(t)}, \ldots, \tilde{X}_{mj}^{(t)})$ is a vector of fuzzy inputs in period t. Let $\tilde{Y}_{rj}^{g} = \sum_{t=1}^{T} \tilde{Y}_{rj}^{g(t)}$ and $\tilde{Y}_{fj}^{b} = \sum_{t=1}^{T} \tilde{Y}_{fj}^{b(t)}$ be the r^{th} desirable parabolic fuzzy output and f^{th} undesirable parabolic fuzzy output of a system, respectively, where

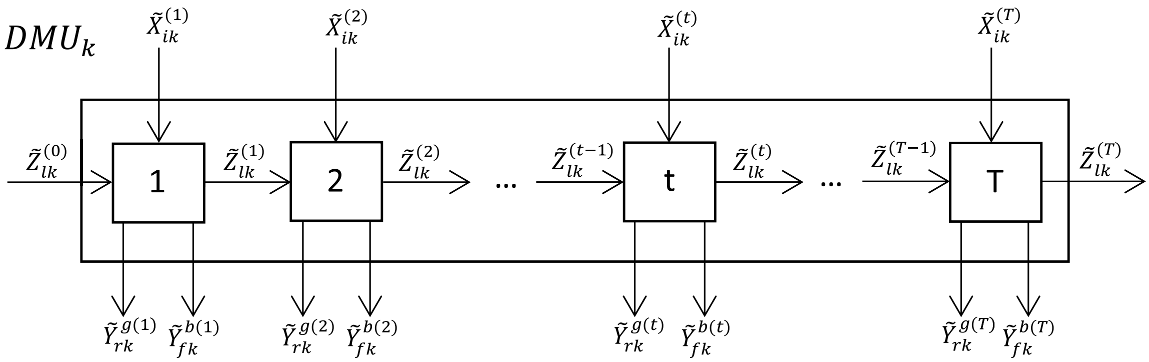

FIGURE 1. Dynamic structure of DMU_k with parabolic fuzzy data over T number of periods.

$\tilde{Y}_{rj}^{g(t)} = (\tilde{Y}_{1j}^{g(t)}, \tilde{Y}_{2j}^{g(t)}, \ldots, \tilde{Y}_{sj}^{g(t)})$, and $\tilde{Y}_{fj}^{b(t)} = (\tilde{Y}_{1j}^{b(t)}, \tilde{Y}_{2j}^{b(t)}, \ldots, \tilde{Y}_{hj}^{b(t)})$ are the respective vectors of r^{th} desirable parabolic fuzzy output and f^{th} undesirable parabolic fuzzy output in period t. Let $\tilde{Z}_{lj}^{(t)} = (\tilde{Z}_{1j}^{(t)}, \tilde{Z}_{2j}^{(t)}, \ldots, \tilde{Z}_{qj}^{(t)})$ be a vector of fuzzy links produced in period t and consumed by period $t+1$, *i.e.* acting as a link between periods t and $t+1$. Let v_i, u_r, w_f be the weights associated with the i^{th} input, r^{th} desirable output and f^{th} undesirable output, respectively, and γ_l, β_l be the weights for l^{th} link when treated as an output and input, respectively.

Then, for any DMU_j, system fuzzy efficiency and t^{th} period fuzzy efficiency are respectively defined as

$$\tilde{E}_j = \frac{\sum_{r=1}^{s} u_r \tilde{Y}_{rj}^{g} + \sum_{l=1}^{q} \gamma_l \tilde{Z}_{lj}^{(T)} - \sum_{f=1}^{h} w_f \tilde{Y}_{fj}^{b}}{\sum_{i=1}^{m} v_i \tilde{X}_{ij} + \sum_{l=1}^{q} \beta_l \tilde{Z}_{lj}^{(0)}}, \quad \forall j \tag{3.1}$$

$$\tilde{E}_j^{(t)} = \frac{\sum_{r=1}^{s} u_r \tilde{Y}_{rj}^{(t)} + \sum_{l=1}^{q} \gamma_l \tilde{Z}_{lj}^{(t)} - \sum_{f=1}^{h} w_f \tilde{Y}_{fj}^{b(t)}}{\sum_{i=1}^{m} v_i \tilde{X}_{ij}^{(t)} + \sum_{l=1}^{q} \beta_l \tilde{Z}_{lj}^{(t-1)}}, \quad \forall j, \ \forall t. \tag{3.2}$$

Using the above definitions of the system and period fuzzy efficiencies, the following parabolic fuzzy dynamic DEA model is presented to evaluate the system fuzzy efficiency of DMU_k:

Model-1

$$\max \ \tilde{E}_k = \frac{\sum_{r=1}^{s} u_r \tilde{Y}_{rk} + \sum_{l=1}^{q} \gamma_l \tilde{Z}_{lk}^{(T)} - \sum_{f=1}^{h} w_f \tilde{Y}_{fk}^{b}}{\sum_{i=1}^{m} v_i \tilde{X}_{ik} + \sum_{l=1}^{q} \beta_l \tilde{Z}_{lk}^{(0)}}$$

$$\text{s.t. } 0 \le \frac{\sum_{r=1}^{s} u_r \tilde{Y}_{rj} + \sum_{l=1}^{q} \gamma_l \tilde{Z}_{lj}^{(T)} - \sum_{f=1}^{h} w_f \tilde{Y}_{fj}^{b}}{\sum_{i=1}^{m} v_i \tilde{X}_{ij} + \sum_{l=1}^{q} \beta_l \tilde{Z}_{lj}^{(0)}} \le 1, \quad j = 1, 2, \ldots, n,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r \tilde{Y}_{rj}^{(t)} + \sum_{l=1}^{q} \gamma_l \tilde{Z}_{lj}^{(t)} - \sum_{f=1}^{h} w_f \tilde{Y}_{fj}^{b(t)}}{\sum_{i=1}^{m} v_i \tilde{X}_{ij}^{(t)} + \sum_{l=1}^{q} \beta_l \tilde{Z}_{lj}^{(t-1)}} \leq 1, \quad j = 1, 2, \ldots, n, \;\; t = 1, 2, \ldots, T,$$

$$u_r \geq \epsilon, \;\; v_i \geq \epsilon, \;\; \gamma_l \geq \epsilon, \;\; \beta_l \geq \epsilon, \;\; \epsilon > 0,$$

where ϵ is a non-Archimedean infinitesimal.

Model-1 maximizes the system efficiency of the k^{th} DMU subject to the condition that the system fuzzy efficiency of every DMU lies between 0 and 1, and every t^{th} period fuzzy efficiency for all DMUs lies between 0 and 1.

The present problem is a fuzzy dynamic DEA problem where data is represented as a fuzzy number and can be solved using a fuzzy ranking approach [7] which possesses a drawback that different fuzzy ranking methods may result in different efficiency scores. Jafarian- Moghaddam and Ghoseiri [18, 19] suggested a fuzzy dynamic multi-objective DEA approach that resulted in solving nonlinear programming problem, which is difficult to solve for large data sets. Therefore, we have applied the α-cut approach to solve the fuzzy dynamic DEA model in the present study which is the most popular approach that often provides a fuzzy efficiency score whose membership function is constructed from α-cuts for each α in [0,1]. By using Kao and Liu [22]'s concept of transforming fuzzy DEA models into a family of crisp DEA models, Model-1 is transformed into two dynamic DEA models given by Model-2 and Model-3 based on the α-cut approach, which is described as under:

If $S(\tilde{X}_{ij})$, $S(\tilde{Y}_{rj}^{g})$, $S(\tilde{Y}_{fj}^{b})$ and $S(\tilde{Z}_{lj}^{(t)})$ denote the support of the i^{th} input, r^{th} desirable output, f^{th} undesirable output and l^{th} link produced in period t of DMU_j respectively. Then, the α-cuts of $\tilde{X}_{ij}$, $\tilde{Y}_{rj}^{g}$, $\tilde{Y}_{fj}^{b}$ and $\tilde{Z}_{lj}^{(t)}$ denoted by $(\tilde{X}_{ij})_\alpha$ $(\forall i, j)$, $(\tilde{Y}_{rj}^{g})_\alpha$ $(\forall r, j)$, $(\tilde{Y}_{fj}^{b})_\alpha$ $(\forall f, j)$, and $(\tilde{Z}_{lj}^{(t)})_\alpha$ $(\forall l, t, j)$, respectively, are defined as

$$\begin{aligned}(\tilde{X}_{ij})_\alpha =& \{X_{ij} \in S(\tilde{X}_{ij}) | \mu_{\tilde{X}_{ij}}(X_{ij}) \geq \alpha\} = [(X_{ij})_\alpha^L, (X_{ij})_\alpha^U] \\ =& \left[\min_{X_{ij}}\{X_{ij} \in S(\tilde{X}_{ij}) | \mu_{\tilde{X}_{ij}}(X_{ij}) \geq \alpha\}, \max_{X_{ij}}\{X_{ij} \in S(\tilde{X}_{ij}) | \mu_{\tilde{X}_{ij}}(X_{ij}) \geq \alpha\}\right], \end{aligned} \tag{3.3}$$

$$\begin{aligned}(\tilde{Y}_{rj}^{g})_\alpha =& \{Y_{rj}^{g} \in S(\tilde{Y}_{rj}^{g}) | \mu_{\tilde{Y}_{rj}^{g}}(Y_{rj}^{g}) \geq \alpha\} = [(Y_{rj}^{g})_\alpha^L, (Y_{rj}^{g})_\alpha^U] \\ =& \left[\min_{Y_{rj}^{g}}\{Y_{rj}^{g} \in S(\tilde{Y}_{rj}^{g}) | \mu_{\tilde{Y}_{rj}^{g}}(Y_{rj}^{g}) \geq \alpha\}, \max_{Y_{rj}^{g}}\{Y_{rj}^{g} \in S(\tilde{Y}_{rj}) | \mu_{\tilde{Y}_{rj}^{g}}(Y_{rj}^{g}) \geq \alpha\}\right], \end{aligned} \tag{3.4}$$

$$\begin{aligned}(\tilde{Y}_{fj}^{b})_\alpha =& \{Y_{fj}^{b} \in S(\tilde{Y}_{fj}^{b}) | \mu_{\tilde{Y}_{fj}^{b}}(Y_{fj}^{b}) \geq \alpha\} = [(Y_{fj}^{b})_\alpha^L, (Y_{fj}^{b})_\alpha^U] \\ =& \left[\min_{Y_{fj}^{b}}\{Y_{fj}^{b} \in S(\tilde{Y}_{fj}^{b}) | \mu_{\tilde{Y}_{fj}^{b}}(Y_{fj}^{b}) \geq \alpha\}, \max_{Y_{fj}^{b}}\{Y_{fj}^{b} \in S(\tilde{Y}_{fj}^{b}) | \mu_{\tilde{Y}_{fj}^{b}}(Y_{fj}^{b}) \geq \alpha\}\right], \end{aligned} \tag{3.5}$$

$$\begin{aligned}\text{and} \;\; (\tilde{Z}_{lj}^{(t)})_\alpha =& \{Z_{lj}^{(t)} \in S(\tilde{Z}_{lj}^{(t)}) | \mu_{\tilde{Z}_{lj}^{(t)}}(Z_{lj}^{(t)}) \geq \alpha\} = [(Z_{lj}^{(t)})_\alpha^L, (Z_{lj}^{(t)})_\alpha^U] \\ =& \left[\min_{Z_{lj}^{(t)}}\{Z_{lj}^{(t)} \in S(\tilde{Z}_{lj}^{(t)}) | \mu_{\tilde{Z}_{lj}^{(t)}}(Z_{lj}^{(t)}) \geq \alpha\}, \max_{Z_{lj}^{(t)}}\{Z_{lj}^{(t)} \in S(\tilde{Z}_{lj}^{(t)}) | \mu_{\tilde{Z}_{lj}^{(t)}}(Z_{lj}^{(t)}) \geq \alpha\}\right]. \end{aligned} \tag{3.6}$$

