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BUILDING A SUSTAINABILITY IN A TWO-ECHELON CLOSED LOOP SUPPLY CHAINS: A MATHEMATICAL APPROACH FOR PERMISSIBLE DELAY IN PAYMENT AND BACKLOGGING

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Abstract. This paper develops a supply chain framework where payments take place with permissible delays. Unlike other studies, the supply chain is modelled as a closed-loop system, in which returning the products is incorporated into classical supply chain models and the full backorder is permitted. Among the most important research questions is the order amount of chain members to maximize Retailer delays and the profit chains. The study determines the portion of the time interval in which inventory system experiences shortage and determines the optimal replenishment time and the frequency. Ultimately, it was shown that, if the delay from the supplier to the retailer increases, the chain profit also increases. For all three proposed models, a closed-form solution is developed and a solution algorithm is presented. Applying the coordinated model considerably increases the total profit earned by the whole SC as well as all SC members. An example of our model is a bottle supplier for the drink producer. Another example is military oriented. Furthermore, the study elaborates the feasibility of the suggested models by means of some numerical examples and discusses the results using sensitivity analysis.

Mathematics Subject Classification. 30-XX, 49-XX.

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1. Introduction

Most suppliers and distributers intend to achieve a long term cooperative relationship in order to guarantee dependable supply as well as long lasting demands, while maximizing the profit, through the current competitive environment of the business world, which is rapidly growing in recent years. As a result, determining the prefect parameters, the most important of which, is optimal order quantity, for every supply chain system composed of vendors and buyers, becomes crucial.

After extensive extensions presented in inventory models by different researchers, three different strategies are found in the literature for financial exchange of purchasing cost between the sellers and buyers. The earliest and simplest way was what Harris introduced in the classical EOQ model. But nowadays, the conditions of the market may impose other strategies. sometimes sellers offer delay in payment to their customers as a marketing strategy to increase sales and reduce inventories (Teng *et al.* [40]; Wu and Chan [42]; Zhang *et al.* [44]). In

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this case, two different scenarios may be used: (3.1) all of the payment can be paid after a predetermined time (called full delayed payment) or (3.2) a percent of the payment should be paid at the delivery moment and the remaining will be received later (partial delayed payment). Bregman [6] found that the large timing of disbursements may have tremendous effects on the order quantity in a classical EOQ model; and Lau and Lau [31] showed that despite of this fact, the impact of timing of disbursements on the overall cost may be small.

Different versions of inventory models with delayed payment have been presented in the literature. For the first time, Goyal [21] proposed permissible delay in payments for a single-item inventory model. Aggarwal and Jaggi [2] extended this EOQ model for deteriorating items. Chang *et al.* [8] established an EOQ model for deteriorating items, in which the buyers are allowed to pay the purchasing cost by delay, if their purchasing cost is greater than a predetermined amount. Delay in payments for the continuous review inventory model was investigated by Salameh *et al.* [36]. Discount policy on purchasing cost along with a delay in payment was considered by Wu and Zhou [43].

Closed-loop supply chain is becoming the focal point of the relevant research area for a variety of reasons. The most important of which are as follows. Primarily, relative legislations is used to impose in majority of developed countries to put organizations in charge of handling their own products and packaging after consumption by the customers. Moreover, the awareness of customers regarding contamination of the environment, due to consumed products, packaging burial and incineration is in growth. It also forces manufacturers to recycle their goods rather than destroy them. The interest of end-users towards buying environment-friendly products is also being intensified Carter and Ellram [7] estimated the market to be over $200 billion, causing extra burden on the manufacturers' shoulders for initiating and enhancing the process of recollecting their consumed products and recapture the lucrative merits as hard as possible. Assimilating environmentally friendly production processes as well as the final products themselves, on one hand, paves the way for the manufacturers to stay in accordance with the governmental legislations, parallel to customers' demand, while, on the other hand, improves their corporate images. Last but not Least, assimilating proactive production approaches is becoming more interesting for organizations, compared with passive approaches, causing them to consider employing a recycling approach strongly.

This study will answer the following questions:

1. What are the optimal replenishment policies for SC members in presence of returnable defective products and the strategy of variant delay payment?
2. Is there a feasible replenishment policy to Increase the capital profit of the chain and create a regular order schedule for all the SC parties at the same time?
3. How can an appropriate incentive mechanism be proposed to induce SC members to accept these optimal policies?
4. What are the effects of returnable defective products on the model?

In order to give an appropriate response to the aforementioned questions, a two-echelon supply chain model including one supplier and one distributor was formed. The mentioned SC is a closed-loop system in which returning the products is incorporated into classic supply chain models and also an incentive policy is taking place and full backorder is permitted, too. Consider a situation where the distributor uses an EOQ model to control the inventory of products where the shortage is being backlogged. When the distributer places an order to the supplier, after a fixed lead time, the issued order will be delivered. Notice that after selling the products to the final customer a fraction of them will be returned to the supplier which leads to reproducing products with acceptable quality. There are two ways for the supplier to provide products, foreign supplier and repairing returned products.

We configured the SC under three separate conditions:

1. A closed party model in which each part of SC will decide only based on their own reliable profit.
2. An open model in which each part will engage in order to maximize capital profit of the whole system.

In the first model each of the members wants to optimize their own profits but in the second one they decide integrately based on whether the length of receiving money is constant or variable.

The contribution of this paper consists of two sections: Firstly, a mathematical model will be conducted in order to concieve the concept of closed loop supply chains and permissible delay. To do that the effect of returnable defective products along with permissible delay in two echelon SC is investigated and a two-layer SC decision structure is modeled by considering backordering for retailer. Secondly, an incentive policy is conducted in order to study the effects of it on the order size and capital profit of the whole SC. The innovative part of this paper is that for the first time an integrated model with delayed payments, closed loop supply chain and partial backorder will be formed.

The model is simulated under seven separate circumstances: no shortage, full returnable defective products, partial returnable defective products, no returnable defective products, full delay payment, partial delay payment, and no delay payment, And there is also the possibility of combining the aforementioned circumstances. Therefore, another contribution of the current work is to allow. adopting any of these assumptions, which is not the case in other works found in the literature.

2. Literature review

In one of the primary studies, Goyal [20] first discussed the idea of both customer and the supplier's joint total cost. Afterwards, this study was expanded by Banerjee [4] through developing a cooperative economic lot size for the supply chain system whilst considering rewarding plans regarding the quantity discount for the buyer. This area of study was further extended by Jamal *et al.* [28] by permitting shortages. Additionally, various strategies like, price discounts, different credit payment possibilities and buy-back deals, to name but a few, are employed for the purpose of coordination. Delay in payments was studied where cooperation existed as well as in the situations with no cooperation by Abad and Jaggi [1], by considering the demand to be price sensitive. Pareto efficiency solution, combined with Nash bargaining concept comprised the core of their analysis utilities. The subjects of study of Chen and Kang [9] were combined vendor–buyer models with deterministic demands in which payment delay was permissible. The model was solved through a presented procedure in which balancing the savings of the vendor and buyer was achieved by extending the period of delay.

A manufacturer–customer model in which credit options are available was developed by Sarmah *et al.* [37], discussing two categories of individual target profits, that are, pre-decided as well as non-pre-decided profits. In the model proposed by Ouyang *et al.* [35], freight rate and terms of trade credit were incorporated and calculated by the quantity of order. Furthermore, Chern *et al.* [11] employed the concept of Nash equilibrium in a vendor–buyer system of supply chain while allowing payments delay. In a complementary research, Chern *et al.* [12], expanded the EOQ model, considering both the vendor and buyer, in a parallel environment. They also paved the way for disclosing the time of customer's replenishment cycle, the deliveries number of items delivered from vendor in each cycle of production, and the offered period of trade from the vendor. To solve this model, they employed the concept of Nash equilibrium. A coordination policy for the supply chain was developed by Duan *et al.* [14] through the approach of payment delay for a group of products, sharing the quality of fixed lifetime. In this investigation, solutions that were analytically manageable, were obtained. Chen and Kang [10] developed an inventory system of integrated models, enjoying the quality of permitted delay in payment, in order to consider a bi-level policy of trade credit in managing the vendor–buyer–customer supply chain. Hemapriya and Uthayakumar [22] consider two echelon supply chain with permissible delay in payments under exponential lead time involving investment for quality improvement and ordering cost reduction. Diabat *et al.* [15] apply a model of economic order quantity (EOQ) in supply chains with partial downstream delayed payment and partial upstream advance payment for a deteriorating item under three conditions: 1) shortage is not allowed, 2) full back ordering is allowed, and 3) partial back ordering is allowed. Heydari *et al.* [23] investigates the coordination of decisions in a two-echelon supply chain (SC) consisting of a supplier delivering a single product to a retailer. Demand is assumed stochastic and credit-dependent. Liao *et al.* [32] develop an economic order quantity for deteriorating items under the condition of permissible delay in payments. Taleizadeh

Table 1. Brief review of the most related papers discussed.

Reference	Echelon	Environmental issue	Trade credit		Backordering	Closed form solution	Decision variable		
			Full	Partial			Shipment quantity	Economic order quantity	Shortage
[3]	Two	–	*	–	–	–	*	*	–
[5]	Two	*	–	–	–	–	*	*	–
[41]	Two	*	–	–	*	*	*	*	*
[27]	Two	–	*	–	–	*	*	*	–
[30]	One	–	*	*	*	*	–	*	*
Present paper	Two	*	*	*	*	*	*	*	*

et al. [39] a VMI model in a two-echelon supply chain, including one vendor and two buyers are considered to develop periodic replenishment (R, T) and continuous replenishment (r, Q) models with partial back-ordering under VMI policy. In partial back-ordering, lost-sales and back-ordering are allowed as this assumption is more pragmatic.

A deterministic economic order of quantity model, in which, closed-loop supply chain analysis approach was employed, was published by Schrady [38]. In his study, a system of multiple cycles of recycling/repair supporting a single production/procurement cycle was investigated and the optimal quantities of lot sizes was calculated. Additionally, this model was generalized by Nahmiasj and Rivera [34] for the case of finite recycling/repair rate.

The literature regarding the permitted returns inventory systems and comprised of multi echelon lacks potent investigations. In this regard, Korugan and Gupta [29] discussed the characteristics of a double-echelon inventory system and its return flows. They described the system behavior through considering finite buffers for a network with open queuing. In the next step, they analyzed the system through an extended methodology.

Some recent papers on supply chain and sustainable supply chain uncertainty have also been developed and evaluated the performance of these chains. Izadikhah *et al.* [24–26]. Goodarzian *et al.* [17] developed a green supply chain network under uncertainty for inventory and purchasing decisions. To cope with uncertain parameters, fuzzy method was used. Goodarzian *et al.* [18] proposed a new supply chain network to decrease the total cost and the delivery time and maximize the reliability of the transportation system. To solve their model, some heuristic methods and meta-heuristic algorithms were provided. Goodarzian *et al.* [19] designed a multi-echelon, multi-product, and multi-period mathematical model for a sustainable supply chain network during COVID-19 pandemic. To solve their model, they suggested three meta-heuristic algorithms. Goodarzian *et al.* [16] developed a multi-objective sustainable medicine SCN that their main aims were to minimize economic and environmental aspects. To solve their model, a hybrid meta-heuristic algorithm is developed.

A comprehensive literature review has been undertaken by Dekker *et al.* [13] respect, a system of inventory with two echelons was studied by Mitra [33], in a generalized environment. In this study, two models were introduced, the first of which was a deterministic model with no shortages accepted. In the second one any shortcomings to satisfy demands of the first stage for final products was allowed to be backordered. Employing the deterministic model, introduced by Mitra [33], in order to consider returning products while optimizing the inventory model, in which, partial backordering is possible.

Based on the above descriptions of literature review and what presented in Table 1, one can clearly realize that a two-echelon closed-loop supply chain model with return product is not developed when supplier offers partial delay payments to the buyer and shortage for buyer is fully backlogged. So the main intention of this study is to develop environmental Vendor Managed Inventory (VMI) model with a constant delay between payment and delivery and shortage. This comprehensive model will be extended for the first time in which the economic order quantity, shipment quantity from the vendor and lack quantities are being determined using closed-form equations.

The organization of the present research is as follows. The third and the fourth sections discuss the methods as well as the numerical examples, respectively. Results and Discussion is provided in the fifth section and sensitivity of the model is illustrated in Section 5. In the last to sections, the managerial insights and suggestions for future research are presented, accordingly.

3. Methods

This section consisted of two parts: "Problem definition" and "Model development". In the first part, the problem explained and parameters and variables introduced. In the second part the model developed.

3.1. Problem definition

A bi-echelon system of inventory, that is, a warehouse and a distributor, is considered (As shown in Fig. 1). Receiving a demand from customer, distributor orders a request to the warehouse. Returnable defective products, which distributor receive from costumers, are also sent back to warehouse to be recovered. Thus, it is logical to conjecture that this sequence takes place in one step. In the condition, that flawed items are stored in the warehouse, the inventory of these parts becomes a section of warehouse inventory, rather than the inventory of distributor. Due to the fact that the rate of return is lower than rate of demand most of the times, satisfying the distributer demand merely through recovering defective parts is not a comprehensive answer for warehouse. Furthermore, warehouse is obliged to cover the demand by outsourcing the needs from external suppliers with unlimited capacity. Therefore, the inventory of the warehouse can be separated into two parts, namely, recoverable items' inventory and the purchased items. It is also supposed that the items of these two inventories enjoy equal quality and value and the time required to repair the defective items is insignificant compared to lead time of orders arriving from other suppliers. Last but not Least, the return rate is assumed to be independent from demand rate.

The distributor is allowed to pay a portion of the price, required for owning the goods at the time of delivery and disburse the rest at a predetermined time in the future. Indeed the payment defer is modeled through credit point of view, in which the purchase fund is covered by a loan from a bank and paid to seller in a specified time in future. Additionally, the strategy of variant delay payment is employed in order to share mutual savings of costs among warehouse (supplier) and distributor. It is also a proper tempting for the customer to join collaborative relationship and guaranteeing the success of both sides of the deal. In one scenario, the income acquired during the time of delay in payment is invested somewhere, providing a revenue by the form of periodical interest, the rate of which is, also, supposed to be lower than risk free interest rate. Finally, in this system, the shortages are completely backordered. Figure 2 illustrates the discussed system of inventory.

Inventory costs for this system include setup cost, inventory maintenance cost for each of the three stages, shortage that is occurred in the first stage as well as the paid interest and received interest for both of the echelons. The goal of the study is to determine the optimum period of refilling for distributor, parallel to the warehouse (supplier) and the shortage distributor experiences.

Examples of the developed model are milk and beer bottle suppliers. In this regard a two echelon supply chain consisted of milk and beer bottle suppliers and the distributers of them can be modeled in which the supplier will gather around used bottles from some of the distributers and return them into the cycle of producing as new bottles after the process of washing and sterilizing them.