Similarly, the α-cuts for fuzzy efficiency scores $\tilde{E}_j$ and $\tilde{E}_j^{(t)}$ are respectively defined as

$$\begin{aligned}(\tilde{E}_j)_\alpha =& \{E_j \in S(\tilde{E}_j) | \mu_{\tilde{E}_j}(E_j) \geq \alpha\} = [(E_j)_\alpha^L, (E_j)_\alpha^U] \\ =& \left[\min_{E_j}\{E_j \in S(\tilde{E}_j) | \mu_{\tilde{E}_j}(E_j) \geq \alpha\}, \max_{E_j}\{E_j \in S(\tilde{E}_j) | \mu_{\tilde{E}_j}(E_j) \geq \alpha\}\right], \end{aligned} \tag{3.7}$$

$$\text{and} \;\; (\tilde{E}_j^{(t)})_\alpha = \{E_j^{(t)} \in S(\tilde{E}_j^{(t)}) | \mu_{\tilde{E}_j^{(t)}}(E_j^{(t)}) \geq \alpha\} = [(E_j^{(t)})_\alpha^L, (E_j^{(t)})_\alpha^U]$$

$$= \left[\min_{E_j^{(t)}}\{E_j^{(t)} \in S(\tilde{E}_j^{(t)})|\mu_{\tilde{E}_j^{(t)}}(E_j^{(t)}) \geq \alpha\}, \max_{E_j^{(t)}}\{E_j^{(t)} \in S(\tilde{E}_j^{(t)})|\mu_{\tilde{E}_j^{(t)}}(E_j^{(t)}) \geq \alpha\}\right]. \tag{3.8}$$

3.1. Proposed models to evaluate system efficiency

Using the α-cuts defined in equations (3.3)–(3.8) and α-cut approach, the following fractional problems are presented to evaluate upper and lower bound of interval system efficiencies for DMU_k, where for α in $(0,1]$, we have

Model-2

$$(E_k)_\alpha^U = \max_{\substack{(X_{ij})_\alpha^L \leq X_{ij} \leq (X_{ij})_\alpha^U \\ (Y_{rj}^{g(t)})_\alpha^L \leq Y_{rj}^{g(t)} \leq (Y_{rj}^{g(t)})_\alpha^U \\ (Y_{fj}^{b(t)})_\alpha^L \leq Y_{fj}^{b(t)} \leq (Y_{fj}^{b(t)})_\alpha^U \\ (Z_{lj}^{(t)})_\alpha^L \leq Z_{lj}^{(t)} \leq (Z_{lj}^{(t)})_\alpha^U \\ \forall i,r,q,l,j}} \left\{ \begin{array}{l} max \;\; E_k = \dfrac{\sum_{r=1}^{s} u_r Y_{rk}^g + \sum_{l=1}^{q} \gamma_l Z_{lk}^{(T)} - \sum_{f=1}^{h} w_f Y_{fk}^b}{\sum_{i=1}^{m} v_i X_{ik} + \sum_{l=1}^{q} \beta_l Z_{lk}^{(0)}} \\ \text{s.t. } \; 0 \leq \dfrac{\sum_{r=1}^{s} u_r Y_{rj}^g + \sum_{l=1}^{q} \gamma_l Z_{lj}^{(T)} - \sum_{f=1}^{h} w_f Y_{fj}^b}{\sum_{i=1}^{m} v_i X_{ij} + \sum_{l=1}^{q} \beta_l Z_{lj}^{(0)}} \leq 1, \forall j, \\ 0 \leq \dfrac{\sum_{r=1}^{s} u_r Y_{rj}^{g(t)} + \sum_{l=1}^{q} \gamma_l Z_{lj}^{(t)} - \sum_{f=1}^{h} w_f Y_{fj}^{b(t)}}{\sum_{i=1}^{m} v_i X_{ij}^{(t)} + \sum_{l=1}^{q} \beta_l Z_{lj}^{(t-1)}} \leq 1, \forall j, \;\; \forall t, \\ u_r \geq \epsilon \; \forall r, \;\; v_i \geq \epsilon \; \forall i, \;\; \gamma_l \geq \epsilon \; \forall l, \;\; \beta_l \geq \epsilon \; \forall l, \;\; \epsilon > 0. \end{array} \right\}$$

Model-3

$$(E_k)_\alpha^L = \min_{\substack{(X_{ij})_\alpha^L \leq X_{ij} \leq (X_{ij})_\alpha^U \\ (Y_{rj}^{g(t)})_\alpha^L \leq Y_{rj}^{g(t)} \leq (Y_{rj}^{g(t)})_\alpha^U \\ (Y_{fj}^{b(t)})_\alpha^L \leq Y_{fj}^{b(t)} \leq (Y_{fj}^{b(t)})_\alpha^U \\ (Z_{lj}^{(t)})_\alpha^L \leq Z_{lj}^{(t)} \leq (Z_{lj}^{(t)})_\alpha^U \\ \forall i,r,q,l,j}} \left\{ \begin{array}{l} max \;\; E_k = \dfrac{\sum_{r=1}^{s} u_r Y_{rk}^g + \sum_{l=1}^{q} \gamma_l Z_{lk}^{(T)} - \sum_{f=1}^{h} w_f Y_{fk}^b}{\sum_{i=1}^{m} v_i X_{ik} + \sum_{l=1}^{q} \beta_l Z_{lk}^{(0)}} \\ \text{s.t. } \; 0 \leq \dfrac{\sum_{r=1}^{s} u_r Y_{rj}^g + \sum_{l=1}^{q} \gamma_l Z_{lj}^{(T)} - \sum_{f=1}^{h} w_f Y_{fj}^b}{\sum_{i=1}^{m} v_i X_{ij} + \sum_{l=1}^{q} \beta_l Z_{lj}^{(0)}} \leq 1, \forall j, \\ 0 \leq \dfrac{\sum_{r=1}^{s} u_r Y_{rj}^{g(t)} + \sum_{l=1}^{q} \gamma_l Z_{lj}^{(t)} - \sum_{f=1}^{h} w_f Y_{fj}^{b(t)}}{\sum_{i=1}^{m} v_i X_{ij}^{(t)} + \sum_{l=1}^{q} \beta_l Z_{lj}^{(t-1)}} \leq 1, \forall j, \;\; \forall t, \\ u_r \geq \epsilon \; \forall r, \;\; v_i \geq \epsilon \; \forall i, \;\; \gamma_l \geq \epsilon \; \forall l, \;\; \beta_l \geq \epsilon \; \forall l, \;\; \epsilon > 0. \end{array} \right\}$$

Further, Pareto's efficiency concept has been used to convert Models-2 and 3 into single objective fractional programming problems as suggested by Kao and Liu [22] in fuzzy DEA literature. To evaluate upper bound system efficiency at α-level, lower bound inputs/undesirable outputs/links (as input) and upper bound desirable outputs/links (as output) are used for the targeted DMU ($(E_k)_\alpha^U$) whereas upper bound inputs/undesirable outputs/links (as input) and lower bound desirable outputs/links (as output) are used for the other DMUs

$((E_j)^L_\alpha, \forall j \neq k)$. A similar concept has been used for periods of targeted DMU and other DMUs, and Model-2 reduces to the following form:

Model-4

$$\max\ (E_k)^U_\alpha = \frac{\sum_{r=1}^{s} u_r (Y^g_{rk})^U_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lk})^U_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fk})^L_\alpha}{\sum_{i=1}^{m} v_i (X_{ik})^L_\alpha + \sum_{l=1}^{q} \beta_l (Z^{(0)}_{lk})^L_\alpha}$$

$$\text{s.t.}\ 0 \leq \frac{\sum_{r=1}^{s} u_r (Y^g_{rk})^U_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lk})^U_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fk})^L_\alpha}{\sum_{i=1}^{m} v_i (X_{ik})^L_\alpha + \sum_{l=1}^{q} \beta_l (Z^{(0)}_{lk})^L_\alpha} \leq 1,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y^g_{rj})^L_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lj})^L_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fj})^U_\alpha}{\sum_{i=1}^{m} v_i (X_{ij})^U_\alpha + \sum_{l=1}^{q} \beta_l (Z^{(0)}_{lj})^U_\alpha} \leq 1, \quad \forall j \neq k,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y^{g(t)}_{rk})^U_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(t)}_{lk})^U_\alpha - \sum_{f=1}^{h} w_f (Y^{b(t)}_{fk})^L_\alpha}{\sum_{i=1}^{m} v_i (X^{(t)}_{ik})^L_\alpha + \sum_{l=1}^{q} \beta_l (Z^{(t-1)}_{lk})^L_\alpha} \leq 1, \quad \forall t,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y^{g(t)}_{rj})^L_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(t)}_{lj})^L_\alpha - \sum_{f=1}^{h} w_f (Y^{b(t)}_{fj})^U_\alpha}{\sum_{i=1}^{m} v_i (X^{(t)}_{ij})^U_\alpha + \sum_{l=1}^{q} \beta_l (Z^{(t-1)}_{lj})^U_\alpha} \leq 1, \quad \forall t, j \neq k,$$

$$u_r \geq \epsilon\ \forall r,\ \ v_i \geq \epsilon\ \forall i,\ \ \gamma_l \geq \epsilon\ \forall l,\ \ \beta_l \geq \epsilon\ \forall l,\ \ \epsilon > 0.$$

Similarly, Model-5 is proposed to evaluate lower bound system efficiency at α-level in which targeted DMU's upper bound inputs/undesirable outputs/links (as input) and lower bound desirable outputs/links (as output) are considered $((E_k)^L_\alpha)$. In contrast, lower bound inputs/undesirable outputs/links (as input) and upper bound desirable outputs/links (as output) are used for other DMUs $((E_j)^U_\alpha, \forall j \neq k)$. A similar concept has been used for periods of all DMUs.

Model-5

$$\max\ (E_k)^L_\alpha = \frac{\sum_{r=1}^{s} u_r (Y^g_{rk})^L_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lk})^L_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fk})^U_\alpha}{\sum_{i=1}^{m} v_i (X_{ik})^U_\alpha + \sum_{l=1}^{q} \beta_l (Z^{(0)}_{lk})^U_\alpha}$$

$$\text{s.t.}\ 0 \leq \frac{\sum_{r=1}^{s} u_r (Y^g_{rk})^L_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lk})^L_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fk})^U_\alpha}{\sum_{i=1}^{m} v_i (X_{ik})^U_\alpha + \sum_{l=1}^{q} \beta_l (Z^{(0)}_{lk})^U_\alpha} \leq 1,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y_{rj}^{g})_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(T)})_{\alpha}^{U} - \sum_{f=1}^{h} w_f (Y_{fj}^{b})_{\alpha}^{L}}{\sum_{i=1}^{m} v_i (X_{ij})_{\alpha}^{L} + \sum_{l=1}^{q} \beta_l (Z_{lj}^{(0)})_{\alpha}^{L}} \leq 1, \quad \forall j \neq k,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y_{rk}^{g(t)})_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(t)})_{\alpha}^{L} - \sum_{f=1}^{h} w_f (Y_{fk}^{b(t)})_{\alpha}^{U}}{\sum_{i=1}^{m} v_i (X_{ik}^{(t)})_{\alpha}^{U} + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(t-1)})_{\alpha}^{U}} \leq 1, \quad \forall t,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y_{rj}^{g(t)})_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(t)})_{\alpha}^{U} - \sum_{f=1}^{h} w_f (Y_{fj}^{b(t)})_{\alpha}^{L}}{\sum_{i=1}^{m} v_i (X_{ij}^{(t)})_{\alpha}^{L} + \sum_{l=1}^{q} \beta_l (Z_{lj}^{(t-1)})_{\alpha}^{L}} \leq 1, \quad \forall t, j \neq k,$$

$$u_r \geq \epsilon \ \forall r, \quad v_i \geq \epsilon \ \forall i, \quad \gamma_l \geq \epsilon \ \forall l, \quad \beta_l \geq \epsilon \ \forall l, \quad \epsilon > 0.$$

By using Charnes-Cooper transformation [5], Models-4 and 5 can be reduced to the following linear problems given by Models-6 and 7, respectively.

Model-6

$$\max \ (E_k)_{\alpha}^{U} = \sum_{r=1}^{s} u_r (Y_{rk}^{g})_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(T)})_{\alpha}^{U} - \sum_{f=1}^{h} w_f (Y_{fk}^{b})_{\alpha}^{L}$$

$$\text{s.t.} \ \sum_{i=1}^{m} v_i (X_{ik})_{\alpha}^{L} + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(0)})_{\alpha}^{L} = 1,$$

$$\sum_{r=1}^{s} u_r (Y_{rk}^{g})_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(T)})_{\alpha}^{U} - \sum_{f=1}^{h} w_f (Y_{fk}^{b})_{\alpha}^{L} \geq 0,$$

$$\sum_{r=1}^{s} u_r (Y_{rk}^{g})_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(T)})_{\alpha}^{U} - \sum_{f=1}^{h} w_f (Y_{fk}^{b})_{\alpha}^{L} - \sum_{i=1}^{m} v_i (X_{ik})_{\alpha}^{L} - \sum_{l=1}^{q} \beta_l (Z_{lk}^{(0)})_{\alpha}^{L} \leq 0,$$

$$\sum_{r=1}^{s} u_r (Y_{rj}^{g})_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(T)})_{\alpha}^{L} - \sum_{f=1}^{h} w_f (Y_{fj}^{b})_{\alpha}^{U} \geq 0, \quad \forall j \neq k,$$

$$\sum_{r=1}^{s} u_r (Y_{rj}^{g})_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(T)})_{\alpha}^{L} - \sum_{f=1}^{h} w_f (Y_{fj}^{b})_{\alpha}^{U} - \sum_{i=1}^{m} v_i (X_{ij})_{\alpha}^{U} + \sum_{l=1}^{q} \beta_l (Z_{lj}^{(0)})_{\alpha}^{U} \leq 0, \quad \forall j \neq k,$$

$$\sum_{r=1}^{s} u_r (Y_{rk}^{g(t)})_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(t)})_{\alpha}^{U} - \sum_{f=1}^{h} w_f (Y_{fk}^{b(t)})_{\alpha}^{L} \geq 0, \quad \forall t,$$

$$\sum_{r=1}^{s} u_r (Y_{rk}^{g(t)})_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(t)})_{\alpha}^{U} - \sum_{f=1}^{h} w_f (Y_{fk}^{b(t)})_{\alpha}^{L} - \sum_{i=1}^{m} v_i (X_{ik}^{(t)})_{\alpha}^{L} - \sum_{l=1}^{q} \beta_l (Z_{lk}^{(t-1)})_{\alpha}^{L} \leq 0, \quad \forall t,$$

$$\sum_{r=1}^{s} u_r (Y_{rj}^{(t)})_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(t)})_{\alpha}^{L} - \sum_{f=1}^{h} w_f (Y_{fj}^{b(t)})_{\alpha}^{U} \geq 0, \quad \forall t, j \neq k,$$

$$\sum_{r=1}^{s} u_r (Y_{rj}^{(t)})_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(t)})_{\alpha}^{L} - \sum_{f=1}^{h} w_f (Y_{fj}^{b(t)})_{\alpha}^{U} - \sum_{i=1}^{m} v_i (X_{ij}^{(t)})_{\alpha}^{U} - \sum_{l=1}^{q} \beta_l (Z_{lj}^{(t-1)})_{\alpha}^{U} \leq 0, \ \forall t, j \neq k,$$

$$u_r \geq \epsilon \ \forall r, \quad v_i \geq \epsilon \ \forall i, \quad \gamma_l \geq \epsilon \ \forall l, \quad \beta_l \geq \epsilon \ \forall l, \quad \epsilon > 0.$$

Model-7

$$
\begin{aligned}
\max\ (E_k)^L_\alpha = {} & \sum_{r=1}^{s} u_r (Y^g_{rk})^L_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lk})^L_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fk})^U_\alpha \\
\text{s.t.} \ & \sum_{i=1}^{m} v_i (X_{ik})^U_\alpha + \sum_{l=1}^{q} \beta_l (Z^{(0)}_{lk})^U_\alpha = 1, \\
& \sum_{r=1}^{s} u_r (Y^g_{rk})^L_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lk})^L_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fk})^U_\alpha \geq 0, \\
& \sum_{r=1}^{s} u_r (Y^g_{rk})^L_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lk})^L_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fk})^U_\alpha - \sum_{i=1}^{m} v_i (X_{ik})^U_\alpha - \sum_{l=1}^{q} \beta_l (Z^{(0)}_{lk})^U_\alpha \leq 0, \\
& \sum_{r=1}^{s} u_r (Y^g_{rj})^U_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lj})^U_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fj})^L_\alpha \geq 0, \quad \forall j \neq k, \\
& \sum_{r=1}^{s} u_r (Y^g_{rj})^U_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(T)}_{lj})^U_\alpha - \sum_{f=1}^{h} w_f (Y^b_{fj})^L_\alpha - \sum_{i=1}^{m} v_i (X_{ij})^L_\alpha + \sum_{l=1}^{q} \beta_l (Z^{(0)}_{lj})^L_\alpha \leq 0, \quad \forall j \neq k, \\
& \sum_{r=1}^{s} u_r (Y^{g(t)}_{rk})^L_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(t)}_{lk})^L_\alpha - \sum_{f=1}^{h} w_f (Y^{b(t)}_{fk})^U_\alpha \geq 0, \quad \forall t, \\
& \sum_{r=1}^{s} u_r (Y^{g(t)}_{rk})^L_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(t)}_{lk})^L_\alpha - \sum_{f=1}^{h} w_f (Y^{b(t)}_{fk})^U_\alpha - \sum_{i=1}^{m} v_i (X^{(t)}_{ik})^U_\alpha - \sum_{l=1}^{q} \beta_l (Z^{(t-1)}_{lk})^U_\alpha \leq 0, \quad \forall t, \\
& \sum_{r=1}^{s} u_r (Y^{(t)}_{rj})^U_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(t)}_{lj})^U_\alpha - \sum_{f=1}^{h} w_f (Y^{b(t)}_{fj})^L_\alpha \geq 0, \quad \forall t, j \neq k, \\
& \sum_{r=1}^{s} u_r (Y^{(t)}_{rj})^U_\alpha + \sum_{l=1}^{q} \gamma_l (Z^{(t)}_{lj})^U_\alpha - \sum_{f=1}^{h} w_f (Y^{b(t)}_{fj})^L_\alpha - \sum_{i=1}^{m} v_i (X^{(t)}_{ij})^L_\alpha - \sum_{l=1}^{q} \beta_l (Z^{(t-1)}_{lj})^L_\alpha \leq 0, \ \forall t, j \neq k, \\
& u_r \geq \epsilon \ \forall r, \ \ v_i \geq \epsilon \ \forall i, \ \ \gamma_l \geq \epsilon \ \forall l, \ \ \beta_l \geq \epsilon \ \forall l, \ \ \epsilon > 0.
\end{aligned}
$$

The optimal objective function values of Models-6 and 7 are respectively known as upper bound $((E_k)^{U^*}_\alpha)$ and lower bound $((E_k)^{L^*}_\alpha)$ system efficiency of DMU_k. The shape of the membership function of system efficiency $\mu_{\tilde{E}_k}$ is approximated by the set of intervals $\{[(E_k)^L_\alpha, (E_k)^U_\alpha] | \alpha \in (0,1]\}$. Since the data for inputs/outputs/links are parabolic fuzzy numbers, so the membership function $\mu_{\tilde{E}_k}$ can be approximated by the parabolic fuzzy numbers whose α-cut is defined by $(\tilde{E}_k)_\alpha = \{[(E_k)^L_\alpha, (E_k)^U_\alpha] | \alpha \in (0,1]\}$.

3.2. Proposed algorithm to evaluate period efficiencies

Models-6 and 7 are linear programming problems that may possess alternative solutions. Therefore, measuring upper and lower bound period efficiencies using non-unique weights obtained from these models may lead to invalid intervals at some α levels. However, there is a need to evaluate valid period interval efficiencies at each α-level. So an algorithm is presented to assess interval period efficiencies of DMU_k for the period p at level α is summarized as below:

Algorithm to evaluate interval period efficiencies

Step 1. Evaluate upper bound $((E_k)^U_\alpha)$ and lower bound $((E_k)^L_\alpha)$ system efficiencies using Models-6 and 7, respectively.

Step 2. To solve Model-9(a) which is the linear form of Model-8(a) that maximizes the period efficiency of t^{th} period of DMU_k ($(E_k^{(t)})_\alpha^U$) at any given α-level while retaining the upper bound system efficiency derived from Model-6 and and considers the DMU_k's upper bounds of desirable outputs/links (as output) and lower bounds of inputs/undesirable outputs/links (as input) for period p. For all other periods except period p ($\forall t \neq p$) of DMU_k and for all periods ($\forall t$) of DMU_j, $j \neq k$, lower bounds of desirable outputs/links (as output) and upper bounds of inputs/undesirable outputs/links (as input) are used.

Model-8(a)

$$\max\ (E_k^{(p)})_\alpha^U = \frac{\sum_{r=1}^{s} u_r (Y_{rk}^{g(p)})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(p)})_\alpha^U - \sum_{f=1}^{h} w_p (Y_{fk}^{b(p)})_\alpha^L}{\sum_{i=1}^{m} v_i (X_{ik}^{(p)})_\alpha^L + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(p-1)})_\alpha^L}$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y_{rk}^{g(p)})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(p)})_\alpha^U - \sum_{f=1}^{h} w_f (Y_{fk}^{b(p)})_\alpha^L}{\sum_{i=1}^{m} v_i (X_{ik}^{(p)})_\alpha^L + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(p-1)})_\alpha^L} \leq 1,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y_{rk}^{g(t)})_\alpha^L + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(t)})_\alpha^L - \sum_{f=1}^{h} w_f (Y_{fk}^{b(t)})_\alpha^U}{\sum_{i=1}^{m} v_i (X_{ik}^{(t)})_\alpha^U + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(t-1)})_\alpha^U} \leq 1, \quad t \neq p,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y_{rj}^{g(t)})_\alpha^L + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(t)})_\alpha^L - \sum_{f=1}^{h} w_f (Y_{fj}^{b(t)})_\alpha^U}{\sum_{i=1}^{m} v_i (X_{ij}^{(t)})_\alpha^U + \sum_{l=1}^{q} \beta_l (Z_{lj}^{(t-1)})_\alpha^U} \leq 1, \quad \forall t, j \neq k,$$

$$\frac{\sum_{r=1}^{s} u_r (Y_{rk}^{g})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(T)})_\alpha^U - \sum_{f=1}^{h} w_f (Y_{fk}^{b})_\alpha^L}{\sum_{i=1}^{m} v_i (X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(0)})_\alpha^L} = (E_k)_\alpha^{U^*},$$

$$u_r \geq \epsilon\ \forall r,\ \ v_i \geq \epsilon\ \forall i,\ \ \gamma_l \geq \epsilon\ \forall l,\ \ \beta_l \geq \epsilon\ \forall l,\ \ \epsilon > 0.$$

By using Charnes-Cooper transformation [5], Model-8(a) is further reduced to the linear form given by Model-9(a).