The parameters and decision variables are introduced hereunder:

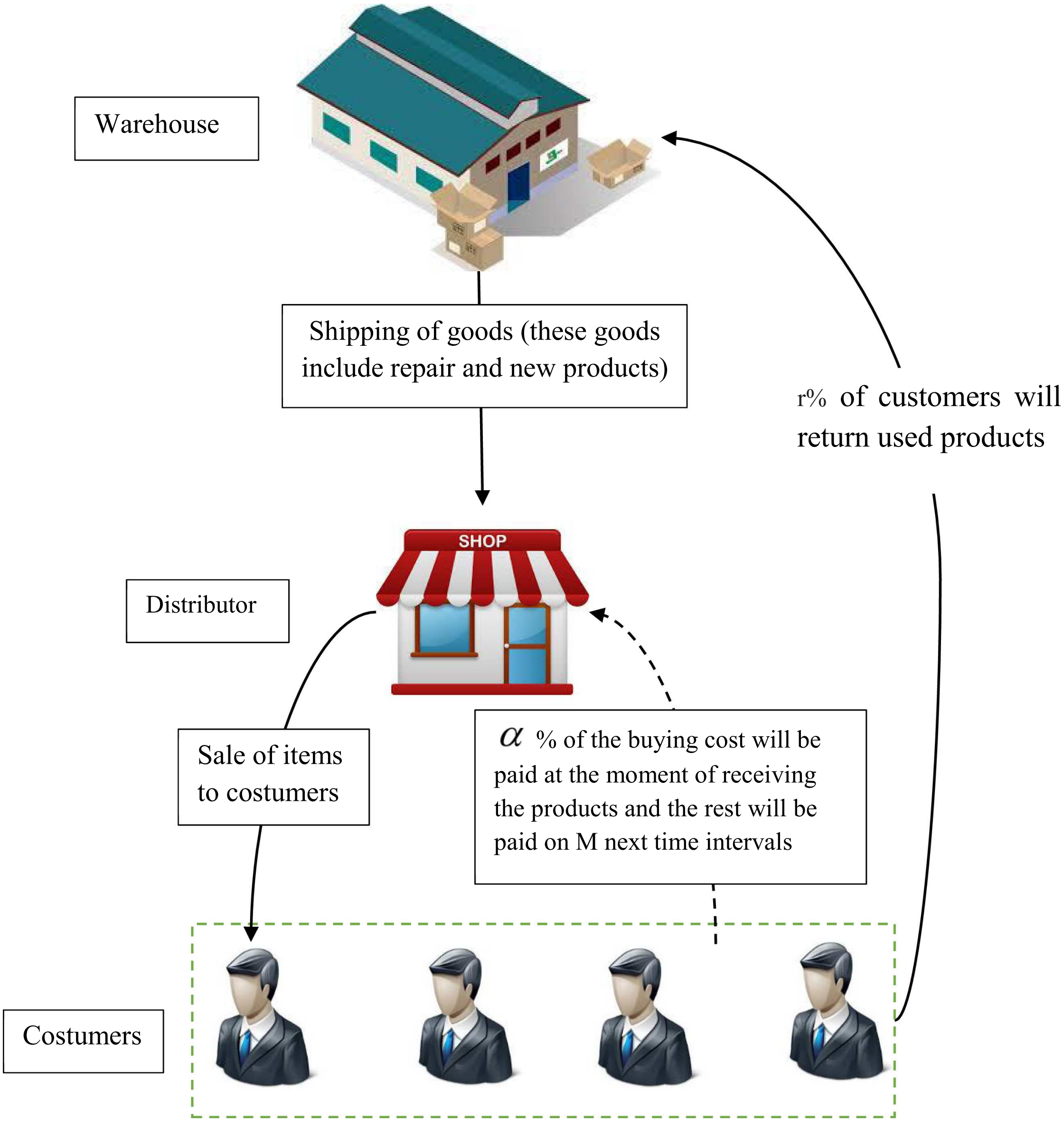

FIGURE 1. Schematic representation of the model presented.

Parameters:

D	The fixed rate of demand per unit time,
A_B	Setting-up cost for distributor (buyer) (\$/order),
A_i^V	The fixed ordering cost for warehouse (supplier) at stage i $(i = 2, 3)$ (\$/order),
h_i	The holding cost of a unit in one period of time at stage i $(i = 1, 2, 3)$ (\$/unit-time),
r	Ratio of returned items in each time period $0 < r < 1$.
C	The price of each purchased item (\$).
P_B	Selling price of each item (\$).

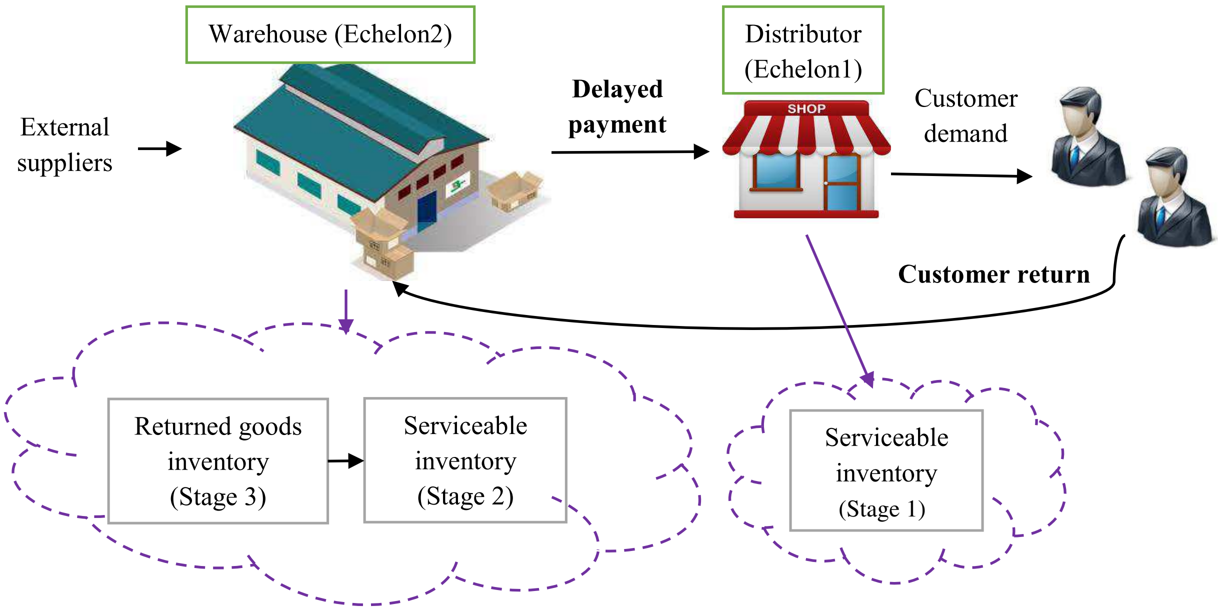

FIGURE 2. A uni-warehouse, uni-distributor system of inventory considering profit.

F	Warehouse's unchanging process cost of dealing by every order (\$) per unit time (year)
M	The permissible delay to settle the remaining of the purchasing cost $(1-\alpha)$.
π	The cost of the shortage of one item (\$) per unit time (year)
α	The ratio of buying cost, paid at the delivering time of items.
I_{vc}	The opportunity cost of the warehouse for each dollar in a period of time.
I_{Be}	The opportunity cost of the distributer for each dollar in a period of time.
I_{Bc}	The interest rate determining the distributer's income for each dollar in a period of time.
I_V	Annual interest rate for computing the warehouse's opportunity interest loss due to the delay payment
M_1	The lengthy postponement period in payment in the combined model of Model 3 $(\Delta > 0,\ M_1 = M + \Delta)$
M_S	The most permitted postponement period in payment accepted by warehouse (supplier) in the combined model of Model 3, where $M_1 < M_s$
Decision Variables:	
B	The number of backordered items
K_i	The ratio of demand, satisfied from the storage in model i $(i = 1, 2, 3)$
Q	The quantity of order
n_i	Shipment quantity from the warehouse to the distributer in each time period, (which is an integer quantity in model i $(i = 1, 2, 3)$.
T_i	The duration of a cycle of inventory in model $i(i = 1,2,3) = \begin{cases} T_{i1} & if\ M \le K_i T_i \\ T_{i2} & if\ M > K_i T_i \end{cases}$
$(*)$	Indicates the optimal value
Dependent Variables:	
TD_i	The whole cost of setup, maintenance, and shortage and demand lost for the distributor in model i $(i = 1, 2, 3)$

TABLE 2. The difference of models, annual expenditure warehouse, distributer and their supply chain.

Model	Sharing information	M is Decision variable?	Condition	Annual total cost of distributer	Annual total cost of warehouse (supplier)	Annual total cost of supply chain
Model 1	No	No	$M < KT$	ATB_{11}	TCV_1	ATC_1
			$M > KT$	ATB_{12}		
Model 2	Yes	No	$M < KT$	ATB_{21}	TCV_2	ATC_2
			$M > KT$	ATB_{22}		
Model 3	Yes	Yes	$M < KT$	ATB_{31}	TCV_3	ATC_3
			$M > KT$	ATB_{32}		

AHC_i The yearly holding cost for each stage i $(i = 1, 2, 3)$
TCV_i Whole cost per unit time for the warehouse (supplier) in model i $(i = 1, 2, 3)$
ACC_{ij} The annual capital cost per year in the model i $(i = 1, 2, 3)$ and case j $(j = 1, 2)$
AIE_{ij} The annual interest earned per year in the model i $(i = 1, 2, 3)$ and case j $(j = 1, 2)$
ATB_{ij} The annual total cost for distributer per year in the model i $(i = 1, 2, 3)$ and case j $(j = 1, 2)$
ATC_i The yearly whole cost for each phase
$ATC_i^{'}$ The equal yearly whole cost for each phase
IPD The periodic investment cost as prepayment
IED The periodic interest transaction due to delay in payment
ATB_i The annual total cost for distributer per year in the model i $(i = 1, 2, 3)$
ATC_i Summation of annual total cost for supplier and retailer in model i $(i = 1, 2, 3)$

3.2. Model development

According to the discussed issues, the interactions among two echelons, combined with acceptable paying delays, the model is as follows.

Table 2 shows the difference of models, annual expenditure warehouse, distributer and their supply chain. In the first step, the yearly cost for the first echelon 1 is discussed and is, thus, a part of the total cost for the echelon 2. The aim of this section is to develop three subsequent models, the first of which, depicts the costs of the warehouse (supplier) and is utilized in the rest of the subsequent models.

3.2.1. Warehouse's (Supplier's) cost model

For the warehouse, the total cost for each time period includes setup, holding, processing and opportunity cost.

The behavior of these types of costs are assessed here under:

(1) Setup cost: according to Figure 3, it can be deducted that the set-ups numbers in every time period of the second and the third stages are $\frac{A_2^V}{n_i T_i}$ and $\frac{A_3^V}{n_i T_i}$, separately, in the ith model $(i = 1, 2, 3)$.
(2) Maintenance cost: for each cycle, the warehouse's inventory level in the second stage, can be determined as the difference of the distributer's total inventory level from that of warehouse's.

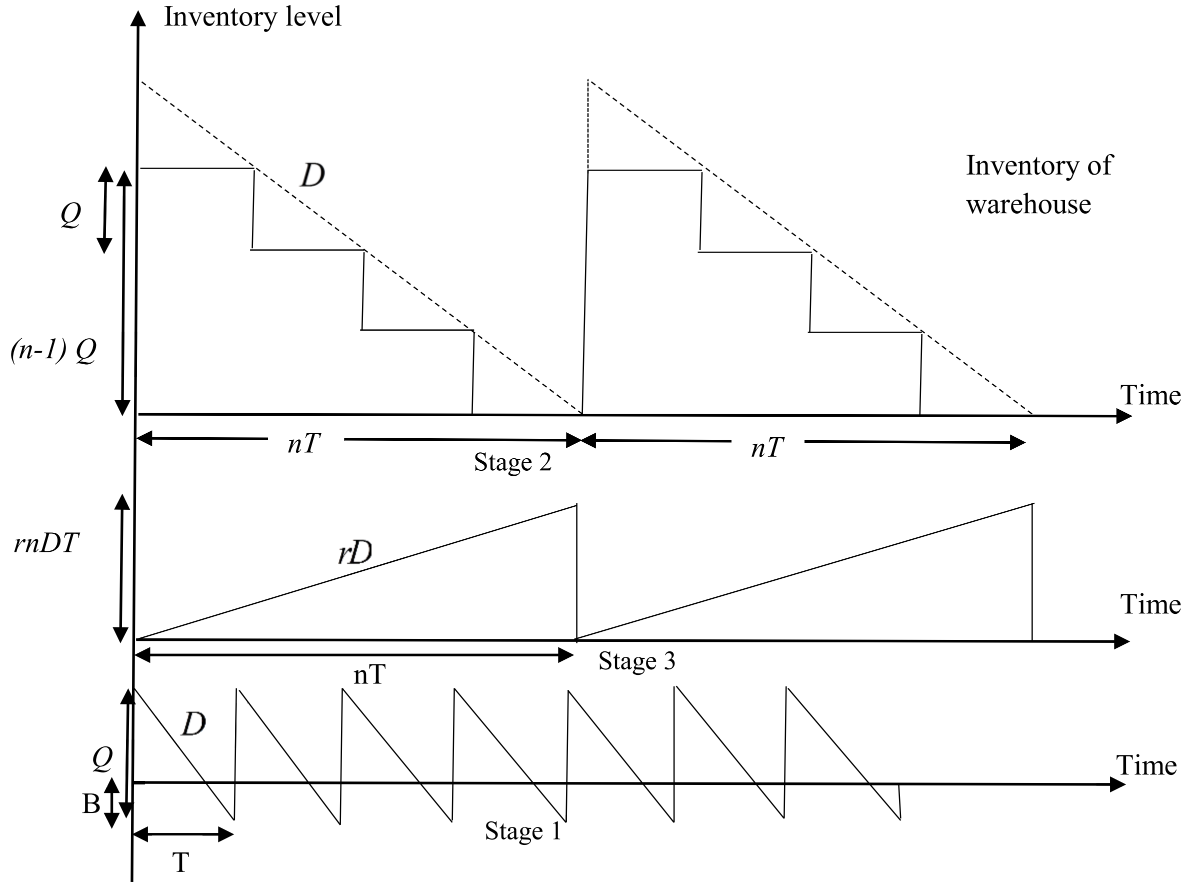

FIGURE 3. Inventory level of warehouse and distributor When $n = 4$.

Thus, in each unit time, the inventory cost of warehouse in stage 2 and 3 are as follows:

$$AHC_2 = \frac{h_2}{n_iT_i}\left[(n_i-1)\,DT_i \times T_i + (n_i-2)\,DT_i \times T_i + \ldots + DT_i \times T_i\right] \tag{3.1}$$

$$= \frac{h_2}{n_iT_i}\left[DT_i^2\left((n_i-1)+(n_i-2)\ldots+1\right)\right] = \frac{h_2}{n_iT_i}\left[DT_i^2\frac{n_i\,(n_i-1)}{2}\right] = h_2\frac{(n_i-1)\,DT_i}{2} \quad \text{for } i \text{ is } 1,2,3$$

$$AHC_3 = h_3\frac{rn_iDT_i}{2} \quad \text{for } i \text{ is } 1,2,3 \tag{3.2}$$

(3) Processing cost: The cost to warehouse faces for dealing with each order per unit time in the model $i(i = 1,2,3)$ is $\frac{F}{T_i}$
(4) Cost of Opportunity interest loss per unit time for the model $i(i = 1,2) = (1-\alpha)\,MCI_Vn_iDT_i$ and for the model $3 = (1-\alpha)\,M_1CI_Vn_3DT_3$

Accordingly, the total cost in each time period, for warehouse, in the model $i(i = 1,2)$ is:

$$TCV_i = \frac{A_2^V}{n_iT_i} + \frac{A_3^V}{n_iT_i} + h_2\frac{(n_i-1)\,DT_i}{2} + h_3\frac{rn_iDT_i}{2} + \frac{F}{T_i} + (1-\alpha)\,CI_VMD. \tag{3.3}$$

Whereas the accumulative cost of each time period in the third model for warehouse is

$$TCV_3 = \frac{A_2^V}{n_3T_3} + \frac{A_3^V}{n_3T_3} + h_2\frac{(n_3-1)\,DT_3}{2} + h_3\frac{rn_3DT_3}{2} + \frac{F}{T_3} + (1-\alpha)\,CI_VM_1D. \tag{3.4}$$

3.2.2. Model 1: Non-integrated warehouse–distributer model

The sum of distributer's cost, in each period, including ordering, holding, backordering, cost of opportunity, and gained interest, are formulated as follows:

(1) Cost of order in each time period: $\frac{A_B}{T_1}$.
(2) Cost of maintenance (excluding interest charges) in each time period: $\frac{1}{2}h_1DK_1^2T_1$.
(3) Cost of shortage for backordering in each time period: $\frac{\pi D}{2}(1-K_1)^2T_1$.
Furthermore $TD_1 = \frac{A_B}{T_1} + \frac{1}{2}h_1DK_1^2T_1 + \frac{\pi D}{2}(1-K_1)^2T_1$.
(4) Cost of opportunity and gained interest: according to the relation of K_1T_1 and M, one of the two situations may occur.

i. $M < K_1T_1$
ii. $M > K_1T_1$

First situation: $M < K_1T_1$

The cost of interest capital in this case has two parts. Since α percent of money goods is paid at the beginning of period and these goods are sold in KT periods, its cost of capital is $\frac{\alpha CI_cDK_1T_1*K_1T_1}{2}$ (part I of Fig. 4, left side) and the second part is cost of $1-\alpha$ percent of money goods that is paid in time M. Its cost of capital is $\frac{(1-\alpha)CI_cD(K_1T_1-M)^2}{2}$ (part II of Fig. 4, left side). So total cost of capital in this case is: (Fig. 4, left side)

$$ACC_{11} = \frac{1}{T_1}\left(\underbrace{\frac{\alpha CI_cDK_1^2T_1^2}{2}}_{\text{I}} + \underbrace{\frac{(1-\alpha)\,CI_cD\,(K_1T_1-M)^2}{2}}_{\text{II}}\right). \tag{3.5}$$

Since at the beginning of the period, $1-\alpha$ percent of backordered money is received and we have it until time M, gained interest is $DT_1\,(1-K_1)\,(1-\alpha)\,P_BI_e*M$ (part I of Fig. 4, right side) and gained interest from selling goods in M duration and receiving $1-\alpha$ percent of money of goods is $\frac{DM(1-\alpha)P_BI_e*M}{2}$ (part II of Fig. 4, right side). Annually gained interest in this case is:

$$AIE_{11} = \frac{1}{T_1}\left(\underbrace{(DT_1\,(1-K_1)\,(1-\alpha)\,P_BMI_e)}_{\text{I}} + \underbrace{\frac{DM^2}{2}\,(1-\alpha)\,P_BI_e}_{\text{II}}\right). \tag{3.6}$$

Case 2. $M > K_1T_1$

Since α percent of money of goods is paid at the beginning of period and these goods are sold in KT duration, its cost of capital is $\frac{\alpha CI_cDK_1T_1*K_1T_1}{2}$. So annually cost of capital is: (Fig. 5, left side)

$$ACC_{12} = \frac{\alpha CI_cDK_1^2T_1}{2}. \tag{3.7}$$

Gained interest in this case consists of three parts. First part: $1-\alpha$ percent of backordered money that happens at the beginning of period and it is in our possession until time M. Gained interest from this money is $(DT_1\,(1-K_1)\,(1-\alpha)\,P_BI_e)\,M$. Second part: since $1-\alpha$ percent of money of goods is received at the buying of goods from costumer and selling continues until time KT, therefor gained interest in this part is

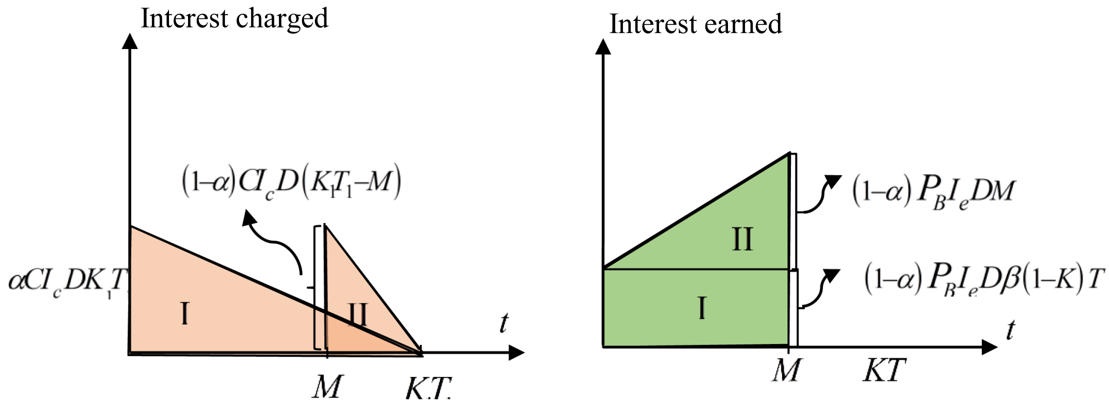

FIGURE 4. Income of interest *vs.* cost of interest for the first case 1.

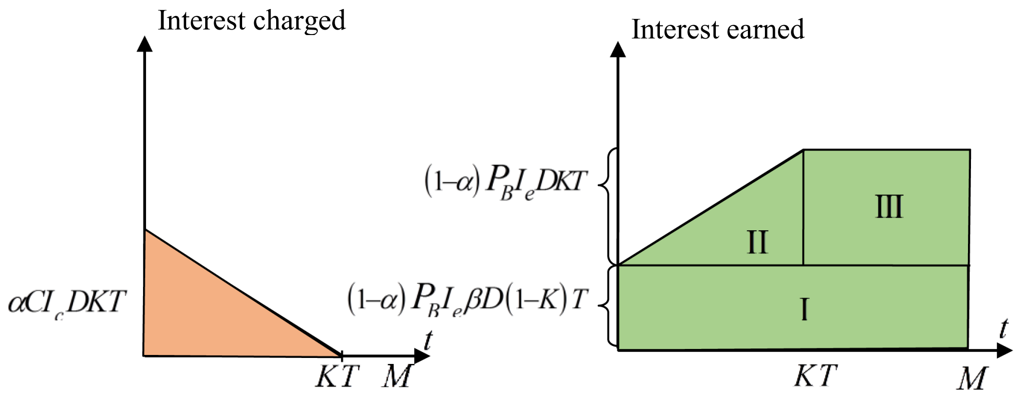

FIGURE 5. Interest earned and interest charged for case 2.

$\frac{DK_1^2T_1^2}{2}(1-\alpha)P_BI_e$ and since such money is in our possession until time M, gained interest from this money is $DK_1T_1(M-K_1T_1)(1-\alpha)P_BI_e$. So gained annually interest is:

$$AIE_{12}=\frac{1}{T_1}\left(\underbrace{(DT_1(1-K_1)(1-\alpha)P_BMI_e)}_{\text{I}}+\underbrace{\frac{DK_1^2T_1^2}{2}(1-\alpha)P_BI_e}_{\text{II}}+\underbrace{DK_1T_1(M-K_1T_1)(1-\alpha)P_BI_e}_{\text{III}}\right). \tag{3.8}$$

The accumulative net cost in each time period or warehouse (supplier) is modeled as follows. Accordingly, regarding the first two cases, it can be inferred respectively that:

$$\begin{aligned}ATB_{11}=TD_1+ACC_{11}-AIE_{11}&=\frac{A_B}{T_1}+\frac{1}{2}h_1DK_1^2T_1+\frac{\pi D}{2}(1-K_1)^2T_1\\&+CI_cD\left(\frac{\alpha K_1^2T_1}{2}+\frac{(1-\alpha)(K_1T_1-M)^2}{2T_1}\right)\\&-(D(1-K_1)(1-\alpha)P_BMI_e)-\frac{DM^2}{2T_1}(1-\alpha)P_BI_e \qquad M<K_1T_1\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
ATB_{12} = TD_1 + ACC_{12} - AIE_{12} &= \frac{A_B}{T_1} + \frac{1}{2}h_1 DK_1^2 T_1 + \frac{\pi D}{2}(1-K_1)^2 T_1 \\
&+ \frac{\alpha CI_c DK_1^2 T_1}{2} - (D(1-K_1)M(1-\alpha)P_B I_e) - \frac{DK_1^2 T_1}{2}(1-\alpha)P_B I_e \\
&- DK_1(M - K_1T_1)(1-\alpha)P_B I_e \quad M > K_1 T_1
\end{aligned} \tag{3.10}
$$

Case 1. $M < K_1 T_1$

The objective function shown in equation (3.9) can be rewritten as follows;

$$\Delta_{11}(K_1, T_1) = \frac{\psi_{11}}{T_1} + (\psi_{12}K_1^2 - 2\psi_{13}K_1 + \psi_{14})T_1 + \psi_{15}K_1 + \psi_{16}. \tag{3.11}$$

Where

$$\psi_{11} = \frac{2A_B + DM^2(1-\alpha)(CI_c - P_B I_e)}{2} \tag{3.12}$$

$$\psi_{12} = \frac{D}{2}(h_1 + \pi + CI_c) \tag{3.13}$$

$$\psi_{13} = \frac{\pi D}{2} \tag{3.14}$$

$$\psi_{14} = \frac{\pi D}{2} \tag{3.15}$$

$$\psi_{15} = ((D(1-\alpha)M)(P_B I_e - CI_c)) \tag{3.16}$$

$$\psi_{16} = -P_B I_e D(1-\alpha)M. \tag{3.17}$$

Moreover equation (3.11) can be rewritten as follows.

$$\Delta_{11}(K_1, T_1) = \frac{\psi_{11}}{T_1} + T_1\gamma(K_1) + \psi_{15}K_1 + \psi_{16}. \tag{3.18}$$

Where $\gamma(K_1) = \psi_{12}K_1^2 - 2\psi_{13}K_1 + \psi_{14}$. Equation (A.1), is the objective function, which is a convex in equation (3.11) the global minimum of this equation can be calculated. This is also true for equations (3.19) and (3.20), (see Appendix A, Eqs. (A.9) and (A.10)). The optimum quantities of are K_1 and T_1, respectively.

$$
\begin{aligned}
&T_{11}^* = T_1^* \\
&= \sqrt{\frac{(h_1 + \pi + CI_c)^2[2A_B + DM^2(1-\alpha)(CI_c - P_B I_e)] - \left(\pi D + [(h_1 + CI_c + \pi)D]((1-\alpha)M)^2(P_B I_e - CI_c)^2\right)}{\pi D(h_1 + CI_c)^2}}
\end{aligned} \tag{3.19}
$$

$$K_{11}^* = K_1^* = \frac{\pi T_1^* - ((1-\alpha)M)(P_B I_e - CI_c)}{T_1^*(h_1 + \pi + CI_c)}. \tag{3.20}$$

Case 2. $M > K_1 T_1$

The objective function shown in equation (3.10) can be rewritten as follows;

$$\Delta_{12}(K_1, T_1) = \frac{\psi_{21}}{T_1} + (\psi_{22}K_1^2 - 2\psi_{23}K_1 + \psi_{24})T_1 + \psi_{25}. \tag{3.21}$$

Where

$$\psi_{21} = A_B \tag{3.22}$$

$$\psi_{22} = \frac{D}{2}\left(h_1 + \pi + \left(\alpha C I_c + (1-\alpha) P_B I_e\right)\right) \tag{3.23}$$

$$\psi_{23} = \frac{\pi D}{2} \tag{3.24}$$

$$\psi_{24} = \frac{\pi D}{2} \tag{3.25}$$

$$\psi_{25} = -P_B I_e D (1-\alpha) M \tag{3.26}$$

Equation (B.1), is the objective function, which is a convex in equation (3.21) and the global minimum of this equation can be calculated. This is also true for equations (3.27) and (3.28), (see Appendix B, Eqs. (B.9) and (B.10)). The optimum quantities of are K_1 and T_1, respectively.

$$T_{12}^{*} = T_1^{*} = \sqrt{\frac{2A_B}{\left(h_1 D + \pi D + \alpha C I_c D + D(1-\alpha) P_B I_e\right)\left(\frac{\pi}{h_1+\pi+\alpha C I_c+(1-\alpha)P_B I_e}\right)^2 - 2\pi D\left(\frac{\pi}{h_1+\pi+\alpha C I_c+(1-\alpha)P_B I_e}\right) + \pi D}} \tag{3.27}$$

$$K_{12}^{*} = K_1^{*} = \frac{\psi_{23}}{\psi_{22}} = \frac{\pi}{h_1 + \pi + \alpha C I_c + (1-\alpha) P_B I_e}\cdot \tag{3.28}$$

At this point, the optimal solution (K_1^*, T_1^*) can be obtained through establishing in the Appendix H
It is undeniable that the optimal n_1^* must be capable of satisfying subsequent conditions, for the minimal TCV_1,

$$TCV_1\left(T_1^*, n_1^* - 1\right) \geq TCV_1\left(T_1^*, n_1^*\right) \text{ And } TCV_1\left(T_1^*, n_1^* + 1\right) \geq TCV_1\left(T_1^*, n_1^*\right). \tag{3.29}$$

Hence, the whole cost per year in the first model is

$$ATC_1\left(K_1^*, T_1^*, n_1^*\right) = ATB_1\left(K_1^*, T_1^*\right) + TCV_1\left(T_1^*, n_1^*\right). \tag{3.30}$$

3.2.3. Model 2: Combined model with a postponed payment

This model is developed for identifying proper time interval of replenishment for the distributer while minimizing the whole cost per time unit, under the circumstances that warehouse and the distributer enjoy mutual interactions of sharing information, paving the way for constructing a strategic long-term alliance. In the first model, each member of chain minimizes its costs separately, but in this case minimization happens from viewpoint of both members. In fact, cost of chains that includes supplier(warehouse) and vendor costs is added together then minimize it.

The model is given in terms of two cases as follows.

Case 1. $M < K_2 T_2$
Cost function ATC_{21} is obtained from adding supplier (warehouse) costs TCV_{21} and retailer costs ATB_{21} that if $K_1 = K_2$ and $T_1 = T_2$, ATB_{11} and TCV_{11} will become ATB_{21} and TCV_{21} respectively. So equations are as follows:

$$\begin{aligned}ATC_{21} = ATB_{21} + TCV_{21} &= \frac{A_B}{T_2} + \frac{1}{2} h_1 D K_2^2 T_2 + \frac{\pi D}{2}(1-K_2)^2 T_2 + C I_c D\left(\frac{\alpha K_2^2 T_2}{2} + \frac{(1-\alpha)(K_2 T_2 - M)^2}{2T_2}\right) \\ &- \left(D(1-K_2)(1-\alpha) P_B M I_e\right) - \frac{D M^2}{2T_2}(1-\alpha) P_B I_e \\ &+ \left(\frac{A_2^v}{n_2 T_2} + \frac{A_3^v}{n_2 T_2} + h_2 \frac{(n-1) D T_2}{2} + h_3 \frac{r n D T_2}{2} + \frac{F}{T_2} + (1-\alpha) M C I_V D\right).\end{aligned} \tag{3.31}$$

Case 2. $M > K_2T_2$

Like previous case chain member costs are added. Since $M > K_2T_2$, these costs for supplier is TCV_{22} and for vendor is ATB_{22}. So equations are as follows:

$$ATC_{22} = ATB_{22} + TCV_{22} = \frac{A}{T_2} + \left(\frac{1}{2}h_1 D + \frac{\pi D}{2} + \frac{\alpha C I_c D}{2} + \frac{D(1-\alpha)P_B I_e}{2}\right) K_2^2 T_2 - 2K_2T_2\left(\frac{\pi D}{2}\right) + \left(\frac{\pi D}{2}\right) T_2$$
$$- (DM(1-\alpha)P_B I_e) + \left(\frac{A_2}{n_2T_2} + \frac{A_3}{n_2T_2} + h_2\frac{(n-1)DT_2}{2} + h_3\frac{rnDT_2}{2} + \frac{F}{T_2} + (1-\alpha)MCI_V D\right). \qquad (3.32)$$

Case 1. $M < K_2T_2$

Fixing n_2, the objective functions shown in equation (3.31) can be rewritten as follows;

$$\Delta_{21}(K_2,T_2) = \frac{\psi_{31}}{T_2} + (\psi_{32}K_2^2 - 2\psi_{33}K_2 + \psi_{34})T_2 + \psi_{35}K_2 + \psi_{36}. \qquad (3.33)$$

Where

$$\psi_{31} = \left(A_b + F + \frac{A_2^V}{n_2} + \frac{A_3^V}{n_2} + \frac{DM^2(1-\alpha)(CI_c - P_BI_e)}{2}\right) \qquad (3.34)$$
$$\psi_{32} = \frac{D}{2}(h_1 + \pi + CI_c) \qquad (3.35)$$
$$\psi_{33} = \frac{\pi D}{2} \qquad (3.36)$$
$$\psi_{34} = \left(\frac{\pi D}{2} + h_2\frac{(n_2-1)D}{2} + h_3\frac{rn_2D}{2}\right) \qquad (3.37)$$
$$\psi_{35} = D(1-\alpha)MP_BI_e \qquad (3.38)$$
$$\psi_{36} = -P_BI_eD(1-\alpha)M + (1-\alpha)MCI_VD. \qquad (3.39)$$

Moreover equation (3.33) can be rewritten as follows.

$$\Delta_{21}(K_2,T_2) = \frac{\psi_{31}}{T_2} + T_2\gamma(K_2)\Psi_{35}K_2 + \Psi_{36}. \qquad (3.40)$$

Where $\gamma(K_2) = \psi_{32}K_2^2 - 2\psi_{33}K_2 + \psi_{34}$,

Equation (C.1), is the objective function, which is a convex in equation (3.33) and the global minimum this equation can be calculated. This is also true for equations (3.41) and (3.42), (see Appendix C, Eqs. (C.9) and (C.10)). The optimum quantities of are K_2 and T_2, respectively.

$$T_{21}^* = T_2^* = \sqrt{\frac{2D(h_1+\pi+CI_c)\left(A_b + F + \frac{A_2^V}{n_2} + \frac{A_3^V}{n_2} + \frac{DM^2(1-\alpha)(CI_c-P_BI_e)}{2}\right) - [D(1-\alpha)M(P_BI_e - CI_c)]^2}{D^2(h_1+\pi+CI_c)(\pi + h_2(n_2-1) + h_3rn_2) - \pi^2D^2}} \qquad (3.41)$$

$$K_{21}^* = K_2^* = \frac{\pi}{(h_1+\pi+CI_c)} - \frac{2D(1-\alpha)M(P_BI_e - CI_c)}{(h_1+\pi+CI_c)}$$
$$\sqrt{\frac{D^2(h_1+\pi+CI_c)(\pi + h_2(n_2-1) + h_3rn_2) - \pi^2D^2}{8D(h_1+\pi+CI_c)\left(A_B + F + \frac{A_2^V}{n_2} + \frac{A_3^V}{n_2} + \frac{DM^2(1-\alpha)(CI_c-P_BI_e)}{2}\right) - [2D(1-\alpha)M(P_BI_e - CI_c)]^2}}. \qquad (3.42)$$