Model-9(a)

$$\max\ (E_k^{(p)})_\alpha^U = \sum_{r=1}^{s} u_r (Y_{rk}^{g(p)})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(p)})_\alpha^U - \sum_{f=1}^{h} w_p (Y_{fk}^{b(p)})_\alpha^L$$

$$\text{s.t.} \sum_{i=1}^{m} v_i (X_{ik}^{(p)})_\alpha^L + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(p-1)})_\alpha^L = 1,$$

$$\sum_{r=1}^{s} u_r \left(Y_{rk}^{g(p)}\right)_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l \left(Z_{lk}^{(p)}\right)_{\alpha}^{U} - \sum_{f=1}^{h} w_f \left(Y_{fk}^{b(p)}\right)_{\alpha}^{L} \geq 0,$$

$$\sum_{r=1}^{s} u_r \left(Y_{rk}^{g(p)}\right)_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l \left(Z_{lk}^{(p)}\right)_{\alpha}^{U} - \sum_{f=1}^{h} w_f \left(Y_{fk}^{b(p)}\right)_{\alpha}^{L} - \sum_{i=1}^{m} v_i \left(X_{ik}^{(p)}\right)_{\alpha}^{L} - \sum_{l=1}^{q} \beta_l \left(Z_{lk}^{(p-1)}\right)_{\alpha}^{L} \leq 0,$$

$$\sum_{r=1}^{s} u_r \left(Y_{rk}^{g(t)}\right)_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l \left(Z_{lk}^{(t)}\right)_{\alpha}^{L} - \sum_{f=1}^{h} w_f \left(Y_{fk}^{b(t)}\right)_{\alpha}^{U} \geq 0, \quad \forall t \neq p,$$

$$\sum_{r=1}^{s} u_r \left(Y_{rk}^{g(t)}\right)_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l \left(Z_{lk}^{(t)}\right)_{\alpha}^{L} - \sum_{f=1}^{h} w_f \left(Y_{fk}^{b(t)}\right)_{\alpha}^{U} - \sum_{i=1}^{m} v_i \left(X_{ik}^{(t)}\right)_{\alpha}^{U} - \sum_{l=1}^{q} \beta_l \left(Z_{lk}^{(t-1)}\right)_{\alpha}^{U} \leq 0, \quad \forall t \neq p,$$

$$\sum_{r=1}^{s} u_r \left(Y_{rj}^{g(t)}\right)_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l \left(Z_{lj}^{(t)}\right)_{\alpha}^{L} - \sum_{f=1}^{h} w_f \left(Y_{fj}^{b(t)}\right)_{\alpha}^{U} \geq 0, \quad \forall t, j \neq k,$$

$$\sum_{r=1}^{s} u_r \left(Y_{rj}^{g(t)}\right)_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l \left(Z_{lj}^{(t)}\right)_{\alpha}^{L} - \sum_{f=1}^{h} w_f \left(Y_{fj}^{b(t)}\right)_{\alpha}^{U} - \sum_{i=1}^{m} v_i \left(X_{ij}^{(t)}\right)_{\alpha}^{U} - \sum_{l=1}^{q} \beta_l \left(Z_{lj}^{(t-1)}\right)_{\alpha}^{U} \leq 0, \quad \forall t, j \neq k,$$

$$\sum_{r=1}^{s} u_r \left(Y_{rk}^{g}\right)_{\alpha}^{U} + \sum_{l=1}^{q} \gamma_l \left(Z_{lk}^{(T)}\right)_{\alpha}^{U} - \sum_{f=1}^{h} w_f \left(Y_{fk}^{b}\right)_{\alpha}^{L} - \left(E_k\right)_{\alpha}^{U^*} * \left(\sum_{i=1}^{m} v_i \left(X_{ik}\right)_{\alpha}^{L} + \sum_{l=1}^{q} \beta_l \left(Z_{lk}^{(0)}\right)_{\alpha}^{L}\right) = 0,$$

$$u_r \geq \epsilon \, \forall r, \quad v_i \geq \epsilon \, \forall i, \quad \gamma_l \geq \epsilon \, \forall l, \quad \beta_l \geq \epsilon \, \forall l, \quad \epsilon > 0.$$

The optimal objective function value of Model-9(a) is known as upper bound period efficiency ($(E_k^{(p)})_{\alpha}^{U^*}$) at α-level of DMU_k for period p.

Step 3. To evaluate lower bound period efficiency of period p of DMU_k ($(E_k^{(p)})_{\alpha}^{L}$) using Model-9(b) which is reduced from Model-8(b) that minimizes $(E_k^{(p)})_{\alpha}^{L}$ at any given α-level and retains the lower bound system efficiency ($(E_k)_{\alpha}^{L^*}$) obtained from Model-7 and considers the DMU_k's lower bounds of desirable outputs/links (as output) and upper bounds of inputs/undesirable outputs/links (as input) for period p. On contrary, for all other periods except p ($\forall t \neq p$) of DMU_k and for all periods ($\forall t$) of DMU_j, $j \neq k$, upper bounds of desirable outputs/links (as output) and lower bounds of inputs/undesirable outputs/links (as input) are used.

Model-8(b)

$$\max \;\; (E_k^{(p)})_{\alpha}^{L} = \frac{\sum_{r=1}^{s} u_r \left(Y_{rk}^{g(p)}\right)_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l \left(Z_{lk}^{(p)}\right)_{\alpha}^{L} - \sum_{f=1}^{h} w_p \left(Y_{fk}^{b(p)}\right)_{\alpha}^{U}}{\sum_{i=1}^{m} v_i \left(X_{ik}^{(p)}\right)_{\alpha}^{U} + \sum_{l=1}^{q} \beta_l \left(Z_{lk}^{(p-1)}\right)_{\alpha}^{U}}$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r \left(Y_{rk}^{g(p)}\right)_{\alpha}^{L} + \sum_{l=1}^{q} \gamma_l \left(Z_{lk}^{(p)}\right)_{\alpha}^{L} - \sum_{f=1}^{h} w_f \left(Y_{fk}^{b(p)}\right)_{\alpha}^{U}}{\sum_{i=1}^{m} v_i \left(X_{ik}^{(p)}\right)_{\alpha}^{U} + \sum_{l=1}^{q} \beta_l \left(Z_{lk}^{(p-1)}\right)_{\alpha}^{U}} \leq 1,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y_{rk}^{g(t)})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(t)})_\alpha^U - \sum_{f=1}^{h} w_f (Y_{fk}^{b(t)})_\alpha^L}{\sum_{i=1}^{m} v_i (X_{ik}^{(t)})_\alpha^L + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(t-1)})_\alpha^L} \leq 1, \quad t \neq p,$$

$$0 \leq \frac{\sum_{r=1}^{s} u_r (Y_{rj}^{g(t)})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(t)})_\alpha^U - \sum_{f=1}^{h} w_f (Y_{fj}^{b(t)})_\alpha^L}{\sum_{i=1}^{m} v_i (X_{ij}^{(t)})_\alpha^L + \sum_{l=1}^{q} \beta_l (Z_{lj}^{(t-1)})_\alpha^L} \leq 1, \quad \forall t, j \neq k,$$

$$\frac{\sum_{r=1}^{s} u_r (Y_{rk}^{g})_\alpha^L + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(T)})_\alpha^L - \sum_{f=1}^{h} w_f (Y_{fk}^{b})_\alpha^U}{\sum_{i=1}^{m} v_i (X_{ik})_\alpha^U + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(0)})_\alpha^U} = (E_k)_\alpha^{L^*},$$

$$u_r \geq \epsilon \; \forall r, \;\; v_i \geq \epsilon \; \forall i, \;\; \gamma_l \geq \epsilon \; \forall l, \;\; \beta_l \geq \epsilon \; \forall l, \;\; \epsilon > 0.$$

By using Charnes-Cooper transformation [5], Model-8(b) is reduced to the following linear model given by Model-9(b).

Model-9(b)

$$\max \; (E_k^{(p)})_\alpha^L = \sum_{r=1}^{s} u_r (Y_{rj}^{g(p)})_\alpha^L + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(p)})_\alpha^L - \sum_{f=1}^{h} w_p (Y_{fj}^{b(p)})_\alpha^U$$

$$\text{s.t.} \; \sum_{i=1}^{m} v_i (X_{ij}^{(p)})_\alpha^U + \sum_{l=1}^{q} \beta_l (Z_{lj}^{(p-1)})_\alpha^U = 1$$

$$\sum_{r=1}^{s} u_r (Y_{rk}^{g(p)})_\alpha^L + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(p)})_\alpha^L - \sum_{f=1}^{h} w_f (Y_{fk}^{b(p)})_\alpha^U \geq 0,$$

$$\sum_{r=1}^{s} u_r (Y_{rk}^{g(p)})_\alpha^L + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(p)})_\alpha^L - \sum_{f=1}^{h} w_f (Y_{fk}^{b(p)})_\alpha^U - \sum_{i=1}^{m} v_i (X_{ik}^{(p)})_\alpha^U - \sum_{l=1}^{q} \beta_l (Z_{lk}^{(p-1)})_\alpha^U \leq 0,$$

$$\sum_{r=1}^{s} u_r (Y_{rk}^{g(t)})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(t)})_\alpha^U - \sum_{f=1}^{h} w_f (Y_{fk}^{b(t)})_\alpha^L \geq 0, \;\; \forall t \neq p,$$

$$\sum_{r=1}^{s} u_r (Y_{rk}^{g(t)})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(t)})_\alpha^U - \sum_{f=1}^{h} w_f (Y_{fk}^{b(t)})_\alpha^L - \sum_{i=1}^{m} v_i (X_{ik}^{(t)})_\alpha^L - \sum_{l=1}^{q} \beta_l (Z_{lk}^{(t-1)})_\alpha^L \leq 0, \;\; \forall t \neq p,$$

$$\sum_{r=1}^{s} u_r (Y_{rj}^{g(t)})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(t)})_\alpha^U - \sum_{f=1}^{h} w_f (Y_{fj}^{b(t)})_\alpha^L \geq 0, \;\; \forall t, j \neq k,$$

$$\sum_{r=1}^{s} u_r (Y_{rj}^{g(t)})_\alpha^U + \sum_{l=1}^{q} \gamma_l (Z_{lj}^{(t)})_\alpha^U - \sum_{f=1}^{h} w_f (Y_{fj}^{b(t)})_\alpha^L - \sum_{i=1}^{m} v_i (X_{ij}^{(t)})_\alpha^L - \sum_{l=1}^{q} \beta_l (Z_{lj}^{(t-1)})_\alpha^L \leq 0, \;\; \forall t, j \neq k,$$

$$\sum_{r=1}^{s} u_r (Y_{rk}^{g})_\alpha^L + \sum_{l=1}^{q} \gamma_l (Z_{lk}^{(T)})_\alpha^L - \sum_{f=1}^{h} w_f (Y_{fk}^{b})_\alpha^U - (E_k)_\alpha^{L^*} * \left(\sum_{i=1}^{m} v_i (X_{ik})_\alpha^U + \sum_{l=1}^{q} \beta_l (Z_{lk}^{(0)})_\alpha^U \right) = 0,$$

$$u_r \geq \epsilon \, \forall r, \;\; v_i \geq \epsilon \, \forall i, \;\; \gamma_l \geq \epsilon \, \forall l, \;\; \beta_l \geq \epsilon \, \forall l, \;\; \epsilon > 0.$$

The optimal objective function value of Model-9(b) is known as lower bound period efficiency $((E_k^{(p)})_\alpha^{L^*})$ of DMU_k for period p. The shape of the membership function of period efficiency $\mu_{\tilde{E}_k^{(p)}}$ of DMU_k for period p is approximated by the set of intervals $\{[(E_k^{(p)})_\alpha^L, (E_k^{(p)})_\alpha^U] | \alpha \in (0,1]\}$. It is further noted that the shape of membership function at different α-levels may constitute a PFN.