Case 2. $M > K_2T_2$

Fixing n_2 The objective function shown in equation (3.32) can be rewritten as follows;

$$\Delta_{22}(K_2, T_2) = \frac{\psi_{41}}{T_2} + (\psi_{42}K_2^2 - 2\psi_{43}K_2 + \psi_{44})T_2 + \psi_{45}. \tag{3.43}$$

Where

$$\psi_{41} = \left(A_b + F + \frac{A_2^V}{n_2} + \frac{A_3^V}{n_2}\right) \tag{3.44}$$

$$\psi_{42} = \left(\frac{1}{2}h_1 D + \frac{\pi D}{2} + \frac{\alpha C I_c D}{2} + \frac{D\,(1-\alpha)\,P_B I_e}{2}\right) \tag{3.45}$$

$$\psi_{43} = \left(\frac{\pi D}{2}\right) \tag{3.46}$$

$$\psi_{44} = \left(\frac{\pi D}{2} + h_2\frac{(n_2-1)\,D}{2} + h_3\frac{r n_2 D}{2}\right) \tag{3.47}$$

$$\psi_{45} = -\,(DM\,(1-\alpha)\,P_B I_e) + (1-\alpha)\,MCI_V D. \tag{3.48}$$

Moreover equation (3.43) can be rewritten as follows.

$$\Delta_{22}(K_2, T_2) = \frac{\psi_{41}}{T_2} + T_2\gamma(K_2) + \psi_{45}. \tag{3.49}$$

Where $\gamma(K_2) = \psi_{42}K_2^2 - 2\psi_{43}K_2 + \psi_{44}$,
Equation (D.1), is the objective function, which is a convex in equation (3.43) the global minimum this equation can be calculated. This is also true for equations (3.50) and (3.51), (see Appendix D, Eqs. (D.9) and (D.10)). The optimum quantities of are K_2 and T_2, respectively.

$$K_{22}^* = K_2^* = \frac{\psi_{43}}{\psi_{42}} = \frac{\pi}{h_1 + \pi + \alpha C I_c + (1-\alpha)\,P_B I_e} \tag{3.50}$$

$$T_{22}^* = T_2^* = T_2^*\,(K_2) = \sqrt{\frac{2\left(A_B + F + \frac{A_2^V}{n_2} + \frac{A_3^V}{n_2}\right)}{(\pi D + h_2\,(n_2-1)\,D + h_3 r n_2 D) - \pi D\left(\frac{\pi}{h_1+\pi+\alpha C I_c+(1-\alpha)P_B I_e}\right)}} \tag{3.51}$$

To find out optimal values of K_2^*, T_2^* and n_2^* the following algorithm is provided.

Solution procedure for model 2
See appendix F.

3.2.4. Model 3: Combined model with a negotiation procedure

Generally, the infrastructures of information sharing, paves the way for lowering the total cost in integrated environment, compared to a non-integrated environment.

Nevertheless, it must be mentioned that the distributer may experience some increase in the prices after integration, due to increase in procurement costs. On the other hand, however, the warehouse (supplier) allocates a considerably longer interval to distributer for delaying the payment from him/her, the terms of which is discussed through negotiations. This interval of delay motivates the distributer, through reducing his/her total cost, to enter a cooperative relationship for constructing a win–win management environment of supply chain. In other words, the information sharing cooperation helps both warehouse (supplier) as well as the distributer to benefit in the form of lowered costs.

In this regard, consider $(M <)\,M_n$ is the interval of delay, the distributer gets from warehouse (supplier), as discussed above. As a result, the whole cost per each time unit can be modeled in to categories, as follows

Case 1. $(M < K_3T_3)$

Like case one, cost function of model two ATC_{31} is written from viewpoint of whole members of supply chain and it is obtained from adding supplier (warehouse) costs TCV_{31} and vendor costs ATB_{31}. So equations are as follows:

$$\begin{aligned} ATC_{31} = ATB_{31} + TCV_{31} = \frac{A_B}{T_3} + \frac{1}{2}h_1DK_3^2T_3 + \frac{\pi D}{2}(1-K_3)^2T_3 + CI_cD\left(\frac{\alpha K_3^2T_3}{2} + \frac{(1-\alpha)(K_3T_3-M_n)^2}{2T_3}\right) \\ -(D(1-K_3)(1-\alpha)P_BM_nI_e) - \frac{DM_n^2}{2T_3}(1-\alpha)P_BI_e \\ +\left(\frac{A_2^V}{n_3T_3} + \frac{A_3^V}{n_3T_3} + h_2\frac{(n_3-1)DT_2}{2} + h_3\frac{rn_3DT_2}{2} + \frac{F}{T_3} + (1-\alpha)M_nCI_VD\right). \end{aligned} \tag{3.52}$$

Case 2. $(M > K_3T_3)$

In this case we want to minimize total costs of chain members. So we add obtained costs of supplier TCV_{31} and costs of vendor ATB_{31} and minimize them.

$$\begin{aligned} ATC_{32} = ATB_{32} + TCV_{32} = \frac{A_B}{T_3} + \left(\frac{1}{2}h_1D + \frac{\pi D}{2} + \frac{\alpha CI_cD}{2} + \frac{D(1-\alpha)P_BI_e}{2}\right)K_3^2T_3 - 2K_3T_3\left(\frac{\pi D}{2}\right) + \left(\frac{\pi D}{2}\right)T_3 \\ -(DM_n(1-\alpha)P_BI_e) + \left(\frac{A_2^V}{n_3T_3} + \frac{A_3^V}{n_3T_3} + h_2\frac{(n_3-1)DT_3}{2} + h_3\frac{rn_3DT_3}{2} + \frac{F}{T_3} + (1-\alpha)M_nCI_VD\right). \end{aligned} \tag{3.53}$$

The model of coordination, while the supply chain strategy includes flexible delay interval for payment is as follows:

$M_1 = M + \Delta$, $\Delta > 0$.

With a fixed quantity for n_3, K_3 and T_3, the first derivation of both ATC_{31} and ATC_{32} with respect to Δ is as follows:

$$\begin{aligned} \frac{dATC_{31}}{d\Delta} = -D(1-\alpha)(P_BI_e - CI_V)(1-K_3) - \frac{(1-\alpha)DM_n}{T_3}(P_BI_e - CI_c) < 0 \\ + \frac{(1-\alpha)D(M_n)^2}{2T_3}(CI_c - P_BI_e) - (1-\alpha)M_nCI_cDK_3 \\ -(D(1-K_3)(1-\alpha)P_BM_nI_e) + (1-\alpha)M_nCI_VD \end{aligned} \tag{3.54}$$

$$\frac{dATC_{32}}{d\Delta} = ((1-\alpha)CI_VD) - ((1-\alpha)P_BI_eD) < 0. \tag{3.55}$$

From equations (3.47) and (3.48), it is easily concluded that the total cost represents a decreasing conduct in this model as the warehouse (supplier) decides to prolong the payment period of the distributer. At this point, the profit of the distributer and warehouse (supplier) from cost savings are defined respectively as:

$$BS = TB_1 - TB_3$$
$$VS = VB_1 - VB_3.$$

Altogether, a study on the variations of the vendor's conduct as well as distributer's, through comparing BS *vs.* VS, sheds light on some crucial aspects of the model. The parameter α is assumed to be the coefficient of compromise for sharing combined cost savings between distributer and warehouse (supplier), and is specified through satisfying $VS = \alpha * BS$, $\alpha > 0$. The condition in which $\alpha = 0$, illustrates the situation in which, the distributer experiences all possible savings, while, $\alpha = 1$ stands for the sharing environment with equal cost decrease for both warehouse (supplier) as well as the distributer. Between these two extremes, the larger quantities of coefficient α illustrate conditions, saving most of the costs for the warehouse (supplier). The smaller quantities of coefficient α, however, depicts a cooperative atmosphere, in which, profits gained as saving the costs are inclined towards the distributer.

Table 3. The optimal solutions taken from the three models.

	n^*	K^*	T^*	α^*	M_n^*/M_S	ATB	ATV	ATC	$PATC(\%)$
Model 1	6	0.6515	0.1263	0.5	1/12	152.41	497.51	649.93	–
Model 2	3	0.6525	0.2480	0.5	1/12	189.68	425.82	615.51	5.3
Model 3(M_n)	3	0.6494	0.2474	0	0.35	146.19	448.28	594.47	8.53
Model 4(M_S)	3	0.6494	0.2474	0	0.548	117.67	462.54	580.22	10.73

The realization of the savings is regulated by the length of the payment delay interval. With this in mind, the smaller the correlation, the longer the delay interval is. But, a vendor normally has the most allowed postponement period as M_n for the buyer in the real world. Thus, M_n plays the role of an upper limit for delay interval decision variable in Model 3. Taking into account the negotiation coefficient α and the maximum length of delay interval M_n, a solving procedure is developed to optimize T_3^*, K_3^* and n_3^*.

Solving procedure:

Step 1: The parameters α, ξ, M_n, and δ are set equal to zero. In order for the parameters to gain positive real numbers, ξ and δ should be minimal.

Step 2: the minimum quantity of $ATV_1\,(n_1^*, T_1^*)$ and $ATB_1\,(K_1^* T_1^*)$ are determined in the first Model by letting $M_n = M$, while, M is predetermined delay interval.

Step 3: Quantities of n_3^*, K_3^* and T_3^* are determined based on M_n, through using Theorems 1 and 2 in Model 2. Calculate TV_3, TB_3, $VS\,(= TV_1\,(T_1^*, n_1^*) - TV_3)$, and $BS\,(= TB_1\,(T_1^*, n_1^*) - TB_3)$.

Step 4: If $|VS - \alpha * BS| < \xi$, the termination criteria are triggered, and the final answers will be K_3^*, T_3^*, n_3^*, $M_1^*\,(= M_1)$ and TC_3; otherwise, procedure is iterated through Step 3 while, $M_n = M_n + \Delta$.

Step 5: When $M_n^* > M_S$, let $M_n^* = M_S$, and the optimal quantities of n_3^*, T_3^*, and TC_3 are determined.

When the condition $M_n^* < M_S$ is met, it can be concluded that the negotiation to achieve collaboration is successful through a payment delay interval shorter than M_n. This situation can be recognized from a larger α. The parameter M_n^* is the delay interval, determined in the negotiation. Although, it must be considered that M_n^* could exceed the maximum permissible delay interval M_n. At this point Step 5 is will be performed.

4. Numerical example

We use a numerical example to show the suggested three models with the intention of demonstrate the difference among them. with regard to the related literature, we accepted some parameters from the paper of Teng (2002). Let $A = 10$, $A_2^V = 120$, $A_3^V = 50$, $F = 10$, $D = 3600$, $M = \frac{1}{12}$, $I_c = 0.06$, $I_e = I_v = 0.04$, $h_1 = 0.5$, $h_2 = 0.15$, $h_3 = 0.05$, $C = 0.5$, $P_B = 1$, set $M_S = 0.35$ $\delta = 10^{-3}$, $\xi = 10^{-3}$ and $\alpha = 1$. Table 3 provides the optimal solutions to the three models through deploying the following parameters.

5. Results and discussion

As it can be inferred from the Table 3, the cost of the first model in which retailer and the warehouse (supplier) are separately trying to optimize their costs, is higher than second and third mode, while, the whole cost of the second model is higher than the third. In the first model, the retailer does not consider the costs of the warehouse (supplier), due to lack of exchanging information, and she optimizes T^* and K^* according to her own costs. Therefore, her costs are lower than the second model in which information exchange takes place. On the other hand, since the warehouse merely optimizes his own n^* and has no part in optimizing T^*, the total cost is higher than the second and third models. As a matter of fact, it can be pointed out that the calculated quantity for the total cost is not the lowest cost, since the model is optimized in two steps and warehouse's cost is not considered for the optimization of T^* and K^*. However, in the second model, in which information exchange exists, and distributer's cost as well as warehouse's are included simultaneously, the final total cost is 5.3% decreased, compared to the first model. Also, according to the table, T^* is increased in the second model compared to the first model, while n^* is decreased. This represents that, in each cycle, distributer's order quantity is increased, while the frequency of the order is decreased, which lowers the second level setup costs (second and third steps) as well as the delivery costs.

Even though the total cost of the second model is decreased compared to first model, the first level of the second model's cost is increased, in comparison with first model. This issue lowers distributers interest for information exchange. To convince the distributer, the warehouse (supplier) considers better terms for the delay in payment and the portion of the purchase cost to be delayed, compared to the first model, in order to provide financial feasibility to exchange information with the distributer. This structure is established in the third model. The total cost of the third model shows 8.3% decrease compared to the first model. Furthermore, second level costs 49.23 and first level costs 6.22 units decrease compared to the first model. Additionally, delay time has reached 0.35 unit for this model. Last but not least, the portion of the purchase cost that distributer is obligated to pay when receiving the products, has reached to zero.

In the fourth model, the constraint, $M_n < M_s$ is not taken into account, resulting in the profit of 35 units for both levels, while $\alpha = 1$ and the total cost is decreased 10.73%, compared to the first model and has reached 580.22.

5.1. Sensitive analyze and managerial insight

Sensitivity analysis for the negotiation coefficient α is also done for more examining of the models in this section.

In this section, in order to gain some managerial insights, several sensitivity analyses are performed on some key parameters of the model for the first example. For this purpose, the parameters h_2, h_3, A_2^V, A_3^V, A_b, h_1, α, and D are changed. The effects of the changes are shown in Table 4 and also the succeeding conclusions are attained. The effect of some important parameters on decision variables is also given in Table 5

As shown in Table 4 increasing in holding costs at level 2 (2nd and 3rd phases) leads to increase in the difference between total costs of models 1 and 3, this illustrates the fact that as holding costs in level 2 increases the necessity of information exchange exceeds as well and the warehouse (supplier) should convince the first level (distributer) to decrease his costs and total cost as a result.

According to Table 5 and sensitivity analysis, summary of the results for giving insights to the retailers are as bellow:

- An increase in h_1, decreases the values of Q^*, T^*. This relationship is due to the fact that the retailer, in order to deal with the increase in the inventory holding cost rate, follows a policy with a lower level of inventory (through decreasing the order quantity and the period of the positive inventory level).
- An increase in α, leads to decreases in T^* and Q^*, but increase in B^* with a slight change.
- By increasing of I_{Bc}, the values of T^* and Q^* are increased, while the value of B^* is decreased.
- By increasing of I_{Be}, the values of T^* and B^* are increased
- By increasing of M, the values of T^* and B^* are increased

Figure 7 shows the influence of increase h_2 on objective function, also focuses on the cooperation between retailer and vendor. Because with this increase, the total cost of model 3 increases with a less gradient.

Figures 8 and 9 show the influence of this parameter on objective function of retailer and vendor. In Figure 8, the objective function of retailer in model 1 does not change by increasing h_2, because exchange information does not conduct in this model and determining optimum K and T is conducted with respect to the costs of retailer. However, costs of retailer is increasing in two other models. But as shown in Figure 9, increase h2 cause to increase objective function of retailer in all three models but this increase carries out with more gradient in model 1. However it can say, although the increase in h2 does not cause to increase cost of the first level of model 1, the whole cost of chain strongly increase compared to two other models. Therefore, the more increase

Table 4. Effects of Percent decrease and increase some parameters on models.