3.3. Schematic view of the proposed PFDDEA approach

Figure 2 clearly depicts the schematic view of the proposed PFDDEA approach for measuring system and period efficiencies of the DMUs when input-output data are available as PFNs.

4. Relationship between system and period efficiencies

This section presents a relationship between upper bound (lower bound) system efficiency and upper bound (lower bound) period efficiencies.

4.1. Relationship between upper bound system and upper bound period efficiencies

Let $(u_r^* \, \forall r; \; w_f^* \, \forall f; \; v_i^* \, \forall i; \; \gamma_l^* \, \forall l; \; \beta_l^* \, \forall l)$ be the optimal weights derived from Model-9(a) at level α, $0 \leq \alpha \leq 1$. Since,

$$\sum_{t=1}^{T} \left[\sum_{r=1}^{s} u_r^* (Y_{rk}^{g(t)})_\alpha^U + \sum_{l=1}^{q} \gamma_l^* (Z_{lk}^{(t)})_\alpha^U - \sum_{f=1}^{h} w_f^* (Y_{fk}^{b(t)})_\alpha^L - \sum_{i=1}^{m} v_i^* (X_{ik}^{(t)})_\alpha^L - \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(t-1)})_\alpha^L \right]$$
$$= \sum_{r=1}^{s} u_r^* (Y_{rk}^{g})_\alpha^U + \sum_{t=1}^{T-1} \sum_{l=1}^{q} \gamma_l^* (Z_{lk}^{(t)})_\alpha^U + \sum_{l=1}^{q} \gamma_l^* (Z_{lk}^{(T)})_\alpha^U - \sum_{f=1}^{h} w_f^* (Y_{fk}^{b})_\alpha^L - \sum_{i=1}^{m} v_i^* (X_{ik})_\alpha^L - \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(0)})_\alpha^L - \sum_{t=2}^{T} \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(t-1)})_\alpha^L$$

Dividing on both sides by $\sum_{i=1}^{m} v_i^* (X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(0)})_\alpha^L$, we get

$$\sum_{t=1}^{T} \left[\left((E_k^{(t)})_\alpha^{U^*} - 1 \right) W_\alpha^{(t)U} \right] = \left((E_k)_\alpha^{U^*} - 1 \right) + \Delta_\alpha^U \tag{4.1}$$

$$\text{where } W_\alpha^{(t)U} = \left(\frac{\sum_{i=1}^{m} v_i^* (X_{ik}^{(t)})_\alpha^L + \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(t-1)})_\alpha^L}{\sum_{i=1}^{m} v_i^* (X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(0)})_\alpha^L} \right) \text{ and } \Delta_\alpha^U = \frac{\sum_{t=1}^{T-1} \sum_{l=1}^{q} \gamma_l^* (Z_{lk}^{(t)})_\alpha^U - \sum_{t=2}^{T} \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(t)})_\alpha^L}{\sum_{i=1}^{m} v_i^* (X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(0)})_\alpha^L} \tag{4.2}$$

$$\text{Now, } \sum_{t=1}^{T} W_\alpha^{(t)U} = \sum_{t=1}^{T} \left(\frac{\sum_{i=1}^{m} v_i^* (X_{ik}^{(t)})_\alpha^L + \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(t-1)})_\alpha^L}{\sum_{i=1}^{m} v_i^* (X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(0)})_\alpha^L} \right) = \frac{\sum_{i=1}^{m} v_i^* (X_{ik})_\alpha^L + \sum_{t=1}^{T} \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(t-1)})_\alpha^L}{\sum_{i=1}^{m} v_i^* (X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^* (Z_{lk}^{(0)})_\alpha^L}$$

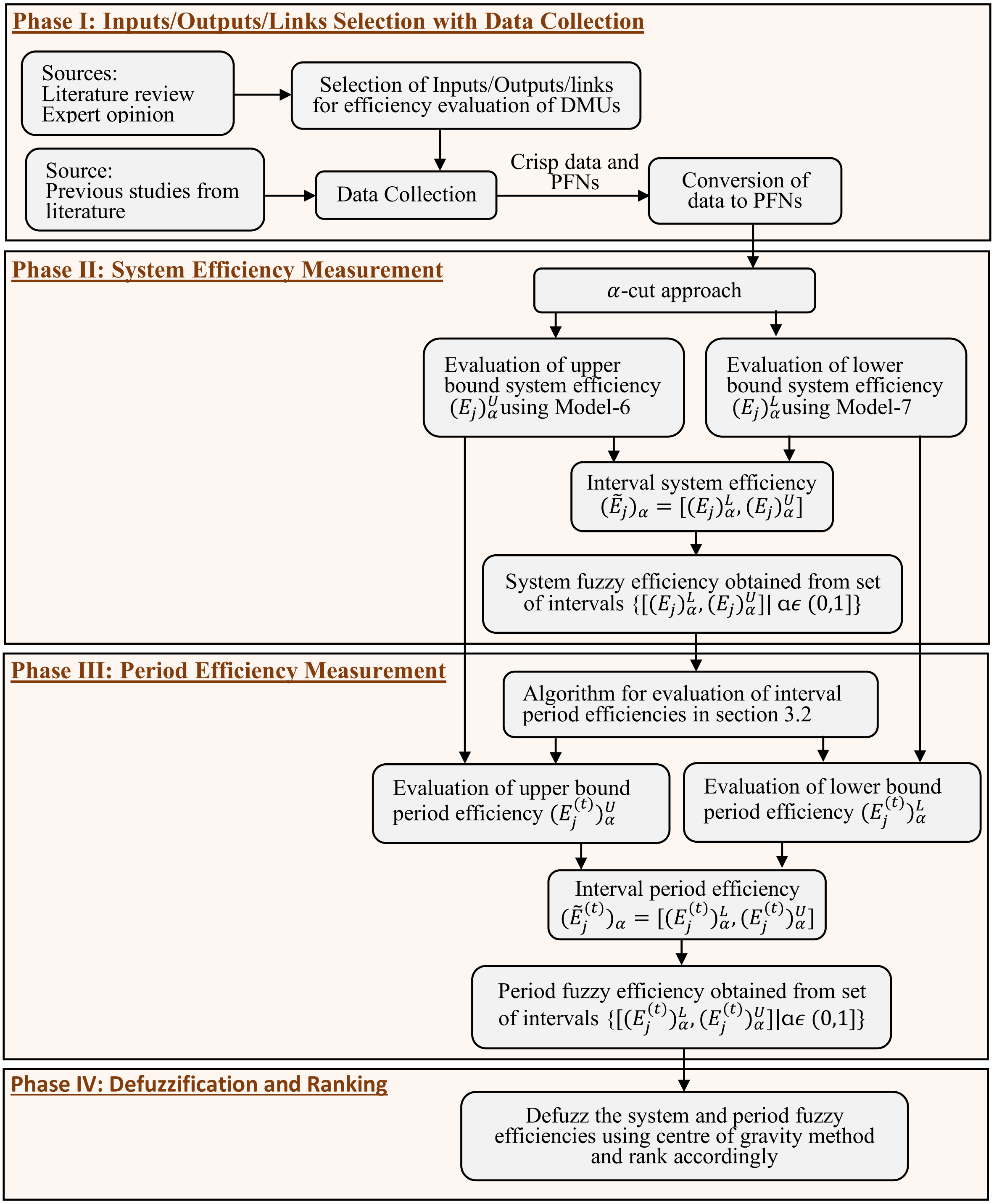

FIGURE 2. Schematic view of the proposed parabolic fuzzy dynamic DEA model.

$$= \frac{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(0)})_\alpha^L + \sum_{t=2}^{T}\sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(t-1)})_\alpha^L}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(0)})_\alpha^L}$$

$$= 1 + \frac{\sum_{t=2}^{T}\sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(t-1)})_\alpha^L}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(0)})_\alpha^L} = 1 + \Delta_{1\alpha}^U \tag{4.3}$$

If $\Delta_\alpha^U = 0$ and $\Delta_{1\alpha}^U = 0$, then by using the equations (4.1), (4.2), and (4.3), it is evident that complement of upper bound system efficiency $((E_k)_\alpha^{U^*} - 1)$ is linear combination of the complement of upper bound period efficiencies $((E_k^{(t)})_\alpha^{U^*} - 1)$ at level α, $0 \le \alpha \le 1$.

Remark 2. From equations (4.1)–(4.3), we have

$$\sum_{t=1}^{T} \left((E_k^{(t)})_\alpha^{U^*}\right) W_\alpha^{(t)U} = (E_k)_\alpha^{U^*} + (\Delta_\alpha^U - 1) + \sum_{t=1}^{T} W_\alpha^{(t)U} = (E_k)_\alpha^{U^*} + \Delta_\alpha^U + \Delta_{1\alpha}^U = (E_k)_\alpha^{U^*} + \Delta_{2\alpha}^U$$

where, $\Delta_{2\alpha}^U = \Delta_\alpha^U + \Delta_{1\alpha}^U$

$$= \frac{\sum_{t=1}^{T-1}\sum_{l=1}^{q} \gamma_l^*(Z_{lk}^{(t)})_\alpha^U - \sum_{t=2}^{T}\sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(t)})_\alpha^L}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(0)})_\alpha^L} + \frac{\sum_{t=2}^{T}\sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(t-1)})_\alpha^L}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(0)})_\alpha^L}$$

$$= \frac{\sum_{t=1}^{T-1}\sum_{l=1}^{q} \gamma_l^*(Z_{lk}^{(t)})_\alpha^U - \sum_{t=2}^{T}\sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(t)})_\alpha^L + \sum_{t=2}^{T}\sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(t-1)})_\alpha^L}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(0)})_\alpha^L} = \frac{\sum_{t=1}^{T}\sum_{l=1}^{q} \gamma_l^*(Z_{lk}^{(t)})_\alpha^U}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^L + \sum_{l=1}^{q} \beta_l^*(Z_{lk}^{(0)})_\alpha^L}.$$

It implies that upper bound system efficiency $((E_k)_\alpha^{U^*})$ is linear combination of upper bound period efficiencies $((E_k^{(t)})_\alpha^{U^*}, \forall t)$ at level α $(0 \le \alpha \le 1)$, provided $\Delta_{2\alpha}^U = 0$. Also, if $\Delta_{1\alpha}^U = 0$ (& $\Delta_{2\alpha}^U = 0$), $(E_k)_\alpha^{U^*}$ can be expressed as weighted average of $(E_k^{(t)})_\alpha^{U^*}, (\forall t)$ at level α.

4.2. Relationship between lower bound system efficiency and lower bound period efficiencies

Let $(u_r^* \ \forall r;\ w_f^* \ \forall f;\ v_i^* \ \forall i;\ \gamma_l^* \ \forall l;\ \beta_l^* \ \forall l)$ be the optimal weights derived from Model-9(b) at level α, $0 \le \alpha \le 1$.

Since,

$$\sum_{t=1}^{T}\left[\sum_{r=1}^{s} u_r^*(Y_{rk}^{g(t)})_\alpha^L + \sum_{l=1}^{q}\gamma_l^*(Z_{lk}^{(t)})_\alpha^L - \sum_{f=1}^{h} w_f^*(Y_{fk}^{b(t)})_\alpha^U - \sum_{i=1}^{m} v_i^*(X_{ik}^{(t)})_\alpha^U - \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t-1)})_\alpha^U\right]$$
$$= \sum_{r=1}^{s} u_r^*(Y_{rk}^{g})_\alpha^L + \sum_{t=1}^{T-1}\sum_{l=1}^{q}\gamma_l^*(Z_{lk}^{(t)})_\alpha^L + \sum_{l=1}^{q}\gamma_l^*(Z_{lk}^{(T)})_\alpha^L - \sum_{f=1}^{h} w_f^*(Y_{fk}^{b})_\alpha^U - \sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U -$$
$$\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U - \sum_{t=2}^{T}\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t-1)})_\alpha^U.$$

Dividing on both sides by $\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U$, we get

$$\sum_{t=1}^{T}\left[\left((E_k^{(t)})_\alpha^{L^*} - 1\right) W_\alpha^{(t)L}\right] = \left((E_k)_\alpha^{L^*} - 1\right) + \Delta_\alpha^L \tag{4.4}$$

$$\text{where } W_\alpha^{(t)L} = \left(\frac{\sum_{i=1}^{m} v_i^*(X_{ik}^{(t)})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t-1)})_\alpha^U}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U}\right) \text{ and } \Delta_\alpha^L = \frac{\sum_{t=1}^{T-1}\sum_{l=1}^{q}\gamma_l^*(Z_{lk}^{(t)})_\alpha^L - \sum_{t=2}^{T}\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t)})_\alpha^U}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U} \tag{4.5}$$

$$\text{Now, } \sum_{t=1}^{T} W_\alpha^{(t)L} = \sum_{t=1}^{T}\left(\frac{\sum_{i=1}^{m} v_i^*(X_{ik}^{(t)})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t-1)})_\alpha^U}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U}\right) = \frac{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U + \sum_{t=1}^{T}\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t-1)})_\alpha^U}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U}$$
$$= \frac{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U + \sum_{t=2}^{T}\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t-1)})_\alpha^U}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U}$$
$$= 1 + \frac{\sum_{t=2}^{T}\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t-1)})_\alpha^U}{\sum_{i=1}^{m} v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U} = 1 + \Delta_{1\alpha}^L. \tag{4.6}$$

If $\Delta_\alpha^L = 0$ and $\Delta_{1\alpha}^L = 0$, then by using the equations (4.4)–(4.6), complement of lower bound system efficiency $((E_k)_\alpha^{L^*} - 1)$ can be written as linear combination of the complement of lower bound period efficiencies $((E_k^{(t)})_\alpha^{L^*} - 1)$.

Remark 3. From equations (4.4)–(4.5), we have

$$\sum_{t=1}^{T}\left((E_k^{(t)})_\alpha^{L^*}\right) W_\alpha^{(t)L} = (E_k)_\alpha^{L^*} + (\Delta_\alpha^L - 1) + \sum_{t=1}^{T} W_\alpha^{(t)L} = (E_k)_\alpha^{L^*} + \Delta_\alpha^L + \Delta_1^L = (E_k)_\alpha^{L^*} + \Delta_{2\alpha}^L$$

where, $\Delta_{2\alpha}^L = \Delta_\alpha^L + \Delta_{1\alpha}^L$

$$= \frac{\sum_{t=1}^{T-1}\sum_{l=1}^{q}\gamma_l^*(Z_{lk}^{(t)})_\alpha^L - \sum_{t=2}^{T}\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t)})_\alpha^U}{\sum_{i=1}^{m}v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U} + \frac{\sum_{t=2}^{T}\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t-1)})_\alpha^U}{\sum_{i=1}^{m}v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U}$$

$$= \frac{\sum_{t=1}^{T-1}\sum_{l=1}^{q}\gamma_l^*(Z_{lk}^{(t)})_\alpha^L - \sum_{t=2}^{T}\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t)})_\alpha^U + \sum_{t=2}^{T}\sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(t-1)})_\alpha^U}{\sum_{i=1}^{m}v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U} = \frac{\sum_{t=1}^{T-1}\sum_{l=1}^{q}\gamma_l^*(Z_{lk}^{(t)})_\alpha^L}{\sum_{i=1}^{m}v_i^*(X_{ik})_\alpha^U + \sum_{l=1}^{q}\beta_l^*(Z_{lk}^{(0)})_\alpha^U}.$$

It indicates that lower bound system efficiency $((E_k)_\alpha^{L^*})$ is linear combination of lower bound period efficiencies $((E_k^{(t)})_\alpha^{L^*}, \forall t)$, at level α $(0 \le \alpha \le 1)$ provided $\Delta_{2\alpha}^L = 0$. Also, if $\Delta_{1\alpha}^L = 0$ (& $\Delta_{2\alpha}^L = 0$), $(E_k)_\alpha^{L^*}$ can be expressed as weighted average of $(E_k^{(t)})_\alpha^{L^*}, \forall t$ at level α.

5. Case study in the Indian banking sector

This section presents an application of the proposed approach to the Indian banking sector. A free-float market capitalization scheme is used to select 11 banks (See Tab. 1) for efficiency evaluation over two financial periods, 2019-2020 and 2020-2021. Each bank utilizes three inputs (Total employees $(\tilde{X}_{1j}^{(t)})$, Total loanable funds $(\tilde{X}_{2j}^{(t)})$ and Fixed assets $(X_{3j}^{(t)})$) to yield two desirable outputs (Performing loans $(\tilde{Y}_{1j}^{g(t)})$ and Net profit $(Y_{2j}^{g(t)})$), and one undesirable output (Loss due to non-performing assets $(Y_{1j}^{b(t)})$) in each period t, and the consecutive periods (t and $t+1$) are connected through one link (unused assets $(\tilde{Z}_{1j}^{(t)})$). The description of all the input/output variables and links is as follows:

- Total employees (TE): These are the total number of employees working in a bank, including officers, clerks, and sub-staff members.
- Total loanable funds (TLF): Total Loanable funds are the sum of deposits and borrowings of a bank.
- Fixed assets (FA): These comprise premises, fixed assets under construction and other assets.
- Performing loans (PL): These are calculated by subtracting the gross NPAs from advances where NPAs are the non-performing assets.
- Net profit (NP): It is evaluated by subtracting total expenses, provisions and contingencies from the total income of a bank.
- Loss due to non-performing assets (LNPAs): It is the amount held as a provision for non-performing assets (NPAs) by the bank.
- Unused assets (UA): These are calculated by subtracting fixed capital, loans, required reserves, and security investments from total assets [3].

5.1. Data and imprecision

Data for 11 banks for the periods 2018-2019, 2019-2020, and 2020-2021 are collected from RBI [33] and are depicted in Tables 2, 3, and 4, respectively. The data for all variables except total employees are in Rs. Crores. Although the original data is collected in a crisp form but discussions with the bank experts infer that due to the uncertainty, data for some variables can be considered in the form of PFNs. Therefore, the data imprecision is taken in the present study, which is summarized below:

Data for two inputs (TE and TLF), one desirable output (PL) and link (UA) are taken in the form of PFNs and crisp data are considered for the remaining inputs/desirable outputs. A variation of 2% and 1% has been

TABLE 1. List of banks included in the case study.

Bank code	Bank names	Bank code	Bank names
FB	Federal Bank Ltd.	BB	Bandhan Bank Ltd.
RBL	RBL Bank Ltd.	ICICI	ICICI Bank Ltd.
KMB	Kotak Mahindra Bank Ltd.	IB	Indusind Bank Ltd.
HDFC	HDFC Bank Ltd.	AB	Axis Bank Ltd.
PNB	Punjab National Bank	SBI	State Bank of India
BoB	Bank of Baroda		

TABLE 2. Initial links.

DMU	$z_{1j}^{(2019)}$	DMU	$z_{1j}^{(2019)}$	DMU	$z_{1j}^{(2019)}$	DMU	$z_{1j}^{(2019)}$
FB	(15811.3, 15971.0, 16130.8)	HDFC	(144084.4, 145539.8, 146995.2)	BB	(4619.9, 4666.6, 4713.2)	AB	(165253.2, 166922.5, 168591.7)
RBL	(11231.9, 11345.4, 11458.8)	PNB	(126117.6, 127391.6, 128665.5)	ICICI	(200458.4, 202483.3, 204508.1)	SBI	(609648.2, 615806.3, 621964.4)
KMB	(37714.3, 38095.3, 38476.2)	BoB	(125913.7, 127185.6, 128457.4)	IB	(34039.9, 34383.8, 34727.6)		

taken for defining the left and right spreads of PFNs in the case of TE and UA, respectively. An imprecision of 2% for the left spread and 1.5% for the right spread in a PFN is considered for TLF, whereas 1% for the left spread and 0.5% for the right spread in a PFN for PL is considered. For normalization, the data have been divided by the number of branches of the respective bank before final implementation.

5.2. Results and discussion

The upper and lower bounds of the α-cuts of system fuzzy efficiency $((\tilde{E}_j)_\alpha, \forall j)$ at each α in $(0, 1]$ are evaluated using Models-5 and 7, respectively, and are shown in Table 5 at some α-levels. In a similar manner, the upper and lower bounds of the α-cuts of period efficiency are evaluated by using Models-9(a) and 9(b), and are shown in Tables 6 and 7, respectively. MATLAB software is used to solve the linear programming models and evaluate efficiencies at each α. Tables 5, 6 and 7 show that FB is the only bank with the highest upper and lower bound system and periods' efficiencies equal to one for every $\alpha \in [0, 1]$. Every other bank has shown upper/lower bound system inefficiency, which is affected by the inefficiency present in either period 1 or period 2 or both of the selected periods. Careful observation of Table 6 depicts that KMB and BB have reached efficiency value 1 for period 1 (both upper and lower efficiencies) but attained inefficiency in period 2 (both upper and lower). Similarly, in Table 7, HDFC and BB are the two banks that have achieved efficiency value 1 in period 2 for each $\alpha \in [0, 1]$; however, ineffficiency in period 1 for both lower and upper bound efficiencies. Due to the inefficiencies present in either period 1 or period 2, the banks, namely, KMB, BB, HDFC and IB, lack in contributing efficiency score 1 in system efficiency. Moreover, the shapes of the membership functions of the system and period fuzzy efficiencies are predicted and identified using the system and period interval efficiencies obtained for $\alpha \in [0, 1]$ in Tables 5, 6, and 7.

Figures 3 and 4 represent the graphical representation of the α-cuts, namely, $(\tilde{E}_j)_\alpha$ and $(\tilde{E}_j^{(t)})_\alpha$. The figures depict that the shape of the membership functions of the system and period fuzzy efficiencies can be approximated as PFNs which are presented in Table 8.

Moreover, the system and period fuzzy efficiencies of 11 banks are defuzzified using the centroid/centre of gravity method [32] as defined in equation (5.1) and are ranked accordingly.

$$d(\tilde{A}) = \frac{\int_x x.\mu_{\tilde{A}}(x)\mathrm{d}x}{\int_x \mu_{\tilde{A}}(x)\mathrm{d}x} \tag{5.1}$$

where $\tilde{A}$ is a fuzzy number with membership function $\mu_{\tilde{A}}$ and $d(\tilde{A})$ denotes the defuzzified value of $\tilde{A}$.

TABLE 3. Data for financial year 2020.