Parameter	**Percent**	**Model 1**	**Model 3**	**Difference**
h_2, h_3	-0.5	535.8758	492.0547	43.8211
	-0.25	598.2698	549.1842	49.0856
	0	649.9302	594.4761	55.4541
	0.25	695.6659	631.7691	63.8968
	0.5	734.0152	659.7897	74.2255
	1	803.0448	696.4151	106.6297
A_2^V, A_3^V	-0.5	518.8316	463.4637	55.3679
	-0.25	589.9913	534.3017	55.6896
	0	649.9302	594.4761	55.4541
	0.25	703.4571	647.9507	55.5064
	0.5	751.5465	695.1968	56.3497
	1	837.1331	780.9336	56.1995
A_b, h_1	-0.5	595.1075	514.6708	80.4367
	-0.25	623.5133	558.6192	64.8941
	0	649.9302	594.4761	55.4541
	0.25	676.0425	626.189	49.8535
	0.5	701.5434	654.7644	46.779
	1	744.898	704.4147	40.4833
D	-0.5	460.0456	425.5772	34.4684
	-0.25	563.13	517.7552	45.3748
	0	649.9302	594.4761	55.4541
	0.25	726.2481	661.3189	64.9292
	0.5	795.1729	721.1451	74.0278
	1	917.3827	825.9543	91.4284
α	-0.5	648.8423	594.4761	54.3662
	0	649.9302	594.4761	55.4541
	0.5	651.0169	594.4761	56.5408
	1	652.1026	594.4761	57.6265

Table 5. Effects of decrease and increase some parameters on decision variable.

Parameters	*Type of Change*	*Optimal Values*			
		K	n_r	n_s	Q
h_1	*Decreasing*	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
	Increasing	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$
I_{Bc}	*Decreasing*	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
	Increasing	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$
I_{Be}	*Decreasing*	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$
	Increasing	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
M	*Decreasing*	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$
	Increasing	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
α	*Decreasing*	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
	Increasing	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$

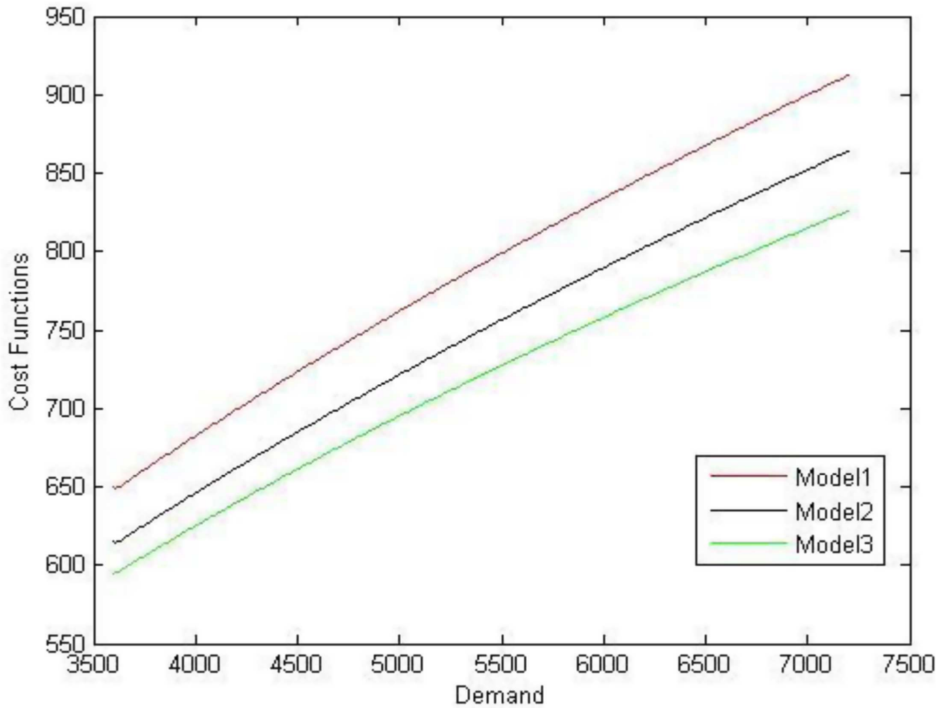

Figure 6. Effects of demand on the Cost function of three models.

in costs at level 2, the strong agreement should take among managers at level 2 with level 1 in order to increase whole cost of chain with less gradient.

From Table 4, by increasing the initiation cost of phases 2 and 3 the cost differences of models 1 and 3 increases but it's not too much in comparison with the holding cost. Table 4 illustrates that an increase in A_b, h_1 retailers decreases the difference of costs between models 1 and 3 but in reality companies and factories at a lower level (2,3,...) are bigger and have bigger costs compared with upper levels. Table of Sensitivity analysis illustrates the fact that increasing in return rate of defective goods increases the difference between models 1 and 3 and it leads to a more necessity for information exchange. Also demand ratio increases using model 3 is more copacetic.

Figure 6 shows that increase in demand causes an increase of costs in all three models, but this increase has more gradient in model 1 compared to the two other models and the third model has the least increase gradient. So, in large organizations, it is very important to have strong integrity, coordination and consistency with chain members.

The performed sensitivity analyses reveal that decreasing the length of the permissible delay to settle the remaining of the purchasing cost, leads to increasing in the total cost of the retailer. Therefore, the retailer tends to receive bigger periods for the permissible delay from the supplier. From this fact, it can be suggested that the managers should try to determine the permissible delay periods as bigger as possible. Moreover, when the permissible delay periods increases, the total cost of the retailer decreases. This finding indicates that it is better for the retailer to choose the suppliers who offer more

the length of the permissible delay to settle the remaining of the purchasing cost. However, the suppliers would obtain less profit in such a situation. Hence, in some cases that supplier can not decrease the price of goods, increasing the length of delayed payment causes to decreasing costs of retailer and attracting more costumers and retailer also should deal with suppliers whom have longer length of delayed payment.

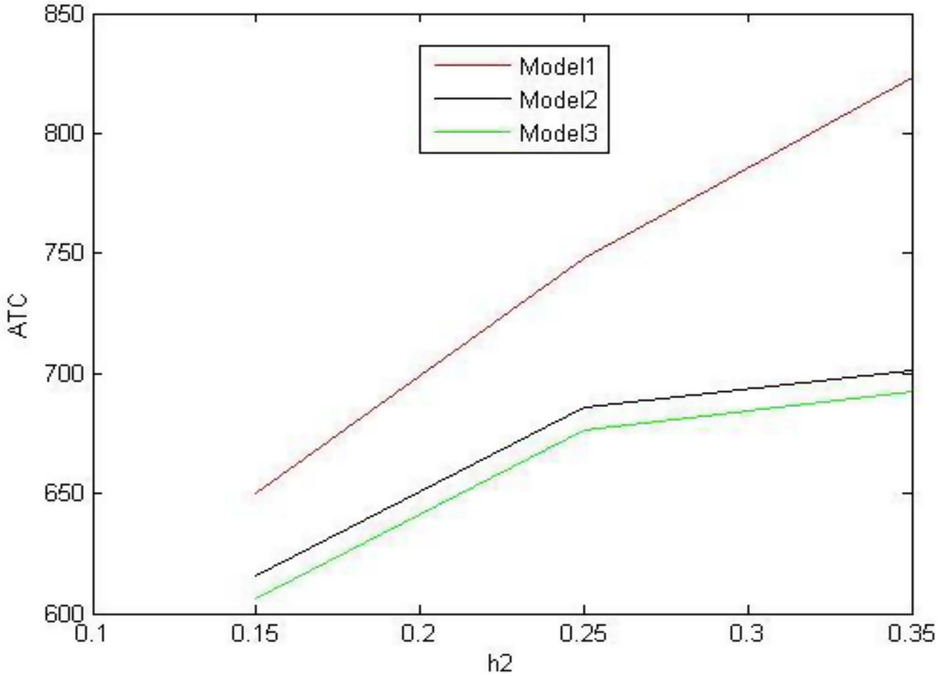

Figure 7. Effects of h2 on the Cost function of three models.

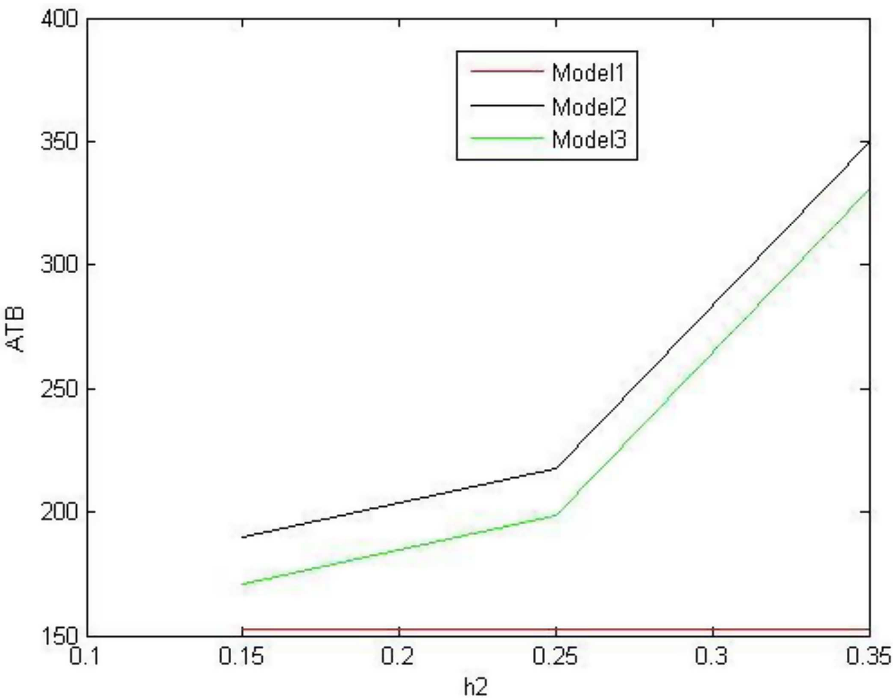

Figure 8. Effects of h2 on the ATB of three models.

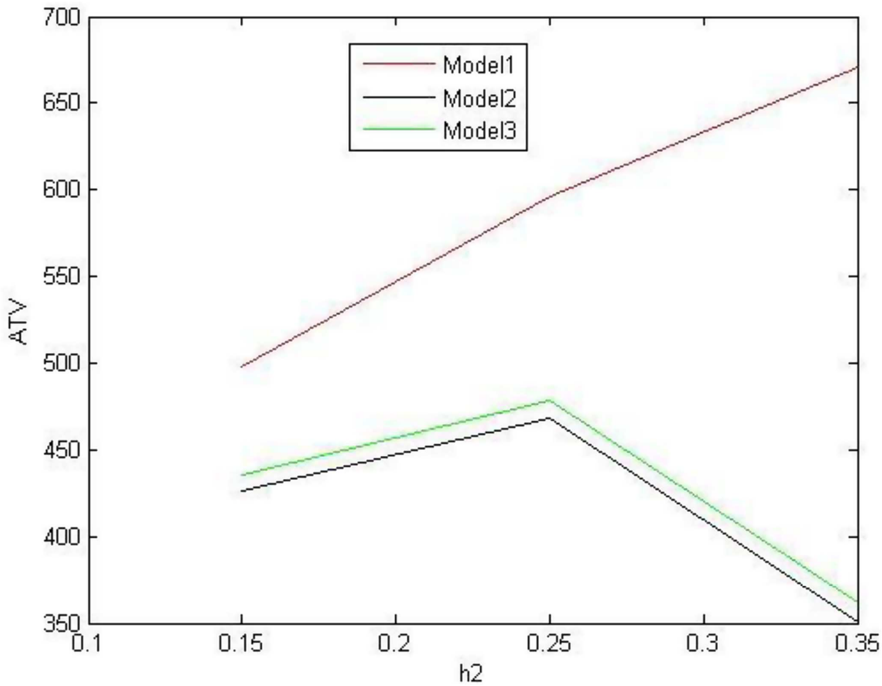

Figure 9. Effects of h2 on the ATV of three models.

Whenever the ratio of buying cost, paid at the delivering time of items increases, the total cost of the retailer increases as well. Hence, it is recommended that the retailer purchases his/her items from the suppliers who request less amount of buying cost, paid at the delivering time of items. In such a case, the retailer receives a smaller amount of loan from the bank, and consequently will incur a fewer capital cost.

As a summary of managerial insights, it can be claimed that the method presented in the current study provides a useful and powerful tool for the managers to capture their trade credit inventory problem in the real-life situations. Such a claim is accepted, because using this flexible framework, it is easy to handle all problems with integrated or non-integrated model and under no trade credit, full delayed payment, partial delayed payment, no shortage, full backordering, returned items, no returned items, partial returned items, and even any combination of all cases.

6. Conclusion and future research

In this paper, a two-echelon closed-loop supply chain with coordination policy under variant partial permissible delay and backlogging is investigated. For this purpose, three mathematical programming models are developed. Convexity of the objective function of the models is proved and an optimal solution finding procedure is provided for each model. In order to analyze the problem and validate the proposed models, a numerical experiment is designed and executed.

This paper for the first time in the literature, takes into account closed loop supply chains, partial permissible delay and full backordering, simultaneously. In addition, this new contribution is along with considering three different situations: (1) Decentralized decision making model in which each SC member decides based on its own profit and (2) Centralized decision making model in which all SC members collaborate to increase the overall SC profitability. In the first model, each member decides by its own with the aim of increasing its profit. But, in the next two models, all members decides corporately since the delivery time of money is fixed or variable. The new presented framework is comprehensive and flexible enough to capture a variety of real-life situations

Our results show that when all members of the supply chain determine their optimal decisions coordinately, the total cost of the supply chain is significantly reduced. However, following the third policy (*i.e.* cooperation) is a strategic decisions for the retailer when the interest obtained from delay in payment is more than the cost incurred by increasing order quantity. In other words, the retailer is willing to adopt a cooperation policy, if where sum of the benefits is more than the cost of chaining the optimal ordering decision. To obtain key factors of the model, a sensitivity analysis is performed. Furthermore, an out and out project is solved with the proposed model and numerical results were close enough to prove that the model is working properly.

For future studies, the researchers can work on partial backorders, money inflation, deterioration rate, and Extending robust optimization. Incorporating discount schemes pricing policies could be another background for future works.

Appendix A.

Proving convexity of the objective function and determining T_1^* and K_1^* for case 1.

First partial derivation of the function $\Delta_{11}(K_1, T_1)$ with respect to T_1 is as follows;

$$\frac{\partial \Delta_{11}(K_1, T_1)}{\partial T_1} = -\frac{\psi_{11}}{T_1^2} + \gamma(K_1). \tag{A.1}$$

Equating the outcome to zero will result in;

$$T_1^* = T_1^*(K_1) = \sqrt{\frac{\psi_{11}}{\gamma(K_1)}}. \tag{A.2}$$

In which $\gamma(K_1) = \psi_{12}K_1^2 - 2\psi_{13}K_1 + \psi_{14}$, and the relative discriminant is derived from the equation (A.2).

$$\begin{aligned}\Delta = b^2 - 4ac = 4\psi_{13}^2 - 4\psi_{12}\psi_{14} = 4\left(\frac{\pi D}{2}\right)^2 - 4\left(\frac{D}{2}(h_1 + \pi + CI_c)\right)\left(\frac{\pi D}{2}\right) \\ = -\left(\pi D^2 (h_1 + CI_c)\right) < 0.\end{aligned} \tag{A.3}$$

Considering the fact that Δ is negative in all situations, no root can be determined for $\gamma(K_1)$, promising that it will be never equal to zero. Additionally, since $\gamma(0) = \frac{\pi D}{2} > 0$, it is logical to say $\gamma(K_1)$ owns a positive quantity in interval of [0, 1] in all conditions. As a result, equation (A.2) determines a unique $T^* = T^*(K_1) = \sqrt{\frac{\psi_{11}}{\gamma(K_1)}}$ minimizing $\Delta_{11}(K_1, T_1)$, for any possible value of K_1. Substituting the above equation into $\Delta_{11}(K_1, T_1)$ in equation (3.18) results in;

$$\Delta_{11}(K_1) = \Delta_{11}(T^*(K_1), K_1) = \frac{\psi_{11}}{\sqrt{\frac{\psi_{11}}{\gamma(K_1)}}} + \gamma(K_1)\sqrt{\frac{\psi_{11}}{\gamma(K_1)}} + \psi_{15}K_1 + \psi_{16} = 2\sqrt{\psi_{11}\gamma(K_1)} + \psi_{15}K_1 + \psi_{16}. \tag{A.4}$$

The outcome of the equation, represents the probable minimal cost for every quantity of K_1. In this regard, the crucial point that $\Delta_{11}(K_1)$ is continuous in the closed interval of [0, 1] has to be taken into account. This is the reason for the fact that it has one or more local minimal quantity, the least of which is the optimum answer for the cost function. Furthermore, for finding the optimum answer, first and second derivations of $\Delta_{11}(K_1)$

with respect to K_1 are determined as discussed in (A.5) and (A.6).

$$\frac{d\Delta_{11}(K_1)}{dK_1} = \sqrt{\psi_{11}}\frac{\gamma'(K_1)}{\sqrt{\gamma(K_1)}} + \psi_{15} \tag{A.5}$$

$$\frac{d^2\Delta_{11}(K_1)}{dK^2} = \frac{\sqrt{\psi_{11}}\left[2\gamma''(K_1)\gamma(K_1) - \left(\gamma'(K_1)\right)^2\right]}{2\left(\gamma(K_1)\right)^{\frac{3}{2}}}. \tag{A.6}$$