DMU	$X_{1j}^{(2020)}$	$X_{2j}^{(2020)}$	$X_{3j}^{(2020)}$	$Y_{1j}^{g(2020)}$	$Y_{2j}^{g(2020)}$	$Y_{1j}^{b(2020)}$	$Z_{1j}^{(2020)}$
FB	(8247, 8497, 8747)	(159409.3, 162662.5, 165102.5)	479.99	(117549.7, 118737.1, 119330.8)	1542.8	1885.3	(19918.1, 20119.3, 20320.5)
RBL	(4766, 4910, 5055)	(73322.58, 74818.96, 75941.24)	469.76	(55323.7, 55882.5, 56161.9)	505.7	947.2	(12375.4, 12500.4, 12625.4)
KMB	(33022, 34023, 35024)	(294797.5, 300813.8, 305326.0)	1623.1	(212574.1, 214721.3, 215794.9)	5947.2	3469.0	(66333.9, 67004.0, 67674.1)
HDFC	(77200, 79540, 81880)	(1266288.2, 1292130.8, 1311512.8)	4431.9	(971242.4, 981052.9, 985958.2)	26257.3	9107.6	(165646.9, 167320.1, 168993.3)
PNB	(45395, 46771, 48147)	(738990.3, 754071.7, 765382.8)	7239.1	(394365.5, 398348.9, 400340.7)	336.2	45843.5	(124436.3, 125693.2, 126950.2)
BoB	(55627, 57312, 58998)	(1018272.6, 1039053.7, 1054639.5)	8889.3	(614531.9, 620739.3, 623843.0)	546.2	47804.8	(183122.8, 184972.6, 186822.3)
BB	(26235, 27030, 27825)	(71991.5, 73460.7, 74562.6)	368.8	(64980.8, 65637.2, 65965.4)	3023.7	603.4	(6477.9, 6543.4, 6608.8)
ICICI	(64254, 66201, 68147)	(915188.4, 933865.7, 947873.7)	8410.3	(598416.3, 604460.9, 607483.2)	7930.8	30905.8	(230283.7, 232609.8, 237935.9)
IB	(20245, 20858, 21471)	(257537.5, 262793.4, 266735.3)	1820.12	(199620.1, 201636.4, 202644.6)	4417.9	3260.2	(38513.3, 38902.3, 39291.3)
AB	(48932, 50415, 51898)	(772297.9, 788059.1, 799879.9)	4312.9	(535778.4, 541190.3, 543896.3)	1627.2	20802.7	(193697.5, 195654.1, 197610.6)
SBI	(164636, 169625, 174614)	(3485150.9, 3556276.4, 3609620.5)	38439.3	(2154435.7, 2176197.7, 2187078.7)	14488.1	97220.5	(685354.2, 692276.9, 699199.7)

Notes. Any crisp number (a) can be written in the form of a parabolic fuzzy number (a, a, a).

TABLE 4. Data for financial year 2021.

DMU	$X_{1j}^{(2021)}$	$X_{2j}^{(2021)}$	$X_{3j}^{(2021)}$	$Y_{1j}^{g(2021)}$	$Y_{2j}^{g(2021)}$	$Y_{1j}^{b(2021)}$	$Z_{1j}^{(2021)}$
FB	(8311, 8563, 8814)	(178078.7, 181713.0, 184438.7)	491.13	(126003.4, 127276.2, 127912.3)	1590.30	2998.11	(30816.3, 31127.6, 31438.9)
RBL	(5159, 5315, 5471)	(82660.2, 84347.2, 85612.4)	466.48	(55460.8, 56021.0, 56301.1)	508.00	1,360.18	(19961.1, 20162.7, 20364.3)
KMB	(34144, 35179, 36214)	(297675.7, 303750.7, 308307.0)	1535.27	(214100.5, 216263.1, 217344.4)	6964.84	4720.34	(64219.5, 64868.1, 65516.8)
HDFC	(79261, 81663, 84065)	(1441136.6, 1470547.5, 1492605.8)	4909.32	(1106573.1, 1117750.6, 1123339.4)	31116.53	10531.18	(220066.9, 222289.8, 224512.7)
PNB	(63519, 65444, 67369)	(1126189.3, 1149172.8, 1166410.4)	11020.90	(564108.6, 569806.7, 572655.7)	2021.62	65127.85	(205271.5, 207344.9, 209418.3)
BoB	(54579, 56233, 57887)	(1013168.0, 1033844.9, 1049352.5)	8016.25	(633233.2, 639629.5, 642827.7)	828.96	44871.11	(190298.7, 192220.9, 194143.1)
BB	(32634, 33623, 34612)	(93033.9, 94932.6, 96356.6)	486.71	(75096.6, 75855.1, 76234.4)	2205.46	2896.73	(5264.8, 5318.00, 5371.2)
ICICI	(65175, 67150, 69125)	(1003670.1, 1024153.1, 1039515.4)	8877.58	(685958.8, 692887.7, 696352.1)	16192.68	31723.76	(251186.5, 253723.72, 256261.0)
IB	(19576, 20169, 20763)	(301377.2, 307527.8, 312140.7)	1809.37	(204732.4, 206800.4, 207834.4)	2836.39	4318.42	(76251.3, 77021.5, 77791.7)
AB	(51683, 53249, 54815)	(833175.7, 850179.2, 862931.9)	4245.03	(592421.3, 598405.4, 601397.4)	6588.50	18188.98	(168164.8, 169863.5, 171562.1)
SBI	(162130, 167043, 171956)	(4016603.3, 4098574.8, 4160053.4)	38419.24	(2299877.7, 2323108.8, 2334724.3)	20410.47	89579.30	(909129.3, 918312.4, 927495.5)

Notes. Any crisp number (a) can be written in the form of a parabolic fuzzy number (a, a, a).

Table 5. α-cuts of system fuzzy efficiency.

DMU	$\alpha = 0$ [L,U]	$\alpha = 0.2$ [L,U]	$\alpha = 0.4$ [L,U]	$\alpha = 0.6$ [L,U]	$\alpha = 0.8$ [L,U]	$\alpha = 1$ [L,U]
FB	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
RBL	[0.896,0.990]	[0.916,0.968]	[0.925,0.959]	[0.931,0.952]	[0.937,0.947]	[0.942,0.942]
KMB	[0.974,1.000]	[0.992,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
HDFC	[0.998,1.000]	[0.999,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
PNB	[0.654,0.723]	[0.669,0.707]	[0.675,0.700]	[0.680,0.695]	[0.684,0.691]	[0.688,0.688]
BoB	[0.765,0.860]	[0.785,0.837]	[0.794,0.828]	[0.800,0.821]	[0.806,0.816]	[0.811,0.811]
BB	[0.999,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
ICICI	[0.834,0.921]	[0.853,0.901]	[0.861,0.893]	[0.867,0.887]	[0.872,0.881]	[0.877,0.877]
IB	[0.999,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
AB	[0.866,0.959]	[0.886,0.937]	[0.894,0.928]	[0.901,0.922]	[0.906,0.916]	[0.911,0.911]
SBI	[0.859,0.977]	[0.882,0.946]	[0.891,0.933]	[0.898,0.924]	[0.905,0.916]	[0.910,0.910]

Notes. In a similar way, the system efficiency can be evaluated at different α-levels.

Table 6. α-cuts of Period 1 fuzzy efficiency.

DMU	$\alpha = 0$ [L,U]	$\alpha = 0.2$ [L,U]	$\alpha = 0.4$ [L,U]	$\alpha = 0.6$ [L,U]	$\alpha = 0.8$ [L,U]	$\alpha = 1$ [L,U]
FB	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
RBL	[0.894,0.989]	[0.914,0.967]	[0.923,0.958]	[0.929,0.951]	[0.935,0.945]	[0.940,0.940]
KMB	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
HDFC	[0.958,1.000]	[0.971,1.000]	[0.978,1.000]	[0.985,1.000]	[0.991,1.000]	[0.997,0.997]
PNB	[0.685,0.754]	[0.700,0.738]	[0.706,0.732]	[0.711,0.727]	[0.715,0.722]	[0.719,0.719]
BoB	[0.736,0.834]	[0.744,0.808]	[0.754,0.796]	[0.761,0.788]	[0.768,0.781]	[0.774,0.774]
BB	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
ICICI	[0.813,0.895]	[0.831,0.876]	[0.838,0.868]	[0.844,0.862]	[0.849,0.858]	[0.853,0.853]
IB	[0.899,1.000]	[0.927,1.000]	[0.939,0.990]	[0.949,0.980]	[0.957,0.971]	[0.964,0.964]
AB	[0.851,0.943]	[0.871,0.922]	[0.879,0.913]	[0.886,0.906]	[0.891,0.901]	[0.896,0.896]
SBI	[0.757,0.901]	[0.790,0.873]	[0.805,0.862]	[0.816,0.853]	[0.827,0.846]	[0.836,0.836]

Table 7. α-cuts of Period 2 fuzzy efficiency.

DMU	$\alpha = 0$ [L,U]	$\alpha = 0.2$ [L,U]	$\alpha = 0.4$ [L,U]	$\alpha = 0.6$ [L,U]	$\alpha = 0.8$ [L,U]	$\alpha = 1$ [L,U]
FB	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
RBL	[0.895,0.987]	[0.915,0.966]	[0.923,0.957]	[0.930,0.951]	[0.935,0.945]	[0.940,0.940]
KMB	[0.755,0.908]	[0.769,0.857]	[0.775,0.838]	[0.785,0.824]	[0.794,0.813]	[0.804,0.804]
HDFC	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
PNB	[0.660,0.730]	[0.674,0.713]	[0.680,0.706]	[0.685,0.701]	[0.689,0.697]	[0.693,0.693]
BoB	[0.811,0.874]	[0.826,0.862]	[0.833,0.857]	[0.837,0.852]	[0.841,0.848]	[0.845,0.845]
BB	[0.845,0.950]	[0.863,0.924]	[0.873,0.914]	[0.881,0.906]	[0.887,0.899]	[0.893,0.893]
ICICI	[0.851,0.937]	[0.869,0.917]	[0.877,0.909]	[0.883,0.902]	[0.888,0.897]	[0.893,0.893]
IB	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]	[1.000,1.000]
AB	[0.873,0.967]	[0.894,0.945]	[0.902,0.936]	[0.909,0.930]	[0.914,0.924]	[0.919,0.919]
SBI	[0.866,0.966]	[0.886,0.936]	[0.895,0.924]	[0.901,0.917]	[0.907,0.916]	[0.911,0.911]

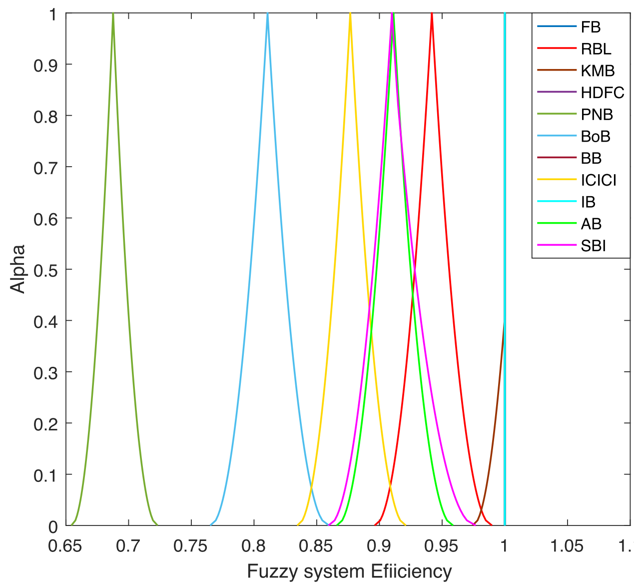

FIGURE 3. Shape of membership function of system efficiency ($\tilde{E}_j$) of DMU_j.

TABLE 8. The values of $\tilde{E}_j$, $\tilde{E}_j^{(1)}$ and $\tilde{E}_j^{(2)}$ approximated as PFNs with defuzzified values and ranking.