It can be inferred, for all possible quantities of K_1, that:

$$\begin{aligned}\frac{d^2\Delta_{11}(K_1)}{dK_1^2} &= \frac{\sqrt{\psi_{11}}\left[2\gamma''(K_1)\gamma(K_1) - \left(\gamma'(K_1)\right)^2\right]}{2\left(\gamma(K_1)\right)^{\frac{3}{2}}} = \frac{\sqrt{\psi_{11}}\left[(2\psi_{12})\left(\psi_{12}K_1^2 - 2\psi_{13}K_1 + \psi_{14}\right) - (\psi_{12}K_1 - \psi_{13})^2\right]}{(\gamma(K_1))^{\frac{3}{2}}} \\ &= \frac{\sqrt{\psi_{11}}\left[2\left(\psi_{12}^2K_1^2 - 2\psi_{12}\psi_{13}K_1 + \psi_{12}\psi_{14}\right) - \left(\psi_{12}^2K_1 - 2\psi_{12}\psi_{13}K_1 + \psi_{13}^2\right)\right]}{(\gamma(K_1))^{\frac{3}{2}}} \\ &= \frac{\sqrt{\psi_{11}}\left[\left(\psi_{12}^2K_1^2 - 2\psi_{12}\psi_{13}K_1 + \psi_{12}\psi_{14}\right) + \psi_{12}\psi_{13} - \psi_{13}^2\right]}{(\gamma(K_1))^{\frac{3}{2}}} \\ &= \frac{\sqrt{\psi_{11}}\left[\psi_{12}\left(\psi_{12}K_1^2 - 2\psi_{13}K_1 + \psi_{14}\right) + (\psi_{12} - \psi_{13})\psi_{13}\right]}{(\gamma(K_1))^{\frac{3}{2}}} = \frac{\sqrt{\psi_{11}}\left[\psi_{12}\gamma(K_1) + (\psi_{12} - \psi_{13})\psi_{13}\right]}{(\gamma(K_1))^{\frac{3}{2}}} > 0.\end{aligned} \tag{A.7}$$

As $\psi_{11}, \psi_{12}, \psi_{13}, \gamma(K_1)$ own positive quantities all the time, and also $(\psi_{12} - \psi_{13}) = \left(\frac{1}{2}h_1D + \frac{\pi D}{2} + \frac{CI_cD}{2}\right) - \frac{\pi D}{2} = \frac{1}{2}\left(h_1D + CI_cD\right) > 0$ is positive, It is easily accepted that $\Delta_{11}(K_1)$ is a convex function. Consequently, a globally optimum answer can be determined from setting the first derivative of $\Delta_{11}(K_1)$ equal to zero, as shown in equation (A.5).

$$\begin{aligned}\frac{d\Delta_{11}(K_1)}{dK_1} &= \sqrt{\psi_{11}}\frac{2\psi_{12}K_1 - 2\psi_{13}}{\sqrt{\psi_{12}K_1^2 - 2\psi_{13}K_1 + \psi_{14}}} + \psi_{15} = \sqrt{\frac{\psi_{11}}{\psi_{12}K_1^2 - 2\psi_{13}K_1 + \psi_{14}}}\left(2\psi_{12}K_1 - 2\psi_{13}\right) + \psi_{15} \\ &= T_1^*\left(2\psi_{12}K_1 - 2\psi_{13}\right) + \psi_{15} = T_1^*\left(D\left(h_1 + \pi + CI_c\right)K_1 - \pi D\right) + \left(CD\left(1-\alpha\right)M\right)\left(I_e - I_c\right) = 0.\end{aligned} \tag{A.8}$$

It can be inferred that;

$$K_1^* = \frac{\pi T_1^* - \left(C\left(1-\alpha\right)M\right)\left(I_e - I_c\right)}{T_1^*\left(h_1 + \pi + CI_c\right)}. \tag{A.9}$$

At this stage, substitution of K_1^* in equation (A.2), after the subsequent simplification, paves the way for obtaining optimal value of the considered period.

$$T_1^* = \sqrt{\frac{\left(h_1 + \pi + CI_c\right)^2\left[2A + DM^2\left(1-\alpha\right)C\left(I_c - I_e\right)\right] - \left(\pi D + \left[\left(h_1 + CI_c + \pi\right)D\right]\left(C\left(1-\alpha\right)M\right)^2\left(I_e - I_c\right)^2\right)}{\pi D\left(h_1 + CI_c\right)^2}}. \tag{A.10}$$

If the solution found by using equations (3.19) and (3.20) results in $M > K_1T_1$, afterward a logical solution is to set $K_1 = \frac{M}{T_1}$. In spite of this, we should then await that not only will K be equivalent to K_1, but also T will change. To determine the optimal T for Case1 when $K_1 = \frac{M}{T_1}$, we can do the following. Substituting $K_1 = \frac{M}{T_1}$ in equation (3.11) gives:

$$\begin{aligned}\Delta_{11}(T_1) &= \frac{\psi_{11}}{T_1} + (\psi_{12}\frac{M^2}{T_1^2} - 2\psi_{13}\frac{M}{T_1} + \psi_{14})T_1 + \psi_{15}\frac{M}{T_1} + \psi_{16} \\ &+ \frac{\psi_{12}M^2 + \psi_{15}M + \psi_{11}}{T_1} + \psi_{14}T_1 + \psi_{16} - 2\psi_{13}M.\end{aligned} \tag{A.11}$$

Considering the derivative with regard to T and setting it equal to 0 gives:

$$\frac{d\Delta_{11}(T_1)}{dT_1} = -\frac{\psi_{12}M^2 + \psi_{15}M + \psi_{11}}{T_1^2} + \psi_{14} = 0. \tag{A.12}$$

This gives:

$$T_1^{'} = \sqrt{\frac{\psi_{12}M^2 + \psi_{15}M + \psi_{11}}{\psi_{14}}}. \tag{A.13}$$

And

$$K_1^{'} = \frac{M}{T_1^{'}}. \tag{A.14}$$

An analogous process may be considered for solving the second case

Appendix B.

Proving convexity of the objective function and determining T_1^* and K_1^* for case2

First partial derivation of the function $\Delta_{12}(K_1, T_1)$ with respect to T_1 is as follows;

$$\frac{\partial \Delta_{12}(K_1, T_1)}{\partial T_1} = -\frac{\psi_{21}}{T_1^2} + \gamma(K_1). \tag{B.1}$$

Equating the outcome to zero will result in;

$$T^* = T^*(K) = \sqrt{\frac{\psi_{21}}{\gamma(K_1)}}. \tag{B.2}$$

In which $\gamma(K_1) = \psi_{22}K_1^2 - 2\psi_{23}K_1 + \psi_{24}$, and the relative discriminant is derived from the equation (B.2).

$$\Delta = b^2 - 4ac = 4\psi_{23}^2 - 4\psi_{22}\psi_{24} == 4\left(\frac{\pi D}{2}\right)^2 - 4\left(\frac{1}{2}h_1 D + \frac{\pi D}{2} + \frac{\alpha C I_c D}{2} + \frac{D(1-\alpha) C I_e}{2}\right)\left(\frac{\pi D}{2}\right)$$
$$= -\left(h_1 D + \alpha C I_c D + D(1-\alpha) C I_e\right)(\pi D) < 0. \tag{B.3}$$

Considering the fact that Δ is negative in all situations, no root can be determined for $\gamma(K_1)$, promising that it will be never equal to zero. Additionally, since $\gamma(0) = \frac{\pi D}{2} > 0$, it is logical to say $\gamma(K_1)$ owns a positive quantity in interval of [0, 1] in all conditions. As a result, equation (B.2) determines a unique $T^* = T^*(K_1) = \sqrt{\frac{\psi_{21}}{\gamma(K_1)}}$ minimizing $\Delta_{12}(K_1, T_1)$, for any possible value of K_1. Substituting equation (3.36) into $\Delta_{12}(K_1, T_1)$ from equation (3.34) gives;

$$\Delta_{12}(K_1) = \frac{\psi_{21}}{\sqrt{\frac{\psi_{21}}{\gamma(K_1)}}} + \sqrt{\frac{\psi_{21}}{\gamma(K_1)}}\gamma(K_1) + \psi_{25} = 2\sqrt{\psi_{21}\gamma(K_1)} + \psi_{25}. \tag{B.4}$$

The outcome of the equation, represents the probable minimal cost for every quantity of K_1. In this regard, the crucial point that $\Delta_{11}(K_1)$ is continuous in the closed interval of [0, 1] has to be taken into account. This is the reason for the fact that it has one or more local minimal quantity, the least of which is the optimum answer for the cost function. Furthermore, for finding the optimum solutions, the first and the second derivations of $\Delta_{12}(K_1)$ with respect to K_1 are determined as discussed in (B.5) and (B.6).

$$\frac{d\Delta_{11}(K_1)}{dK_1} = \sqrt{\psi_{21}}\frac{\gamma^{'}(K_1)}{\sqrt{\gamma(K_1)}} \tag{B.5}$$

$$\frac{d^2\Delta_{11}(K_1)}{dK^2} = \frac{\sqrt{\psi_{21}}\left[2\gamma^{''}(K_1)\gamma(K_1) - \left(\gamma^{'}(K_1)\right)^2\right]}{2\left(\gamma(K_1)\right)^{\frac{3}{2}}}. \tag{B.6}$$

It can be inferred, for all possible quantities of K1, that:

$$\frac{d^2\Delta_{11}(K_1)}{dK_1^2}=\frac{\sqrt{\psi_{21}}\left[2\gamma^{''}(K_1)\gamma(K_1)-\left(\gamma^{'}(K_1)\right)^2\right]}{2\left(\gamma(K_1)\right)^{\frac{3}{2}}}=\frac{\sqrt{\psi_{21}}\left[(2\psi_{22})\left(\psi_{22}K_1^2-2\psi_{23}K_1+\psi_{24}\right)-(\psi_{22}K_1-\psi_{23})^2\right]}{\left(\gamma(K_1)\right)^{\frac{3}{2}}}$$
$$=\frac{\sqrt{\psi_{21}}\left[2\left(\psi_{22}^2K_1^2-2\psi_{22}\psi_{23}K_1+\psi_{22}\psi_{24}\right)-\left(\psi_{22}^2K_1-2\psi_{22}\psi_{23}K_1+\psi_{23}^2\right)\right]}{\left(\gamma(K_1)\right)^{\frac{3}{2}}}$$
$$=\frac{\sqrt{\psi_{21}}\left[\left(\psi_{22}^2K_1^2-2\psi_{22}\psi_{23}K_1+\psi_{22}\psi_{24}\right)+\psi_{22}\psi_{23}-\psi_{23}^2\right]}{\left(\gamma(K_1)\right)^{\frac{3}{2}}}$$
$$=\frac{\sqrt{\psi_{21}}\left[\psi_{22}\left(\psi_{22}K^2-2\psi_{23}K+\psi_{24}\right)+(\psi_{22}-\psi_{23})\psi_{23}\right]}{\left(\gamma(K)\right)^{\frac{3}{2}}}=\frac{\sqrt{\psi_{21}}\left[\psi_{22}\gamma(K)+(\psi_{22}-\psi_{23})\psi_{23}\right]}{\left(\gamma(K)\right)^{\frac{3}{2}}}>0. \quad \text{(B.7)}$$

As $\psi_{11},\psi_{12},\psi_{13},\gamma(K_1)$ own positive quantities all the time, and also $(\psi_{22}-\psi_{23})=\left(\frac{1}{2}h_1D+\frac{\pi D}{2}+\frac{\alpha CI_cD}{2}+\frac{D(1-\alpha)CI_e}{2}\right)-\frac{\pi D}{2}=\left(\frac{1}{2}h_1D+\frac{\alpha CI_cD}{2}+\frac{D(1-\alpha)CI_e}{2}\right)>0$ is positive, it is easily accepted that $\Delta_{11}(K_1)$ is a convex function. Consequently, a globally optimum answer can be determined from setting the first derivative of $\Delta_{11}(K_1)$ equal to zero, as shown in equation (B.5).

$$2\psi_{22}K_1-2\psi_{23}=0. \quad \text{(B.8)}$$

Gives;

$$K_1^*=\frac{\psi_{23}}{\psi_{22}}=\frac{\pi}{h_1+\pi+\alpha CI_c+(1-\alpha)\,CI_e}\cdot \quad \text{(B.9)}$$

At this stage, substitution of K_1^* in equation (B.2), after the subsequent simplification, paves the way for obtaining optimal value of the considered period.

$$T_1^*=T_1^*\left(K_1^*\right)=\sqrt{\frac{2A}{\left(h_1D+\pi D+\alpha CI_cD+D\left(1-\alpha\right)CI_e\right)\left(\frac{\pi}{h_1+\pi+\alpha CI_c+(1-\alpha)CI_e}\right)^2-2\pi D\left(\frac{\pi}{h_1+\pi+\alpha CI_c+(1-\alpha)CI_e}\right)+\pi D}}. \quad \text{(B.10)}$$

If the solution obtained by using equations (3.31) and (3.32) results in $M<K_1T_1$, then a logical solution is to set $K_1=\frac{M}{T_1}$. However, we should then expect that not only will K be equal to K_1, but T will also change. To determine the optimal T for Case2 when $K_1=\frac{M}{T_1}$, we can do the following. Substituting $K_1=\frac{M}{T_1}$ in equation (3.25) gives:

$$\Delta_{12}(T_1)=\frac{\psi_{21}}{T_1}+(\psi_{22}\frac{M^2}{T_1^2}-2\psi_{23}\frac{M}{T_1}+\psi_{24})T_1+\psi_{25}=\frac{\psi_{22}M^2+\psi_{21}}{T_1}+\psi_{24}T_1+\psi_{25}-2\psi_{23}M. \quad \text{(B.11)}$$

Considering the derivative with respect to T and setting it equal to 0 gives:

$$\frac{d\Delta_{12}(T_1)}{dT_1}=-\frac{\psi_{22}M^2+\psi_{21}}{T_1^2}+\psi_{24}=0. \quad \text{(B.12)}$$

This gives:

$$T_1^{'}=\sqrt{\frac{\psi_{22}M^2+\psi_{21}}{\psi_{24}}}\cdot \quad \text{(B.13)}$$

And

$$K_1^{'}=\frac{M}{T_1^{'}}\cdot \quad \text{(B.14)}$$

Appendix C.