DMU	$\tilde{E}_j$	$d(\tilde{E}_j)$	Rank	$\tilde{E}_j^{(1)}$	$d(\tilde{E}_j^{(1)})$	Rank	$\tilde{E}_j^{(2)}$	$d(\tilde{E}_j^{(2)})$	Rank
FB	(1.000,1.000,1.000)	1.0000	1	(1.000,1.000,1.000)	1.0000	1	(1.000,1.000,1.000)	1.0000	1
RBL	(0.896,0.942,0.990)	0.9425	6	(0.894,0.940,0.989)	0.9407	6	(0.895,0.940,0.987)	0.9405	4
KMB	(0.974,1.000,1.000)	0.9888	5	(1.000,1.000,1.000)	1.0000	1	(0.755,0.804,0.908)	0.8178	10
HDFC	(0.998,1.000,1.000)	0.9999	2	(0.958,0.997,1.000)	0.9853	4	(1.000,1.000,1.000)	1.0000	1
PNB	(0.654,0.688,0.723)	0.6883	11	(0.685,0.719,0.754)	0.7193	11	(0.660,0.693,0.730)	0.6939	11
BoB	(0.765,0.811,0.860)	0.8117	10	(0.736,0.774,0.834)	0.7795	10	(0.811,0.845,0.874)	0.8437	9
BB	(0.999,1.000,1.000)	0.9999	2	(1.000,1.000,1.000)	1.0000	1	(0.845,0.893,0.950)	0.8952	7
ICICI	(0.834,0.877,0.921)	0.8773	9	(0.813,0.853,0.895)	0.8534	8	(0.851,0.893,0.937)	0.8935	8
IB	(0.999,1.000,1.000)	0.9999	2	(0.899,0.964,1.000)	0.9567	5	(1.000,1.000,1.000)	1.0000	1
AB	(0.866,0.911,0.959)	0.9117	8	(0.851,0.896,0.943)	0.8965	7	(0.873,0.919,0.967)	0.9195	5
SBI	(0.859,0.910,0.977)	0.9140	7	(0.757,0.836,0.901)	0.8325	9	(0.866,0.911,0.966)	0.9135	6

The defuzzified values of the system and period fuzzy efficiencies and their ranking are also shown in Table 8, where $d(\tilde{E}_j)$ and $d(\tilde{E}_j^{(t)})$ represents the defuzzified values of system fuzzy efficiency ($\tilde{E}_j$) and period fuzzy efficiency ($\tilde{E}_j^{(t)}, \forall t$), respectively. It has been observed that defuzzified system as well as period efficiencies of Federal Bank are 1, hence ranked as most efficient bank whereas Punjab National Bank (PNB) is found to be the least efficient with 11th rank in system and both the periods. Moreover, 3 banks (HDFC, BB, IB) are given rank 2 with defuzzified system efficiency 0.9999. Similarly, 3 banks (FB, KMB, BB) and 3 banks (FB, HDFC, IB) are given rank 1 in period 1 and period 2, respectively.

5.3. Comparison with a static approach

The conventional DEA models tend to ignore the interdependence of periods and measure the performance of DMUs statically. However, the proposed approach evaluates the efficiency in a dynamic environment with periods connected through carry-overs/links. To better analyze the effect of links on the efficiency of DMUs,

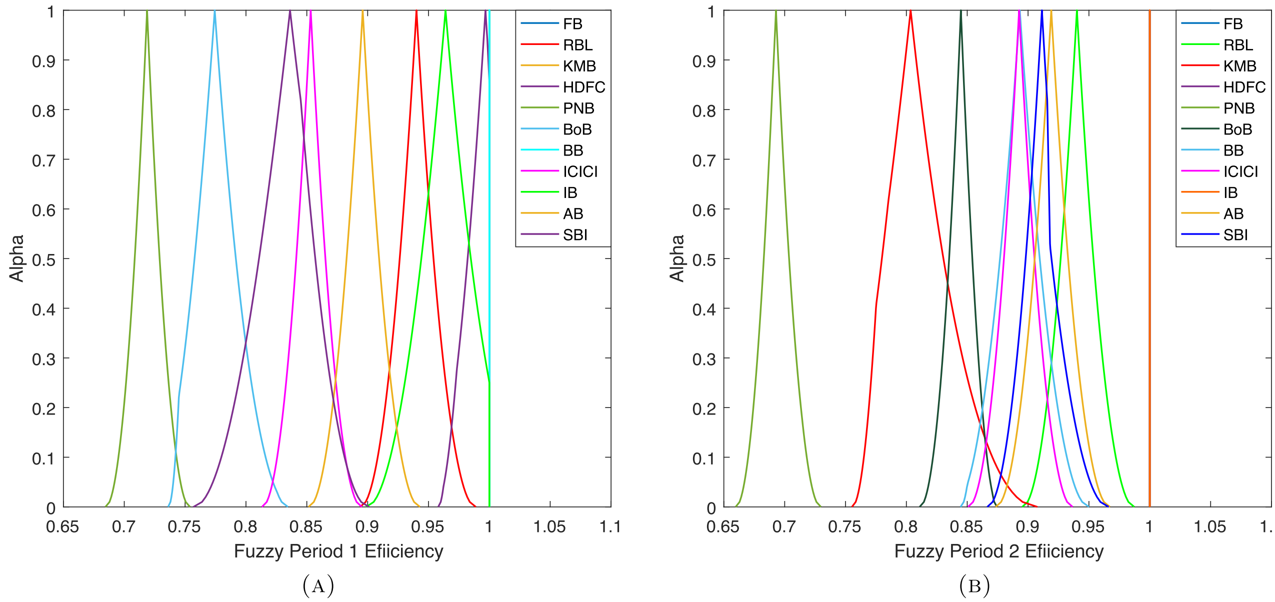

FIGURE 4. Shape of membership function of Period efficiencies ($\tilde{E}_j^{(t)}$) of DMU_j. (A) Shape of membership function of $\tilde{E}_j^{(1)}$. (B) Shape of membership function of $\tilde{E}_j^{(2)}$.

particularly, the impact of unused assets on the performance of selected banks in our case study, a comparative analysis is conducted between the proposed approach (with carry-overs/links) and the static approach (without carry-overs/links) at each level α. The comparison concludes that all banks except FB have shown an increase in system fuzzy efficiency (defuzzified value) with the proposed dynamic approach compared to the static approach. The efficiency of FB remains the same (See Fig. 5). The increase in system efficiency in a dynamic environment represents the positive impact of unused assets on the system efficiency of the selected banks. It implies that carry-overs like unused assets must be included in the production process to get a realistic picture of banks' performance. Thus, the proposed dynamic DEA approach has shown a significant impact of carry-over elements (unused assets) on the banks' efficiencies in India during the selected period. Therefore, in the present scenario, it is evident to use the dynamic approach, particularly the proposed approach, when data is fuzzy.

6. IMPLICATIONS

This section consists of theoretical, managerial and policy implications of the present study.

6.1. Theoretical implications

The present study developed a novel parabolic fuzzy dynamic DEA (PFDDEA) approach that not only handles inputs/outputs/links data as PFNs along with undesirable outputs but also considers the inter-relationships present between two periods in terms of carry-over variables to see the dynamic effect of variables on DMUs' efficiencies. Different weights are assigned to links that act as inputs at one time and outputs at another. The modeling of the proposed approach based on the α-cut approach and Pareto's efficiency concept characterizes two main aspects: system fuzzy efficiency score and algorithmic approach for period fuzzy efficiency. As a theoretical development in the proposed PFDDEA approach, (i) the shapes of the membership functions of the system and period fuzzy efficiencies were predicted and identified, (ii) some conditions were derived that ensure upper (lower) bound system efficiency can be expressed as a linear combination (weighted average) of the upper

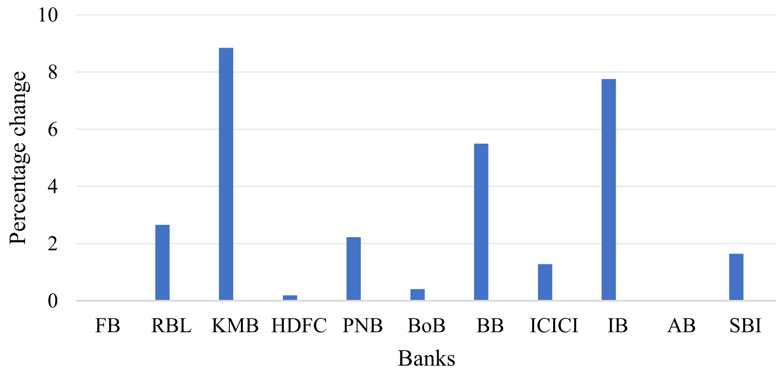

Figure 5. Percentage change in defuzzified values of system efficicney at level $\alpha = 0$ from static to dynamic environment.

(lower) bound period efficiencies at each α in [0,1], and (iii) it can also be applied to problems having other forms of imprecision such as interval numbers, ordinal numbers, and fuzzy numbers (like triangular/trapezoidal/LR fuzzy numbers).

6.2. Managerial implications

For validation and effectiveness, the proposed PFDDEA approach has been applied to the Indian banking sector. The findings of the case study provide practical implications to the bank experts for managerial decisions, which are summarized below:

- FB ranked first with a defuzzified value of system and period fuzzy efficiencies equal to 1 and can act as a benchmark for the other low-ranked banks. PNB ranked last and needs to refer to FB for improvement in the system as well as period efficiencies.
- HDFC, BB and IB possess the second rank for system fuzzy efficiency. BB performed efficiently in period 1, whereas HDFC and IB worked efficiently in period 2.
- The percentage change of lower and upper bound period efficiencies from period 1 to period 2 at level $\alpha =$ 0 is analyzed, which depicts that SBI has shown the highest percentage increase of 14.5% (7%) for lower (upper) bound period efficiency. In contrast, KMB has shown the highest percentage decrease of 24.5% (9%) for lower (upper) bound period efficiency.
- Among all banks, seven banks (RBL, HDFC, BoB, ICICI, IB, AB and SBI) have shown a percentage increase (ranges in [0.15%, 14.5%] for lower and [2.6%, 7.24%] for upper bound efficiencies) whereas three banks (KMB, PNB and BB) have shown percentage decrease (ranges in [3.63%, 24.5%] for lower and [0.16%, 9.24%] for upper bound efficiencies) from period 1 to period 2 using the proposed approach.
- The proposed approach, when compared with the static approach, resulted in an increase in the system efficiency (defuzzified value) for all banks in a dynamic proposed approach (with carry-overs/links) as compared to the static approach (without carry-overs/links). It ranges from 0.01% to 8.85%, with the highest increase in KMB and lowest in AB. It clearly indicates that the carry-over/link (unused assets in the present case study) imparted its positive impact on the system efficiency of each bank which implies that carry-overs like unused assets must be included in the production process to get a realistic picture of banks' performance which is ignored by static approach.

6.3. Policy implications

The case study presented in this paper demonstrated that the proposed PFDDEA model offers more consistent efficiency results when the target industry is characterized by intertemporal dependencies and fuzziness. As a performance analysis tool, the PFDDEA model not only can provide policymakers with a coherent depiction of the industry's dynamic performance over periods but can also handle qualitative performance indicators like customer satisfaction, service quality, etc. Moreover, it can also be used by the decision-makers to know about the need for investment in the early stages (*e.g.*, number of employees, loanable funds, assets in the case study). Lastly, the findings on the impact of unused assets over banks' performance can be utilized by the bank experts in policy making.

7. Conclusion

In the present study, a parabolic fuzzy dynamic DEA approach has been developed to evaluate system and period efficiencies of DMUs in the presence of undesirable outputs and imprecise data in the form of PFNs. An algorithm is presented to evaluate period efficiencies. The α-cut approach has been used to measure efficiencies. Further a relationship between upper bound (lower bound) system and upper bound (lower bound) period efficiencies is derived to indicate the dependence among each other. To prove the validity of the proposed approach, it has been applied to 11 banks in India and the resulting system as well as period efficiencies are derived as PFNs. For ranking the fuzzy efficiencies in dynamic environment, the centre of gravity method has been used to defuzzify the system and period efficiencies and then ranked accordingly. The findings of the case study conclude that FB is the most efficient and PNB is the least efficient among selected DMUs in dynamic environment.

The future scope of the present study is very wide and can be expanded in various directions. The proposed fuzzy dynamic DEA approach has vast implications and withstands its immense applicability in real-life problems. The present study can be extended to dynamic network structures with data uncertainties of different types and applications in other sectors like manufacturing, insurance, supply chain, health care, power plants, etc.

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