Proving convexity of the objective function and determining T_2^* and K_2^* for case1

First partial derivation of the function $\Delta_{21}(K_2, T_2)$ with respect to T_2 is as follows;

$$\frac{\partial \Delta_{21}(K_2, T_2)}{\partial T_2} = -\frac{\psi_{31}}{T_2^2} + \gamma(K_2). \tag{C.1}$$

Equating the outcome to zero will result in;

$$T_2^* = T_2^*(K_2) = \sqrt{\frac{\psi_{31}}{\gamma(K_2)}}. \tag{C.2}$$

In which $\gamma(K_2) = \psi_{32}K_2^2 - 2\psi_{33}K_2 + \psi_{34}$, and the relative discriminant is derived from the equation (C.2).

$$\begin{aligned}\Delta &= b^2 - 4ac = 4\psi_{33}^2 - 4\psi_{32}\psi_{34}\\ &= 4\left(\frac{\pi D}{2}\right)^2 - 4\left(\frac{D}{2}(h_1 + \pi + CI_c)\right)\left(\frac{\pi D}{2} + h_2\frac{(n-1)D}{2} + h_3\frac{rnD}{2}\right)\\ &= -\left(D\left(\pi D + h_2(n-1)D + h_3 rnD\right)(h_1 + CI_c)\right) < 0.\end{aligned} \tag{C.3}$$

Considering the fact that Δ is negative in all situations, no root can be determined for $\gamma(K_2)$, promising that it will be never equal to zero. Additionally, since $\gamma(0) = \left(\frac{\pi D}{2} + h_2\frac{(n-1)D}{2} + h_3\frac{rnD}{2}\right) > 0$, it is logical to say $\gamma(K_2)$ owns a positive quantity in interval of [0, 1] in all conditions. As a result, equation (C.2) determines a unique $T_2^* = T_2^*(K_2) = \sqrt{\frac{\psi_{31}}{\gamma(K_2)}}$ minimizing $\Delta_{21}(K_2, T_2)$, for any possible value of K_2. Substituting the above equation into $\Delta_{21}(K_2, T_2)$ in equation (3.41) results in;

$$\begin{aligned}\Delta_{21}(K_2) = \Delta_{21}(T_2^*(K_2), K_2) &= \frac{\psi_{31}}{\sqrt{\frac{\psi_{31}}{\gamma(K_2)}}} + \gamma(K_2)\sqrt{\frac{\psi_{31}}{\gamma(K_2)}} + \psi_{35}K_2 + \psi_{36}\\ &= 2\sqrt{\psi_{31}\gamma(K_2)} + \psi_{35}K_2 + \psi_{36}.\end{aligned} \tag{C.4}$$

The outcome of the equation, represents the probable minimal cost for every quantity of K_2. In this regard, the crucial point that $\Delta_{21}(K_2)$ is continuous in the closed interval of [0, 1] has to be taken into account. This is the reason for the fact that it has one or more local minimal quantity, the least of which is the optimum answer for the cost function. Furthermore, for finding the optimum answer, first and second derivations of $\Delta_{21}(K_2)$ with respect to K_2 are determined as discussed in (C.5) and (C.6).

$$\frac{d\Delta_{21}(K_2)}{dK_2} = \sqrt{\psi_{31}}\frac{\gamma'(K_2)}{\sqrt{\gamma(K_2)}} + \psi_{35} \tag{C.5}$$

$$\frac{d^2\Delta_{21}(K_2)}{dK_2^2} = \frac{\sqrt{\psi_{31}}\left[2\gamma''(K_2)\gamma(K_2) - \left(\gamma'(K_2)\right)^2\right]}{2\left(\gamma(K_2)\right)^{\frac{3}{2}}}. \tag{C.6}$$

It can be inferred, for all possible quantities of K_2, that:

$$\frac{d^2\Delta_{21}(K_2)}{dK_2^2} = \frac{\sqrt{\psi_{31}}\left[2\gamma''(K_2)\gamma(K_2) - \left(\gamma'(K_2)\right)^2\right]}{2\left(\gamma(K_2)\right)^{\frac{3}{2}}} = \frac{\sqrt{\psi_{31}}\left[(2\psi_{32})\left(\psi_{32}K_2^2 - 2\psi_{33}K_2 + \psi_{34}\right) - (\psi_{32}K_2 - \psi_{33})^2\right]}{(\gamma(K_2))^{\frac{3}{2}}}$$
$$= \frac{\sqrt{\psi_{31}}\left[2\left(\psi_{32}^2K_2^2 - 2\psi_{32}\psi_{33}K_2 + \psi_{32}\psi_{34}\right) - \left(\psi_{32}^2K_2 - 2\psi_{32}\psi_{33}K_2 + \psi_{33}^2\right)\right]}{(\gamma(K_2))^{\frac{3}{2}}}$$
$$= \frac{\sqrt{\psi_{31}}\left[\left(\psi_{32}^2K_2^2 - 2\psi_{32}\psi_{33}K_2 + \psi_{32}\psi_{34}\right) + \psi_{32}\psi_{33} - \psi_{33}^2\right]}{(\gamma(K_2))^{\frac{3}{2}}}$$
$$= \frac{\sqrt{\psi_{31}}\left[\psi_{32}\left(\psi_{32}K_2^2 - 2\psi_{33}K_2 + \psi_{34}\right) + (\psi_{32} - \psi_{33})\psi_{33}\right]}{(\gamma(K_2))^{\frac{3}{2}}} = \frac{\sqrt{\psi_{31}}\left[\psi_{32}\gamma(K_2) + (\psi_{32} - \psi_{33})\psi_{33}\right]}{(\gamma(K_2))^{\frac{3}{2}}} > 0. \tag{C.7}$$

As $\psi_{31}, \psi_{32}, \psi_{33}, \gamma(K_2)$ own positive quantities all the time, and also $(\psi_{32} - \psi_{33}) = \left(\frac{1}{2}h_1D + \frac{\pi D}{2} + \frac{CI_cD}{2}\right) - \frac{\pi D}{2} = \frac{1}{2}\left(h_1D + CI_cD\right) > 0$ is positive, it is easily accepted that $\Delta_{21}(K_2)$ is a convex function. Consequently, a globally optimum answer can be determined from setting the first derivative of $\Delta_{21}(K_2)$ equal to zero, as shown in equation (C.5).

$$\frac{d\Delta_{21}(K_2)}{dK_2} = \sqrt{\psi_{31}}\frac{2\psi_{32}K_2 - 2\psi_{33}}{\sqrt{\psi_{32}K_2^2 - 2\psi_{33}K_2 + \psi_{34}}} + \psi_{35} = \sqrt{\frac{\psi_{31}}{\psi_{32}K_2^2 - 2\psi_{33}K_2 + \psi_{34}}}\left(2\psi_{32}K_2 - 2\psi_{33}\right) + \psi_{35}$$
$$= T_2^*\left(2\psi_{32}K_2 - 2\psi_{33}\right) + \psi_{35} = T_2^*\left(D\left(h_1 + \pi + CI_c\right)K_2 - \pi D\right) + \left(CI_eD\left(1-\alpha\right)M\right) = 0. \tag{C.8}$$

It can be inferred that;

$$K_2^* = \frac{\pi T_2^* - \left(CI_e\left(1-\alpha\right)M\right)}{T_2^*\left(h_1 + \pi + CI_c\right)}. \tag{C.9}$$

At this stage, substitution of K_2^* in equation (C.2), after the subsequent simplification, paves the way for obtaining optimal value of the considered period.

$$T_2^* = T_2^*\left(K_2\right) = \sqrt{\frac{\left(A + F + \frac{A_2}{n_2} + \frac{A_3}{n_2}\right)}{\frac{D}{2}\frac{\left(\pi T_2^* - \left(CI_e(1-\alpha)M\right)\right)^2}{T_2^{*2}(h_1+\pi+CI_c)} - \pi D\left(\frac{\pi T_2^* - \left(CI_e(1-\alpha)M\right)}{T_2^*(h_1+\pi+CI_c)}\right) + \left(\frac{\pi D}{2} + h_2\frac{(n-1)D}{2} + h_3\frac{rnD}{2}\right)}}. \tag{C.10}$$

After simplificatoins we have:

$$T_{21}^* = T_2^* = \sqrt{\frac{2D\left(h_1 + \pi + CI_c\right)\left(A_b + F + \frac{A_2^V}{n_2} + \frac{A_3^V}{n_2} + \frac{DM^2(1-\alpha)C(I_c-I_e)}{2}\right) - \left[\left(CD\left(1-\alpha\right)M\right)\left(I_e - I_c\right)\right]^2}{D^2\left(h_1 + \pi + CI_c\right)\left(\pi + h_2\left(n_2 - 1\right) + h_3rn_2\right) - \pi^2D^2}} \tag{C.11}$$

$$K_{21}^* = K_2^* = \frac{\pi}{\left(h_1 + \pi + CI_c\right)} - \frac{2I_e\left(CD\left(1-\alpha\right)M\right)}{\left(h_1 + \pi + CI_c\right)}$$
$$\sqrt{\frac{D^2\left(h_1 + \pi + CI_c\right)\left(\pi + h_2\left(n_2 - 1\right) + h_3rn_2\right) - \pi^2D^2}{8D\left(h_1 + \pi + CI_c\right)\left(A_B + F + \frac{A_2^V}{n_2} + \frac{A_3^V}{n_2} + \frac{DM^2(1-\alpha)C(I_c-I_e)}{2}\right) - \left[2I_e\left(CD\left(1-\alpha\right)M\right)\right]^2}}. \tag{C.12}$$

If the solution obtained by using equations (C.11) and (C.12) results in $M > K_2T_2$, then a logical solution is to set $K_2 = \frac{M}{T_2}$. But, we should then await that not only will K be equal to K_2, but T will also change. To set the optimal T for Case1 when $K_2 = \frac{M}{T_2}$, we can do the following.

Substituting $K_2 = \frac{M}{T_2}$ in equation (3.41) gives:

$$\Delta_{21}(T_2) = \frac{\psi_{31}}{T_2} + (\psi_{32}\frac{M^2}{T_2^2} - 2\psi_{33}\frac{M}{T_2} + \psi_{34})T_2 + \psi_{35}\frac{M}{T_2} + \psi_{36}$$
$$= \frac{\psi_{32}M^2 + \psi_{35}M + \psi_{31}}{T_2} + \psi_{34}T_2 + \psi_{36} - 2\psi_{33}M. \tag{C.13}$$

Considering the derivative with respect to T and setting it equivalent to 0 gives:

$$\frac{d\Delta_{21}(T_2)}{dT_2} = -\frac{\psi_{32}M^2 + \psi_{35}M + \psi_{31}}{T_2^2} + \psi_{34} = 0. \tag{C.14}$$

This gives:

$$T_2' = \sqrt{\frac{\psi_{32}M^2 + \psi_{35}M + \psi_{31}}{\psi_{34}}} \tag{C.15}$$

And

$$K_2' = \frac{M}{T_2'}. \tag{C.16}$$

An analogous process may be considered for solving the second case.

Appendix D.

Proving convexity of the objective function and determining T_2^* and K_2^* for case2
First partial derivation of the function $\Delta_{22}(K_2, T_2)$ with respect to $T_2$is as follows;

$$\frac{\partial \Delta_{22}(K_2, T_2)}{\partial T_2} = -\frac{\psi_{41}}{T_2^2} + \gamma(K_2). \tag{D.1}$$

Equating the outcome to zero will result in;

$$T^* = T^*(K_2) = \sqrt{\frac{\psi_{41}}{\gamma(K_2)}}. \tag{D.2}$$

In which $\gamma(K_2) = \psi_{42}K_2^2 - 2\psi_{43}K_2 + \psi_{44}$, and the relative discriminant is derived from the equation (D.2).

$$\Delta = b^2 - 4ac = 4\psi_{43}^2 - 4\psi_{42}\psi_{44}$$
$$= 4\left(\frac{\pi D}{2}\right)^2 - 4\left(\frac{1}{2}h_1 D + \frac{\pi D}{2} + \frac{\alpha C I_c D}{2} + \frac{D(1-\alpha) C I_e}{2}\right)\frac{D}{2}(\pi + h_2(n-1) + h_3 rn)$$
$$= -(h_1 D + \alpha C I_c D + D(1-\alpha) C I_e) D (\pi + h_2(n-1) + h_3 rn) < 0. \tag{D.3}$$

Considering the fact that Δ is negative in all situations, no root can be determined for $\gamma(K_2)$, promising that it will be never equal to zero. Additionally, since $\gamma(0) = \frac{D}{2}(\pi + h_2(n-1) + h_3 rn) > 0$, it is logical to say $\gamma(K_2)$ owns a positive quantity in interval of [0, 1] in all conditions. As a result, equation (D.2) determines a unique $T^* = T^*(K_2) = \sqrt{\frac{\psi_{41}}{\gamma(K_2)}}$ minimizing $\Delta_{22}(K_2, T_2)$, for any possible value of K_2. Substituting the above Equation into $\Delta_{22}(K_2, T_2)$ from equation (3.50) gives;

$$\Delta_{22}(K_2) = \frac{\psi_{41}}{\sqrt{\frac{\psi_{41}}{\gamma(K_2)}}} + \sqrt{\frac{\psi_{41}}{\gamma(K_2)}}\gamma(K_2) + \psi_{45} = 2\sqrt{\psi_{41}\gamma(K_2)} + \psi_{45}. \tag{D.4}$$

The outcome of the equation, represents the probable minimal cost for every quantity of K_2. In this regard, the crucial point that $\Delta_{22}(K_2)$ is continuous in the closed interval of [0, 1] has to be taken into account. This is the reason for the fact that it has one or more local minimal quantity, the least of which is the optimum answer for the cost function. Furthermore, for finding the optimum answer, the first and the second derivations of $\Delta_{22}(K_2)$ with respect to K_2 are determined as discussed in (D.5) and (D.6).

$$\frac{d\Delta_{22}(K_2)}{dK_2} = \sqrt{\psi_{41}}\frac{\gamma^{'}(K_2)}{\sqrt{\gamma(K_2)}} \tag{D.5}$$

$$\frac{d^2\Delta_{22}(K_2)}{dK_2^2} = \frac{\sqrt{\psi_{41}}\left[2\gamma^{''}(K_2)\gamma(K_2) - \left(\gamma^{'}(K_2)\right)^2\right]}{2\left(\gamma(K_2)\right)^{\frac{3}{2}}}. \tag{D.6}$$

It can be inferred, for all possible quantities of K_2, that:

$$\begin{aligned}\frac{d^2\Delta_{22}(K_2)}{dK_2^2} &= \frac{\sqrt{\psi_{41}}\left[2\gamma^{''}(K_2)\gamma(K_2) - \left(\gamma^{'}(K_2)\right)^2\right]}{2\left(\gamma(K_2)\right)^{\frac{3}{2}}} = \frac{\sqrt{\psi_{41}}\left[(2\psi_{42})\left(\psi_{42}K_2^2 - 2\psi_{43}K_2 + \psi_{44}\right) - (\psi_{42}K_2 - \psi_{43})^2\right]}{(\gamma(K_2))^{\frac{3}{2}}}\\ &= \frac{\sqrt{\psi_{41}}\left[2\left(\psi_{42}^2K_2^2 - 2\psi_{42}\psi_{43}K_2 + \psi_{42}\psi_{44}\right) - \left(\psi_{42}^2K_2 - 2\psi_{42}\psi_{43}K_2 + \psi_{43}^2\right)\right]}{(\gamma(K_2))^{\frac{3}{2}}}\\ &= \frac{\sqrt{\psi_{41}}\left[\left(\psi_{42}^2K_2^2 - 2\psi_{42}\psi_{43}K_2 + \psi_{42}\psi_{44}\right) + \psi_{42}\psi_{43} - \psi_{43}^2\right]}{(\gamma(K_2))^{\frac{3}{2}}}\\ &= \frac{\sqrt{\psi_{41}}\left[\psi_{42}\left(\psi_{42}K_2^2 - 2\psi_{43}K_2 + \psi_{44}\right) + (\psi_{42} - \psi_{43})\psi_{43}\right]}{(\gamma(K_2))^{\frac{3}{2}}} = \frac{\sqrt{\psi_{41}}\left[\psi_{42}\gamma(K_2) + (\psi_{42} - \psi_{43})\psi_{43}\right]}{(\gamma(K_2))^{\frac{3}{2}}} > 0.\end{aligned} \tag{D.7}$$

As $\psi_{41}, \psi_{42}, \psi_{43}, \gamma(K_2)$ own positive quantities all the time, and also $(\psi_{42} - \psi_{43}) = \left(\frac{1}{2}h_1D + \frac{\pi D}{2} + \frac{\alpha CI_cD}{2} + \frac{D(1-\alpha)CI_e}{2}\right) - \frac{\pi D}{2} = \left(\frac{1}{2}h_1D + \frac{\alpha CI_cD}{2} + \frac{D(1-\alpha)CI_e}{2}\right) > 0$ is positive, it is easily accepted that $\Delta_{22}(K_2)$ is a convex function. Consequently, a globally optimum answer can be determined from setting the first derivative of $\Delta_{22}(K_2)$ equal to zero, as shown in equation (D.5).

$$2\psi_{42}K_2 - 2\psi_{43} = 0. \tag{D.8}$$

Gives;

$$K_2^* = \frac{\psi_{43}}{\psi_{42}} = \frac{\pi}{h_1 + \pi + \alpha CI_c + (1-\alpha)\,CI_e}. \tag{D.9}$$

At this stage, substitution of K_2^* in equation (D.2), after the subsequent simplification, paves the way for obtaining optimal value of the considered period.

$$T_2^* = T_2^*\left(K_2\right) = \sqrt{\frac{2\left(A + F + \frac{A_2}{n} + \frac{A_3}{n}\right)}{\left(\pi D + h_2\left(n-1\right)D + h_3rnD\right) - \pi D\left(\frac{\pi}{h_1+\pi+\alpha CI_c+(1-\alpha)CI_e}\right)}}. \tag{D.10}$$

If the solution obtained by using equations (D.9) and (D.10) results in $M < K_2T_2$, afterward a logical solution is to set $K_2 = \frac{M}{T_2}$. However, we should then expect that not only will K be equal to K_2, but T will also change. To determine the optimal T for Case2 when $K_2 = \frac{M}{T_2}$, we can do the following.

Substituting $K_2 = \frac{M}{T_2}$ in equation (3.55) gives:

$$\Delta_{22}(T_2) = \frac{\psi_{42}M^2 + \psi_{41}}{T_2} + \psi_{44}T_2 + \psi_{45} - 2\psi_{43}M. \tag{D.11}$$

Considering the derivative with regard to T and setting it equal to 0 gives:

$$\frac{d\Delta_{22}(T_2)}{dT_2} = -\frac{\psi_{42}M^2 + \psi_{41}}{T_2^2} + \psi_{44} = 0. \tag{D.12}$$

This gives:

$$T_2^{'} = \sqrt{\frac{\psi_{42}M^2 + \psi_{41}}{\psi_{44}}}. \tag{D.13}$$

And

$$K_2^{'} = \frac{M}{T_2^{'}}. \tag{D.14}$$

Appendix E.

Finding the length of inventory periods

If we assume that $K_{11} = 1$, from equation (3.11), we have:

$$\Delta_{11}^{\#}(K_1, T_1) = \Delta_{11}^{\#}(1, T_1) = \frac{\psi_{11}}{T_1} + (\psi_{12} - 2\psi_{13} + \psi_{14})T_1 + \psi_{15} + \psi_{16}. \tag{E.1}$$

Taking the first derivative of $\Delta_{11}^{\#}(T_1)$ with respect to T_1 and setting it equal to zero gives:

$$\frac{d\Delta_{11}^{\#}(T_1)}{dT_{1-1}} = (\psi_{12} - 2\psi_{13} + \psi_{14}) - \frac{\psi_{11}}{T_1^2}. \tag{E.2}$$

The above equation leads to:

$$T_1^{\#} = \sqrt{\frac{\psi_{11}}{(\psi_{12} - 2\psi_{13} + \psi_{14})}}. \tag{E.3}$$

Similarly for the other cases, $T^{\#}$ can be obtained when $K_{12} = 1$ Therefore, we have:

$$T_{12}^{\#} = \sqrt{\frac{\psi_{21}}{\psi_{22} - 2\psi_{23} + \psi_{24}}}. \tag{E.4}$$

Appendix F.

Solution procedure for model 2

Step1. Obtain $(n_{21}^*, K_{21}^*, T_{21}^*)$ and $(n_{22}^*, K_{22}^*, T_{22}^*)$.

A. Determine $(n_{21}^*, K_{21}^*, T_{21}^*)$ using following sub-steps.

A1.Frist set $n_{21} = 1$ and calculate $\psi_{31-}\psi_{36}$, from equations (3.34) to (3.39).

A2. Obtain $K_{21}\left(n_{21}\right)$ *and* $T_{21}\left(n_{21}\right)$ *via* equations (3.41) and (3.42). If $M < K_{21}\left(n_{21}\right)T_{21}\left(n_{21}\right)$ go to step A3, Otherwise obtain $K_{21}\left(n_{21}\right)$ *and* $T_{21}\left(n_{21}\right)$ *via* equations (B.11) and (B.12) and go to step A3.

A3. If $K_{21}\left(n_{21}\right) \leq 1$, go to step A5; otherwise, go to step A4.

A4. Set $K_{21}\left(n_{21}\right) = 1$ and determine $T_{21}^{\#}\left(n_{21}\right)$ *via* equation (E.4) in Appendix E. If $M < K_{21}\left(n_{21}\right)T_{21}\left(n_{21}\right)$, set $\left(K_{21}\left(n_{21}\right), T_{21}\left(n_{21}\right)\right) = \left(1, T_{21}^{\#}\left(n_{21}\right)\right)$ and go to step A5; if not, set $\left(K_{21}\left(n_{21}\right), T_{21}\left(n_{21}\right)\right) = (1, M)$ and go to step A5.

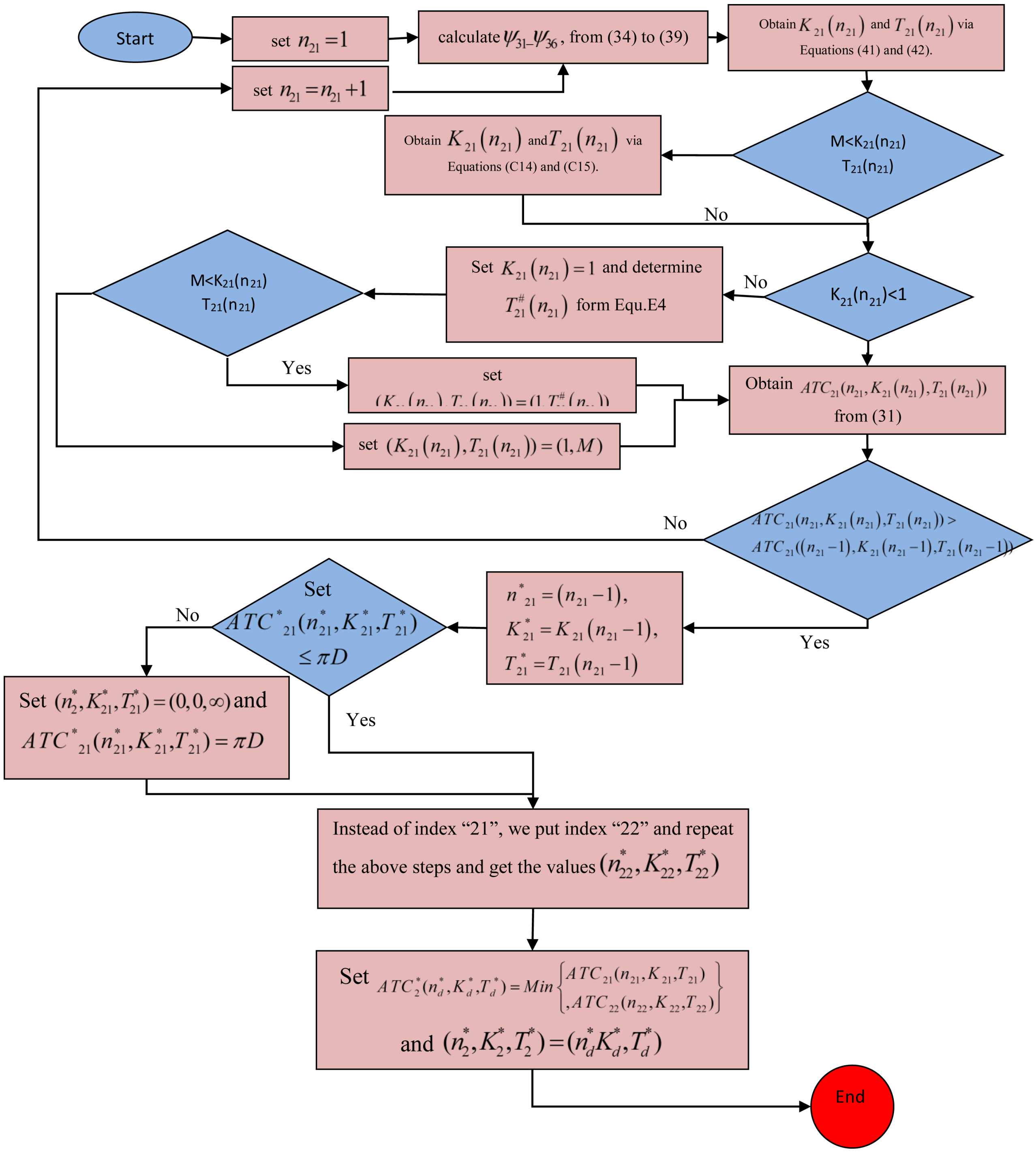

FIGURE F.1. Flowchart for Solution procedure.

A5. Obtain $ATC_{21}(n_{21}, K_{21}(n_{21}), T_{21}(n_{21}))$ *via* equation (3.31),

A6. Set $n_{21} = n_{21} + 1$ and calculate $\psi_{31-}\psi_{36}$, from equations (3.34) to (3.39).

A7. Obtain $K_{21}(n_{21})$ *and* $T_{21}(n_{21})$ *via* equations (3.41) and (3.42). If $M < K_{21}(n_{21}) T_{21}(n_{21})$ go to step A8, Otherwise obtain $K_{21}(n_{21})$ *and* $T_{21}(n_{21})$ *via* equations (C.13) and (C.14) and go to step A8.

A8. If $K_{21}(n_{21}) \leq 1$, go to step A10; otherwise, go to step A9.

A9. Set $K_{21}(n_{21}) = 1$ and determine $T_{21}^{\#}$ *via* equation (E.4) in Appendix E. If $M < K_{21}(n_{21}) T_{21}(n_{21})$, set $(K_{21}(n_{21}), T_{21}(n_{21})) = (1, T_{21}^{\#}(n_{21}))$ and go to step A10; if not, set $(K_{21}(n_{21}), T_{21}(n_{21})) = (1, M)$ and go to step A10.

A10. Obtain $ATC_{21}(n_{21}, K_{21}(n_{21}), T_{21}(n_{21}))$ *via* equation (3.31),

A11. If $ATC_{21}(n_{21}, K_{21}(n_{21}), T_{21}(n_{21})) > ATC_{21}((n_{21}-1), K_{21}(n_{21}-1), T_{21}(n_{21}-1))$ then the optimal values are $n_{21}^* = (n_{21}-1)$, $K_{21}^* = K_{21}(n_{21}-1)$, $T_{21}^* = T_{21}(n_{21}-1)$ and go to step A12. Otherwise, return to step A6.

A12. If $ATC_{21}^*(n_{21}^*, K_{21}^*, T_{21}^*) \leq \pi D$, go to step B; otherwise, go to step A13.

A13. Set $(n_2^*, K_{21}^*, T_{21}^*) = (0, 0, \infty)$ and $ATC_{21}(K_{21}^*, T_{21}^*) = \pi D$, then go to step B.

B. Determine $(n_{22}^*, K_{22}^*, T_{22}^*)$ using following sub-steps.

We calculated the optimum value of $(n_{22}^*, K_{22}^*, T_{22}^*)$ in Thirteen different parts in section B

B1.Frist set $n_{22} = 1$ and calculate $\psi_{41-}\psi_{45}$, from equations (3.44) to (3.48).

B2. Obtain $K_{22}(n_{22})$ *and* $T_{22}(n_{22})$ *via* equations (3.50) and (3.51). If $M < K_{22}(n_{22}) T_{22}(n_{22})$ go to step B3, Otherwise obtain $K_{22}(n_{22})$ *and* $T_{22}(n_{22})$ *via* equations (D.11) and (D.12) and go to step B3.

B3. If $K_{22}(n_{22}) \leq 1$, go to step B5; otherwise, go to step B4.

B4. Set $K_{22}(n_{22}) = 1$ and determine $T_{22}^{\#}(n_{22})$ *via* equation (E5) in Appendix E. If $M < K_{22}(n_{22}) T_{22}(n_{22})$, set $(K_{22}(n_{22}), T_{22}(n_{22})) = (1, T_{22}^{\#}(n_{22}))$ and go to step B5; if not, seta and go to step B5.

B5. Obtain $ATC_{22}(n_{22}, K_{22}(n_{22}), T_{22}(n_{22}))$ *via* equation (3.36),

B6. Set $n_{22} = n_{22} + 1$ and calculate $\psi_{41-}\psi_{45}$, from equations (3.44) to (3.48).

B7. Obtain $K_{22}(n_{22})$ *and* $T_{22}(n_{22})$ *via* equations (3.50) and (3.51). If $M < K_{22}(n_{22}) T_{22}(n_{22})$ go to step B8, Otherwise obtain $K_{22}(n_{22})$ *and* $T_{22}(n_{22})$ *via* equations (D.11) and (D.12) and go to step B8.

B8. If $K_{22}(n_{22}) \leq 1$, go to step B10; otherwise, go to step B9.

B9. Set $K_{22}(n_{22}) = 1$ and determine $T_{22}^{\#}(n_{22})$ i equation (E5) in appendix E. If $M < K_{22}(n_{22}) T_{22}(n_{22})$, set $(K_{22}(n_{22}), T_{22}(n_{22})) = (1, T_{22}^{\#}(n_{22}))$ and go to step B10; if not, set $(K_{22}(n_{22}), T_{22}(n_{22})) = (1, M)$ and go to step B10.

B10. Obtain $ATC_{22}(n_{22}, K_{22}(n_{22}), T_{22}(n_{22}))$ *via* equation (3.33),

B11. If $ATC_{22}(n_{22}, K_{22}(n_{22}), T_{22}(n_{22})) > ATC_{22}((n_{22}-1), K_{22}(n_{22}-1), T_{22}(n_{22}-1))$ then the optimal values are $n_{22}^* = (n_{22}-1)$, $K_{22}^* = K_{22}(n_{22}-1)$, $T_{22}^* = T_{22}(n_{22}-1)$ and go to step B12. Otherwise, return to step B6.

B12. If $ATC_{22}^*(n_{22}^*, K_{22}^*, T_{22}^*) \leq \pi D$, go to step B; otherwise, go to step B13.

B13. Set $(n_{22}^*, K_{22}^*, T_{22}^*) = (0, 0, \infty)$ and $ATC_{22}(K_{22}^*, T_{22}^*) = \pi D$, then go to step 2.

Step2. Determine the optimal policy.

C1-Set $ATC_2^*(n_d^*, K_d^*, T_d^*) = Min\{ATC_{21}(n_{21}, K_{21}, T_{21}), ATC_{22}(n_{22}, K_{22}, T_{22})\}$ and $(n_2^*, K_2^*, T_2^*) = (n_d^* K_d^*, T_d^*)$.

C2-End.

Appendix G.

Solution procedure for model 1

Step1. Obtain (K_{11}^*, T_{11}^*) and (K_{12}^*, T_{12}^*).

A. Determine (K_{11}^*, T_{11}^*) using following sub-steps.

A1. Calculate $\psi_{11-}\psi_{16}$, from equations (3.12) to (3.17).

A2. Obtain $K_{11}\, and\, T_{11}$ *via* equations (3.19) and (3.20). If $M < K_{11}T_{11}$ go to step A3, Otherwise obtain $K_{11}\, and\, T_{11}$ *via* equations (3.21) and (3.22) and go to step A3.

A3. If $K_{11} \leq 1$, go to step A5; otherwise, go to step A4.

A4. Set $K_{11} = 1$ and determine $T_{11}^{\#}$ *via* equation (E.3) in Appendix E. If $M < K_{11}T_{11}$, set $(K_{11}^{*}, T_{11}^{*}) = (1, T_{11}^{\#})$ and go to step A5; if not, set $(K_{11}^{*}, T_{11}^{*}) = (1, M)$ and go to step A5.

A5. Obtain $ATB_{11}(K_{11}^{*}, T_{11}^{*})$ *via* equation (3.9). If $ATB_{11}(K_{11}^{*}, T_{11}^{*}) \leq \pi D$, go to step B; otherwise, go to step A6.

A6. Set $(K_{11}^{*}, T_{11}^{*}) = (0, \infty)$ and $ATB_{11}(K_{11}^{*}, T_{11}^{*}) = \pi D$, then go to step B.

B. Determine (K_{12}^{*}, T_{12}^{*}) using following sub-steps.

We calculated the optimum value of (K_{12}^{*}, T_{12}^{*}) in six different parts in Section B

B1. Calculate $\psi_{21-}\psi_{26}$, from equations (3.22) to (3.26).

B2. Obtain $K_{12}\, and\, T_{12}$ *via* equations (3.27) and (3.28). If $M > K_{12}T_{12}$ go to step B3, Otherwise obtain $K_{12}\, and\, T_{12}$ *via* equations (3.31) and (3.32) and go to step B3.

B3. If $K_{12} \leq 1$, go to step B5; otherwise, go to step B4.

B4. Set $K_{12} = 1$ and determine $T_{12}^{\#}$ *via* equation (E.4) in Appendix E. If $M > K_{12}T_{12}$, set $(K_{11}^{*}, T_{11}^{*}) = (1, T_{11}^{\#})$ and go to step (B.5); if not, set $(K_{11}^{*}, T_{11}^{*}) = (1, M)$ and go to step (B.5).

B5. Obtain $ATB_{12}(K_{12}^{*}, T_{12}^{*})$ *via* equation (3.10). If $ATB_{12}(K_{12}^{*}, T_{12}^{*}) \leq \pi D$, go to step 2; otherwise, go to step B6.

B6. Set $(K_{12}^{*}, T_{12}^{*}) = (0, \infty)$ and $ATB_{12}(K_{12}^{*}, T_{12}^{*}) = \pi D$, then go to step 2.

Step2. Determine the optimal policy.

C1-Set $ATB_{1}^{*}(K_{d}^{*}, T_{d}^{*}) = Min\,\{ATB_{11}(K_{11}^{*}, T_{11}^{*}), ATB_{12}(K_{12}^{*}, T_{12}^{*})\}$ and $(K_{1}^{*}, T_{1}^{*}) = (K_{d}^{*}, T_{d}^{*})$.

C2-End.

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